An Instructor's Solutions Manual to Accompany

SEVENTH EDITION

MECHANICS OF MATERIALS

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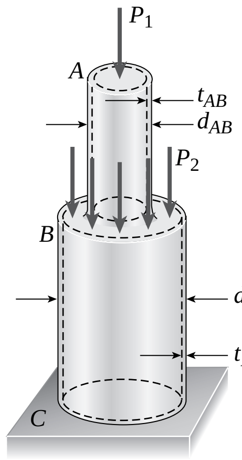
1

Tension, Compression, and Shear

Normal Stress and Strain

Problem 1.2-1 A hollow circular post *ABC* (see figure) supports a load $P_1 = 1700$ lb acting at the top. A second load P_2 is uniformly distributed around the cap plate at *B*. The diameters and thicknesses of the upper and lower parts of the post are $d_{AB} = 1.25$ in., $t_{AB} = 0.5$ in., $d_{BC} = 2.25$ in., and $t_{BC} = 0.375$ in., respectively.

(a) Calculate the normal stress σ_{AB} in the upper part of the post.
(b) If it is desired that the lower part of the post have the same compressive stress as the upper part, what should be the magnitude of the load P_2?
(c) If P_1 remains at 1700 lb and P_2 is now set at 2260 lb, what new thickness of *BC* will result in the same compressive stress in both parts?



Solution 1.2-1

PART (a)

$P_1 = 1700 \qquad d_{AB} = 1.25 \qquad t_{AB} = 0.5$

$d_{BC} = 2.25 \qquad t_{BC} = 0.375$

$$A_{AB} = \frac{\pi[\,d_{AB}^2 - (d_{AB} - 2t_{AB})^2]}{4}$$

$$A_{AB} = 1.178 \qquad \sigma_{AB} = \frac{P_1}{A_{AB}}$$

$$\sigma_{AB} = 1443 \text{ psi} \qquad \leftarrow$$

PART (b)

$$A_{BC} = \frac{\pi[\,d_{BC}^2 - (d_{BC} - 2t_{BC})^2]}{4}$$

$$A_{BC} = 2.209 \qquad P_2 = \sigma_{AB}A_{BC} - P_1$$

$$P_2 = 1488 \text{ lbs} \qquad \leftarrow$$

$$\text{CHECK:} \qquad \frac{P_1 + P_2}{A_{BC}} = 1443 \text{ psi}$$

Part (c)

$$P_2 = 2260 \qquad \frac{P_1 + P_2}{\sigma_{AB}} = A_{BC}$$

$$\frac{P_1 + P_2}{\sigma_{AB}} = 2.744$$

$$(d_{BC} - 2t_{BC})^2$$

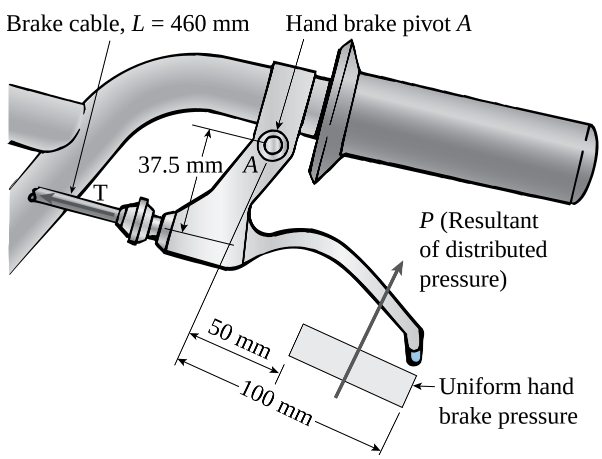
$$= d_{BC}{}^2 - \frac{4}{\pi}\left(\frac{P_1 + P_2}{\sigma_{AB}}\right)$$

$$d_{BC} - 2t_{BC} = \sqrt{d_{BC}{}^2 - \frac{4}{\pi}\left(\frac{P_1 + P_2}{\sigma_{AB}}\right)}$$

$$t_{BC} = \frac{d_{BC} - \sqrt{d_{BC}{}^2 - \frac{4}{\pi}\left(\frac{P_1 + P_2}{\sigma_{AB}}\right)}}{2}$$

$$t_{BC} = 0.499 \text{ inches} \quad \leftarrow$$

Problem 1.2-2 A force P of 70 N is applied by a rider to the front hand brake of a bicycle (P is the resultant of an evenly distributed pressure). As the hand brake pivots at A, a tension T develops in the 460-mm long brake cable ($A_e = 1.075 \text{ mm}^2$) which elongates by $\delta = 0.214$ mm. Find normal stress σ and strain ε in the brake cable.



Solution 1.2-2

$P = 70$ N $\qquad A_e = 1.075 \text{ mm}^2$

$L = 460$ mm $\qquad \delta = 0.214$ mm

Statics: sum moments about A to get $T = 2P$

$$\sigma = \frac{T}{A_e} \qquad \sigma = 103.2 \text{ MPa} \quad \leftarrow$$

$$\varepsilon = \frac{\delta}{L} \qquad \varepsilon = 4.65 \times 10^{-4} \quad \leftarrow$$

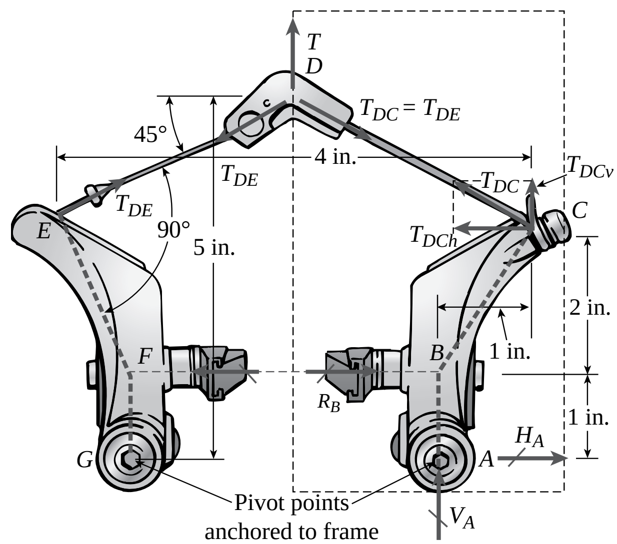
$$E = \frac{\sigma}{\varepsilon} = 1.4 \times 10^5 \text{ MPa}$$

NOTE: (E for cables is approx. 140 GPa)

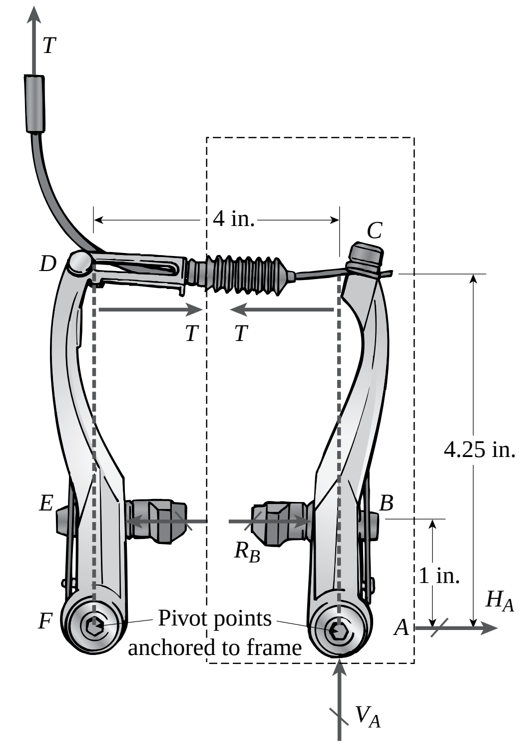
Problem 1.2-3 A bicycle rider would like to compare the effectiveness of cantilever hand brakes [see figure part (a)] versus V brakes [figure part (b)].

(a) Calculate the braking force R_B at the wheel rims for each of the bicycle brake systems shown. Assume that all forces act in the plane of the figure and that cable tension $T = 45$ lbs. Also, what is the average compressive normal stress σ_c on the brake pad ($A = 0.625$ in.^{2})?

(b) For each braking system, what is the stress in the brake cable (assume effective cross-sectional area of 0.00167 in.^{2})? (*HINT:* Because of symmetry, you only need to use the right half of each figure in your analysis.)



(a) Cantilever brakes



(b) V brakes

Solution 1.2-3

$T = 45$ lbs $\qquad A_{pad} = 0.625$ in.^{2}

$A_{cable} = 0.00167$ in.^{2}

(a) Cantilever brakes-braking force R_B & pad pressure

Statics: sum forces at D to get $T_{DC} = T/2$

$$\sum M_A = 0$$

$R_B(1) = T_{DCh}(3) + T_{DCv}(1)$

$T_{DCh} = T_{DCv} \qquad T_{DCh} = T/2$

$R_B = 2T \qquad R_B = 90 \text{ lbs} \quad \leftarrow$

so $R_B = 2T$ vs $4.25T$ for V brakes (below)

$$\sigma_{\text{pad}} = \frac{R_B}{A_{\text{pad}}} \qquad \sigma_{\text{pad}} = 144 \text{ psi} \quad \longleftarrow \qquad \frac{4.25}{2} = 2.125$$

$$\sigma_{\text{cable}} = \frac{T}{A_{\text{cable}}} \qquad \sigma_{\text{cable}} = 26{,}946 \text{ psi} \quad \longleftarrow \qquad \text{(same for V-brakes (below))}$$

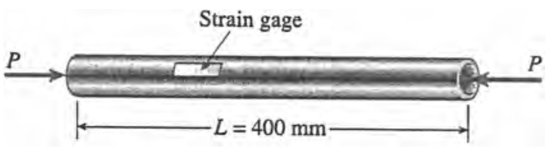
(b) V BRAKES - BRAKING FORCE R_B & PAD PRESSURE

$$\sum M_A = 0 \qquad R_B = 4.25T \qquad R_B = 191.3 \text{ lbs} \quad \longleftarrow$$

$$\sigma_{\text{pad}} = \frac{R_B}{A_{\text{pad}}} \qquad \sigma_{\text{pad}} = 306 \text{ psi} \quad \longleftarrow$$

Problem 1.2-4 A circular aluminum tube of length $L = 400$ mm is loaded in compression by forces P (see figure). The outside and inside diameters are 60 mm and 50 mm, respectively. A strain gage is placed on the outside of the bar to measure normal strains in the longitudinal direction.

(a) If the measured strain in $\epsilon = 550 \times 10^{-6}$, what is the shortening δ of the bar?
(b) If the compressive stress in the bar is intended to be 40 MPa, what should be the load P?



Solution 1.2-4 Aluminum tube in compression

$\varepsilon = 550 \times 10^{-6}$

$L = 400$ mm

$d_2 = 60$ mm

$d_1 = 50$ mm

(a) SHORTENING δ OF THE BAR

$$\delta = \varepsilon L = (550 \times 10^{-6})(400 \text{ mm})$$
$$= 0.220 \text{ mm} \quad \longleftarrow$$

(b) COMPRESSIVE LOAD P

$\sigma = 40$ MPa

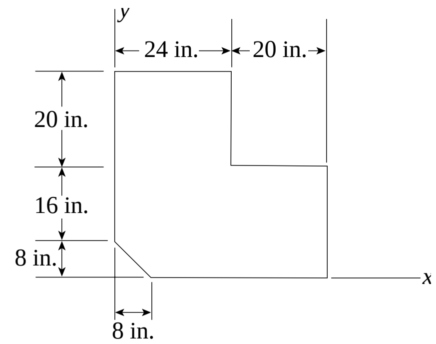
$$A = \frac{\pi}{4}[d_2^2 - d_1^2] = \frac{\pi}{4}[(60 \text{ mm})^2 - (50 \text{ mm})^2]$$

$$P = \sigma A = (40 \text{ MPa})(863.9 \text{ mm}^2)$$
$$= 34.6 \text{ kN} \quad \longleftarrow$$

Problem 1.2-5 The cross section of a concrete corner column that is loaded uniformly in compression is shown in the figure.

(a) Determine the average compressive stress σ_c in the concrete if the load is equal to 3200 k.

(b) Determine the coordinates x_c and y_c of the point where the resultant load must act in order to produce uniform normal stress in the column.



Solution 1.2-5

$$P = 3200 \text{ kips}$$

$$A = (24 + 20)(20 + 16 + 8) - \left(\frac{1}{2}8^2\right) - 20^2$$

$$A = 1.504 \times 10^3 \ \text{in}^2$$

(a) $\sigma_c = \dfrac{P}{A} \qquad \sigma_c = 2.13 \text{ ksi} \quad \leftarrow$

(b) $$x_c = \frac{\left[(24)(20 + 16)(12) + (24 - 8)(8)\left(8 + \dfrac{24 - 8}{2}\right) + (20)(16 + 8)(24 + 10) + \dfrac{1}{2}(8^2)\left(\dfrac{2}{3}8\right)\right]}{A}$$

$x_c = 19.22$ inches $\leftarrow$

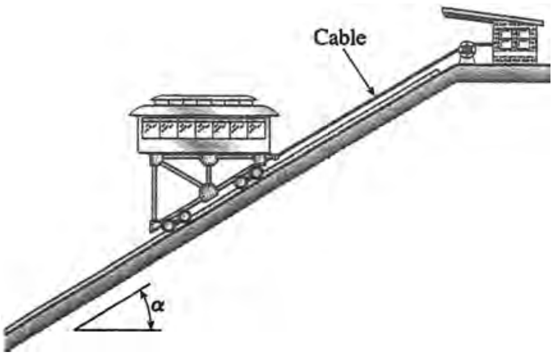
$$y_c = \frac{\left[(24)(20 + 16)\left(8 + \dfrac{20 + 16}{2}\right) + (20)(16 + 8)\left(\dfrac{16 + 8}{2}\right) + (24 - 8)(8)(4) + \dfrac{1}{2}(8^2)\left(\dfrac{2}{3}8\right)\right]}{A}$$

$y_c = 19.22$ inches $\leftarrow$

NOTE: x_c & y_c are the same as expected due to symmetry about a diagonal

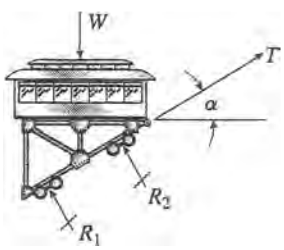
Problem 1.2-6 A car weighing 130 kN when fully loaded is pulled slowly up a steep inclined track by a steel cable (see figure). The cable has an effective cross-sectional area of 490 mm^2, and the angle α of the incline is 30°.

Calculate the tensile stress σ_t in the cable.



Solution 1.2-6 Car on inclined track

FREE-BODY DIAGRAM OF CAR



W = Weight of car

T = Tensile force in cable

α = Angle of incline

A = Effective area of cable

R_1, R_2 = Wheel reactions (no friction force between wheels and rails)

EQUILIBRIUM IN THE INCLINED DIRECTION

$$\Sigma F_T = 0 \quad \nearrow_{+}\swarrow^{-} \quad T - W \sin \alpha = 0$$

$$T = W \sin \alpha$$

TENSILE STRESS IN THE CABLE

$$\sigma_t = \frac{T}{A} = \frac{W \sin \alpha}{A}$$

SUBSTITUTE NUMERICAL VALUES:

$W = 130$ kN $\quad \alpha = 30°$

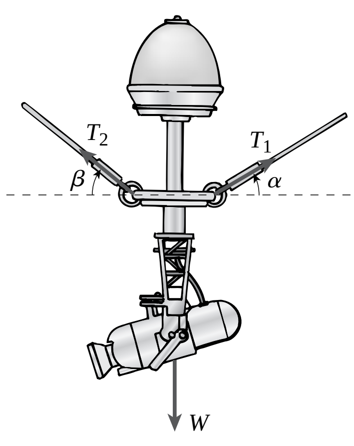
$A = 490$ mm^2

$$\sigma_t = \frac{(130 \text{ kN})(\sin 30°)}{490 \text{ mm}^2}$$

$$= 133 \text{ MPa} \quad \leftarrow$$

Problem 1.2-7 Two steel wires support a moveable overhead camera weighing W = 25 lb (see figure) used for close-up viewing of field action at sporting events. At some instant, wire 1 is at on angle $\alpha = 20°$ to the horizontal and wire 2 is at an angle $\beta = 48°$. Both wires have a diameter of 30 mils. (Wire diameters are often expressed in mils; one mil equals 0.001 in.)

Determine the tensile stresses σ_1 and σ_2 in the two wires.



Solution 1.2-7

NUMERICAL DATA

$W = 25 \text{ lb} \qquad d = 30 \times 10^{-3} \text{ in.}$

$\alpha = 20\dfrac{\pi}{180} \qquad \beta = 48\dfrac{\pi}{180} = \text{radians}$

EQUILIBRIUM EQUATIONS

$\sum F_h = 0 \qquad T_1\cos(\alpha) = T_2\cos(\beta)$

$$T_1 = T_2\frac{\cos(\beta)}{\cos(\alpha)}$$

$\sum F_v = 0 \qquad T_1\sin(\alpha) + T_2\sin(\beta) = W$

$$T_2\left(\frac{\cos(\beta)}{\cos(\alpha)}\sin(\alpha) + \sin(\beta)\right) = W$$

TENSION IN WIRES

$$T_2 = \frac{W}{\left(\dfrac{\cos(\beta)}{\cos(\alpha)}\sin(\alpha) + \sin(\beta)\right)}$$

$T_2 = 25.337 \text{ lb}$

$T_1 = T_2\dfrac{\cos(\beta)}{\cos(\alpha)} \qquad T_1 = 18.042 \text{ lb}$

TENSILE STRESSES IN WIRES

$$A_{wire} = \frac{\pi}{4}d^2$$

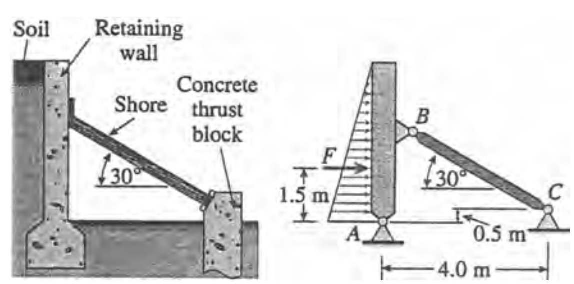
$\sigma_1 = \dfrac{T_1}{A_{wire}} \qquad \sigma_1 = 25.5 \text{ ksi} \quad \longleftarrow$

$\sigma_2 = \dfrac{T_2}{A_{wire}} \qquad \sigma_2 = 35.8 \text{ ksi} \quad \longleftarrow$

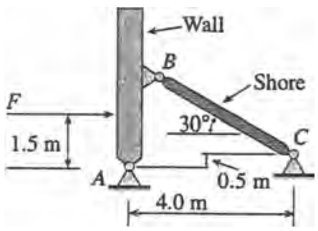
Problem 1.2-8 A long retaining wall is braced by wood shores set at an angle of 30° and supported by concrete thrust blocks, as shown in the first part of the figure. The shores are evenly spaced, 3 m apart.

For analysis purposes, the wall and shores are idealized as shown in the second part of the figure. Note that the base of the wall and both ends of the shores are assumed to be pinned. The pressure of the soil against the wall is assumed to be triangularly distributed, and the resultant force acting on a 3-meter length of the wall is $F = 190$ kN.

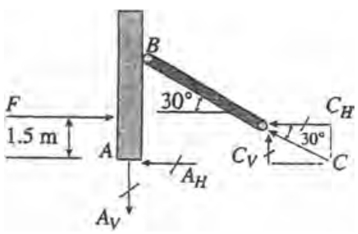
If each shore has a 150 mm × 150 mm square cross section, what is the compressive stress σ_c in the shores?



Solution 1.2-8 Retaining wall braced by wood shores



FREE-BODY DIAGRAM OF WALL AND SHORE



C = compressive force in wood shore

C_H = horizontal component of C

C_V = vertical component of C

$C_H = C\cos 30°$

$C_V = C\sin 30°$

$F = 190$ kN

A = area of one shore

$A = (150\text{ mm})(150\text{ mm})$

$= 22{,}500\text{ mm}^2$

$= 0.0225\text{ m}^2$

SUMMATION OF MOMENTS ABOUT POINT A

$\Sigma M_A = 0$

$-F(1.5\text{ m}) + C_V(4.0\text{ m}) + C_H(0.5\text{ m}) = 0$

or

$$-(190\text{ kN})(1.5\text{ m}) + C(\sin 30°)(4.0\text{ m}) + C(\cos 30°)(0.5\text{ m}) = 0$$

$\therefore C = 117.14$ kN

COMPRESSIVE STRESS IN THE SHORES

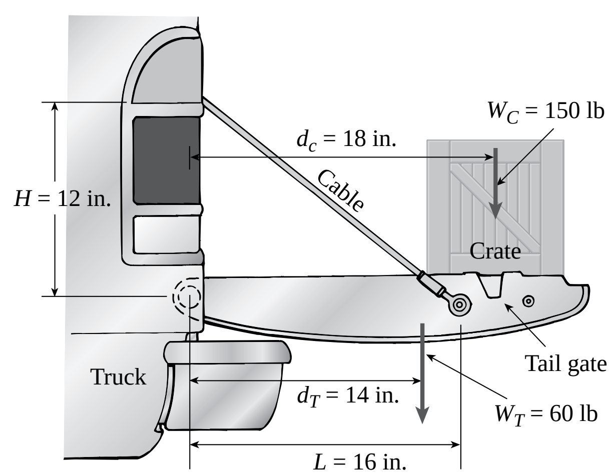
$$\sigma_c = \frac{C}{A} = \frac{117.14\text{ kN}}{0.0225\text{ m}^2}$$

$$= 5.21\text{ MPa} \quad \leftarrow$$

Problem 1.2-9 A pickup truck tailgate supports a crate ($W_C = 150$ lb), as shown in the figure. The tailgate weighs $W_T = 60$ lb and is supported by two cables (only one is shown in the figure). Each cable has an effective cross-sectional area $A_e = 0.017\text{ in}^2$.

(a) Find the tensile force T and normal stress σ in each cable.

(b) If each cable elongates $\delta = 0.01$ in. due to the weight of both the crate and the tailgate, what is the average strain in the cable?



Solution 1.2-9

$W_c = 150 \text{ lb}$

$A_e = 0.017 \text{ in}^2$

$W_T = 60$

$\delta = 0.01$

$d_c = 18$

$d_T = 14$

$H = 12$

$L = 16$

$L_c = \sqrt{L^2 + H^2} \qquad L_c = 20$

$\sum M_{hinge} = 0 \qquad 2T_v L = W_c d_c + W_T d_T$

$$T_v = \frac{W_c d_c + W_T d_T}{2L} \qquad T_v = 110.625 \text{ lb}$$

$$T_h = \frac{L}{H} T_v \qquad T_h = 147.5$$

(a) $T = \sqrt{T_v^2 + T_h^2} \qquad T = 184.4 \text{ lb} \quad \leftarrow$

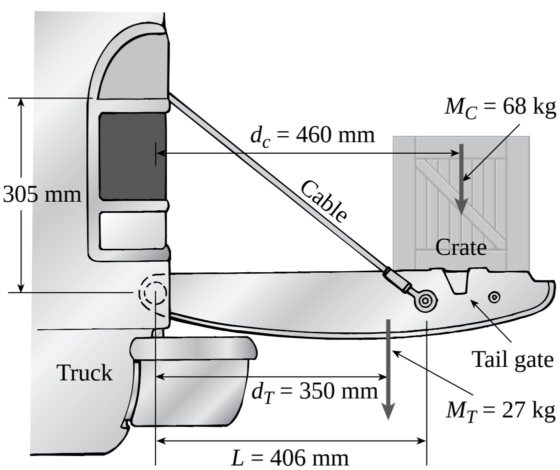
$$\sigma_{cable} = \frac{T}{A_e} \qquad \sigma_{cable} = 10.8 \text{ ksi} \quad \leftarrow$$

(b) $\varepsilon_{cable} = \dfrac{\delta}{L_c} \qquad \varepsilon_{cable} = 5 \times 10^{-4} \quad \leftarrow$

Problem 1.2-10 Solve the preceding problem if the mass of the tail gate is $M_T = 27$ kg and that of the crate is $M_C = 68$ kg. Use dimensions $H = 305$ mm, $L = 406$ mm, $d_C = 460$ mm, and $d_T = 350$ mm. The cable cross-sectional area is $A_e = 11.0 \text{ mm}^2$.

(a) Find the tensile force T and normal stress σ in each cable.

(b) If each cable elongates $\delta = 0.25$ mm due to the weight of both the crate and the tailgate, what is the average strain in the cable?



Solution 1.2-10

$M_c = 68$

$M_T = 27\ \text{kg} \qquad g = 9.81\dfrac{m}{s^2}$

$W_c = M_c g \qquad W_T = M_T g$

$W_c = 667.08 \qquad W_T = 264.87$

$N = kg\dfrac{m}{s^2}$

$A_e = 11.0\ \text{mm}^2 \qquad \delta = 0.25$

$d_c = 460 \qquad d_T = 350$

$H = 305 \qquad L = 406$

$L_c = \sqrt{L^2 + H^2} \qquad L_c = 507.8\ \text{mm}$

$\sum M_{hinge} = 0 \qquad 2T_v L = W_c d_c + W_T d_T$

$T_v = \dfrac{W_c d_c + W_T d_T}{2L} \qquad T_v = 492.071\ \text{N}$

$T_h = \dfrac{L}{H} T_v \qquad T_h = 655.019\ \text{N}$

(a) $T = \sqrt{T_v^2 + T_h^2} \qquad T = 819\ \text{N} \quad \leftarrow$

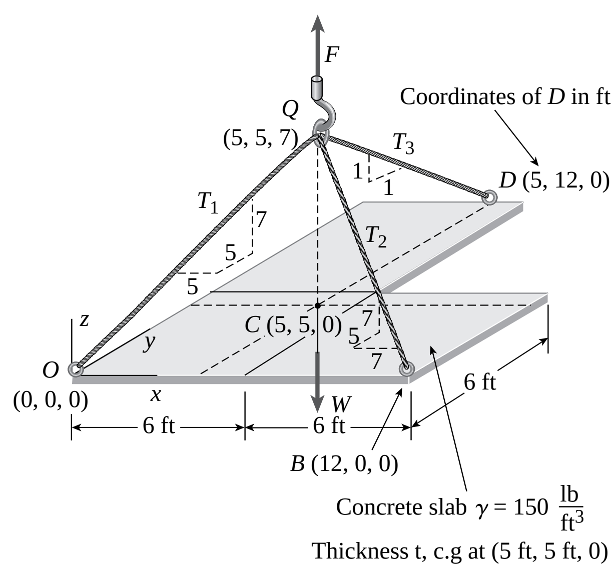
$\sigma_{cable} = \dfrac{T}{A_e} \qquad \sigma_{cable} = 74.5\ \text{MPa} \quad \leftarrow$

(b) $\varepsilon_{cable} = \dfrac{\delta}{L_c} \qquad \varepsilon_{cable} = 4.92 \times 10^{-4} \quad \leftarrow$

Problem ★1.2-11 An L-shaped reinforced concrete slab 12 ft × 12 ft (but with a 6 ft × 6 ft cutout) and thickness $t = 9.0$ in. is lifted by three cables attached at O, B and D, as shown in the figure. The cables are combined at point Q, which is 7.0 ft above the top of the slab and directly above the center of mass at C. Each cable has an effective cross-sectional area of $A_e = 0.12\ \text{in}^2$.

(a) Find the tensile force T_i ($i = 1, 2, 3$) in each cable due to the weight W of the concrete slab (ignore weight of cables).

(b) Find the average stress σ_i in each cable. (See Table H-1 in Appendix H for the weight density of reinforced concrete.)



Solution 1.2-11

CABLE LENGTHS

$L_1 = \sqrt{5^2 + 5^2 + 7^2}$ $\quad L_1 = 9.95$

$5^2 + 5^2 + 7^2 = 99 \quad L_1 = \sqrt{99}$

$L_2 = \sqrt{5^2 + 7^2 + 7^2}$ $\quad L_2 = 11.091$

$5^2 + 7^2 + 7^2 = 123 \quad L_2 = \sqrt{123}$

$L_3 = \sqrt{7^2 + 7^2}$ $\quad L_3 = 9.899$

$7^2 + 7^2 = 98 \quad L_3 = 7\sqrt{2}$

(a) SOLUTION FOR CABLE FORCES USING STATICS (3 EQU, 3 UNKNOWNS)

$$T_1 = \frac{7\sqrt{99}}{144} \qquad T_1 = 0.484 \qquad \delta_1 = \frac{T_1 L_1}{EA}$$

$$T_2 = \frac{5\sqrt{123}}{144} \qquad T_2 = 0.385 \qquad \delta_2 = \frac{T_2 L_2}{EA}$$

$$T_3 = \frac{5\sqrt{2}}{12} \qquad T_3 = 0.589 \qquad \delta_3 = \frac{T_3 L_3}{EA}$$

$$\sum T_{verti} = 0$$

$$T_1\frac{7}{\sqrt{99}} + T_2\frac{7}{\sqrt{123}} + T_3\frac{1}{\sqrt{2}} = 1 \quad \text{CHECK}$$

For unit force in Z-direction

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} \frac{-5}{\sqrt{99}} & \frac{7}{\sqrt{123}} & 0 \\ \frac{-5}{\sqrt{99}} & \frac{-5}{\sqrt{123}} & \frac{1}{\sqrt{2}} \\ \frac{7}{\sqrt{99}} & \frac{7}{\sqrt{123}} & \frac{1}{\sqrt{2}} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} 0.484 \\ 0.385 \\ 0.589 \end{pmatrix} \qquad T_u = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}$$

$$\text{check:} \quad T_1\frac{7}{\sqrt{99}} + T_2\frac{7}{\sqrt{123}} + T_3\frac{1}{\sqrt{2}} = 1$$

NOTE: preferred solution uses sum of moments about a line as follows –

1. sum about x-axis to get T3v, then T3
2. sum about y-axis to get T2v, then T2
3. sum vertical forces to get T1v, then T1 OR sum forces in x-dir to get T1x in terms of T2x

SLAB WEIGHT & C.G.

$$W = 150\,(12^2 - 6^2)\frac{9}{12} \qquad W = 1.215 \times 10^4$$

$$x_{cg} = \frac{2A3 + A(6 + 3)}{3A}$$

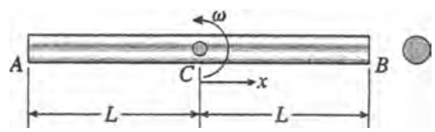
$$x_{cg} = 5 \qquad \text{same for ycg} \qquad y_{cg} = x_{cg}$$

Multiply unit forces by W

$$T = T_u W \qquad T = \begin{pmatrix} 5877 \\ 4679 \\ 7159 \end{pmatrix} \text{lb} \quad \leftarrow$$

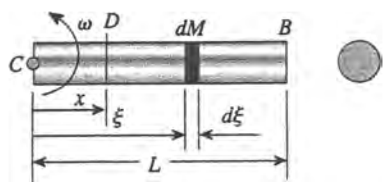
$$\text{(b)} \quad \sigma = \frac{T}{0.12} \qquad \sigma = \begin{pmatrix} 49.0 \text{ ksi} \\ 39.0 \text{ ksi} \\ 60.0 \text{ ksi} \end{pmatrix} \text{psi} \quad \leftarrow$$

Problem *1.2-12 A round bar *ACB* of length $2L$ (see figure) rotates about an axis through the midpoint *C* with constant angular speed ω (radians per second). The material of the bar has weight density γ.



(a) Derive a formula for the tensile stress σ_x in the bar as a function of the distance x from the midpoint *C*.
(b) What is the maximum tensile stress σ_{max}?

Solution 1.2-12 Rotating Bar



ω = angular speed (rad/s)

A = cross-sectional area

γ = weight density

$\frac{\gamma}{g}$ = mass density

We wish to find the axial force F_x in the bar at Section D, distance x from the midpoint *C*.

The force F_x equals the inertia force of the part of the rotating bar from *D* to *B*.

Consider an element of mass dM at distance ξ from the midpoint *C*. The variable ξ ranges from x to L.

$$dM = \frac{\gamma}{g} A\, dj$$

dF = Inertia force (centrifugal force) of element of mass dM

$$dF = (dM)(j\omega^2) = \frac{\gamma}{g} \quad A\omega^2 j dj$$

$$F_x = \int_D^B dF = \int_x^L \frac{\gamma}{g} A\omega^2 j dj = \frac{\gamma A \omega^2}{2g}(L^2 - x^2)$$

(a) Tensile stress in bar at distance x

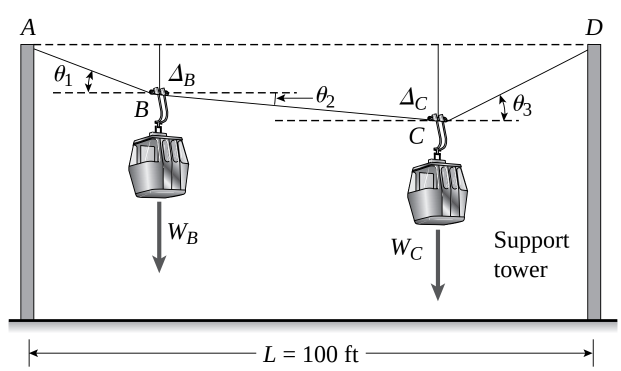
$$\sigma_x = \frac{F_x}{A} = \frac{\gamma\omega^2}{2g}(L^2 - x^2) \quad \leftarrow$$

(b) Maximum tensile stress

$$x = 0 \quad \sigma_{max} = \frac{\gamma\omega^2 L^2}{2g} \quad \leftarrow$$

Problem 1.2-13 Two gondolas on a ski lift are locked in the position shown in the figure while repairs are being made elsewhere. The distance between support towers is L = 100 ft. The length of each cable segment under gondola weights W_B = 450 lb and W_C = 650 lb are D_{AB} = 12 ft, D_{BC} = 70 ft, and D_{CD} = 20 ft. The cable sag at *B* is Δ_B = 3.9 ft and that at C(ΔC) is 7.1 ft. The effective cross-sectional area of the cable is A_e = 0.12 in^2.

(a) Find the tension force in each cable segment; neglect the mass of the cable.
(b) Find the average stress (σ) in each cable segment.



Solution 1.2-13

$W_B = 450$

$Wc = 650$ lb

$\Delta_B = 3.9$ ft

$\Delta_C = 7.1$ ft

$L = 100$ ft

$D_{AB} = 12$ ft

$D_{BC} = 70$ ft

$D_{CD} = 20$ ft

$D_{AB} + D_{BC} + D_{CD} = 102$ ft

$A_e = 0.12\ \text{in}^2$

COMPUTE INITIAL VALUES OF THETA ANGLES (RADIANS)

$$\theta_1 = \operatorname{asin}\left(\frac{\Delta_B}{D_{AB}}\right) \qquad \theta_1 = 0.331$$

$$\theta_2 = \operatorname{asin}\left(\frac{\Delta_C - \Delta_B}{D_{BC}}\right) \qquad \theta_2 = 0.046$$

$$\theta_3 = \operatorname{asin}\left(\frac{\Delta_C}{D_{CD}}\right) \qquad \theta_3 = 0.363$$

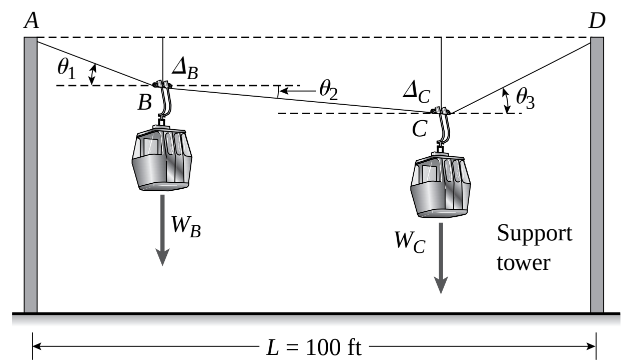
(a) STATICS AT B & C

$$-T_{AB}\cos(\theta_1) + T_{BC}\cos(\theta_2) = 0$$

$$T_{AB}\sin(\theta_1) - T_{BC}\sin(\theta_2) = W_B$$

$$-T_{BC}\cos(\theta_2) + T_{CD}\cos(\theta_3) = 0$$

$$T_{BC}\sin(\theta_2) - T_{CD}\sin(\theta_3) = W_C$$



CONTRAINT EQUATIONS

$$D_{AB}\cos(\theta_1) + D_{BC}\cos(\theta_2) + D_{CD}\cos(\theta_3) = L$$

$$D_{AB}\sin(\theta_1) + D_{BC}\sin(\theta_2) = D_{CD}\sin(\theta_3)$$

SOLVE SIMULTANEOUS EQUATIONS NUMERICALLY FOR TENSION FORCE IN EACH CABLE SEGMENT

$T_{AB} = 1620$ lb $\quad T_{CB} = 1536$ lb $\quad T_{CD} = 1640$ lb $\quad \leftarrow$

CHECK EQUILIBRIUM AT B & C

$$T_{AB}\sin(\theta_1) - T_{BC}\sin(\theta_2) = 450$$

$$T_{BC}\sin(\theta_2) - T_{CD}\sin(\theta_3) = 650$$

(b) COMPUTE STRESSES IN CABLE SEGMENTS

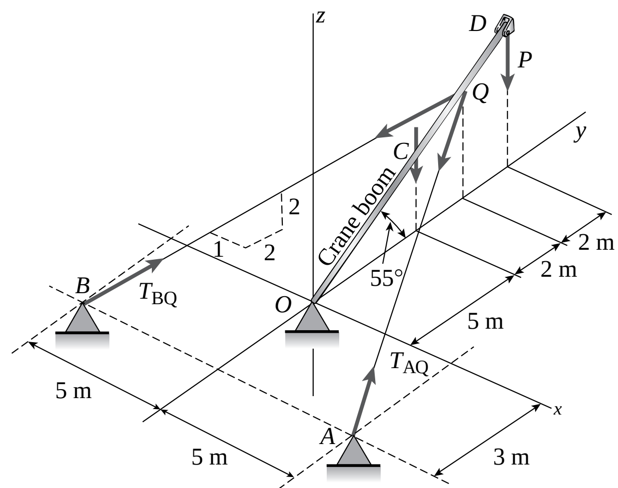
$$\sigma_{AB} = \frac{T_{AB}}{A_e} \qquad \sigma_{BC} = \frac{T_{BC}}{A_e} \qquad \sigma_{CD} = \frac{T_{CD}}{A_e}$$

$\sigma_{AB} = 13.5$ ksi $\quad \sigma_{BC} = 12.8$ ksi

$\sigma_{CD} = 13.67$ ksi $\quad \leftarrow$

Problem 1.2-14 A crane boom of mass 450 kg with its center of mass at *C* is stabilized by two cables *AQ* and *BQ* ($A_e = 304\ \text{mm}^2$ for each cable) as shown in the figure. A load $P = 20$ kN is supported at point *D*. The crane boom lies in the *y–z* plane.

(a) Find the tension forces in each cable: T_{AQ} and T_{BQ} (kN); neglect the mass of the cables, but include the mass of the boom in addition to load *P*.
(b) Find the average stress (σ) in each cable.



Solution 1.2-14

Data $\quad M_{boom} = 450\ \text{kg}$

$g = 9.81\,\dfrac{\text{m}}{\text{s}^2} \qquad W_{boom} = M_{boom}\,g$

$W_{boom} = 4415\ \text{N}$

$P = 20\ \text{kN}$

$A_e = 304\ \text{mm}^2$

(a) symmetry: $T_{AQ} = T_{BQ}$

$\sum M_x = 0$

$$2T_{AQZ}(3000) = W_{boom}(5000) + P(9000)$$

$$T_{AQZ} = \frac{W_{boom}(5000) + P(9000)}{2(3000)}$$

$$T_{AQ} = \sqrt{\frac{2^2 + 2^2 + 1^2}{2}}\,T_{AQz}$$

$$T_{AQ} = 50.5\ \text{kN} = T_{BQ} \quad \leftarrow$$

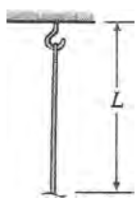
(b) $\sigma = \dfrac{T_{AQ}}{A_e} \qquad \sigma = 166.2\ \text{MPa} \quad \leftarrow$

Mechanical Properties of Materials

Problem 1.3-1 Imagine that a long steel wire hangs vertically from a high-altitude balloon.

(a) What is the greatest length (feet) it can have without yielding if the steel yields at 40 ksi?
(b) If the same wire hangs from a ship at sea, what is the greatest length? (Obtain the weight densities of steel and sea water from Table H-1, Appendix H.)

Solution 1.3-1 Hanging wire of length *L*



W = total weight of steel wire

γ_S = weight density of steel

$= 490 \text{ lb/ft}^3$

γ_w = weight density of sea water

$= 63.8 \text{ lb/ft}^3$

A = cross-sectional area of wire

$\sigma_{max} = 40$ ksi (yield strength)

(a) Wire hanging in air

$$W = \gamma_S AL$$

$$\sigma_{max} = \frac{W}{A} = \gamma_S L$$

$$L_{max} = \frac{\sigma_{max}}{\gamma_S} = \frac{40{,}000 \text{ psi}}{490 \text{ lb/ft}^3}(144 \text{ in.}^2/\text{ft}^2)$$

$$= 11{,}800 \text{ ft} \quad \leftarrow$$

(b) Wire hanging in sea water

F = tensile force at top of wire

$$F = (\gamma_S - \gamma_W)AL \qquad \sigma_{max} = \frac{F}{A} = (\gamma_S - \gamma_W)L$$

$$L_{max} = \frac{\sigma_{max}}{\gamma_S - \gamma_W}$$

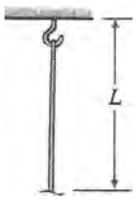
$$= \frac{40{,}000 \text{ psi}}{(490 - 63.8)\text{ lb/ft}^3}(144 \text{ in.}^2/\text{ft}^2)$$

$$= 13{,}500 \text{ ft} \quad \leftarrow$$

Problem 1.3-2 Imagine that a long wire of tungsten hangs vertically from a high-altitude balloon.

(a) What is the greatest length (meters) it can have without breaking if the ultimate strength (or breaking strength) is 1500 MPa?
(b) If the same wire hangs from a ship at sea, what is the greatest length? (Obtain the weight densities of tungsten and sea water from Table H-1, Appendix H.)

Solution 1.3-2 Hanging wire of length *L*



W = total weight of tungsten wire

γ_T = weight density of tungsten

$= 190 \text{ kN/m}^3$

γ_W = weight density of sea water

$= 10.0 \text{ kN/m}^3$

A = cross-sectional area of wire

σ_{max} = 1500 MPa (breaking strength)

(a) WIRE HANGING IN AIR

$$W = \gamma_T A L$$

$$\sigma_{max} = \frac{W}{A} = \gamma_T L$$

$$L_{max} = \frac{\sigma_{max}}{\gamma_T} = \frac{1500\text{MPa}}{190 \text{ kN/m}^3}$$

$$= 7900 \text{ m} \quad \leftarrow$$

(b) WIRE HANGING IN SEA WATER

F = tensile force at top of wire

$$F = (\gamma_T - \gamma_W)AL$$

$$\sigma_{max} = \frac{F}{A} = (\gamma_T - \gamma_W)L$$

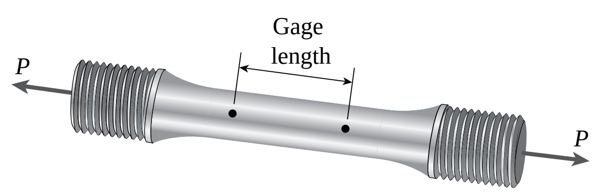
$$L_{max} = \frac{\sigma_{max}}{\gamma_T - \gamma_W}$$

$$= \frac{1500\text{MPa}}{(190 - 10.0) \text{ kN/m}^3}$$

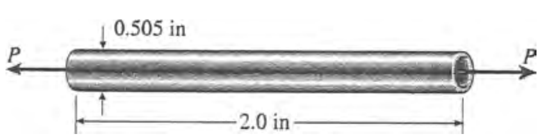
$$= 8300 \text{ m} \quad \leftarrow$$

Problem 1.3-3 Three different materials, designated *A*, *B*, and *C*, are tested in tension using test specimens having diameters of 0.505 in. and gage lengths of 2.0 in. (see figure). At failure, the distances between the gage marks are found to be 2.13, 2.48, and 2.78 in., respectively. Also, at the failure cross sections the diameters are found to be 0.484, 0.398, and 0.253 in., respectively.

Determine the percent elongation and percent reduction in area of each specimen, and then, using your own judgment, classify each material as brittle or ductile.



Solution 1.3-3 Tensile tests of three materials



$$\text{Percent elongation} = \frac{L_1 - L_0}{L_0}(100) = \left(\frac{L_1}{L_0} - 1\right)100$$

$L_0 = 2.0$ in.

$$\text{Percent elongation} = \left(\frac{L_1}{2.0} - 1\right)(100) \qquad \text{(Eq. 1)}$$

where L_1 is in inches.

$$\text{Percent reduction in area} = \frac{A_0 - A_1}{A_0}(100)$$

$$= \left(1 - \frac{A_1}{A_0}\right)(100)$$

d_0 = initial diameter $\qquad d_1$ = final diameter

$$\frac{A_1}{A_0} = \left(\frac{d_1}{d_0}\right)^2 \qquad d_0 = 0.505 \text{ in.}$$

Percent reduction in area

$$= \left[1 - \left(\frac{d_1}{0.505}\right)^2\right](100) \qquad \text{(Eq. 2)}$$

where d_1 is in inches.

Material	L_1 (in.)	d_1 (in.)	% Elongation (Eq. 1)	% Reduction (Eq. 2)	Brittle or Ductile?
A	2.13	0.484	6.5%	8.1%	Brittle
B	2.48	0.398	24.0%	37.9%	Ductile
C	2.78	0.253	39.0%	74.9%	Ductile

Problem 1.3-4 The *strength-to-weight ratio* of a structural material is defined as its load-carrying capacity divided by its weight. For materials in tension, we may use a characteristic tensile stress (as obtained from a stress-strain curve) as a measure of strength. For instance, either the yield stress or the ultimate stress could be used, depending upon the particular application. Thus, the strength-to-weight ratio $R_{S/W}$ for a material in tension is defined as

$$R_{S/W} = \frac{\sigma}{\gamma}$$

in which σ is the characteristic stress and γ is the weight density. Note that the ratio has units of length.

Using the ultimate stress σ_U as the strength parameter, calculate the strength-to-weight ratio (in units of meters) for each of the following materials: aluminum alloy 6061-T6, Douglas fir (in bending), nylon, structural steel ASTM-A572, and a titanium alloy. (Obtain the material properties from Tables H-1 and H-3 of Appendix H. When a range of values is given in a table, use the average value.)

Solution 1.3-4 Strength-to-weight ratio

The ultimate stress σ_U for each material is obtained from Table H-3, Appendix H, and the weight density γ is obtained from Table H-1.

The strength-to-weight ratio (meters) is

$$R_{SW} = \frac{\sigma_U(\text{MPa})}{\gamma(\text{kN/m}^3)}(10^3)$$

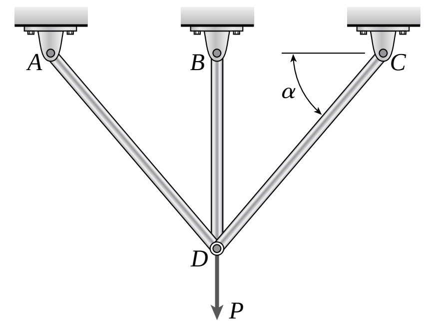
Values of σ_U, γ, and $R_{S/W}$ are listed in the table.

	σ_U (MPa)	γ (kN/m^3)	$R_{S/W}$ (m)
Aluminum alloy 6061-T6	310	26.0	11.9×10^3
Douglas fir	65	5.1	12.7×10^3
Nylon	60	9.8	6.1×10^3
Structural steel ASTM-A572	500	77.0	6.5×10^3
Titanium alloy	1050	44.0	23.9×10^3

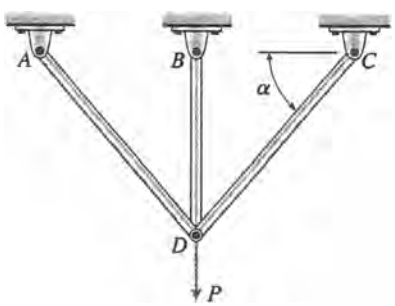
Titanium has a high strength-to-weight ratio, which is why it is used in space vehicles and high-performance airplanes. Aluminum is higher than steel, which makes it desirable for commercial aircraft. Some woods are also higher than steel, and nylon is about the same as steel.

Problem 1.3-5 A symmetrical framework consisting of three pin-connected bars is loaded by a force P (see figure). The angle between the inclined bars and the horizontal is $\alpha = 48°$. The axial strain in the middle bar is measured as 0.0713.

Determine the tensile stress in the outer bars if they are constructed of aluminum alloy having the stress-strain diagram shown in Fig. 1-13. (Express the stress in USCS units.)



Solution 1.3-5 Symmetrical framework

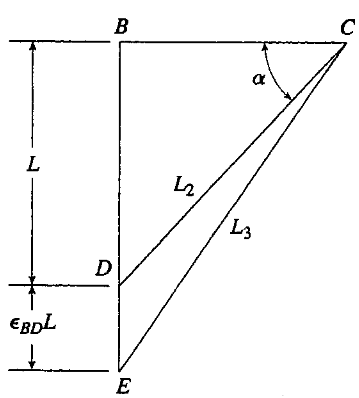


Aluminum alloy

$\alpha = 48°$

$\varepsilon_{BD} = 0.0713$

Use stress-strain diagram of Figure 1-13



L = length of bar BD

L_1 = distance BC

$= L \cot \alpha = L(\cot 48°) = 0.9004\ L$

L_2 = length of bar CD

$= L \csc \alpha = L(\csc 48°) = 1.3456\ L$

Elongation of bar BD = distance $DE = \varepsilon_{BD}L$

$\varepsilon_{BD}L = 0.0713\ L$

L_3 = distance CE

$$L_3 = \sqrt{L_1^2 + (L + \varepsilon_{BD}L)^2}$$

$$= \sqrt{(0.9004L)^2 + L^2(1 + 0.0713)^2}$$

$$= 1.3994\ L$$

δ = elongation of bar CD

$\delta = L_3 - L_2 = 0.0538L$

Strain in bar CD

$$= \frac{\delta}{L_2} = \frac{0.0538L}{1.3456L} = 0.0400$$

From the stress-strain diagram of Figure 1-13:

$\sigma \approx 31$ ksi $\quad \longleftarrow$

Problem 1.3-6 A specimen of a methacrylate plastic is tested in tension at room temperature (see figure), producing the stress-strain data listed in the accompanying table.

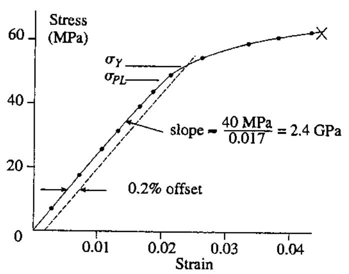
Plot the stress-strain curve and determine the proportional limit, modulus of elasticity (i.e., the slope of the initial part of the stress-strain curve), and yield stress at 0.2% offset. Is the material ductile or brittle?

STRESS-STRAIN DATA FOR PROBLEM 1.3-6

Stress (MPa)	Strain
8.0	0.0032
17.5	0.0073
25.6	0.0111
31.1	0.0129
39.8	0.0163
44.0	0.0184
48.2	0.0209
53.9	0.0260
58.1	0.0331
62.0	0.0429
62.1	Fracture

Solution 1.3-6 Tensile test of a plastic

Using the stress-strain data given in the problem statement, plot the stress-strain curve:



σ_{PL} = proportional limit $\quad \sigma_{PL} \approx 47$ MPa $\leftarrow$

Modulus of elasticity (slope) ≈ 2.4 GPa $\leftarrow$

σ_Y = yield stress at 0.2% offset

$\sigma_Y \approx 53$ MPa $\leftarrow$

Material is *brittle*, because the strain after the proportional limit is exceeded is relatively small. $\leftarrow$

Problem 1.3-7 The data shown in the accompanying table were obtained from a tensile test of high-strength steel. The test specimen had a diameter of 0.505 in. and a gage length of 2.00 in. (see figure for Prob. 1.3-3). At fracture, the elongation between the gage marks was 0.12 in. and the minimum diameter was 0.42 in.

Plot the conventional stress-strain curve for the steel and determine the proportional limit, modulus of elasticity (i.e., the slope of the initial part of the stress-strain curve), yield stress at 0.1% offset, ultimate stress, percent elongation in 2.00 in., and percent reduction in area.

TENSILE-TEST DATA FOR PROBLEM 1.3-7

Load (lb)	Elongation (in.)
1,000	0.0002
2,000	0.0006
6,000	0.0019
10,000	0.0033
12,000	0.0039
12,900	0.0043
13,400	0.0047
13,600	0.0054
13,800	0.0063
14,000	0.0090
14,400	0.0102
15,200	0.0130
16,800	0.0230
18,400	0.0336
20,000	0.0507
22,400	0.1108
22,600	Fracture

Solution 1.3-7 Tensile test of high-strength steel

$d_0 = 0.505$ in. $\qquad L_0 = 2.00$ in.

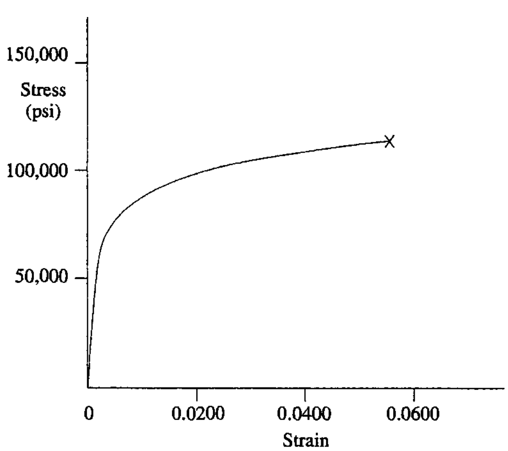
$$A_0 = \frac{\pi d_0^2}{4} = 0.200 \text{ in.}^2$$

CONVENTIONAL STRESS AND STRAIN

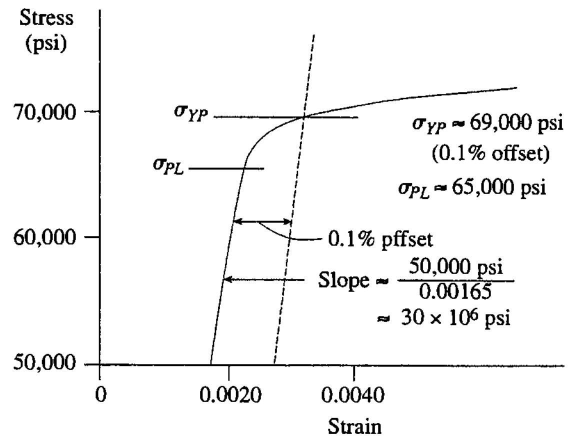
$$\sigma = \frac{P}{A_0} \qquad \varepsilon = \frac{\delta}{L_0}$$

Load P (lb)	Elongation δ (in.)	Stress σ (psi)	Strain ε
1,000	0.0002	5,000	0.00010
2,000	0.0006	10,000	0.00030
6,000	0.0019	30,000	0.00100
10,000	0.0033	50,000	0.00165
12,000	0.0039	60,000	0.00195
12,900	0.0043	64,500	0.00215
13,400	0.0047	67,000	0.00235
13,600	0.0054	68,000	0.00270
13,800	0.0063	69,000	0.00315
14,000	0.0090	70,000	0.00450
14,400	0.0102	72,000	0.00510
15,200	0.0130	76,000	0.00650
16,800	0.0230	84,000	0.01150
18,400	0.0336	92,000	0.01680
20,000	0.0507	100,000	0.02535
22,400	0.1108	112,000	0.05540
22,600	Fracture	113,000	

STRESS-STRAIN DIAGRAM



ENLARGEMENT OF PART OF THE STRESS-STRAIN CURVE



RESULTS

Proportional limit $\approx$ 65,000 psi $\leftarrow$

Modulus of elasticity (slope) $\approx 30 \times 10^6$ psi $\leftarrow$

Yield stress at 0.1% offset $\approx$ 69,000 psi $\leftarrow$

Ultimate stress (maximum stress)

$\approx$ 113,000 psi $\leftarrow$

Percent elongation in 2.00 in.

$$= \frac{L_1 - L_0}{L_0}(100)$$

$$= \frac{0.12 \text{ in.}}{2.00 \text{ in.}}(100) = 6\% \qquad \leftarrow$$

Percent reduction in area

$$= \frac{A_0 - A_1}{A_0}(100)$$

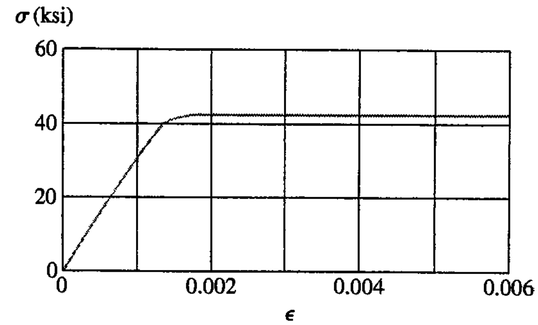
$$= \frac{0.200 \text{ in.}^2 - \frac{\pi}{4}(0.42 \text{ in.})^2}{0.200 \text{ in.}^2}(100)$$

$$= 31\% \qquad \leftarrow$$

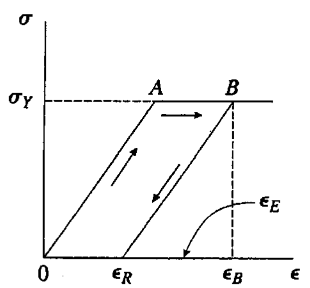
Elasticity, Plasticity, and Creep

Problem 1.4-1 A bar made of structural steel having the stress-strain diagram shown in the figure has a length of 48 in. The yield stress of the steel is 42 ksi and the slope of the initial linear part of the stress-strain curve (modulus of elasticity) is 30×10^3 ksi. The bar is loaded axially until it elongates 0.20 in., and then the load is removed.

How does the final length of the bar compare with its original length? (*Hint*: Use the concepts illustrated in Fig. 1-18b.)



Solution 1.4-1 Steel bar in tension



$L = 48$ in.

Yield stress $\sigma_Y = 42$ ksi

Slope $= 30 \times 10^3$ ksi

$\delta = 0.20$ in.

STRESS AND STRAIN AT POINT B

$\sigma_B = \sigma_Y = 42$ ksi

$$\varepsilon_B = \frac{\delta}{L} = \frac{0.20 \text{ in.}}{48 \text{ in.}} = 0.00417$$

ELASTIC RECOVERY ε_E

$$\varepsilon_E = \frac{\sigma_B}{\text{Slope}} = \frac{42 \text{ ksi}}{30 \times 10^3 \text{ ksi}} = 0.00140$$

RESIDUAL STRAIN ε_R

$$\varepsilon_R = \varepsilon_B - \varepsilon_E = 0.00417 - 0.00140$$
$$= 0.00277$$

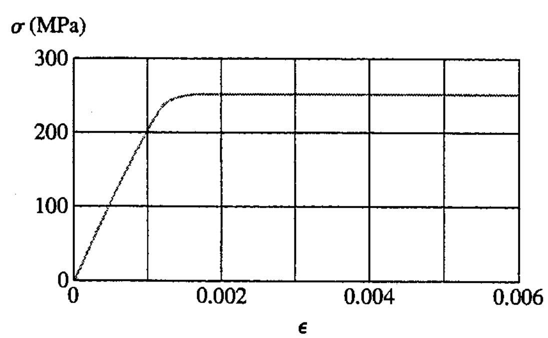
PERMANENT SET

$$\varepsilon_R L = (0.00277)(48 \text{ in.})$$
$$= 0.13 \text{ in.}$$

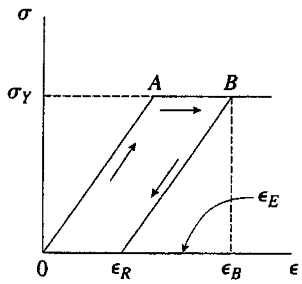
Final length of bar is 0.13 in. greater than its original length. ←

Problem 1.4-2 A bar of length 2.0 m is made of a structural steel having the stress-strain diagram shown in the figure. The yield stress of the steel is 250 MPa and the slope of the initial linear part of the stress-strain curve (modulus of elasticity) is 200 GPa. The bar is loaded axially until it elongates 6.5 mm, and then the load is removed.

How does the final length of the bar compare with its original length? (*Hint*: Use the concepts illustrated in Fig. 1-18b.)



Solution 1.4-2 Steel bar in tension



$L = 2.0 \text{ m} = 2000 \text{ mm}$

Yield stress $\sigma_Y = 250$ MPa

Slope $= 200$ GPa

$\delta = 6.5$ mm

STRESS AND STRAIN AT POINT B

$\sigma_B = \sigma_Y = 250$ MPa

$$\varepsilon_B = \frac{\delta}{L} = \frac{6.5 \text{ mm}}{2000 \text{ mm}} = 0.00325$$

ELASTIC RECOVERY ε_E

$$\varepsilon_E = \frac{\sigma_B}{\text{Slope}} = \frac{250 \text{ MPa}}{200 \text{ GPa}} = 0.00125$$

RESIDUAL STRAIN ε_R

$$\varepsilon_R = \varepsilon_B - \varepsilon_E = 0.00325 - 0.00125$$
$$= 0.00200$$

Permanent set $= \varepsilon_R L = (0.00200)(2000 \text{ mm})$
$= 4.0$ mm

Final length of bar is 4.0 mm greater than its original length. ←

Problem 1.4-3 An aluminum bar has length $L = 5$ ft and diameter $d = 1.25$ in. The stress-strain curve for the aluminum is shown in Fig. 1-13 of Section 1.3. The initial straight-line part of the curve has a slope (modulus of elasticity) of 10×10^6 psi. The bar is loaded by tensile forces $P = 39$ k and then unloaded.

(a) What is the permanent set of the bar?
(b) If the bar is reloaded, what is the proportional limit? (*Hint*: Use the concepts illustrated in Figs. 1-18b and 1-19.)

Solution 1.4-3

(a) PERMAMENT SET

Numerical data $L = 60$ in

$d = 1.25$ in $P = 39$ kips

STRESS AND STRAIN AT PT B

$$\sigma_B = \frac{P}{\frac{\pi}{4} d^2}$$

$\sigma_B = 31.78$ ksi

From Figure 1-13 $\varepsilon_B = 0.05$

ELASTIC RECOVERY

$$\varepsilon_E = \frac{\sigma_B}{10(10)^3} \qquad \varepsilon_E = 3.178 \times 10^{-3}$$

RESIDUAL STRAIN

$\varepsilon_E = \varepsilon_B - \varepsilon_E$ $\varepsilon_R = 0.047$

PERMANENT SET

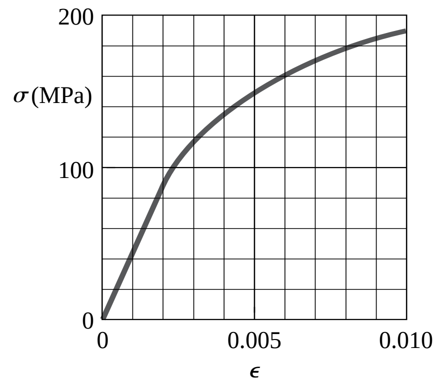
$\varepsilon_R L = 2.81$ in. ←

(b) PROPORTIONAL LIMIT WHEN RELOADED

$\sigma_B = 31.8$ ksi ←

Problem 1.4-4 A circular bar of magnesium alloy is 750 mm long. The stress-strain diagram for the material is shown in the figure. The bar is loaded in tension to an elongation of 6.0 mm, and then the load is removed.

(a) What is the permanent set of the bar?
(b) If the bar is reloaded, what is the proportional limit? (*Hint*: Use the concepts illustrated in Figs. 1-18b and 1-19.)



Solution 1.4-4

NUMERICAL DATA $L = 750$ mm $\delta = 6$ mm

STRESS AND STRAIN AT PT B

$\varepsilon_B = \dfrac{\delta}{L}$ $\varepsilon_B = 8 \times 10^{-3}$ $\sigma_B = 180$ MPa

ELASTIC RECOVERY

$\text{slope} = \dfrac{178}{0.004}$ $\text{slope} = 4.45 \times 10^4$

$\varepsilon_E = \dfrac{\sigma_B}{\text{slope}}$

$\varepsilon_E = 4.045 \times 10^{-3}$

RESIDUAL STRAIN

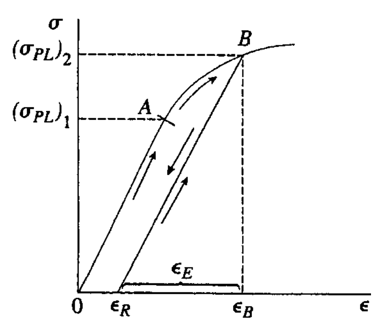
$\varepsilon_R = \varepsilon_B - \varepsilon_E$ $\varepsilon_R = 3.955 \times 10^{-3}$

(a) PERMANENT SET

$\varepsilon_R L = 2.97$ mm $\leftarrow$

(b) PROPORTIONAL LIMIT WHEN RELOADED

$\sigma_B = 180$ MPa $\leftarrow$



Problem 1.4-5 A wire of length $L = 4$ ft and diameter $d = 0.125$ in. is stretched by tensile forces $P = 600$ lb. The wire is made of a copper alloy having a stress-strain relationship that may be described mathematically by the following equation:

$$\sigma = \frac{18{,}000\epsilon}{1 + 300\epsilon} \qquad 0 \le \epsilon \le 0.03 \qquad (\sigma = \text{ksi})$$

in which ϵ is nondimensional and σ has units of kips per square inch (ksi).

(a) Construct a stress-strain diagram for the material.
(b) Determine the elongation of the wire due to the forces P.
(c) If the forces are removed, what is the permanent set of the bar?
(d) If the forces are applied again, what is the proportional limit?

Solution 1.4-5 Wire stretched by forces *P*

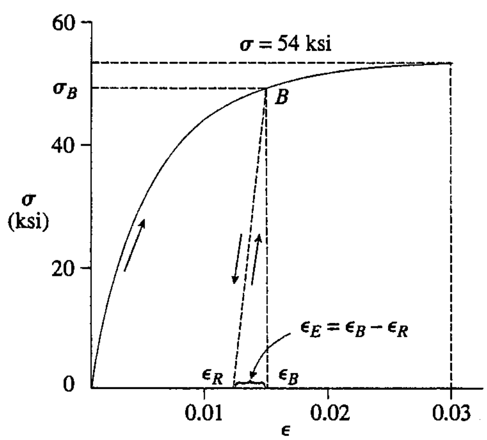
$L = 4 \text{ ft} = 48 \text{ in.} \qquad d = 0.125 \text{ in.}$

$P = 600 \text{ lb}$

COPPER ALLOY

$$\sigma = \frac{18{,}000\varepsilon}{1 + 300\varepsilon} \quad 0 \le \varepsilon \le 0.03 \; (\sigma = \text{ksi}) \quad \text{(Eq. 1)}$$

(a) STRESS-STRAIN DIAGRAM (From Eq. 1)



INITIAL SLOPE OF STRESS-STRAIN CURVE

Take the derivative of σ with respect to ε:

$$\frac{d\sigma}{d\varepsilon} = \frac{(1 + 300\varepsilon)(18{,}000) - (18{,}000)(300)\sigma}{(1 + 300\varepsilon)^2}$$

$$= \frac{18{,}000}{(1 + 300\varepsilon)^2}$$

$$\text{At } \varepsilon = 0, \quad \frac{d\sigma}{d\varepsilon} = 18{,}00 \text{ ksi}$$

$\therefore$ Initial slope = 18,000 ksi

ALTERNATIVE FORM OF THE STRESS-STRAIN RELATIONSHIP

Solve Eq. (1) for ε in terms of σ:

$$\varepsilon = \frac{\sigma}{18{,}000 - 300\sigma} \quad 0 \le \sigma \le 54 \text{ ksi} \; (\sigma = \text{ksi}) \quad \text{(Eq. 2)}$$

This equation may also be used when plotting the stress-strain diagram.

(b) ELONGATION δ OF THE WIRE

$$\sigma = \frac{P}{A} = \frac{600 \text{ lb}}{\frac{\pi}{4}(0.125 \text{ in.})^2} 48{,}900 \text{ psi} = 48.9 \text{ ksi}$$

From Eq. (2) or from the stress-strain diagram:

$\varepsilon = 0.0147$

$\delta = \varepsilon L = (0.0147)(48 \text{ in.}) = 0.71 \text{ in.} \quad \leftarrow$

STRESS AND STRAIN AT POINT B (see diagram)

$\sigma_B = 48.9 \text{ ksi} \quad \varepsilon_B = 0.0147$

ELASTIC RECOVERY ε_E

$$\varepsilon_E = \frac{\sigma_B}{\text{Slope}} = \frac{48.9 \text{ ksi}}{18{,}000 \text{ ksi}} = 0.00272$$

RESIDUAL STRAIN ε_R

$\varepsilon_R = \varepsilon_B - \varepsilon_E = 0.0147 - 0.0027 = 0.0120$

(c) Permanent set $= \varepsilon_R L = (0.0120)(48 \text{ in.})$

$= 0.58 \text{ in.} \quad \leftarrow$

(d) Proportional limit when reloaded $= \sigma_B$

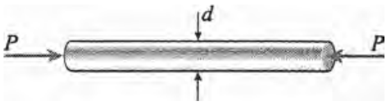
$\sigma_B = 49 \text{ ksi} \quad \leftarrow$

Linear Elasticity, Hooke's Law, and Poisson's Ratio

When solving the problems for Section 1.5, assume that the material behaves linearly elastically.

Problem 1.5-1 A high-strength steel bar used in a large crane has diameter $d = 2.00$ in. (see figure). The steel has modulus of elasticity $E = 29 \times 10^6$ psi and Poisson's ratio $\nu = 0.29$. Because of clearance requirements, the diameter of the bar is limited to 2.001 in. when it is compressed by axial forces.

What is the largest compressive load P_{max} that is permitted?



Solution 1.5-1 Steel bar in compression

STEEL BAR

$d = 2.00$ in. Max. $\Delta d = 0.001$ in.

$E = 29 \times 10^6$ psi $\nu = 0.29$

LATERAL STRAIN

$$\varepsilon' = \frac{\Delta d}{d} = \frac{0.001 \text{ in.}}{2.00 \text{ in.}} = 0.0005$$

AXIAL STRAIN

$$\varepsilon = -\frac{\varepsilon'}{\nu} = -\frac{0.0005}{0.29} = -0.001724$$

(shortening)

AXIAL STRESS

$$\sigma = E\varepsilon = (29 \times 10^6 \text{ psi})(-0.001724)$$
$$= -50.00 \text{ ksi (compression)}$$

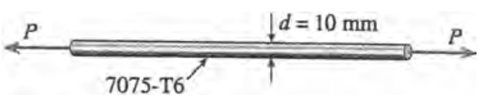
Assume that the yield stress for the high-strength steel is greater than 50 ksi. Therefore, Hooke's law is valid.

MAXIMUM COMPRESSIVE LOAD

$$P_{max} = \sigma A = (50.00 \text{ ksi})\left(\frac{\pi}{4}\right)(2.00 \text{ in.})^2$$
$$= 157 \text{ k} \quad \leftarrow$$

Problem 1.5-2 A round bar of 10 mm diameter is made of aluminum alloy 7075-T6 (see figure). When the bar is stretched by axial forces P, its diameter decreases by 0.016 mm.

Find the magnitude of the load P. (Obtain the material properties from Appendix H.)



Solution 1.5-2 Aluminum bar in tension

$d = 10$ mm $\Delta d = 0.016$ mm

(Decrease in diameter)

7075-T6

From Table H-2: $E = 72$ GPa $\nu = 0.33$

From Table H-3: Yield stress $\sigma_Y = 480$ MPa

LATERAL STRAIN

$$\varepsilon' = \frac{\Delta d}{d} = \frac{-0.016\text{mm}}{10\text{mm}} = -0.0016$$

AXIAL STRAIN

$$\varepsilon = -\frac{\varepsilon'}{\nu} = \frac{0.0016}{0.33}$$
$$= 0.004848 \text{ (Elongation)}$$

AXIAL STRESS

$$\sigma = E\varepsilon = (72 \text{ GPa})(0.004848)$$
$$= 349.1 \text{ MPa (Tension)}$$

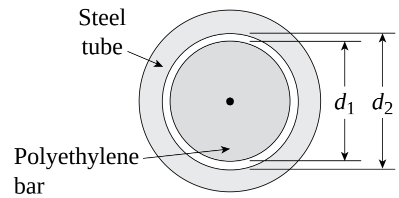
Because $\sigma < \sigma_Y$, Hooke's law is valid.

LOAD P (TENSILE FORCE)

$$P = \sigma A = (349.1 \text{ MPa})\left(\frac{\pi}{4}\right)(10 \text{ mm})^2$$
$$= 27.4 \text{ kN} \quad \leftarrow$$

Problem 1.5-3 A polyethylene bar having diameter $d_1 = 4.0$ in. is placed inside a steel tube having inner diameter $d_2 = 4.01$ in. (see figure). The polyethylene bar is then compressed by an axial force P.

At what value of the force P will the space between the nylon bar and the steel tube be closed? (For nylon, assume $E = 400$ ksi and $v = 0.4$.)



Solution 1.5-3

NUMERICAL DATA

$d_1 = 4$ in $\quad d_2 = 4.01$ in. $\quad E = 200$ ksi

$v = 0.4 \quad \Delta d_1 = 0.01$ in

$$A_1 = \frac{\pi}{4}d_1^2 \qquad A_2 = \frac{\pi}{4}d_2^2 \qquad A_1 = 12.566 \text{ in}^2$$

$A_2 = 12.629 \text{ in}^2$

LATERAL STRAIN

$$\varepsilon_p = \frac{\Delta d_1}{d_1} \qquad \varepsilon_p = \frac{0.01}{4} \qquad \varepsilon_p = 2.5 \times 10^{-3}$$

NORMAL STRAIN

$$\varepsilon_1 = \frac{-\varepsilon_p}{v} \qquad \varepsilon_1 = -6.25 \times 10^{-3}$$

AXIAL STRESS

$\sigma_1 = E\,\varepsilon_1 \qquad \sigma_1 = -1.25$ ksi

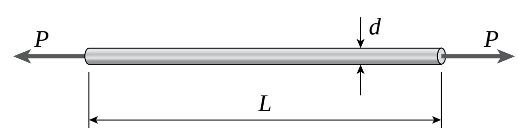
COMPRESSION FORCE

$P = EA_1\varepsilon_1$

$P = -15.71$ kips $\quad \leftarrow$

Problem 1.5-4 A prismatic bar with a circular cross section is loaded by tensile forces $P = 65$ kN (see figure). The bar has length $L = 1.75$ m and diameter $d = 32$ mm. It is made of aluminum alloy with modulus of elasticity $E = 75$ GPa and Poisson's ratio $\nu = 1/3$.

Find the increase in length of the bar and the percent decrease in its cross-sectional area.



Solution 1.5-4

NUMERICAL DATA

$P = 65\text{ kN} \qquad \nu = \frac{1}{3}$

$d = 32\text{ mm} \quad L = 1.75(1000)\text{ mm}$

$E = 75\text{ GPa}$

INITIAL AREA OF CROSS SECTION

$A_i = \frac{\pi}{4}d^2 \qquad A_i = 804.248\text{ mm}^2$

AXIAL STRAIN

$\varepsilon = \frac{P}{EA_i} \qquad \varepsilon = 1.078 \times 10^{-3}$

INCREASE IN LENGTH

$\Delta L = \varepsilon L \qquad \Delta L = 1.886\text{ mm} \quad \leftarrow$

LATERAL STRAIN

$\varepsilon_p = -\nu\varepsilon \qquad \varepsilon_p = -3.592 \times 10^{-4}$

DECREASE IN DIAMETER

$\Delta d = |\varepsilon_p d| \qquad \Delta d = 0.011\text{ mm}$

FINAL AREA OF CROSS SECTION

$A_f = \frac{\pi}{4}(d - \Delta d)^2$

$A_f = 803.67\text{ mm}^2$

$\%\text{ decrease in x-sec area} = \frac{A_f - A_i}{A_i}(100) \quad \leftarrow$

$= -0.072 \quad \leftarrow$

Problem 1.5-5 A bar of monel metal as in the figure (length $L = 9$ in., diameter $d = 0.225$ in.) is loaded axially by a tensile force P. If the bar elongates by 0.0195 in., what is the decrease in diameter d? What is the magnitude of the load P? Use the data in Table H-2, Appendix H.

Solution 1.5-5

NUMERICAL DATA

$E = 25000\text{ ksi}$

$\nu = 0.32$

$L = 9\text{ in.}$

$\delta = 0.0195\text{ in.}$

$d = 0.225\text{ in.}$

NORMAL STRAIN

$\varepsilon = \frac{\delta}{L} \quad \varepsilon = 2.167 \times 10^{-3}$

LATERAL STRAIN

$\varepsilon_p = -\nu\varepsilon \quad \varepsilon_p = -6.933 \times 10^{-4}$

DECREASE IN DIAMETER

$\Delta d = \varepsilon_p d$

$\Delta d = -1.56 \times 10^{-4}\text{ in.} \quad \leftarrow$

INITIAL CROSS SECTIONAL AREA

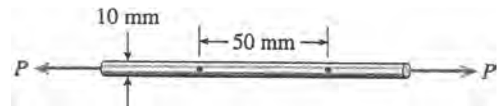
$A_i = \frac{\pi}{4}d^2 \quad A_i = 0.04\text{ in.}^2$

MAGNITUDE OF LOAD P

$P = EA_i\varepsilon$

$P = 2.15\text{ kips} \quad \leftarrow$

Problem 1.5-6 A tensile test is peformed on a brass specimen 10 mm in diameter using a gage length of 50 mm (see figure). When the tensile load P reaches a value of 20 kN, the distance between the gage marks has increased by 0.122 mm.



(a) What is the modulus of elasticity E of the brass?
(b) If the diameter decreases by 0.00830 mm, what is Poisson's ratio?

Solution 1.5-6 Brass specimen in tension

$d = 10$ mm Gage length $L = 50$ mm

$P = 20$ kN $\delta = 0.122$ mm $\Delta d = 0.00830$ mm

AXIAL STRESS

$$\sigma = \frac{P}{A} = \frac{20\text{ k}}{\frac{\pi}{4}(10\text{ mm})^2} = 254.6\text{ MPa}$$

Assume σ is below the proportional limit so that Hooke's law is valid.

AXIAL STRAIN

$$\varepsilon = \frac{\delta}{L} = \frac{0.122\text{ mm}}{50\text{ mm}} = 0.002440$$

(a) MODULUS OF ELASTICITY

$$E = \frac{\sigma}{\varepsilon} = \frac{254.6\text{ MPa}}{0.002440} = 104\text{ Gpa} \quad \leftarrow$$

(b) POISSON'S RATIO

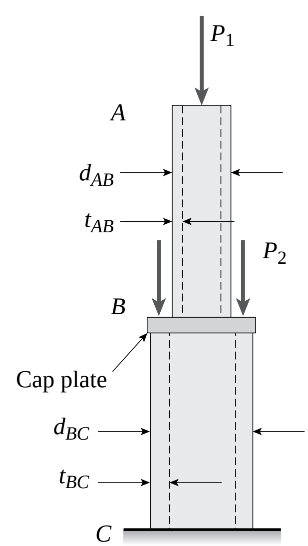
$$\varepsilon' = \nu\varepsilon$$

$$\Delta d = \varepsilon' d = \nu\varepsilon d$$

$$\nu = \frac{\Delta d}{\varepsilon d} = \frac{0.00830\text{ mm}}{(0.002440)(10\text{ mm})} = 0.34 \quad \leftarrow$$

Problem 1.5-7 A hollow, brass circular pipe ABC (see figure) supports a load $P_1 = 26.5$ kips acting at the top. A second load $P_2 = 22.0$ kips is uniformly distributed around the cap plate at B. The diameters and thicknesses of the upper and lower parts of the pipe are $d_{AB} = 1.25$ in., $t_{AB} = 0.5$ in., $d_{BC} = 2.25$ in., and $t_{AB} = 0.375$ in., respectively. The modulus of elasticity is 14,000 ksi. When both loads are fully applied, the wall thickness of pipe BC increases by $200\ 3\ 10^{-6}$ in.

(a) Find the increase in the inner diameter of pipe segment BC.
(b) Find Poisson's ratio for the brass.
(c) Find the increase in the wall thickness of pipe segment AB and the increase in the inner diameter of AB.



Solution 1.5-7

NUMERICAL DATA

$P_1 = 26.5$ kips

$P_2 = 22$ kips

$d_{AB} = 1.25$ in.

$t_{AB} = 0.5$ in.

$d_{BC} = 2.25$ in.

$t_{BC} = 0.375$ in.

$E = 14000$ ksi

$\Delta t_{BC} = 200 \times 10^{-6}$

(a) INCREASE IN THE INNER DIAMETER OF PIPE SEGMENT BC

$$\varepsilon_{pBC} = \frac{\Delta t_{BC}}{t_{BC}} \qquad \varepsilon_{pBC} = 5.333 \times 10^{-4}$$

$$\Delta d_{BCinner} = \varepsilon_{pBC}(d_{BC} - 2t_{BC})$$

$$\Delta\, d_{BCinner} = 8 \times 10^{-4} \text{ inches} \qquad \longleftarrow$$

(b) POISSON'S RATIO FOR THE BRASS

$$A_{BC} = \frac{\pi}{4}\left[\, d_{BC}^{\,2} - (\, d_{BC} - 2t_{BC})^2\right]$$

$$A_{BC} = 2.209 \text{ in.}^2$$

$$\varepsilon_{BC} = \frac{-(P_1 + P_2)}{(EA_{BC})} \qquad \varepsilon_{BC} = -1.568 \times 10^{-3}$$

$$\nu_{brass} = \frac{-\varepsilon_{pBC}}{\varepsilon_{BC}} \qquad \nu_{brass} = 0.34$$

(agrees with App. H (Table H-2))

(c) INCREASE IN THE WALL THICKNESS OF PIPE SEGMENT AB AND THE INCREASE IN THE INNER DIAMETER OF AB

$$A_{AB} = \frac{\pi}{4}\left[\, d_{AB}^{\,2} - (\, d_{AB} - 2t_{AB})^2\right]$$

$$\varepsilon_{AB} = \frac{-P_1}{EA_{AB}} \qquad \varepsilon_{AB} = -1.607 \times 10^{-3}$$

$$\varepsilon_{pAB} = -\nu_{brass}\varepsilon_{AB} \qquad \varepsilon_{pAB} = 5.464 \times 10^{-4}$$

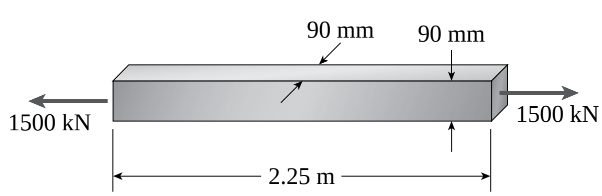
$$\Delta t_{AB} = \varepsilon_{pAB} t_{AB} \qquad \Delta t_{AB} = 2.73 \times 10^{-4} \text{ in.} \qquad \longleftarrow$$

$$\Delta d_{ABinner} = \varepsilon_{pAB}(d_{AB} - 2t_{AB})$$

$$\Delta d_{ABinner} = 1.366 \times 10^{-4} \text{ inches}$$

Problem 1.5-8 A brass bar of length 2.25 m with a square cross section of 90 mm on each side is subjected to an axial tensile force of 1500 kN (see figure). Assume that $E = 110$ GPa and $\nu = 0.34$.

Determine the increase in volume of the bar.



Solution 1.5-8

NUMERICAL DATA

$E = 110\ \text{GPa} \quad \nu = 0.34 \quad P = 1500\ \text{kN}$

$b = 90\ \text{mm} \quad L = 2250\ \text{mm}$

INITIAL VOLUME

$Vol_i = Lb^2$

$Vol_i = 1.822 \times 10^7\ \text{mm}^3$

NORMAL STRESS AND STRAIN

$\sigma = \dfrac{P}{b^2} \qquad \sigma = 185\ \text{MPa}$ (less than yield so Hooke's Law applies)

$\varepsilon = \dfrac{\sigma}{E} \qquad \varepsilon = 1.684 \times 10^{-3}$

LATERAL STRAIN

$\varepsilon_p = \nu\varepsilon \quad \varepsilon_p = 5.724 \times 10^{-4}$

CHANGE IN DIMENSIONS

$\Delta b = \varepsilon_p b \qquad \Delta b = 0.052\ \text{mm}$

$\Delta L = \varepsilon L \qquad \Delta L = 3.788\ \text{mm}$

FINAL LENGTH AND WIDTH

$L_f = L + \Delta L \quad L_f = 2.254 \times 10^3\ \text{mm}$

$b_f = b - \Delta b \qquad b_f = 89.948\ \text{mm}$

FINAL VOLUME

$Vol_f = L_f b_f^{\,2} \quad Vol_f = 1.823 \times 10^7\ \text{mm}^3$

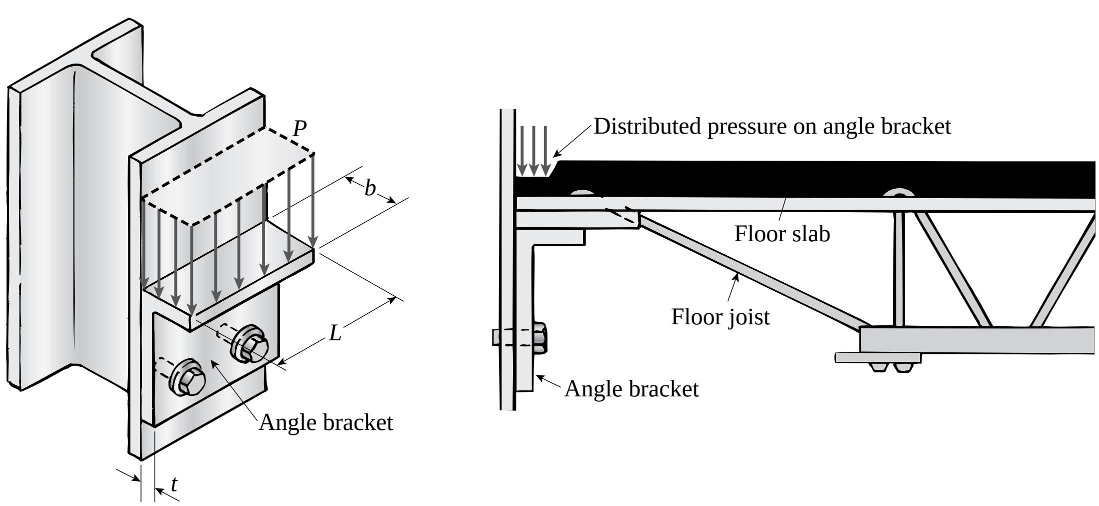
INCREASE IN VOLUME

$\Delta V = Vol_f - Vol \quad \Delta V = 9789\ \text{mm}^3$

Shear Stress and Strain

Problem 1.6-1 An angle bracket having thickness $t = 0.75$ in. is attached to the flange of a column by two 5/8-inch diameter bolts (see figure). A uniformly distributed load from a floor joist acts on the top face of the bracket with a pressure $p = 275$ psi. The top face of the bracket has length $L = 8$ in. and width $b = 3.0$ in.

Determine the average bearing pressure σ_b between the angle bracket and the bolts and the average shear stress τ_{aver} in the bolts. (Disregard friction between the bracket and the column.)



Solution 1.6-1

NUMERICAL DATA

$t = 0.75 \text{ in.} \qquad L = 8 \text{ in.}$

$b = 3. \text{ in.} \qquad p = \dfrac{275}{1000} \text{ ksi} \qquad d = \dfrac{5}{8} \text{ in.}$

BEARING FORCE

$F = pbL \qquad F = 6.6 \text{ kips}$

SHEAR AND BEARING AREAS

$A_S = \dfrac{\pi}{4} d^2 \qquad A_S = 0.307 \text{ in.}^2$

$A_b = dt \qquad A_b = 0.469 \text{ in.}^2$

BEARING STRESS

$\sigma_b = \dfrac{F}{2A_b} \qquad \sigma_b = 7.04 \text{ ksi} \quad \leftarrow$

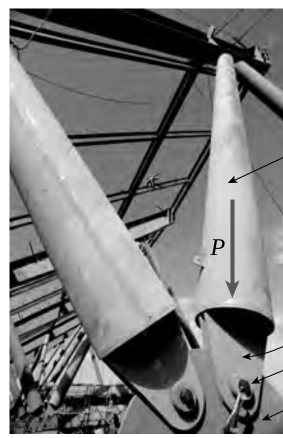
SHEAR STRESS

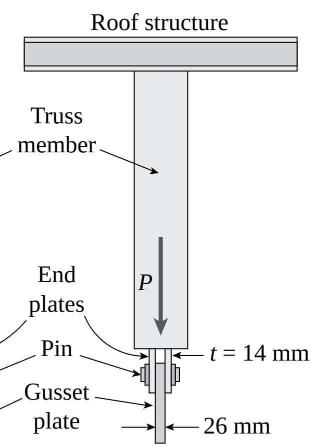
$\tau_{ave} = \dfrac{F}{2A_S} \qquad \tau_{ave} = 10.76 \text{ ksi} \quad \leftarrow$

Problem 1.6-2 Truss members supporting a roof are connected to a 26-mm-thick gusset plate by a 22 mm diameter pin as shown in the figure and photo. The two end plates on the truss members are each 14 mm thick.

(a) If the load $P = 80$ kN, what is the largest bearing stress acting on the pin?
(b) If the ultimate shear stress for the pin is 190 MPa, what force P_{ult} is required to cause the pin to fail in shear?

(Disregard friction between the plates.)





Solution 1.6-2

NUMERICAL DATA

$t_{ep} = 14 \text{ mm}$

$t_{gp} = 26 \text{ mm}$

$P = 80 \text{ kN}$

$d_p = 22 \text{ mm}$

$\tau_{ult} = 190 \text{ MPa}$

(a) BEARING STRESS ON PIN

$\sigma_b = \dfrac{P}{d_p t_{gp}}$ gusset plate is thinner than ($2\ t_{ep}$) so gusset plate controls

$\sigma_b = 139.9 \text{ MPa} \quad \leftarrow$

(b) ULTIMATE FORCE IN SHEAR

Cross sectional area of pin

$A_p = \dfrac{\pi d_p^2}{4}$

$A_p = 380.133 \text{ mm}^2$

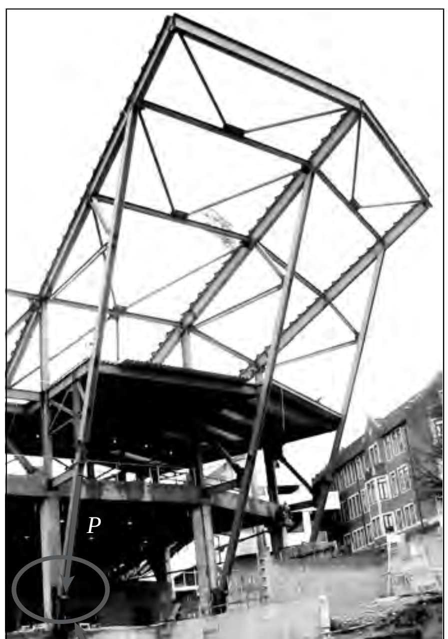
$P_{ult} = 2\tau_{ult} A_p \qquad P_{ult} = 144.4 \text{ kN} \quad \leftarrow$

Problem 1.6-3 The upper deck of a football stadium is supported by braces each of which transfers a load $P = 160$ kips to the base of a column [see figure part (a)]. A cap plate at the bottom of the brace distributes the load P to four flange plates ($t_f = 1$ in.) through a pin ($d_p = 2$ in.) to two gusset plates ($t_g = 1.5$ in.) [see figure parts (b) and (c)].

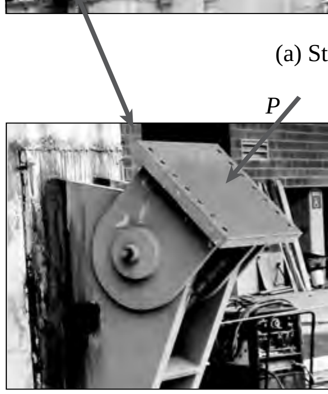
Determine the following quantities.

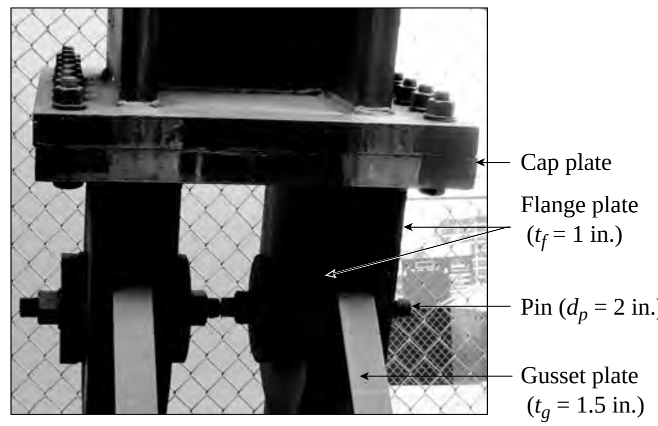
(a) The average shear stress τ_{aver} in the pin.
(b) The average bearing stress between the flange plates and the pin (σ_{bf}), and also between the gusset plates and the pin (σ_{bg}).

(Disregard friction between the plates.)

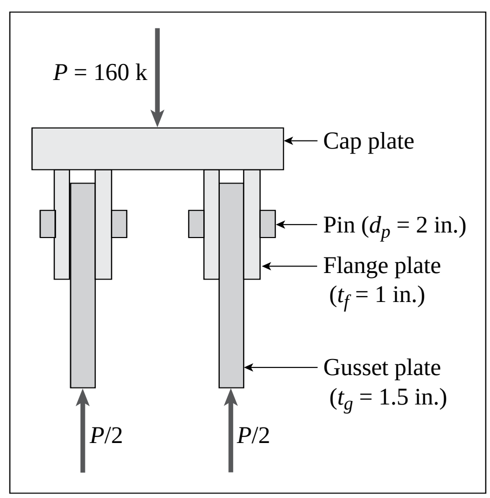


(a) Stadium brace





(b) Detail at bottom of brace



(c) Section through bottom of brace

Solution 1.6-3

NUMERICAL DATA

$P = 160 \text{ kips} \qquad d_p = 2 \text{ in.}$

$t_g = 1.5 \text{ in.} \qquad t_f = 1 \text{ in.}$

(a) SHEAR STRESS ON PIN

$$\tau = \frac{V}{\left(\frac{\pi d_p^2}{4}\right)} \qquad \tau = \frac{\frac{P}{4}}{\left(\frac{\pi d_p^2}{4}\right)}$$

$$\tau = 12.73 \text{ ksi} \quad \leftarrow$$

(b) BEARING STRESS ON PIN FROM FLANGE PLATE

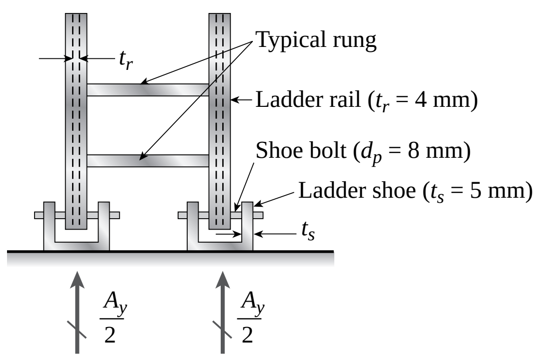
$$\sigma_{bf} = \frac{\frac{P}{4}}{d_p t_f} \qquad \sigma_{bf} = 20 \text{ ksi} \quad \leftarrow$$

BEARING STRESS ON PIN FROM GUSSET PLATE

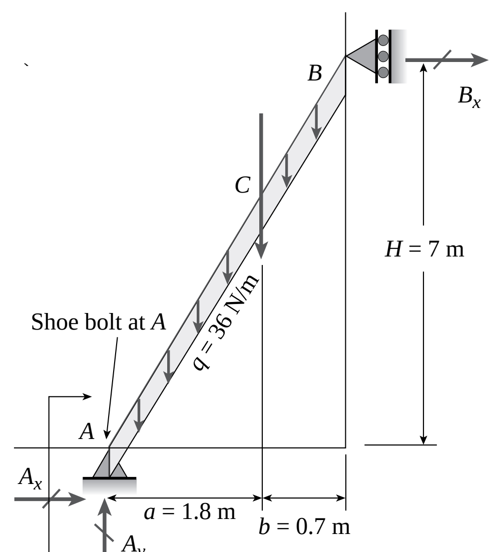
$$\sigma_{bg} = \frac{\frac{P}{2}}{d_p t_g} \qquad \sigma_{bg} = 26.7 \text{ ksi} \quad \leftarrow$$

Problem 1.6-4 The inclined ladder *AB* supports a house painter (82 kg) at *C* and the self weight (q = 36 N/m) of the ladder itself. Each ladder rail (t_r = 4 mm) is supported by a shoe (t_s = 5 mm) which is attached to the ladder rail by a bolt of diameter d_p = 8 mm.

(a) Find support reactions at *A* and *B*.
(b) Find the resultant force in the shoe bolt at *A*.
(c) Find maximum average shear (τ) and bearing (σ_b) stresses in the shoe bolt at *A*.



Section at base



Assume no slip at *A*

Solution 1.6-4

NUMERICAL DATA

$t_r = 4 \text{ mm} \qquad t_s = 5 \text{ mm}$

$d_p = 8 \text{ mm} \qquad P = 82 \text{ kg } (9.81 \text{ m/s}^2)$

$P = 804.42 \text{ N}$

$a = 1.8 \text{ m} \qquad b = 0.7 \text{ m} \qquad H = 7 \text{ m} \qquad q = 36 \frac{\text{N}}{\text{m}}$

(a) SUPPORT REACTIONS

$$L = \sqrt{(a + b)^2 + H^2} \qquad L = 7.433 \text{ m}$$

$$L_{AC} = \frac{a}{a + b} L \qquad L_{AC} = 5.352 \text{ m}$$

$$L_{CB} = \frac{b}{a + b} L \qquad L_{CB} = 2.081 \text{ m}$$

$$L_{AC} + L_{CB} = 7.433 \text{ m}$$

Sum moments about A

$$B_x = \frac{Pa + qL\left(\frac{a+b}{2}\right)}{-H}$$

$B_x = -255 \text{ N}$ ←

$A_x = -B_x \qquad A_y = P + qL$

$A_y = 1072 \text{ N}$ ←

(b) Resultant force in shoe bolt at A

$A_{resultant} = \sqrt{A_x^2 + A_y^2}$

$A_{resultant} = 1102 \text{ N}$ ←

(c) Maximum shear and bearing stresses in shoe bolt at A

$d_p = 8 \text{ mm} \qquad t_s = 5 \text{ mm} \qquad t_r = 4 \text{ mm}$

Shear area $\quad A_s = \frac{\pi}{4} d_p^2 \qquad A_s = 50.265 \text{ mm}^2$

Shear stress $\quad \tau = \frac{\frac{A_{resultant}}{2}}{2A_s} \qquad \tau = 5.48 \text{ MPa}$ ←

Bearing area $\quad A_b = 2d_p t_s \qquad A_b = 80 \text{ mm}^2$

Bearing stress $\quad \sigma_{bshoe} = \frac{\frac{A_{resultant}}{2}}{A_b}$

$\sigma_{bshoe} = 6.89 \text{ MPa}$ ←

Check bearing stress in ladder rail

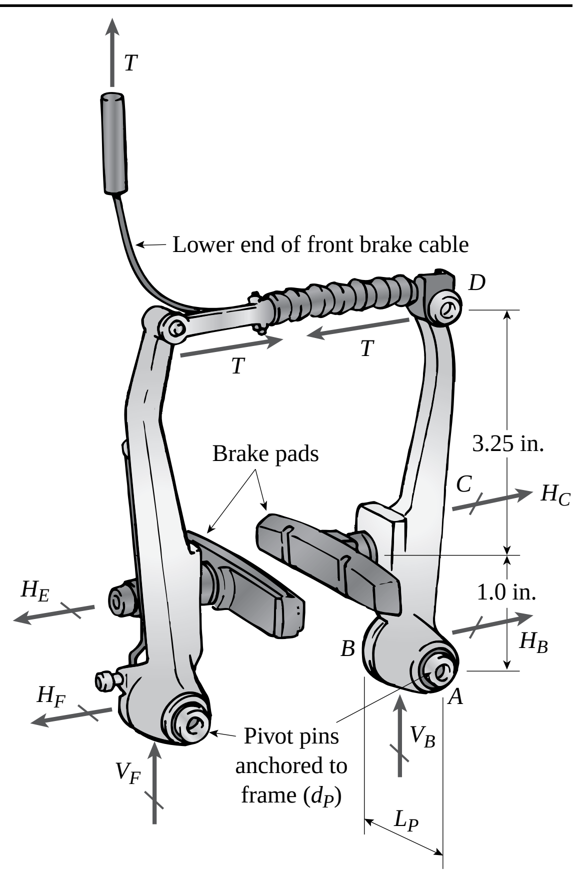
$\sigma_{brail} = \frac{\frac{A_{resultant}}{2}}{d_p t_r} \qquad \sigma_{brail} = 17.22 \text{ MPa}$

Problem 1.6-5 The force in the brake cable of the V-brake system shown in the figure is $T = 45$ lb. The pivot pin at A has diameter $d_p = 0.25$ in. and length $L_p = 5/8$ in.

Use dimensions show in the figure. Neglect the weight of the brake system.

(a) Find the average shear stress τ_{aver} in the pivot pin where it is anchored to the bicycle frame at B.

(b) Find the average bearing stress $\sigma_{b,aver}$ in the pivot pin over segment AB.



Solution 1.6-5

NUMERICAL DATA

$d_p = 0.25\text{ in.}\qquad L = \frac{5}{8}\text{ in.}\qquad CD = 3.25\text{ in.}$

$BC = 1\text{ in.}\qquad T = 45\text{ lb}$

EQUILIBRIUM - FIND HORIZONTAL FORCES AT B AND C [VERTICAL REACTION $V_B = 0$]

$\sum M_B = 0\qquad H_C = \frac{T(BC + CD)}{BC}$

$H_C = 191.25\text{ lb}\qquad \sum F_H = 0$

$H_B = T - H_C\qquad H_B = -146.25\text{ lb}$

(a) FIND THE AVE SHEAR STRESS τ_{ave} IN THE PIVOT PIN WHERE IT IS ANCHORED TO THE BICYCLE FRAME AT B:

$A_S = \frac{\pi d_p^2}{4}\qquad A_s = 0.049\text{ in.}^2$

$\tau_{ave} = \frac{|H_B|}{A_S}\qquad \tau_{ave} = 2979\text{ psi}\quad \leftarrow$

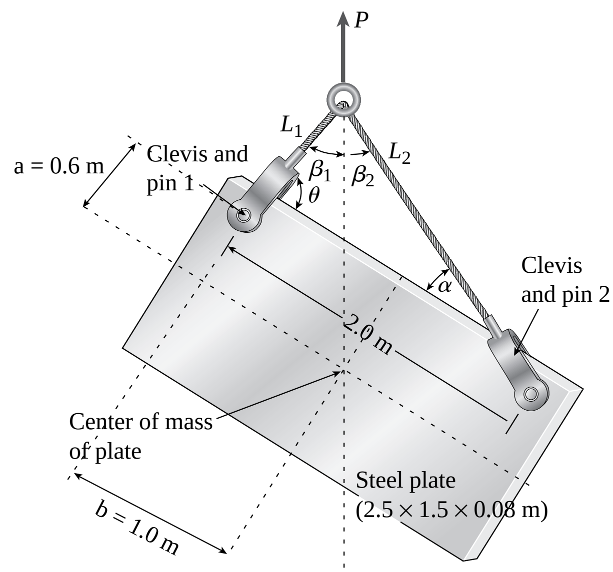
(b) FIND THE AVE BEARING STRESS $\sigma_{b,ave}$ IN THE PIVOT PIN OVER SEGMENT AB.

$A_b = d_p L\qquad A_b = 0.156\text{ in.}^2$

$\sigma_{b,ave} = \frac{|H_B|}{A_b}\qquad \sigma_{b,ave} = 936\text{ psi}\quad \leftarrow$

Problem 1.6-6 A steel plate of dimensions $2.5 \times 1.5 \times 0.08$ m and weighing 23.1kN is hoisted by steel cables with lengths $L_1 = 3.2$ m and $L_2 = 3.9$ m that are each attached to the plate by a clevis and pin (see figure). The pins through the clevises are 18 mm in diameter and are located 2.0 m apart. The orientation angles are measured to be $\theta = 94.4°$ and $\alpha = 54.9°$.

For these conditions, first determine the cable forces T_1 and T_2, then find the average shear stress τ_{aver} in both pin 1 and pin 2, and then the average bearing stress σ_b between the steel plate and each pin. Ignore the mass of the cables.



Solution 1.6-6

NUMERICAL DATA

$L_1 = 3.2\text{ m}\qquad L_2 = 3.9\text{ m}\qquad \alpha = 54.9\left(\frac{\pi}{180}\right)\text{ rad.}$

$\theta = 94.4\left(\frac{\pi}{180}\right)\text{ rad.}$

$a = 0.6\text{ m}\qquad b = 1\text{ m}$

$W = 77.0(2.5 \times 1.5 \times 0.08)\qquad W = 23.1\text{ kN}$

(77 = wt density of steel, kN/m^3)

SOLUTION APPROACH

STEP (1) $d = \sqrt{a^2 + b^2}\qquad d = 1.166\text{ m}$

STEP (2) $\theta_1 = \text{atan}\left(\frac{a}{b}\right)\qquad \theta_1\frac{180}{\pi} = 30.964\text{ degrees}$

STEP (3)-Law of cosines

$$H = \sqrt{d^2 + L_1^2 - 2dL_1\cos(\theta + \theta_1)}$$

$$H = 3.99\text{ m}$$

STEP (4) $\beta_1 = \mathrm{acos}\left(\frac{L_2^2 + H^2 - d^2}{2L_1H}\right)$

$\beta_1\frac{180}{\pi} = 13.789$ degrees

STEP (5) $\beta_2 = \mathrm{acos}\left(\frac{L_2^2 + H^2 - d^2}{2L_2H}\right)$

$\beta_2\frac{180}{\pi} = 16.95$ degrees

STEP (6)

Check $(\beta_1 + \beta_2 + \theta + \alpha)\frac{180}{\pi}$

$= 180.039$ degrees

Statics

$T_1\sin(\beta_1) = T_2\sin(\beta_2)$

$T_1 = T_2\left(\frac{\sin(\beta_2)}{\sin(\beta_1)}\right)$

$T_1\cos(\beta_1) + T_2\cos(\beta_2) = W$

$$T_2 = \frac{W}{\cos(\beta_1)\frac{\sin(\beta_2)}{\sin(\beta_1)} + \cos(\beta_2)}$$

$T_2 = 10.77$ kN $\leftarrow$

$T_1 = T_2\left(\frac{\sin(\beta_2)}{\sin(\beta_1)}\right)$ $\quad T_1 = 13.18$ kN $\leftarrow$

$T_1\cos(\beta_1) + T_2\cos(\beta_2) = 23.1$ < checks

SHEAR & BEARING STRESSES

$d_p = 18$ mm $\quad t = 100$ mm

$A_S = \frac{\pi}{4}d_p^2 \quad A_b = td_p$

$\tau_{1ave} = \frac{\frac{T_1}{2}}{A_S}$ $\quad \tau_{1ave} = 25.9$ MPa $\leftarrow$

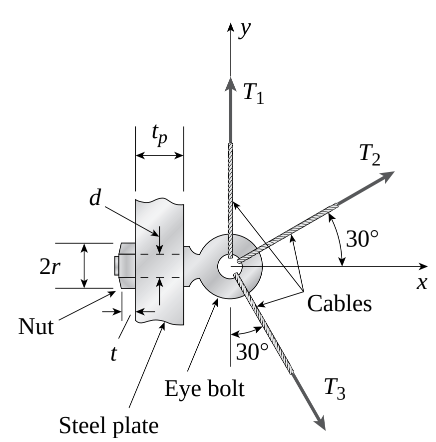
$\tau_{2ave} = \frac{\frac{T_2}{2}}{A_S}$ $\quad \tau_{2ave} = 21.2$ MPa $\leftarrow$

$\sigma_{b1} = \frac{T_1}{A_b}$ $\quad \sigma_{b1} = 7.32$ MPa $\leftarrow$

$\sigma_{b2} = \frac{T_2}{A_b}$ $\quad \sigma_{b2} = 5.99$ MPa $\leftarrow$

Problem 1.6-7 A special-purpose eye bolt of shank diameter $d = 0.50$ in. passes through a hole in a steel plate of thickness $t_p = 0.75$ in. (see figure) and is secured by a nut with thickness $t = 0.25$ in. The hexagonal nut bears directly against the steel plate. The radius of the circumscribed circle for the hexagon is $r = 0.40$ in. (which means that each side of the hexagon has length 0.40 in.). The tensile forces in three cables attached to the eye bolt are $T_1 = 800$ lb., $T_2 = 550$ lb., and $T_3 = 1241$ lb.

(a) Find the resultant force acting on the eye bolt.
(b) Determine the average bearing stress σ_b between the hexagonal nut on the eye bolt and the plate.
(c) Determine the average shear stress τ_{aver} in the nut and also in the steel plate.



Solution 1.6-7

CABLE FORCES

$T_1 = 800$ lb $\quad T_2 = 550$ lb $\quad T_3 = 1241$ lb

(a) RESULTANT

$$P = T_2\frac{\sqrt{3}}{2} + T_3 0.5 \qquad P = 1097 \text{ lb} \quad \leftarrow$$

(b) AVE. BEARING STRESS

$A_b = 0.2194$ in.^{2} $\quad$ hexagon (Case 25, App. D)

$$\sigma_b = \frac{P}{A_b} \qquad \sigma_b = 4999 \text{ psi} \quad \leftarrow$$

(c) AVE. SHEAR THROUGH NUT

$d = 0.5$ in. $\quad t = 0.25$ in.

$$A_{sn} = \pi dt \qquad A_{sn} = 0 \qquad \tau_{nut} = \frac{P}{A_{sn}}$$

$$\tau_{nut} = 2793 \text{ psi} \quad \leftarrow$$

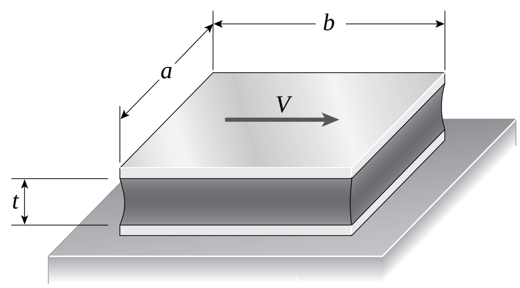
SHEAR THROUGH PLATE $\quad t_p = 0.75 \quad r = 0.40$

$$A_{spl} = 6rt_p \qquad A_{spl} = 2$$

$$\tau_{pl} = \frac{P}{A_{spl}} \qquad \tau_{pl} = 609 \text{ psi} \quad \leftarrow$$

Problem 1.6-8 An elastomeric bearing pad consisting of two steel plates bonded to a chloroprene elastomer (an artificial rubber) is subjected to a shear force *V* during a static loading test (see figure). The pad has dimensions $a = 125$ mm and $b = 240$ mm, and the elastomer has thickness $t = 50$ mm. When the force *V* equals 12 kN, the top plate is found to have displaced laterally 8.0 mm with respect to the bottom plate.

What is the shear modulus of elasticity *G* of the chloroprene?

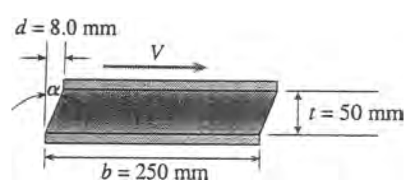


Solution 1.6-8

NUMERICAL DATA

$V = 12$ kN $\quad a = 125$ mm

$b = 240$ mm $\quad t = 50$ mm $\quad d = 8$ mm



AVERAGE SHEAR STRESS

$$\tau_{ave} = \frac{V}{ab} \qquad \tau_{ave} = 0.4 \text{ MPa}$$

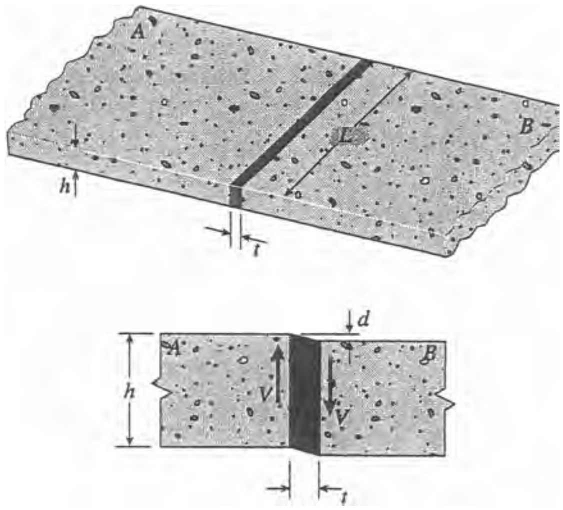
AVERAGE SHEAR STRAIN $\quad \gamma_{ave} = \frac{d}{t} \qquad \gamma_{ave} = 0.16$

SHEAR MODULUS G $\quad G = \frac{\tau_{ave}}{\gamma_{ave}}$

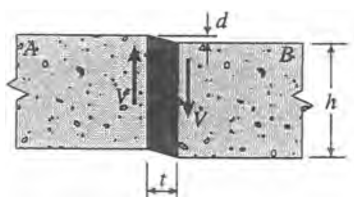
$$G = 2.5 \text{ MPa} \quad \leftarrow$$

Problem 1.6-9 A joint between two concrete slabs A and B is filled with a flexible epoxy that bonds securely to the concrete (see figure). The height of the joint is $h = 4.0$ in., its length is $L = 40$ in., and its thickness is $t = 0.5$ in. Under the action of shear forces V, the slabs displace vertically through the distance $d = 0.002$ in. relative to each other.

(a) What is the average shear strain γ_{aver} in the epoxy?
(b) What is the magnitude of the forces V if the shear modulus of elasticity G for the epoxy is 140 ksi?



Solution 1.6-9 Epoxy joint between concrete slabs



$h = 4.0$ in. $\quad t = 0.5$ in.

$L = 40$ in. $\quad d = 0.002$ in.

$G = 140$ ksi

(a) AVERAGE SHEAR STRAIN

$$\gamma_{aver} = \frac{d}{t} = 0.004 \quad \leftarrow$$

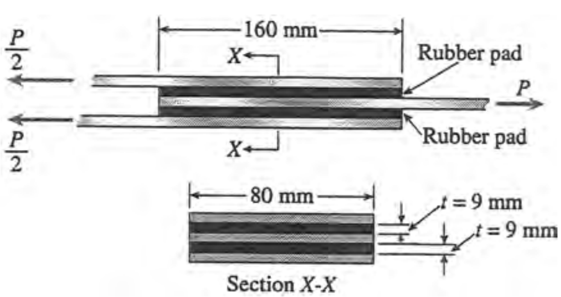
(b) SHEAR FORCES V

Average shear stress: $\tau_{aver} = G\gamma_{aver}$

$$V = \tau_{aver}(hL) = G\gamma_{aver}(hL)$$
$$= (140 \text{ ksi})(0.004)(4.0 \text{ in.})(40 \text{ in.})$$
$$= 89.6 \text{ k} \quad \leftarrow$$

Problem 1.6-10 A flexible connection consisting of rubber pads (thickness $t = 9$ mm) bonded to steel plates is shown in the figure. The pads are 160 mm long and 80 mm wide.

(a) Find the average shear strain γ_{aver} in the rubber if the force $P = 16$ kN and the shear modulus for the rubber is $G = 1250$ kPa.
(b) Find the relative horizontal displacement δ between the interior plate and the outer plates.



Solution 1.6-10 Rubber pads bonded to steel plates

Rubber pads: $t = 9$ mm

Length $L = 160$ mm

Width $b = 80$ mm

$G = 1250$ kPa

$P = 16$ kN

(a) SHEAR STRESS AND STRAIN IN THE RUBBER PADS

$$\tau_{\text{aver}} = \frac{P/2}{bL} = \frac{8\text{kN}}{(80\text{ mm})(160\text{ mm})} = 625\text{ kPa}$$

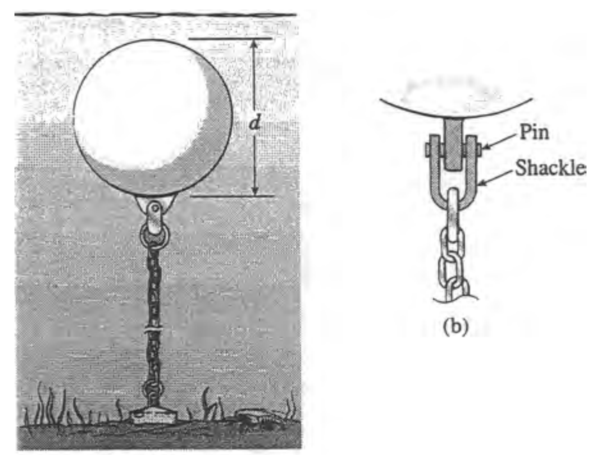
$$\gamma_{\text{aver}} = \frac{\tau_{\text{aver}}}{G} = \frac{625\text{ kPa}}{1250\text{ kPa}} = 0.50 \quad \leftarrow$$

(b) HORIZONTAL DISPLACEMENT

$$\delta = \gamma_{\text{aver}} t = (0.50)(9\text{ mm}) = 4.50\text{ mm} \quad \leftarrow$$

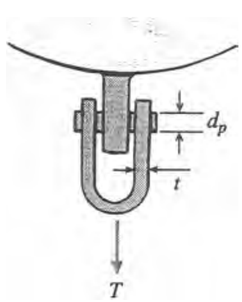
Problem 1.6-11 A spherical fiberglass buoy used in an underwater experiment is anchored in shallow water by a chain [see part (a) of the figure]. Because the buoy is positioned just below the surface of the water, it is not expected to collapse from the water pressure. The chain is attached to the buoy by a shackle and pin [see part (b) of the figure]. The diameter of the pin is 0.5 in. and the thickness of the shackle is 0.25 in. The buoy has a diameter of 60 in. and weighs 1800 lb on land (not including the weight of the chain).

(a) Determine the average shear stress τ_{aver} in the pin.
(b) Determine the average bearing stress σ_b between the pin and the shackle.



(a) (b)

Solution 1.6-11 Submerged buoy



d = diameter of buoy

= 60 in.

T = tensile force in chain

d_p = diameter of pin

= 0.5 in.

t = thickness of shackle

= 0.25 in.

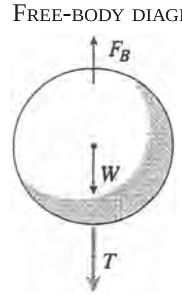
W = weight of buoy

= 1800 lb

γ_W = weight density of sea water

= 63.8 lb/ft^3

FREE-BODY DIAGRAM OF BUOY



F_B = buoyant force of water pressure (equals the weight of the displaced sea water)

V = volume of buoy

$$= \frac{\pi d^3}{6} = 65.45\text{ ft}^3$$

$$F_B = \gamma_W V = 4176\text{ lb}$$

EQUILIBRIUM

$T = F_B - W = 2376 \text{ lb}$

(a) AVERAGE SHEAR STRESS IN PIN

A_p = area of pin

$$A_p = \frac{\pi}{4} d_p^2 = 0.1963 \text{ in.}^2$$

$$\tau_{aver} = \frac{T}{2A_p} = 6050 \text{ psi} \quad \leftarrow$$

(b) BEARING STRESS BETWEEN PIN AND SHACKLE

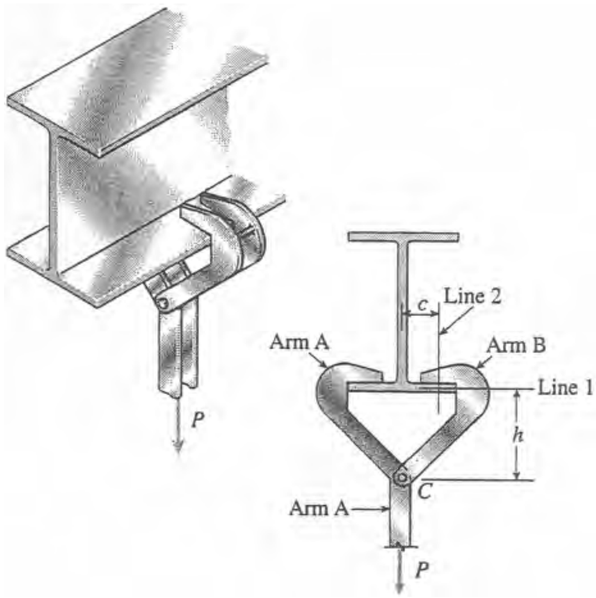
$A_b = 2d_p t = 0.2500 \text{ in.}^2$

$$\sigma_b = \frac{T}{A_b} = 9500 \text{ psi} \quad \leftarrow$$

Problem 1.6-12 The clamp shown in the figure is used to support a load hanging from the lower flange of a steel beam. The clamp consists of two arms (*A* and *B*) joined by a pin at *C*. The pin has diameter $d = 12$ mm. Because arm *B* straddles arm *A*, the pin is in double shear.

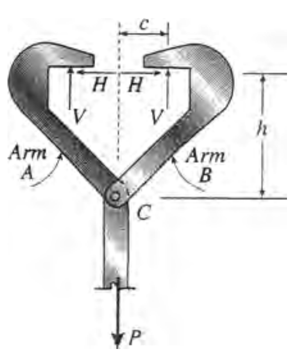
Line 1 in the figure defines the line of action of the resultant horizontal force *H* acting between the lower flange of the beam and arm *B*. The vertical distance from this line to the pin is $h = 250$ mm. Line 2 defines the line of action of the resultant vertical force *V* acting between the flange and arm *B*. The horizontal distance from this line to the centerline of the beam is $c = 100$ mm. The force conditions between arm *A* and the lower flange are symmetrical with those given for arm *B*.

Determine the average shear stress in the pin at *C* when the load $P = 18$ kN.



Solution 1.6-12 Clamp supporting a load *P*

FREE-BODY DIAGRAM OF CLAMP



$h = 250$ mm

$c = 100$ mm

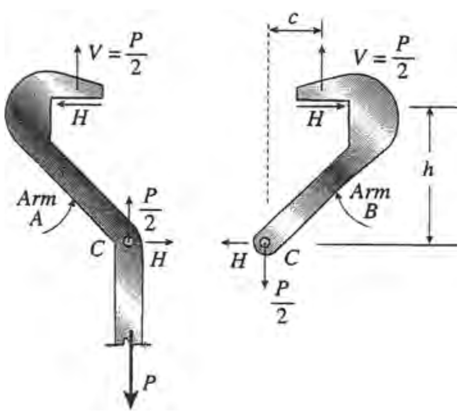
$P = 18$ kN

From vertical equilibrium:

$$V = \frac{P}{2} = 9 \text{ kN}$$

d = diameter of pin at C = 12 mm

FREE-BODY DIAGRAMS OF ARMS A AND B

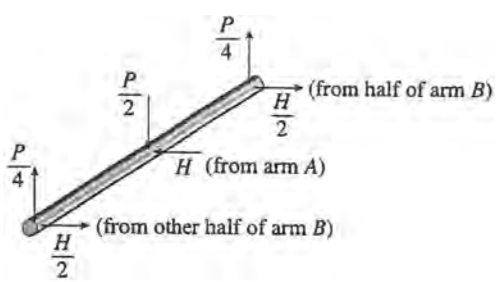


$$\Sigma M_C = 0$$

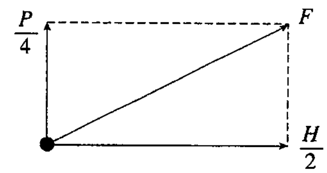
$$V_C - Hh = 0$$

$$H = \frac{V_C}{h} = \frac{P_c}{2h} = 3.6 \text{ kN}$$

FREE-BODY DIAGRAM OF PIN



SHEAR FORCE F IN PIN



$$F = \sqrt{\left(\frac{P}{4}\right)^2 + \left(\frac{H}{2}\right)^2}$$

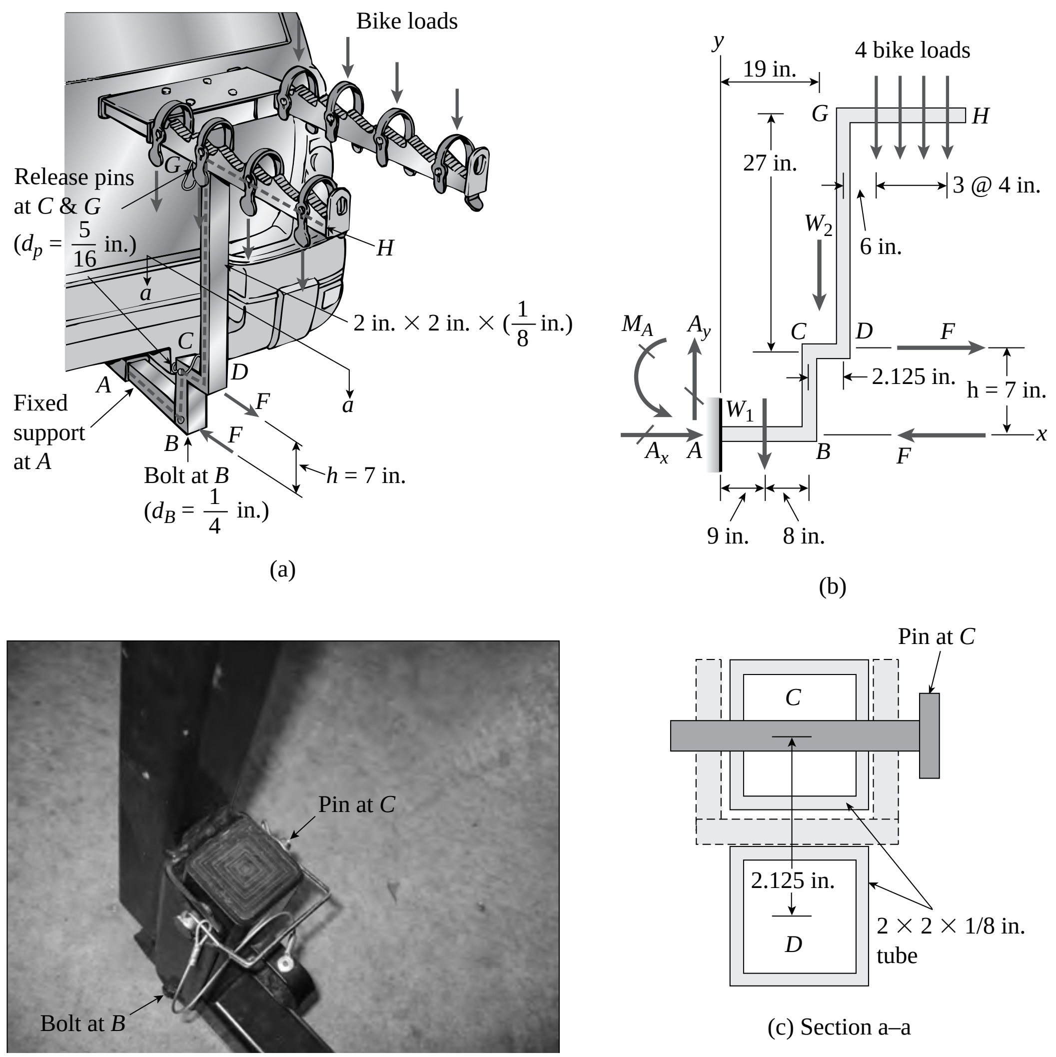
$$= 4.847 \text{ kN}$$

AVERAGE SHEAR STRESS IN THE PIN

$$\tau_{\text{aver}} = \frac{F}{A_{\text{pin}}} = \frac{F}{\frac{\pi a^2}{4}} = 42.9 \text{ MPa} \quad \leftarrow$$

Problem ★1.6-13 A hitch-mounted bicycle rack is designed to carry up to four 30-lb. bikes mounted on and strapped to two arms GH [see bike loads in the figure part (a)]. The rack is attached to the vehicle at A and is assumed to be like a cantilever beam $ABCDGH$ [figure part (b)]. The weight of fixed segment AB is $W_1 = 10$ lb, centered 9 in. from A [see the figure part (b)] and the rest of the rack weighs $W_2 = 40$ lb, centered 19 in. from A. Segment $ABCDG$ is a steel tube, 2×2 in., of thickness $t = 1/8$ in. Segment $BCDGH$ pivots about a bolt at B of diameter $d_B = 0.25$ in. to allow access to the rear of the vehicle without removing the hitch rack. When in use, the rack is secured in an upright position by a pin at C (diameter of pin $d_p = 5/16$ in.) [see photo and figure part (c)]. The overturning effect of the bikes on the rack is resisted by a force couple Fh at BC.

(a) Find the support reactions at A for the fully loaded rack;
(b) Find forces in the bolt at B and the pin at C.
(c) Find average shear stresses τ_{aver} in both the bolt at B and the pin at C.
(d) Find average bearing stresses σ_b in the bolt at B and the pin at C.



(a)

(b)

(c) Section a–a

Solution *1.6-13

NUMERICAL DATA

$t = \frac{1}{8}$ in. $\quad$ b = 2 in.

h = 7 in. $\quad W_1 = 10$ lb $\quad W_2 = 40$ lb

P = 30 lb $\quad d_B = 0.25$ in. $\quad d_p = \frac{5}{16}$ in.

(a) REACTIONS AT A

$A_x = 0 \quad \leftarrow$

$A_y = W_1 + W_2 + 4P \quad \leftarrow$

$A_y = 170$ lb $\quad \leftarrow$

$L_1 = 17 + 2.125 + 6 \qquad L_1 = 25$ in.

(dist from A to 1st bike)

$M_A = W_1(9) + W_2(19) + P(4L_1 + 4 + 8 + 12)$

$M_A = 4585$ in.-lb

(b) FORCES IN BOLT AT B & PIN AT C

$\Sigma F_y = 0 \qquad B_y = W_2 + 4P \qquad B_y = 160$ lb $\quad \leftarrow$

$\Sigma M_B = 0$

RHFB

$$B_x = \frac{\begin{array}{l}[W_2(19 - 17) + P(6 + 2.125) \\ \quad + P(8.125 + 4) + P(8.125 + 8) \\ \quad + P(8.125 + 12)]\end{array}}{h}$$

$$B_x = 254 \text{ lb} \quad \longleftarrow \qquad C_x = -B_x$$

$$B_{res} = \sqrt{{B_x}^2 + {B_y}^2} \qquad B_{res} = 300 \text{ lb} \quad \longleftarrow$$

(c) Average shear stresses τ_{ave} in both the bolt at B and the pin at C

$$A_{sB} = 2\frac{\pi d_B^2}{4} \qquad A_{sB} = 0.098 \text{ in}^2$$

$$\tau_B = \frac{B_{res}}{A_{sB}} \qquad \tau_B = 3054 \text{ psi} \quad \longleftarrow$$

$$A_{sC} = 2\frac{\pi d_p^2}{4} \qquad A_{sC} = 0.153 \text{ in}^2$$

$$\tau_C = \frac{B_x}{A_{sC}} \qquad \tau_C = 1653 \text{ psi} \quad \longleftarrow$$

(d) Bearing stresses σ_B in the bolt at B and the pin at C

$$t = 0.125 \text{ in}$$

$$A_{bB} = 2td_B \qquad A_{bB} = 0.063 \text{ in}^2$$

$$\sigma_{bB} = \frac{B_{res}}{A_{bB}} \qquad \sigma_{bB} = 4797 \text{ psi} \quad \longleftarrow$$

$$A_{bC} = 2td_p \qquad A_{bC} = 0.078 \text{ in}^2$$

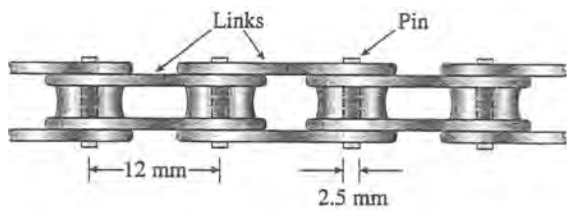
$$\sigma_{bC} = \frac{C_x}{A_{bC}} \qquad \sigma_{bC} = 3246 \text{ psi} \quad \longleftarrow$$

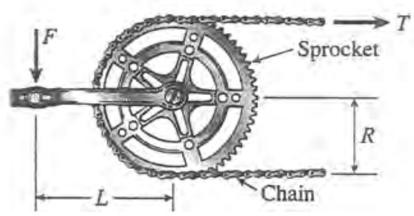
Problem 1.6-14 A bicycle chain consists of a series of small links, each 12 mm long between the centers of the pins (see figure). You might wish to examine a bicycle chain and observe its construction. Note particularly the pins, which we will assume to have a diameter of 2.5 mm.

In order to solve this problem, you must now make two measurements on a bicycle (see figure): (1) the length L of the crank arm from main axle to pedal axle, and (2) the radius R of the sprocket (the toothed wheel, sometimes called the chainring).

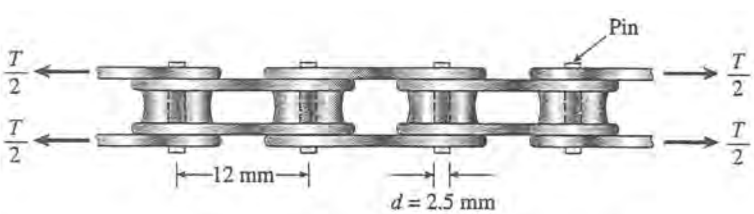
(a) Using your measured dimensions, calculate the tensile force T in the chain due to a force $F = 800$ N applied to one of the pedals.

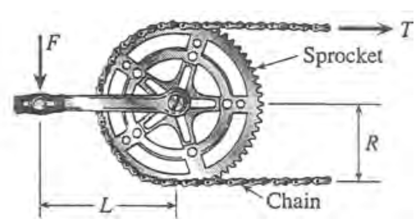
(b) Calculate the average shear stress τ_{aver} in the pins.





Solution 1.6-14 Bicycle chain





F = force applied to pedal = 800 N

L = length of crank arm

R = radius of sprocket

MEASUREMENTS (FOR AUTHOR'S BICYCLE)

(1) $L = 162$ mm (2) $R = 90$ mm

(a) TENSILE FORCE T IN CHAIN

$$\Sigma M_{\text{axle}} = 0 \qquad FL = TR \qquad T = \frac{FL}{R}$$

Substitute numerical values:

$$T = \frac{(800\ N)(162\text{ mm})}{90\text{ mm}} = 1440\text{ N} \qquad \leftarrow$$

(b) SHEAR STRESS IN PINS

$$\tau_{\text{aver}} = \frac{T/2}{A_{\text{pin}}} = \frac{T}{2\dfrac{\pi d^2}{(4)}} = \frac{2T}{\pi d^2}$$

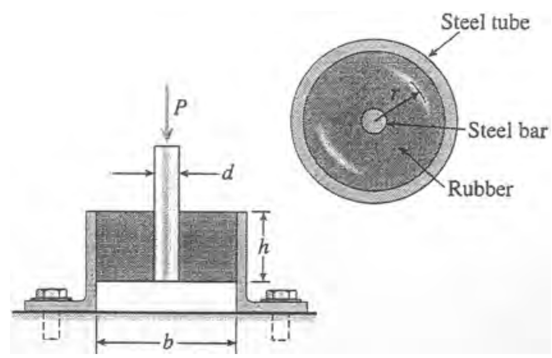
$$= \frac{2FL}{\pi d^2 R}$$

Substitute numerical values:

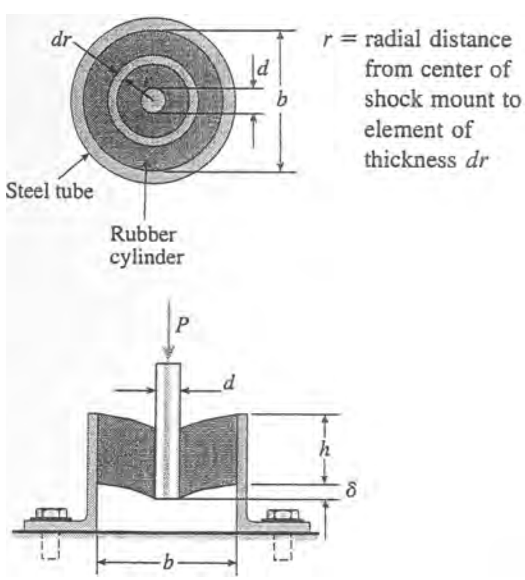
$$\tau_{\text{aver}} = \frac{2(800\text{ N})(162\text{ mm})}{\pi(2.5\text{ mm})^2(90\text{ mm})} = 147\text{ MPa} \qquad \leftarrow$$

Problem 1.6-15 A shock mount constructed as shown in the figure is used to support a delicate instrument. The mount consists of an outer steel tube with inside diameter b, a central steel bar of diameter d that supports the load P, and a hollow rubber cylinder (height h) bonded to the tube and bar.

(a) Obtain a formula for the shear τ in the rubber at a radial distance r from the center of the shock mount.
(b) Obtain a formula for the downward displacement δ of the central bar due to the load P, assuming that G is the shear modulus of elasticity of the rubber and that the steel tube and bar are rigid.



Solution 1.6-15 Shock mount



r = radial distance from center of shock mount to element of thickness dr

(a) SHEAR STRESS τ AT RADIAL DISTANCE r

A_s = shear area at distance r $= 2\pi rh$

$$\tau = \frac{P}{A_s} = \frac{P}{2\pi rh} \qquad \leftarrow$$

(b) DOWNWARD DISPLACEMENT δ

γ = shear strain at distance r

$$\gamma = \frac{\tau}{G} = \frac{P}{2\pi rhG}$$

$d\delta$ = downward displacement for element dr

$$d\delta = \gamma dr = \frac{Pdr}{2\pi rhG}$$

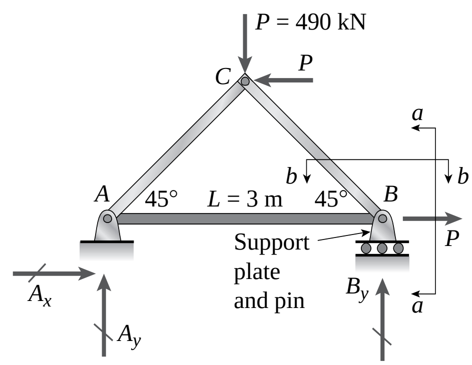
$$\delta = \int d\delta = \int_{d/2}^{b/2} \frac{Pdr}{2\pi rhG}$$

$$\delta = \frac{P}{2\pi hG}\int_{d/2}^{b/2}\frac{dr}{r} = \frac{P}{2\pi hG}\,[\text{In } r]_{d/2}^{b/2}$$

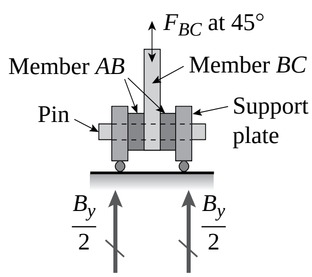
$$\delta = \frac{P}{2\pi hG}\ln\frac{b}{d} \qquad \leftarrow$$

Problem 1.6-16 The steel plane truss shown in the figure is loaded by three forces P, each of which is 490 kN. The truss members each have a cross-sectional area of 3900 mm^2 and are connected by pins each with a diameter of $d_p = 18$ mm. Members AC and BC each consist of one bar with thickness of $t_{AC} = t_{BC} = 19$ mm. Member AB is composed of two bars [see figure part (b)] each having thickness $t_{AB}/2 = 10$ mm and length $L = 3$ m. The roller support at B, is made up of two support plates, each having thickness $t_{sp}/2 = 12$ mm.

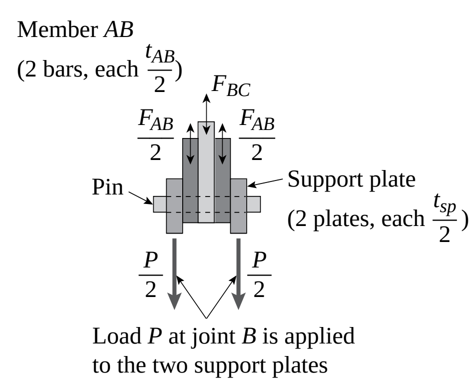
(a) Find support reactions at joints A and B and forces in members AB, BC, and AB.
(b) Calculate the largest average shear stress $\tau_{p,\max}$ in the pin at joint B, disregarding friction between the members; see figures parts (b) and (c) for sectional views of the joint.
(c) Calculate the largest average bearing stress $\sigma_{b,\max}$ acting against the pin at joint B.



(a)



(b) Section a–a at joint B (Elevation view)



(c) Section b–b at joint B (Plan view)

Solution 1.6-16

NUMERICAL DATA

$L = 3000 \text{ mm} \qquad P = 490 \text{ kN}$

$d_p = 18 \text{ mm} \qquad A = 3900 \text{ mm}^2$

$t_{AC} = 19 \text{ mm} \qquad t_{BC} = t_{AC}$

$t_{AB} = 20 \text{ mm} \qquad t_{sp} = 24 \text{ mm}$

(a) SUPPORT REACTIONS AND MEMBER FORCES

$\sum F_x = 0 \qquad A_x = 0 \quad \leftarrow$

$$\sum M_A = 0 \qquad B_y = \frac{1}{L}\left(P\frac{L}{2} - P\frac{L}{2}\right)$$

$B_y = 0 \quad \leftarrow$

$\sum F_y = 0 \qquad A_y = P$

$A_y = 490 \text{ kN} \quad \leftarrow$

METHOD OF JOINTS

$F_{AB} = P \qquad F_{BC} = 0 \quad \leftarrow$

$F_{AC} = -\sqrt{2}P$

$F_{AB} = 490 \text{ kN} \quad \leftarrow$

$F_{AC} = -693 \text{ kN} \quad \leftarrow$

(b) MAX. SHEAR STRESS IN PIN AT B

$$A_s = \frac{\pi d_p^2}{4} \qquad A_s = 254.469 \text{ mm}^2$$

$$\tau_{p\max} = \frac{\frac{F_{AB}}{2}}{A_s} \qquad \tau_{p\max} = 963 \text{ MPa} \quad \leftarrow$$

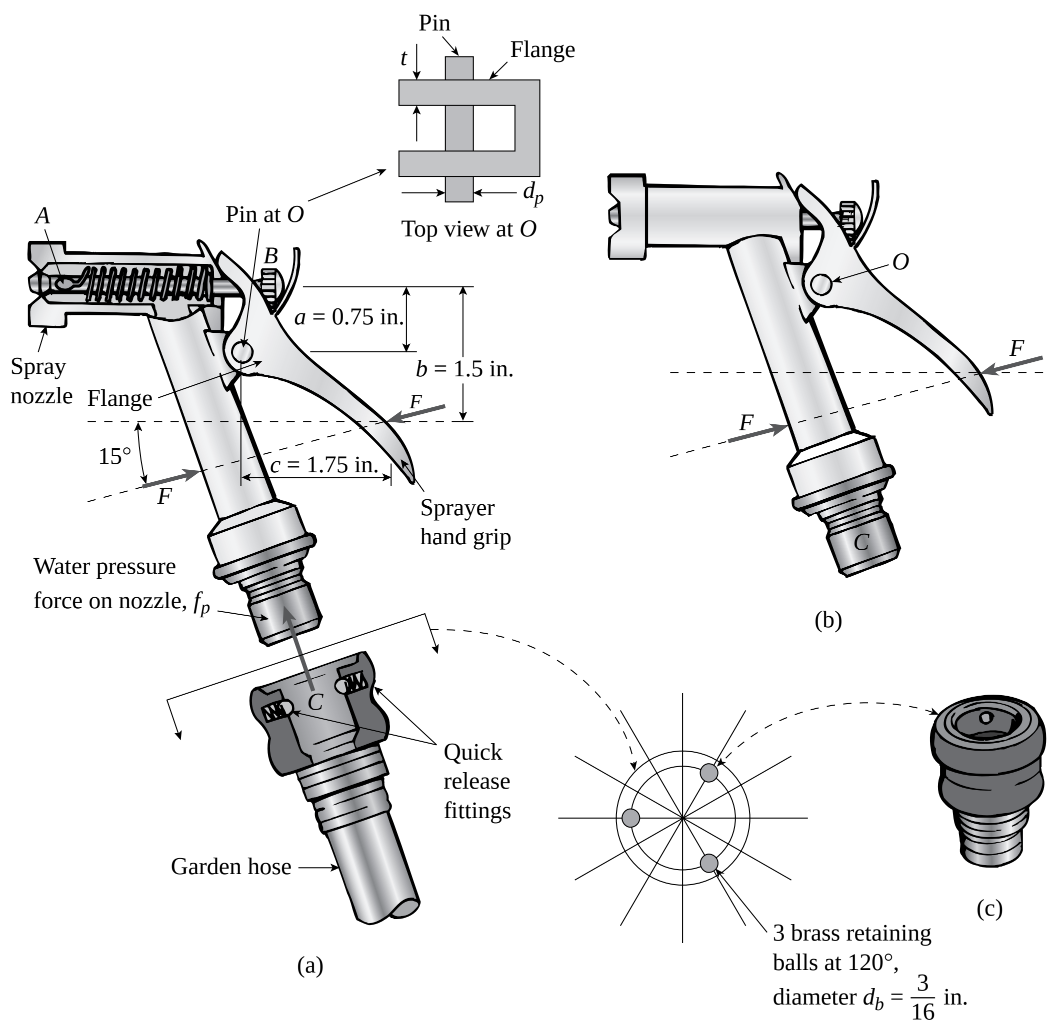
(c) MAX. BEARING STRESS IN PIN AT B ($t_{ab} < t_{sp}$ SO BEARING STRESS ON AB WILL BE GREATER)

$$A_b = d_p \frac{t_{AB}}{2}$$

$$\sigma_{b\max} = \frac{\frac{F_{AB}}{2}}{A_b} \qquad \sigma_{b\max} = 1361 \text{ MPa} \quad \leftarrow$$

Problem 1.6-17 A spray nozzle for a garden hose requires a force $F = 5$ lb. to open the spring-loaded spray chamber AB. The nozzle hand grip pivots about a pin through a flange at O. Each of the two flanges has thickness $t = 1/16$ in., and the pin has diameter $d_p = 1/8$ in. [see figure part (a)]. The spray nozzle is attached to the garden hose with a quick release fitting at B [see figure part (b)]. Three brass balls (diameter $d_b = 3/16$ in.) hold the spray head in place under water pressure force $f_p = 30$ lb. at C [see figure part (c)]. Use dimensions given in figure part (a).

(a) Find the force in the pin at O due to applied force F.
(b) Find average shear stress τ_{aver} and bearing stress σ_b in the pin at O.



Solution 1.6-17

NUMERICAL DATA

$F = 5\text{ lb}$ $\quad t = \frac{1}{16}\text{ in.}$ $\quad d_p = \frac{1}{8}\text{ in.}$ $\quad d_b = \frac{3}{16}\text{ in.}$

$f_p = 30\text{ lb}$ $\quad d_N = \frac{5}{8}\text{ in.}$ $\quad \theta = 15\frac{\pi}{180}\text{ rad.}$

$a = 0.75\text{ in}$ $\quad b = 1.5\text{ in}$ $\quad c = 1.75\text{ in}$

(a) FIND THE FORCE IN THE PIN AT O DUE TO APPLIED FORCE F

$$\sum M_o = 0$$

$$F_{AB} = \frac{[F\cos(\theta)(b - a)] + F\sin(\theta)(c)}{a}$$

$$F_{AB} = 7.849\text{ lb}$$

$$\sum F_H = 0 \qquad O_x = F_{AB} + F\cos(\theta)$$

$$O_y = F\sin(\theta)$$

$$O_x = 12.68\text{ lb} \qquad O_y = 1.294\text{ lb}$$

$$O_{res} = \sqrt{O_x^2 + O_y^2} \qquad O_{res} = 12.74\text{ lb} \quad \leftarrow$$

(b) FIND AVERAGE SHEAR STRESS τ_{ave} AND BEARING STRESS σ_b IN THE PIN AT O

$$A_s = 2\frac{\pi d_p^2}{4} \qquad \tau_O = \frac{O_{res}}{A_s} \qquad \tau_O = 519\text{ psi} \quad \leftarrow$$

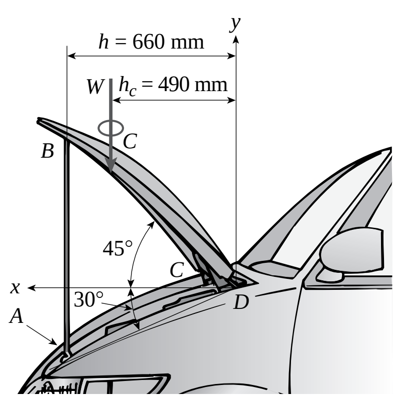
$$A_b = 2td_p \qquad \sigma_{bO} = \frac{O_{res}}{A_b} \qquad \sigma_{bO} = 816\text{ psi} \quad \leftarrow$$

(c) FIND THE AVERAGE SHEAR STRESS τ_{ave} IN THE BRASS RETAINING BALLS AT B DUE TO WATER PRESSURE FORCE F_p

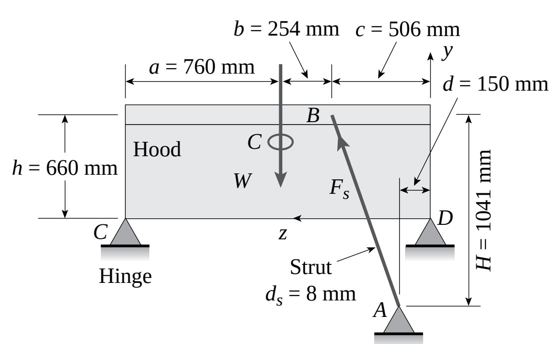
$$A_s = 3\frac{\pi d_b^2}{4} \qquad \tau_{ave} = \frac{f_p}{A_s} \qquad \tau_{ave} = 362\text{ psi} \quad \leftarrow$$

Problem 1.6-18 A single steel strut *AB* with diameter $d_s = 8$ mm. supports the vehicle engine hood of mass 20 kg which pivots about hinges at *C* and *D* [see figures (a) and (b)]. The strut is bent into a loop at its end and then attached to a bolt at *A* with diameter $d_b = 10$ mm. Strut *AB* lies in a vertical plane.

(a) Find the strut force F_s and average normal stress σ in the strut.
(b) Find the average shear stress τ_{aver} in the bolt at *A*.
(c) Find the average bearing stress σ_b on the bolt at *A*.



(a)



(b)

Solution 1.6-18

NUMERICAL DATA

$d_s = 8$ mm $\quad d_b = 10$ mm $\quad m = 20$ kg

$a = 760$ mm $\quad b = 254$ mm

$c = 506$ mm $\quad d = 150$ mm

$h = 660$ mm $\quad h_c = 490$ mm

$$H = h\left(\tan\left(30\frac{\pi}{180}\right) + \tan\left(45\frac{\pi}{180}\right)\right)$$

$H = 1041$ mm

$W = m\,(9.81\text{m/s}^2) \quad W = 196.2$ N

$$\frac{a + b + c}{2} = 760 \text{ mm}$$

VECTOR r_{AB}

$$r_{AB} = \begin{pmatrix} 0 \\ H \\ c - d \end{pmatrix} \qquad r_{AB} = \begin{pmatrix} 0 \\ 1.041 \times 10^3 \\ 356 \end{pmatrix}$$

UNIT VECTOR e_{AB}

$$e_{AB} = \frac{r_{AB}}{|r_{AB}|} \qquad e_AB = \begin{pmatrix} 0 \\ 0.946 \\ 0.324 \end{pmatrix} \qquad |e_{AB}| = 1$$

$$W = \begin{pmatrix} 0 \\ -W \\ 0 \end{pmatrix} \qquad W = \begin{pmatrix} 0 \\ -196.2 \\ 0 \end{pmatrix}$$

$$r_{DC} = \begin{pmatrix} h_c \\ h_c \\ b + c \end{pmatrix} \qquad r_{DC} = \begin{pmatrix} 490 \\ 490 \\ 760 \end{pmatrix}$$

$$\sum M_D \qquad M_D = r_{DB} \times F_s e_{AB} + W \times r_{DC}$$

(ignore force at hinge C since it will vanish with moment about line DC)

$$F_{sx} = 0 \qquad F_{sy} = \frac{H}{\sqrt{H^2 + (c - d)^2}} F_s$$

$$F_{sz} = \frac{c - d}{\sqrt{H^2 + (c - d)^2}} F_s$$

where

$$\frac{H}{\sqrt{H^2 + (c - d)^2}} = 0.946$$

$$\frac{c - d}{\sqrt{H^2 + (c - d)^2}} = 0.324$$

(a) FIND THE STRUT FORCE F_S AND AVERAGE NORMAL STRESS σ IN THE STRUT

$$\sum M_{lineDC} = 0 \qquad F_{sy} = \frac{|W|h_c}{h}$$

$F_{sy} = 145.664$

$$F_s = \frac{F_{sy}}{\dfrac{H}{\sqrt{H^2 + (c - d)^2}}} \qquad F_s = 153.9 \text{ N} \quad \leftarrow$$

$$A_{strut} = \frac{\pi}{4} d_s^2 \qquad A_{strut} = 50.265 \text{ mm}^2$$

$$\sigma = \frac{F_s}{A_{strut}} \qquad \sigma = 3.06 \text{ MPa} \quad \leftarrow$$

(b) FIND THE AVERAGE SHEAR STRESS τ_{ave} IN THE BOLT AT A

$d_b = 10$ mm

$$A_s = \frac{\pi}{4} d_b^2 \qquad A_s = 78.54 \text{ mm}^2$$

$$\tau_{ave} = \frac{F_s}{A_s} \qquad \tau_{ave} = 1.96 \text{ Mpa} \quad \leftarrow$$

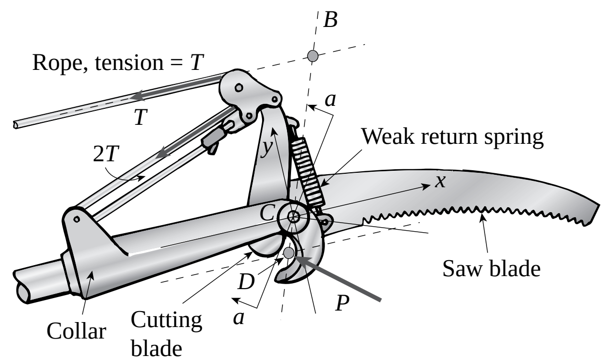
(c) FIND THE BEARING STRESS σ_b ON THE BOLT AT A

$A_b = d_s d_b \qquad A_b = 80 \text{ mm}^2$

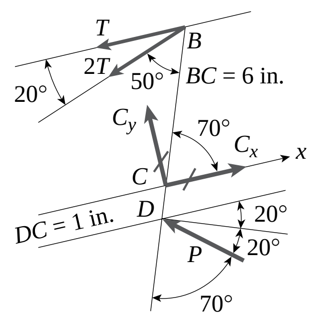
$$\sigma_b = \frac{F_s}{A_b} \qquad \sigma_b = 1.924 \text{ MPa} \quad \leftarrow$$

Problem 1.6-19 The top portion of a pole saw used to trim small branches from trees is shown in the figure part (a). The cutting blade *BCD* [see figure parts (a) and (c)] applies a force *P* at point *D*. Ignore the effect of the weak return spring attached to the cutting blade below *B*. Use properties and dimensions given in the figure.

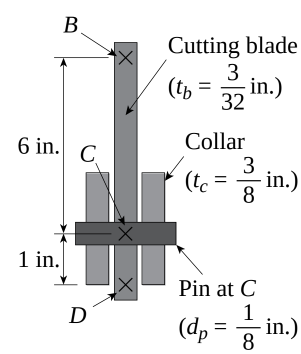
(a) Find the force *P* on the cutting blade at *D* if the tension force in the rope is $T = 25$ lb (see free body diagram in part (b)].
(b) Find force in the pin at *C*.
(c) Find average shear stress τ_{ave} and bearing stress σ_b in the support pin at *C* [see Section *a–a* through cutting blade in figure part (c)].



(a) Top part of pole saw



(b) Free-body diagram



(c) Section *a–a*

Solution 1.6-19

Numerical properties

$$d_p = \frac{1}{8}\text{ in} \qquad t_b = \frac{3}{32}\text{ in} \qquad t_c = \frac{3}{8}\text{ in}$$

$$T = 25\text{ lb} \qquad d_{BC} = 6\text{ in}$$

$$d_{CD} = 1\text{ in} \qquad \alpha = \frac{\pi}{180}\text{ rad/deg}$$

(a) Find the cutting force P on the cutting blade at D if the tension force in the rope is T = 25 lb:

$$\sum M_c = 0$$

$$\begin{aligned} M_C = {} & T(6\sin(70\,\alpha)) \\ & + 2T\cos(20\alpha)(6\sin(70\alpha)) \\ & - 2T\sin(20\alpha)(6\cos(70\alpha)) \\ & - P\cos(20\alpha)(1) \end{aligned}$$

Solve above equation for P

$$P = \frac{[T(6\sin(70\alpha)) + 2T\cos(20\alpha)\,6\sin(70\alpha)) - 2T\sin(20\alpha)(6\cos(70\alpha))]}{\cos(20\alpha)}$$

$$P = 395\text{ lbs} \quad \leftarrow$$

(b) Find force in the pin at *C*

Solve for forces on pin at C

$$\sum F_x = 0 \qquad C_x = T + 2T\cos(20\alpha) + P\cos(40\alpha)$$

$$C_x = 374\text{ lbs} \quad \leftarrow$$

$$\sum F_y = 0 \qquad C_y = 2T\sin(20\alpha) - P\sin(40\alpha)$$

$$C_y = -237\text{ lbs} \quad \leftarrow$$

RESULTANT AT C

$$C_{res} = \sqrt{C_x^2 + C_y^2} \qquad C_{res} = 443 \text{ lbs} \quad \longleftarrow$$

(c) Find maximum shear and bearing stresses in the support pin at C (see section a-a through saw).

SHEAR STRESS - PIN IN DOUBLE SHEAR

$$A_s = \frac{\pi}{4} d_p^2 \qquad A_s = 0.012 \text{ in}^2$$

$$\tau_{ave} = \frac{C_{res}}{2A_s} \qquad \tau_{ave} = 18.04 \text{ ksi}$$

BEARING STRESSES ON PIN ON EACH SIDE OF COLLAR

$$\sigma_{bC} = \frac{\frac{C_{res}}{2}}{d_p t_c} \qquad \sigma_{bC} = 4.72 \text{ ksi} \quad \longleftarrow$$

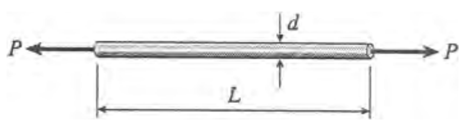
BEARING STRESS ON PIN AT CUTTING BLADE

$$\sigma_{bcb} = \frac{C_{res}}{d_p t_b} \qquad \sigma_{bcb} = 37.8 \text{ ksi} \quad \longleftarrow$$

Allowable Stresses and Allowable Loads

Problem 1.7.1 A bar of solid circular cross section is loaded in tension by forces P (see figure). The bar has length $L = 16.0$ in. and diameter $d = 0.50$ in. The material is a magnesium alloy having modulus of elasticity $E = 6.4 \times 10^6$ psi. The allowable stress in tension is $\sigma_{allow} = 17{,}000$ psi, and the elongation of the bar must not exceed 0.04 in.

What is the allowable value of the forces P?



Solution 1.7-1 Magnesium bar in tension

$L = 16.0$ in. $\qquad d = 0.50$ in.

$E = 6.4 \times 10^6$ psi

$\sigma_{allow} = 17{,}000$ psi $\qquad \delta_{max} = 0.04$ in.

MAXIMUM LOAD BASED UPON ELONGATION

$$\varepsilon_{max} = \frac{\delta_{max}}{L} = \frac{0.04\text{in.}}{16 \text{ in.}} \, 0.00250$$

$$\sigma_{max} = E\varepsilon_{max} = (6.4 \times 10^6 \text{ psi})(0.00250)$$

$$= 16{,}000 \text{ psi}$$

$$P_{max} = \sigma_{max} A = (16.000 \text{ psi})\left(\frac{\pi}{4}\right)(0.50 \text{ in.})^2$$

$$= 3140 \text{ lb}$$

MAXIMUM LOAD BASED UPON TENSILE STRESS

$$P_{max} = \sigma_{allow} A = (17{,}000 \text{ psi})\left(\frac{\pi}{4}\right)(0.50 \text{ in.})^2$$

$$= 3340 \text{ Ib}$$

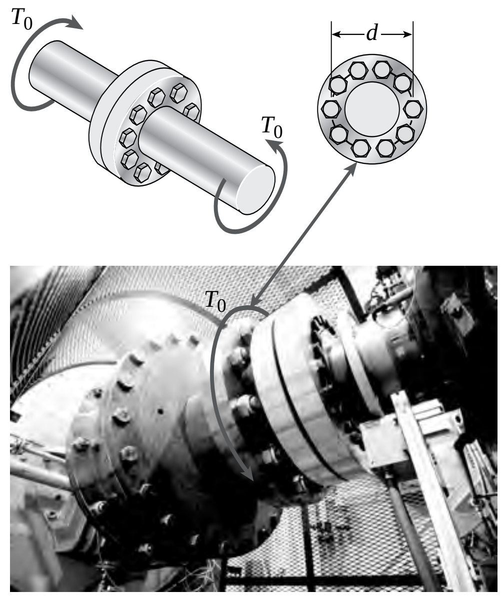
ALLOWABLE LOAD

Elongation governs.

$P_{allow} = 3140$ lb $\quad \longleftarrow$

Problem 1.7-2 A torque T_0 is transmitted between two flanged shafts by means of ten 20-mm bolts (see figure and photo). The diameter of the bolt circle is $d = 250$ mm.

If the allowable shear stress in the bolts is 90 MPa, what is the maximum permissible torque? (Disregard friction between the flanges.)



Solution 1.7-2 Shafts with flanges

NUMERICAL DATA

$r = 10 \quad d = 250$ mm

^ bolts ^ flange

$A_s = \pi r^2$

$A_s = 314.159\ m^2$

$\tau_a = 85$ MPa

MAX. PERMISSIBLE TORQUE

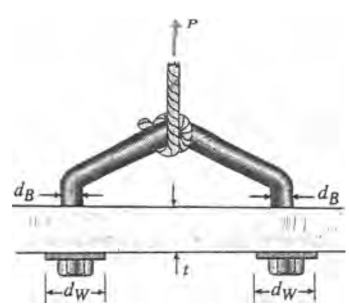
$$T_{max} = \tau_a A_s \left(r \frac{d}{2} \right)$$

$$T_{max} = 3.338 \times 10^7\ N \cdot mm$$

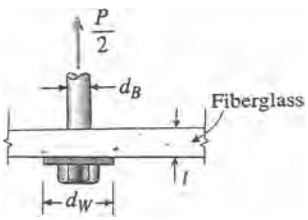
$$T_{max} = 33.4\ kN \cdot m \quad \leftarrow$$

Problem 1.7-3 A tie-down on the deck of a sailboat consists of a bent bar bolted at both ends, as shown in the figure. The diameter d_B of the bar is $^1/_4$ in., the diameter d_W of the washers is $^7/_8$ in., and the thickness t of the fiberglass deck is $^3/_8$ in.

If the allowable shear stress in the fiberglass is 300 psi, and the allowable bearing pressure between the washer and the fiberglass is 550 psi, what is the allowable load P_{allow} on the tie-down?



Solution 1.7-3 Bolts through fiberglass



$$d_B = \frac{1}{4}\text{ in.}$$

$$d_W = \frac{7}{8}\text{ in.}$$

$$t = \frac{3}{8}\text{ in.}$$

Allowable load based upon shear stress in fiberglass

$\tau_{\text{allow}} = 300$ psi

Shear area $A_s = \pi d_W t$

$$\frac{P_1}{2} = \tau_{\text{allow}}\, A_s = \tau_{\text{allow}}\,(\pi d_W t)$$

$$= (300\text{ psi})(\pi)\left(\frac{7}{8}\text{ in.}\right)\left(\frac{3}{8}\text{ in.}\right)$$

$$\frac{P_1}{2} = 309.3\text{ lb}$$

$P_1 = 619$ lb

Allowable load based upon bearing pressure

$\sigma_b = 550$ psi

Bearing area $A_b = \frac{\pi}{4}(d_W^2 - d_B^2)$

$$\frac{P_2}{2} = \sigma_b A_b = (550\text{ psi})\left(\frac{\pi}{4}\right)\left[\left(\frac{7}{8}\text{ in.}\right)^2 - \left(\frac{1}{4}\text{ in.}\right)^2\right]$$

$$= 303.7\text{ lb}$$

$P_2 = 607$ lb

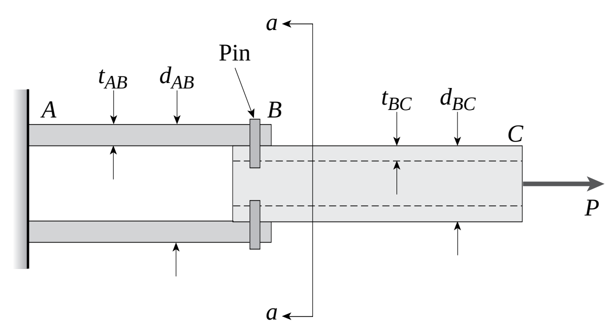
Allowable load

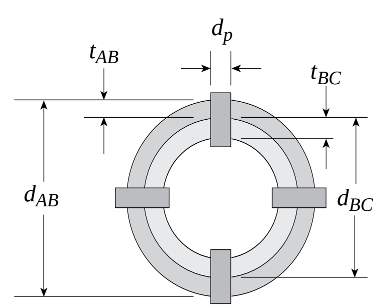
Bearing pressure governs.

$P_{\text{allow}} = 607$ lb $\leftarrow$

Problem 1.7-4 Two steel tubes are joined at B by four pins ($d_p = 11$ mm), as shown in the cross section a–a in the figure. The outer diameters of the tubes are $d_{AB} = 40$ mm and $d_{BC} = 28$ mm. The wall thicknesses are $t_{AB} = 6$ mm and $t_{BC} = 7$ mm. The yield stress in tension for the steel is $\sigma_Y = 200$ MPa and the ultimate stress in *tension* is $\sigma_U = 340$ MPa. The corresponding yield and ultimate values in *shear* for the pin are 80 MPa and 140 MPa, respectively. Finally, the yield and ultimate values in *bearing* between the pins and the tubes are 260 MPa and 450 MPa, respectively. Assume that the factors of safety with respect to yield stress and ultimate stress are 4 and 5, respectively.

(a) Calculate the allowable tensile force P_{allow} considering tension in the tubes.
(b Recompute P_{allow} for shear in the pins.
(c) Finally, recompute P_{allow} for bearing between the pins and the tubes. Which is the controlling value of P?





Section a–a

Solution 1.7-4

Yield and ultimate stresses (all in MPa)

TUBES:

$\sigma_Y = 200 \qquad \sigma_u = 340 \qquad FSy = 4$

PIN (SHEAR):

$\tau_Y = 80 \qquad \tau_u = 140 \qquad FSu = 5$

PIN (BEARING):

$\sigma_{bY} = 260 \qquad \sigma_{bu} = 450$

tubes and pin dimensions (mm)

$d_{AB} = 40 \qquad t_{AB} = 6$

$d_{BC} = d_{AB} - 2t_{AB} \qquad d_{BC} = 28$

$t_{BC} = 7 \qquad d_p = 11$

(a) P_{allow} CONSIDERING TENSION IN THE TUBES

$$A_{netAB} = \frac{\pi}{4}\left[d_{AB}^2 - (d_{AB} - 2t_{AB})^2 - 4\,d_p t_{AB}\right]$$

$$A_{netAB} = 433.45 \text{ mm}^2$$

$$A_{netBC} = \frac{\pi}{4}\left[d_{BC}^2 - (d_{BC} - 2t_{BC})^2 - 4\,d_p t_{BC}\right]$$

$$A_{netAB} = 219.911 \text{ mm}^2 \quad \text{use smaller}$$

$$P_{aT1} = \frac{\sigma_Y}{FSy} A_{netBC} \qquad P_{aT1} = 1.1 \times 10^4 \text{ N}$$

$$P_{aT1} = 11.0 \text{ kN} \quad \leftarrow$$

$$P_{aT2} = \frac{\sigma_u}{FSu} A_{netBC} \qquad P_{aT2} = 1.495 \times 10^4$$

(b) P_{allow} CONSIDERING SHEAR IN THE PINS

$$A_s = \frac{\pi}{4} d_p^2 \qquad A_s = 95.033 \text{ mm}^2 \text{ (one pin)}$$

$$P_{aS1} = (4A_s)\frac{\tau_Y}{FSy}$$

$$P_{aS1} = 7.60 \text{ kN} \quad \leftarrow$$

$$P_{aS2} = (4A_s)\frac{\tau_u}{FSu} \qquad P_{aS2} = 10.64 \text{ kN}$$

(c) P_{allow} CONSIDERING BEARING IN THE PINS

$$A_{bAB} = 4d_p t_{AB}$$

$$A_{bAB} = 264 \text{ mm}^2 \qquad < \text{smaller controls}$$

$$A_{bBC} = 4d_p t_{BC} \qquad A_{bBC} = 308 \text{ mm}^2$$

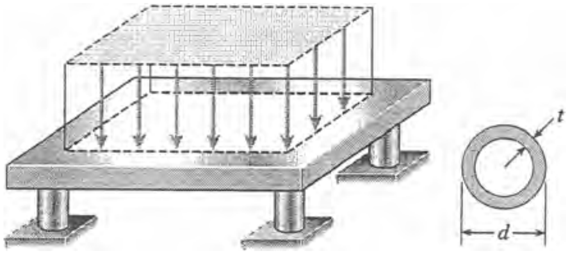
$$P_{ab1} = A_{bAB}\left(\frac{\sigma_{bY}}{FSy}\right) \qquad P_{ab1} = 1.716 \times 10^4$$

$$P_{ab1} = 17.16 \text{ kN} \quad \leftarrow$$

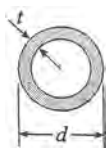
$$P_{ab2} = A_{bAB}\left(\frac{\sigma_{bu}}{FSu}\right) \qquad P_{ab2} = 23.8 \text{ kN}$$

Problem 1.7-5 A steel pad supporting heavy machinery rests on four short, hollow, cast iron piers (see figure). The ultimate strength of the cast iron in compression is 50 ksi. The outer diameter of the piers is $d = 4.5$ in. and the wall thickness is $t = 0.40$ in.

Using a factor of safety of 3.5 with respect to the ultimate strength, determine the total load P that may be supported by the pad.



Solution 1.7-5 Cast iron piers in compression



Four piers

$\sigma_U = 50$ ksi

$n = 3.5$

$$\sigma_{\text{allow}} = \frac{\sigma_U}{n} = \frac{50 \text{ ksi}}{3.5} = 14.29 \text{ ksi}$$

$d = 4.5$ in.

$t = 0.4$ in.

$d_0 = d - 2t = 3.7$ in.

$$A = \frac{\pi}{4}(d^2 - d_0^2) = \frac{\pi}{4}[(4.5 \text{ in.})^2 - (3.7 \text{ in.})^2]$$

$$= 5.152 \text{ in.}^2$$

P_1 = allowable load on one pier

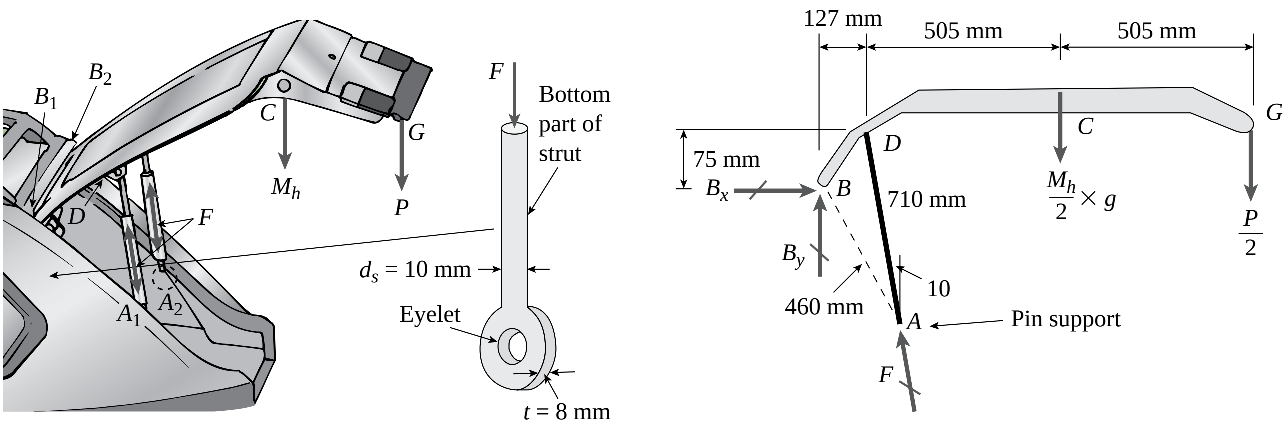
$$= \sigma_{\text{allow}} A = (14.29 \text{ ksi})(5.152 \text{ in.}^2)$$

$$= 73.62 \text{ k}$$

Total load $P = 4P_1 = 294$ k $\leftarrow$

Problem 1.7-6 The rear hatch of a van [*BDCF* in figure part (a)] is supported by two hinges at B_1 and B_2 and by two struts A_1B_1 and A_2B_2 (diameter $d_s = 10$ mm) as shown in figure part (b). The struts are supported at A_1 and A_2 by pins, each with diameter $d_p = 9$ mm and passing through an eyelet of thickness $t = 8$ mm at the end of the strut [figure part (b)]. If a closing force $P = 50$ N is applied at G and the mass of the hatch $M_h = 43$ kg is concentrated at C:

(a) What is the force F in each strut? [Use the free-body diagram of one half of the hatch in the figure part (c)]
(b) What is the maximum permissible force in the strut, F_{allow}, if the allowable stresses are as follows: compressive stress in the strut, 70 MPa; shear stress in the pin, 45 MPa; and bearing stress between the pin and the end of the strut, 110 MPa.



(a) (b) (c)

Solution 1.7-6

Numerical data

$M_h = 43$ kg $\quad \sigma_a = 70$ MPa

$\tau_a = 45$ MPa $\quad \sigma_{ba} = 110$ MPa

$d_s = 10$ mm $\quad d_p = 9$ mm $\quad t = 8$ mm

$P = 50$ N $\quad g = 9.81 \dfrac{\text{m}}{\text{s}^2}$

(a) Force F in each strut from statics (sum moments about B)

$$\alpha = 10\frac{\pi}{180} \quad F_V = F\cos(\alpha) \quad F_H = F\sin(\alpha)$$

$$\sum M_B = 0$$

$$F_V(127) + F_H(75)$$

$$= \frac{M_h}{2} g\,(127 + 505) + \frac{P}{2}[127 + 2(505)]$$

$$F(127\cos(\alpha) + 75\sin(\alpha))$$

$$= \frac{M_h}{2} g\,(127 + 505) + \frac{P}{2}[127 + 2(505)]$$

$$F = \frac{\frac{M_h}{2} g\,(127 + 505) + \frac{P}{2}[127 + 2(505)]}{(127\cos(\alpha) + 75\sin(\alpha))}$$

$$F = 1.171 \text{ kN} \quad \leftarrow$$

(b) MAX. PERMISSIBLE FORCE F IN EACH STRUT F_{max} IS SMALLEST OF THE FOLLOWING

$$F_{a1} = \sigma_a \frac{\pi}{4} d_s^{\,2} \qquad F_{a1} = 5.50 \text{ kN}$$

$$F_{a2} = \tau_a \frac{\pi}{4} d_p^{\,2}$$

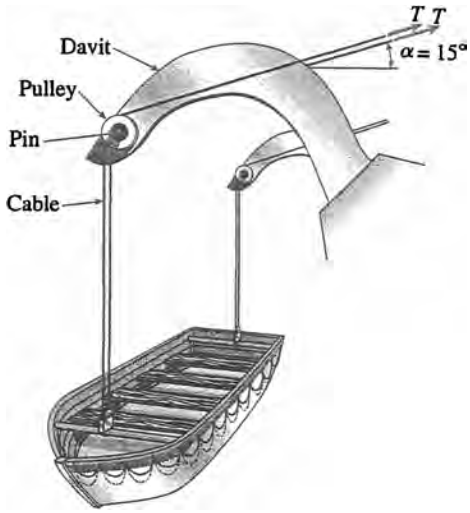
$$F_{a2} = 2.86 \text{ kN} \quad \leftarrow \qquad \frac{F_{a2}}{F} = 2.445$$

$$F_{a3} = \sigma_{ba} d_p t \qquad F_{a3} = 7.92 \text{ kN}$$

Problem 1.7-7 A lifeboat hangs from two ship's davits, as shown in the figure. A pin of diameter $d = 0.80$ in. passes through each davit and supports two pulleys, one on each side of the davit.

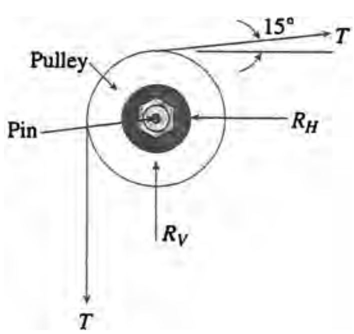
Cables attached to the lifeboat pass over the pulleys and wind around winches that raise and lower the lifeboat. The lower parts of the cables are vertical and the upper parts make an angle $\alpha = 15°$ with the horizontal. The allowable tensile force in each cable is 1800 lb, and the allowable shear stress in the pins is 4000 psi.

If the lifeboat weighs 1500 lb, what is the maximum weight that should be carried in the lifeboat?



Solution 1.7-7 Lifeboat supported by four cables

FREE-BODY DIAGRAM OF ONE PULLEY



Pin diameter $d = 0.80$ in.

T = tensile force in one cable

$T_{allow} = 1800$ lb

$\tau_{allow} = 4000$ psi

W = weight of lifeboat

$= 1500$ lb

$\Sigma F_{horiz} = 0 \qquad R_H = T\cos 15° = 0.9659T$

$\Sigma F_{vert} = 0 \qquad R_V = T - T\sin 15° = 0.7412T$

V = shear force in pin

$V = \sqrt{(R_H)^2 + (R_v)^2} = 1.2175T$

ALLOWABLE TENSILE FORCE IN ONE CABLE BASED UPON SHEAR IN THE PINS

$$V_{\text{allow}} = \tau_{\text{allow}}\, A_{\text{pin}} = (4000 \text{ psi})\left(\frac{\pi}{4}\right)(0.80 \text{ in.})^2$$

$$= 2011 \text{ lb}$$

$$V = 1.2175T \qquad T_1 = \frac{V_{\text{allow}}}{1.2175} = 1652 \text{ lb}$$

ALLOWABLE FORCE IN ONE CABLE BASED UPON TENSION IN THE CABLE

$$T_2 = T_{\text{allow}} = 1800 \text{ lb}$$

MAXIMUM WEIGHT

Shear in the pins governs.

$$T_{\max} = T_1 = 1652 \text{ lb}$$

Total tensile force in four cables

$$= 4T_{\max} = 6608 \text{ lb}$$

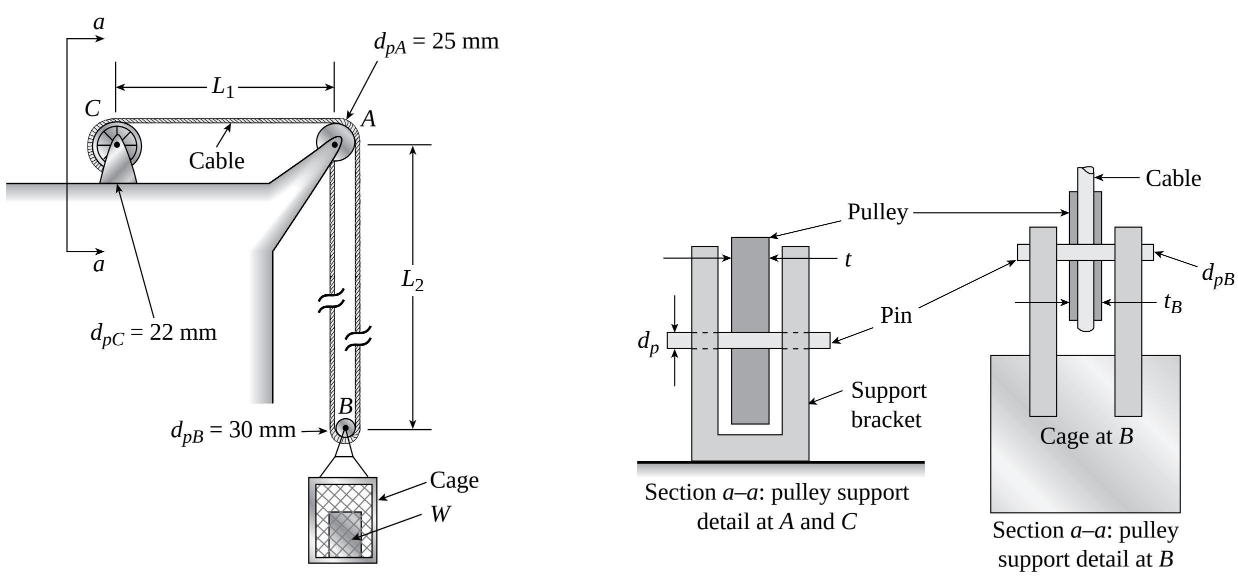
$$W_{\max} = 4T_{\max} - W$$

$$= 6608 \text{ lb} - 1500 \text{ lb}$$

$$= 5110 \text{ lb} \quad \leftarrow$$

Problem 1.7-8 A cable and pulley system in figure part (a) supports a cage of mass 300 kg at B. Assume that this includes the mass of the cables as well. The thickness of each the three steel pulleys is $t = 40$ mm. The pin diameters are $d_{pA} = 25$ mm, $d_{pB} = 30$ mm and $d_{pC} = 22$ mm [see figure, parts (a) and part (b)].

(a) Find expressions for the resultant forces acting on the pulleys at A, B, and C in terms of cable tension T.
(b) What is the maximum weight W that can be added to the cage at B based on the following allowable stresses? Shear stress in the pins is 50 MPa; bearing stress between the pin and the pulley is 110 MPa.



Section a–a: pulley support detail at A and C

Section a–a: pulley support detail at B

(a) (b)

Solution 1.7-8

NUMERICAL DATA

$M = 300\ \text{kg} \qquad g = 9.81\dfrac{m}{s^2}$

$\tau_a = 50\ \text{MPa} \qquad \sigma_{ba} = 110\ \text{MPa}$

$t_A = 40\ \text{mm} \qquad t_B = 40\ \text{mm}$

$t_C = 50 \qquad d_{pA} = 25\ \text{mm}$

$d_{pB} = 30 \qquad d_{pC} = 22\ \text{mm}$

(a) RESULTANT FORCES F ACTING ON PULLEYS A, B & C

$F_A = \sqrt{2}\,T \qquad F_B = 2T$

$F_C = T \qquad T = \dfrac{Mg}{2} + \dfrac{W_{max}}{2}$

$W_{max} = 2T - Mg$

From statics at B

(b) MAX. LOAD W THAT CAN BE ADDED AT B DUE TO τ_a & σ_{ba} IN PINS AT A, B & C

PULLEY AT A

$$\tau_A = \frac{F_A}{A_s}$$

DOUBLE SHEAR

$F_A = \tau_a A_s \qquad \sqrt{2}\,T = \tau_a A_s$

$$\frac{Mg}{2} + \frac{W_{max}}{2} = \frac{\tau_a A_s}{\sqrt{2}}$$

$$W_{max1} = \frac{2}{\sqrt{2}}\left(\tau_a A_s\right) - Mg$$

$$W_{max1} = \frac{2}{\sqrt{2}}\left(\tau_a 2\frac{\pi}{4} d_p A^2\right) - Mg$$

$$\frac{W_{max1}}{Mg} = 22.6$$

$W_{max1} = 66.5\ \text{kN} \quad \leftarrow$

(shear at A controls)

OR check bearing stress

$$W_{max2} = \frac{2}{\sqrt{2}}\left(\sigma_{ba} A_b\right) - Mg$$

$$W_{max2} = \frac{2}{\sqrt{2}}\left(\sigma_{ba} t_A d_{pA}\right) - Mg$$

$W_{max2} = 152.6\ \text{kN} \quad$ (bearing at A)

PULLEY AT B $\qquad 2T = \tau_a A_s$

$$W_{max3} = \frac{2}{2}(\tau_a A_s) - Mg$$

$$W_{max3} = \left[\tau_a\left(2\frac{\pi}{4} d_{pB}^2\right)\right] - Mg$$

$W_{max3} = 67.7\ \text{kN} \quad$ (shear at B)

$$W_{max4} = \frac{2}{2}(\sigma_{ba} A_b) - Mg$$

$W_{max4} = \sigma_{ba} t_B d_{pB} - Mg$

$W_{max4} = 129.1\ \text{kN} \quad$ (bearing at B)

PULLEY AT C $\qquad T = \tau_a A_s$

$W_{max5} = 2(\tau_a A_s) - Mg$

$$W_{max5} = \left[2\tau_a\left(2\frac{\pi}{4} d_{pC}^2\right)\right] - Mg$$

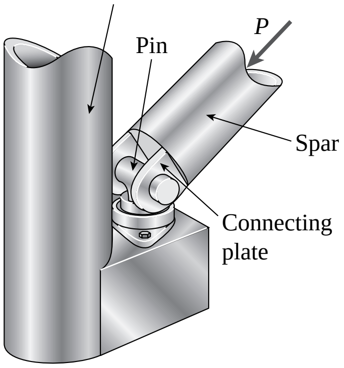
$W_{max5} = 7.3 \times 10^4 \qquad W_{max5} = 73.1\ \text{kN} \quad$ (shear at C)

$W_{max6} = 2\sigma_{ba} t_C d_{pC} - Mg$

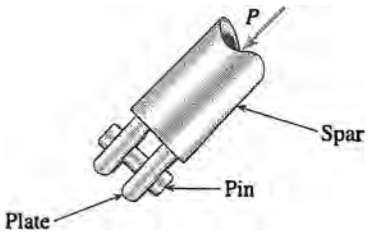
$W_{max6} = 239.1\ \text{kN} \quad$ (bearing at C)

Problem 1.7-9 A ship's spar is attached at the base of a mast by a pin connection (see figure). The spar is a steel tube of outer diameter $d_2 = 3.5$ in. and inner diameter $d_1 = 2.8$ in. The steel pin has diameter $d = 1$ in., and the two plates connecting the spar to the pin have thickness $t = 0.5$ in. The allowable stresses are as follows: compressive stress in the spar, 10 ksi; shear stress in the pin, 6.5 ksi; and bearing stress between the pin and the connecting plates, 16 ksi.

Determine the allowable compressive force P_{allow} in the spar.



Solution 1.7-9



NUMERICAL DATA

$d_2 = 3.5$ in. $d_1 = 2.8$ in.

$d_p = 1$ in. $t = 0.5$ in.

$\sigma_a = 10$ ksi $\tau_a = 6.5$ ksi $\sigma_{ba} = 16$ ksi

COMPRESSIVE STRESS IN SPAR

$$P_{a1} = \sigma_a \frac{\pi}{4}(d_2^2 - d_1^2) \qquad P_{a1} = 34.636 \text{ kips}$$

SHEAR STRESS IN PIN

$$P_{a2} = \tau_a\left(2\frac{\pi}{4}d_p^2\right)$$

$$P_{a2} = 10.21 \text{ kips} < \text{controls} \quad \leftarrow$$

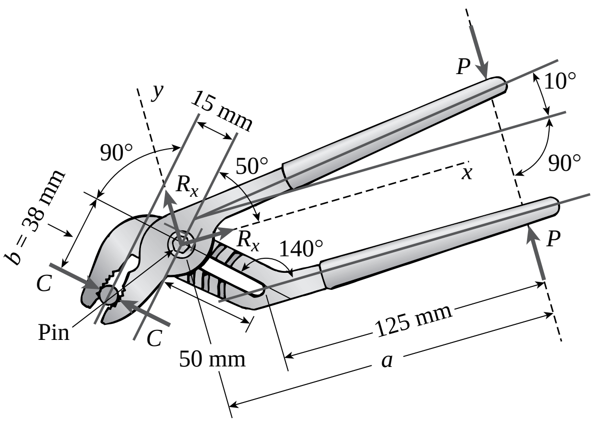
^double shear

BEARING STRESS BETWEEN PIN & CONECTING PLATES

$$P_{a3} = \sigma_{ba}(2d_p t) \qquad P_{a3} = 16 \text{ kips}$$

Problem 1.7-10 What is the maximum possible value of the clamping force C in the jaws of the pliers shown in the figure if the ultimate shear stress in the 5-mm diameter pin is 340 MPa?

What is the maximum permissible value of the applied load P if a factor of safety of 3.0 with respect to failure of the pin is to be maintained?



Solution 1.7-10

NUMERICAL DATA

$FS = 3 \qquad \tau_u = 340\ \text{MPa} \qquad \tau_a = \dfrac{\tau_u}{FS}$

$\alpha = 40\dfrac{\pi}{180}\ \text{rad} \qquad d = 5\ \text{mm}$

$\tau_a = \dfrac{\sqrt{R_x^2 + R_y^2}}{A_s}$ < pin at C in single shear

$R_x = -C\cos(\alpha) \qquad R_y = P + C\sin(\alpha)$

$a = 50\cos(\alpha) + 125 \qquad a = 163.302\ \text{mm}$

$b = 38\ \text{mm}$

STATICS $\quad \sum M_{pin} = 0 \qquad C = \dfrac{P(a)}{b}$

$R_x = -\dfrac{P(a)}{b}\cos(\alpha) \qquad R_y = P\left[1 + \dfrac{a}{b}\sin(\alpha)\right]$

$$P\sqrt{\left[-\frac{a}{b}\cos(\alpha)\right]^2 + \left[1 + \frac{a}{b}\sin(\alpha)\right]^2} = \tau_a A_s$$

$A_s = \dfrac{\pi}{4}d^2$

$\tau_a = \dfrac{\tau_u}{FS} \qquad \tau_a = 113.333\ \text{MPa}$

Find P_{max}

$$P_{max} = \frac{\tau_a A_s}{\sqrt{\left[-\frac{a}{b}\cos(\alpha)\right]^2 + \left[1 + \frac{a}{b}\sin(\alpha)\right]^2}}$$

$P_{max} = 445\ \text{N} \quad \longleftarrow$

here $\dfrac{a}{b} = 4.297$ < a/b = **mechanical advantage**

FIND MAX. CLAMPING FORCE

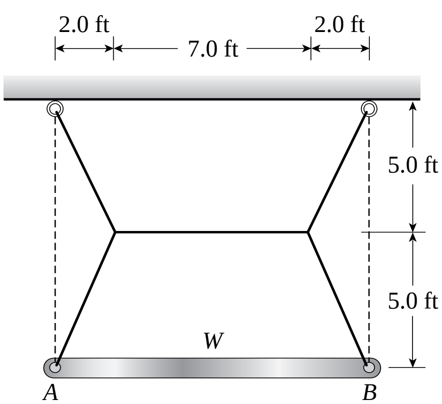
$C_{ult} = P_{max}FS\left(\dfrac{a}{b}\right) \qquad C_{ult} = 5739\ \text{N} \quad \longleftarrow$

$P_{ult} = P_{max}FS \qquad P_{ult} = 1335$

$\dfrac{C_{ult}}{P_{ult}} = 4.297$

Problem 1.7-11 A metal bar *AB* of weight *W* is suspended by a system of steel wires arranged as shown in the figure. The diameter of the wires is 5/64 in., and the yield stress of the steel is 65 ksi.

Determine the maximum permissible weight W_{max} for a factor of safety of 1.9 with respect to yielding.



Solution 1.7-11

NUMERICAL DATA

$d = \dfrac{5}{64}\ \text{in.} \qquad \sigma_Y = 65\ \text{ksi} \qquad FS_y = 1.9$

$\sigma_a = \dfrac{\sigma_Y}{FS_y} \qquad \sigma_a = 34.211\ \text{ksi}$

FORCES IN WIRES AC, EC, BD, FD

$\sum F_V = 0$ at A, B, E or F

$F_W = \dfrac{\sqrt{2^2 + 5^2}}{5} \times \dfrac{W}{2} \qquad \dfrac{\sqrt{2^2 + 5^2}}{10} = 0.539$

$W_{max} = 0.539\ \sigma_a \times A$

$$W_{max} = 0.539\left(\frac{\sigma_Y}{FS_y}\right)\left(\frac{\pi}{4}d^2\right)$$

$$W_{max} = 0.305 \text{ kips} \quad \leftarrow$$

CHECK ALSO FORCE IN WIRE CD

$\sum F_H = 0$ at C or D

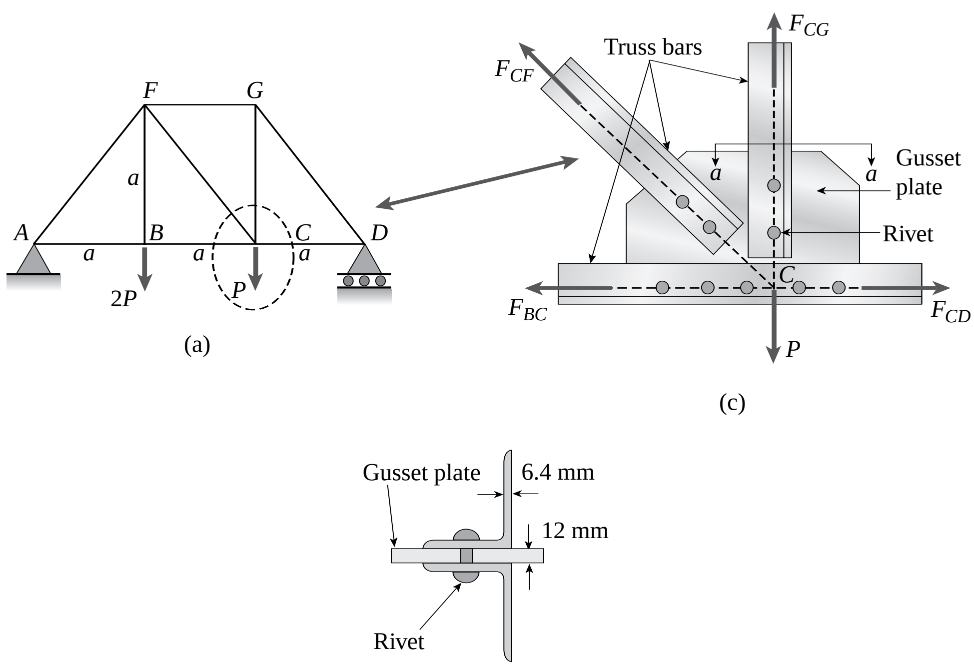
$$F_{CD} = 2\left(\frac{2}{\sqrt{2^2+5^2}}F_w\right)$$

$$F_{CD} = 2\left[\frac{2}{\sqrt{2^2+5^2}}\left(\frac{\sqrt{2^2+5^2}}{5} \times \frac{W}{2}\right)\right]$$

$$F_{CD} = \frac{2}{5}W \qquad \text{less than } F_{AC} \text{ so AC controls}$$

Problem 1.7-12 A plane truss is subjected to loads $2P$ and P at joints B and C, respectively, as shown in the figure part (a). The truss bars are made of two L102 × 76 × 6.4 steel angles [see Table E-5(b): cross sectional area of the two angles, $A = 2180 \text{ mm}^2$, figure part (b)] having an ultimate stress in tension equal to 390 MPa. The angles are connected to an 12 mm-thick gusset plate at C [figure part (c)] with 16-mm diameter rivets; assume each rivet transfers an equal share of the member force to the gusset plate. The ultimate stresses in shear and bearing for the rivet steel are 190 MPa and 550 MPa, respectively.

Determine the allowable load P_{allow} if a safety factor of 2.5 is desired with respect to the ultimate load that can be carried. (Consider tension in the bars, shear in the rivets, bearing between the rivets and the bars, and also bearing between the rivets and the gusset plate. Disregard friction between the plates and the weight of the truss itself.)



(a)

(c)

(b) Section a–a

Solution 1.7-12

NUMERICAL DATA

$A = 2180\ \text{mm}^2$

$t_g = 12\ \text{mm} \qquad d_r = 16\ \text{mm} \qquad t_{ang} = 6.4\ \text{mm}$

$\sigma_u = 390\ \text{MPa} \qquad \tau_u = 190\ \text{MPa}$

$\sigma_{bu} = 550\ \text{MPa} \qquad FS = 2.5$

$$\sigma_a = \frac{\sigma_u}{FS} \qquad \tau_a = \frac{\tau_u}{FS} \qquad \sigma_{ba} = \frac{\sigma_{bu}}{FS}$$

MEMBER FORCES FROM TRUSS ANALYSIS

$$F_{BC} = \frac{5}{3}P \qquad F_{CD} = \frac{4}{3}P \qquad F_{CF} = \frac{\sqrt{2}}{3}P$$

$$\frac{\sqrt{2}}{3} = 0.471 \qquad F_{CG} = \frac{4}{3}P$$

P_{allow} FOR TENSION ON NET SECTION IN TRUSS BARS

$A_{net} = A - 2d_r t_{ang} \qquad A_{net} = 1975\ \text{mm}^2$

$$\frac{A_{net}}{A} = 0.906$$

$F_{allow} = \sigma_a A_{net}$ < allowable force in a member so BC controls since it has the largest member force for this loading

$$P_{allow} = \frac{3}{5}F_{BCmax} \qquad P_{allow} = \frac{3}{5}(\sigma_a A_{net})$$

$P_{allow} = 184.879\ \text{kN}$

Next, P_{allow} for shear in rivets (all are in double shear)

$A_s = 2\frac{\pi}{4}d_r^2$ < for one rivet in DOUBLE shear

$\frac{F_{max}}{N} = \tau_a A_s$ N = number of rivets in a particular member (see drawing of conn. detail)

$P_{BC} = 3\left(\frac{3}{5}\right)(\tau_a A_s) \qquad P_{BC} = 55.0\ \text{kN}$

$P_{CF} = 2\left(\frac{3}{\sqrt{2}}\right)(\tau_a A_s) \qquad P_{CF} = 129.7\ \text{kN}$

$P_{CG} = 2\left(\frac{3}{4}\right)(\tau_a A_s)$

$P_{CG} = 45.8\ \text{kN} \quad \longleftarrow$ < so shear in rivets in CG & CD controls P_{allow} here

$P_{CD} = 2\left(\frac{3}{4}\right)(\tau_a A_s) \qquad P_{CD} = 45.8\ \text{kN} \quad \longleftarrow$

Next, P_{allow} for bearing of rivets on truss bars $A_b = 2d_r t_{ang}$ < rivet bears on each angle in two angle pairs

$$\frac{F_{max}}{N} = \sigma_{ba} A_b$$

$P_{BC} = 3\left(\frac{3}{5}\right)(\sigma_{ba} A_b) \qquad P_{BC} = 81.101\ \text{kN}$

$P_{CF} = 2\left(\frac{3}{\sqrt{2}}\right)(\sigma_{ba} A_b) \qquad P_{CF} = 191.156\ \text{kN}$

$P_{CG} = 2\left(\frac{3}{4}\right)(\sigma_{ba} A_b) \qquad P_{CG} = 67.584\ \text{kN}$

$P_{CD} = 2\left(\frac{3}{4}\right)(\sigma_{ba} A_b) \qquad P_{CD} = 67.584\ \text{kN}$

Finally, P_{allow} for bearing of rivets on gusset plate

$A_b = d_r t_g$

(bearing area for each rivert on gusset plate)

$t_g = 12\ \text{mm} < 2t_{ang} = 12.8\ \text{mm}$

so gusset will control over angles

$P_{BC} = 3\left(\frac{3}{5}\right)(\sigma_{ba} A_b) \qquad P_{BC} = 76.032\ \text{kN}$

$P_{CF} = 2\left(\frac{3}{\sqrt{2}}\right)(\sigma_{ba} A_b) \qquad P_{CF} = 179.209\ \text{kN}$

$P_{CG} = 2\left(\frac{3}{4}\right)(\sigma_{ba} A_b) \qquad P_{CG} = 63.36\ \text{kN}$

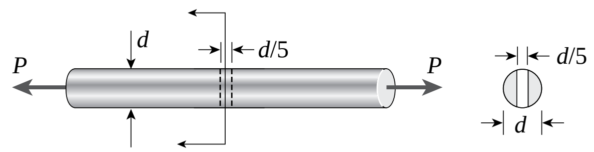
$P_{CD} = 2\left(\frac{3}{4}\right)(\sigma_{ba} A_b) \qquad P_{CD} = 63.36\ \text{kN}$

So, shear in rivets controls: $P_{allow} = 45.8\ \text{kN} \quad \longleftarrow$

Problem 1.7-13 A solid bar of circular cross section (diameter d) has a hole of diameter $d/5$ drilled laterally through the center of the bar (see figure). The allowable average tensile stress on the net cross section of the bar is σ_{allow}.

(a) Obtain a formula for the allowable load P_{allow} that the bar can carry in tension.

(b) Calculate the value of P_{allow} if the bar is made of brass with diameter $d = 1.75$ in. and $\sigma_{allow} = 12$ ksi.
(*Hint*: Use the formulas of Case 15 Appendix D.)



Solution 1.7-13

NUMERICAL DATA

$d = 1.75$ in $\quad \sigma_a = 12$ ksi

(a) FORMULA FOR P_{ALLOW} IN TENSION

From Case 15, Appendix D:

$$A = 2r^2\left(\alpha - \frac{ab}{r^2}\right) \qquad r = \frac{d}{2} \qquad a = \frac{d}{10}$$

$$\alpha = \mathrm{acos}\left(\frac{a}{r}\right) \qquad r = 0.875 \text{ in.} \qquad a = 0.175 \text{ in.}$$

$$\alpha\frac{180}{\pi} = 78.463 \text{ degrees}$$

$$b = \sqrt{r^2 - a^2}$$

$$b = \sqrt{\left[\left(\frac{d}{2}\right)^2 - \left(\frac{d}{10}\right)^2\right]}$$

$$b = \sqrt{\left(\frac{6}{25}d^2\right)} \qquad b = \frac{d}{5}\sqrt{6}$$

$$P_a = \sigma_a A$$

$$P_a = \sigma_a\left[\frac{1}{2}d^2\left(\mathrm{acos}\left(\frac{1}{5}\right) - \frac{2}{25}\sqrt{6}\right)\right]$$

$$\frac{\mathrm{acos}\left(\frac{1}{5}\right) - \frac{2}{25}\sqrt{6}}{2} = 0.587 \qquad \frac{\pi}{4} = 0.785$$

$$P_a = \sigma_a(0.587\,d^2) \quad \leftarrow$$

$$\frac{0.587}{0.785} = 0.748$$

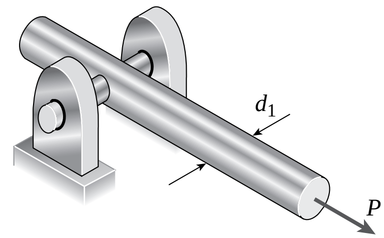
(b) EVALUATE NUMERICAL RESULT

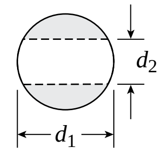
$d = 1.75$ in. $\quad \sigma_a = 12$ ksi

$P_a = 21.6$ kips $\quad \leftarrow$

Problem 1.7-14 A solid steel bar of diameter $d_1 = 60$ mm has a hole of diameter $d_2 = 32$ mm drilled through it (see figure). A steel pin of diameter d_2 passes through the hole and is attached to supports.

Determine the maximum permissible tensile load P_{allow} in the bar if the yield stress for shear in the pin is $\tau_Y = 120$ MPa, the yield stress for tension in the bar is $\sigma_Y = 250$ MPa and a factor of safety of 2.0 with respect to yielding is required. (*Hint*: Use the formulas of Case 15, Appendix D.)





Solution 1.7-14

NUMERICAL DATA

$d_1 = 60$ mm $\quad d_2 = 32$ mm

$\tau_Y = 120$ MPa $\quad \sigma_Y = 250$ MPa

$FS_y = 2$

ALLOWABLE STRESSES

$$\tau_a = \frac{\tau_Y}{FS_y} \qquad \tau_a = 60 \text{ MPa}$$

$$\sigma_a = \frac{\sigma_Y}{FS_y} \qquad \sigma_a = 125 \text{ MPa}$$

From Case 15, Appendix D: $\quad r = \dfrac{d_1}{2}$

$$A = 2r^2\left(\alpha - \frac{ab}{r^2}\right) \qquad \alpha = \text{arc cos}\frac{d_2/2}{d_1/2} = \text{arc cos}\frac{d_2}{d_1}$$

$$a = \frac{d_2}{2} \qquad b = \sqrt{r^2 - a^2}$$

SHEAR AREA (DOUBLE SHEAR)

$$A_s = 2\left(\frac{\pi}{4}d_2^{\,2}\right) \qquad A_s = 1608 \text{ mm}^2$$

NET AREA IN TENSION (FROM CASE 15, APP. D)

$$A_{net} = 2\left(\frac{d_1}{2}\right)^2 \left[\text{acos}\left(\frac{d_2}{d_1}\right) - \frac{\frac{d_2}{2}\left[\sqrt{\left(\frac{d_1}{2}\right)^2 - \left(\frac{d_2}{2}\right)^2}\right]}{\left(\frac{d_1}{2}\right)^2}\right]$$

$A_{net} = 1003 \text{ mm}^2$

P_{allow} in tension: smaller of values based on either shear or tension allowable stress x appropriate area

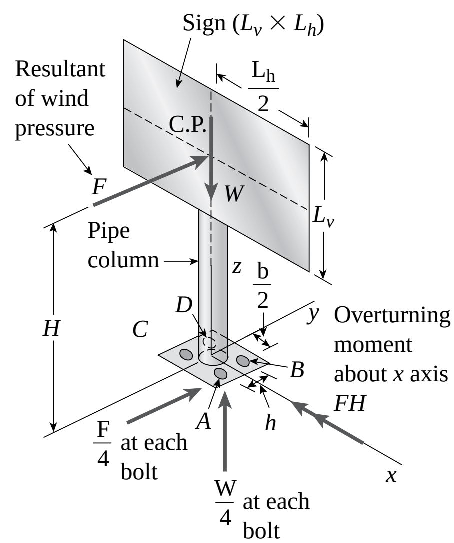
$P_{a1} = \tau_a A_s \qquad P_{a1} = 96.5\text{kN} <$ shear governs $\quad \leftarrow$

$P_{a2} = \sigma_a A_{net} \qquad P_{a2} = 125.4\text{kN}$

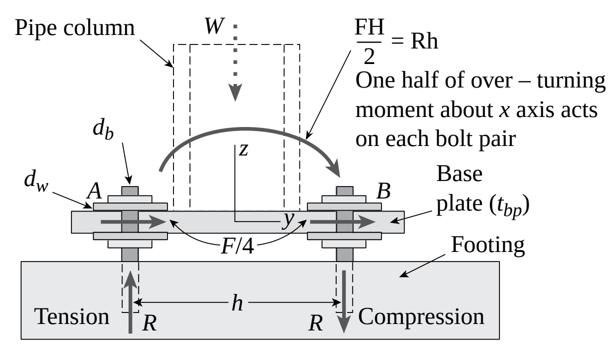
Problem 1.7-15 A sign of weight W is supported at its base by four bolts anchored in a concrete footing. Wind pressure p acts normal to the surface of the sign; the resultant of the uniform wind pressure is force F at the center of pressure. The wind force is assumed to create equal shear forces $F/4$ in the y-direction at each bolt [see figure parts (a) and (c)]. The overturning effect of the wind force also causes an uplift force R at bolts A and C and a downward force $(-R)$ at bolts B and D [see figure part (b)]. The resulting effects of the wind, and the associated ultimate stresses for each stress condition, are: normal stress in each bolt ($\sigma_u = 60$ ksi); shear through the base plate ($\tau_u = 17$ ksi); horizontal shear and bearing on each bolt ($\tau_{hu} = 25$ ksi and $\sigma_{bu} = 75$ ksi); and bearing on the bottom washer at B (or D) ($\sigma_{bw} = 50$ ksi).

Find the maximum wind pressure $p_{\max}$ (psf) that can be carried by the bolted support system for the sign if a safety factor of 2.5 is desired with respect to the ultimate wind load that can be carried.

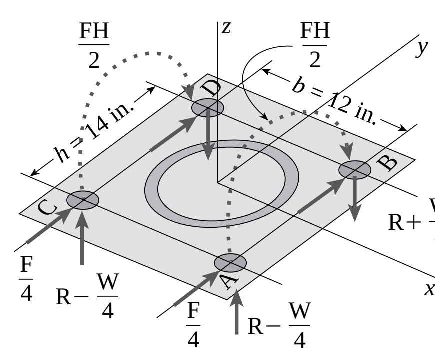
Use the following numerical data: bolt $d_b = \frac{3}{4}$ in.; washer $d_w = 1.5$ in.; base plate $t_{bp} = 1$ in.; base plate dimensions $h = 14$ in. and $b = 12$ in.; $W = 500$ lb; $H = 17$ ft; sign dimensions ($L_v = 10$ ft. $\times$ $L_h = 12$ ft.); pipe column diameter $d = 6$ in., and pipe column thickness $t = 3/8$ in.



(a)



(b)



(c)

Solution 1.7-15

NUMERICAL DATA

$\sigma_u = 60$ ksi $\quad \tau_u = 17$ ksi $\quad \tau_{hu} = 25$ ksi

$\sigma_{bu} = 75$ ksi $\quad \sigma_{bw} = 50$ ksi $\quad FS_u = 2.5$

$d_b = \frac{3}{4}$ in. $\quad d_w = 1.5$ in. $\quad t_{bp} = 1$ in.

$h = 14$ in. $\quad b = 12$ in. $\quad d = 6$ in. $\quad t = \frac{3}{8}$ in.

$W = 0.500$ kips $\quad H = 17(12) \quad H = 204$ in.

$L_v = 10(12) \quad L_h = 12(12) \quad L_v = 120$ in.

$L_h = 144$ in.

ALLOWABLE STRESSES (ksi)

$$\sigma_a = \frac{\sigma_u}{FS_u} \qquad \sigma_a = 24 \qquad \tau_a = \frac{\tau_u}{FS_u}$$

$$\tau_a = 6.8 \qquad \tau_{ha} = \frac{\tau_{hu}}{FS_u} \qquad \tau_{ha} = 10$$

$$\sigma_{ba} = \frac{\sigma_{bu}}{FS_u} \qquad \sigma_{ba} = 30 \qquad \sigma_{bwa} = \frac{\sigma_{bw}}{FS_u}$$

$$\sigma_{bwa} = 20$$

FORCES F AND R IN TERMS OF p_{max}

$$F = p_{max} L_v L_h \qquad R = \frac{FH}{2h}$$

$$R = p_{max} \frac{L_v L_h H}{2h}$$

(1) COMPUTE p_{max} BASED ON NORMAL STRESS IN EACH BOLT (GREATER AT B & D)

$$\sigma = \frac{R + \frac{W}{4}}{\frac{\pi}{4} d_b^2} \qquad R_{max} = \sigma_a\left(\frac{\pi}{4} d_b^2\right) - \frac{W}{4}$$

$$p_{max1} = \frac{\sigma_a\left(\frac{\pi}{4} d_b^2\right) - \frac{W}{4}}{\frac{L_v L_h H}{2h}}$$

$p_{max1} = 11.98$ psf $\quad \leftarrow \quad$ controls

(2) COMPUTE p_{max} BASED ON SHEAR THROUGH BASE PLATE (GREATER AT B & D)

$$\tau = \frac{R + \frac{W}{4}}{\pi d_w t_{bp}}$$

$$R_{max} = \tau_a(\pi d_w t_{bp}) - \frac{W}{4}$$

$$p_{max2} = \frac{\tau_a\left(\pi d_w t_{bp}\right) - \frac{W}{4}}{\frac{L_v L_h H}{2h}}$$

$p_{max2} = 36.5$ psf

(3) COMPUTE p_{max} BASED ON HORIZONTAL SHEAR ON EACH BOLT

$$\tau_h = \frac{\frac{F}{4}}{\left(\frac{\pi}{4} d_b^2\right)} \qquad F_{max} = 4\tau_{ha}\left(\frac{\pi}{4} d_b^2\right)$$

$$p_{max3} = \frac{\tau_{ha}(\pi d_b^2)}{L_v L_h}$$

$$p_{max3} = 147.3 \text{ psf}$$

(4) COMPUTE p_{max} BASED ON HORIZONTAL BEARING ON EACH BOLT

$$\sigma_b = \frac{\frac{F}{4}}{(t_{bp} d_b)} \qquad F_{max} = 4\sigma_{ba}(t_{bp} d_b)$$

$$p_{max4} = \frac{4\sigma_{ba}(t_b p d_b)}{L_v L_h}$$

$$p_{max4} = 750 \text{ psf}$$

(5) COMPUTE p_{max} BASED ON BEARING UNDER THE TOP WASHER AT A (OR C) AND THE BOTTOM WASHER AT B (OR D)

$$\sigma_{bw} = \frac{R + \frac{W}{4}}{\frac{\pi}{4}\left(d_w^2 - d_b^2\right)}$$

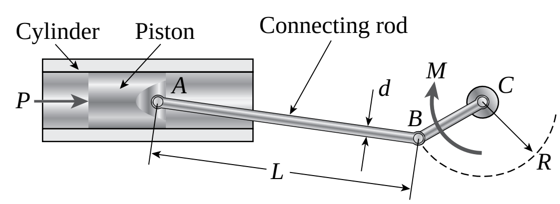
$$R_{max} = \sigma_{bwa}\left[\frac{\pi}{4}\left(d_w^2 - d_b^2\right)\right] - \frac{W}{4}$$

$$p_{max5} = \frac{\sigma_{bwa}\left[\frac{\pi}{4}(d_w^2 - d_b^2)\right] - \frac{W}{4}}{\frac{L_v L_h H}{2h}}$$

$$p_{max5} = 30.2 \text{ psf}$$

So, normal/stress in bolts controls; $p_{max} = 11.98$ psf

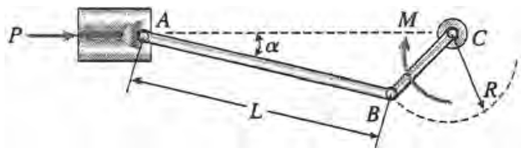
Problem 1.7-16 The piston in an engine is attached to a connecting rod *AB*, which in turn is connected to a crank arm *BC* (see figure). The piston slides without friction in a cylinder and is subjected to a force *P* (assumed to be constant) while moving to the right in the figure. The connecting rod, which has diameter *d* and length *L*, is attached at both ends by pins. The crank arm rotates about the axle at *C* with the pin at *B* moving in a circle of radius *R*. The axle at *C*, which is supported by bearings, exerts a resisting moment *M* against the crank arm.



(a) Obtain a formula for the maximum permissible force P_{allow} based upon an allowable compressive stress σ_c in the connecting rod.

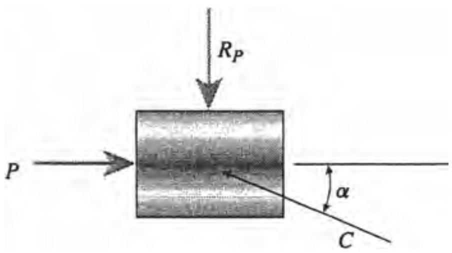
(b) Calculate the force P_{allow} for the following data: $\sigma_c = 160$ MPa, $d = 9.00$ mm, and $R = 0.28L$.

Solution 1.7-16



d = diameter of rod AB

FREE-BODY DIAGRAM OF PISTON



P = applied force (constant)

C = compressive force in connecting rod

RP = resultant of reaction forces between cylinder and piston (no friction)

$$\sum F_{\text{horiz}} = 0 \underset{+}{\rightarrow} \underset{-}{\leftarrow}$$

$$P - C\cos\alpha = 0$$

$$P = C\cos\alpha$$

MAXIMUM COMPRESSIVE FORCE C IN CONNECTING ROD

$$C_{\max} = \sigma_c A_c$$

in which A_c = area of connecting rod

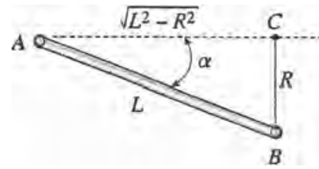
$$A_c = \frac{\pi d^2}{4}$$

MAXIMUM ALLOWABLE FORCE P

$$P = C_{\max}\cos\alpha$$
$$= \sigma_c A_c \cos\alpha$$

The maximun allowable force P occurs when $\cos\alpha$ has its smallest value, which means that α has its largest value.

LARGEST VALUE OF α



The largest value of α occurs when point B is the farthest distance from line AC. The farthest distance is the radius R of the crank arm.

Therefore,

$$\overline{BC} = R$$

Also, $\overline{AC} = \sqrt{L^2 - R^2}$

$$\cos\alpha = \frac{\sqrt{L^2 - R^2}}{L} = \sqrt{1 - \left(\frac{R}{L}\right)^2}$$

(a) MAXIMUM ALLOWABLE FORCE P

$$P_{\text{allow}} = \sigma_c A_c \cos\alpha$$
$$= \sigma_c\left(\frac{\pi d^2}{4}\right)\sqrt{1 - \left(\frac{R}{L}\right)^2} \quad \leftarrow$$

(b) SUBSTITUTE NUMERICAL VALUES

$\sigma_c = 160$ MPa $\quad d = 9.00$ mm

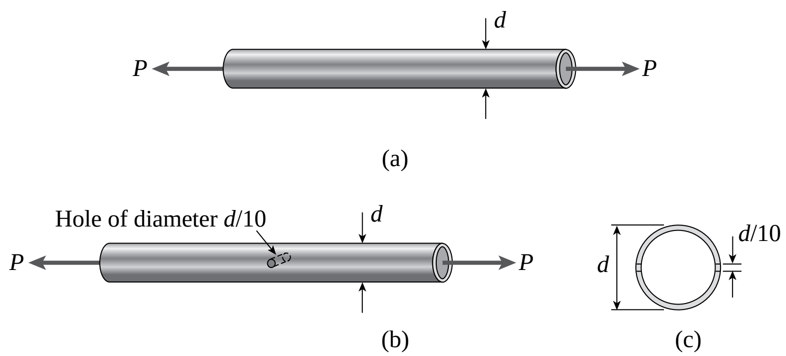
$R = 0.28L \quad R/L = 0.28$

$P_{\text{allow}} = 9.77$ kN $\quad \leftarrow$

Design for Axial Loads and Direct Shear

Problem 1.8-1 An aluminum tube is required to transmit an axial tensile force $P = 33$ k [see figure part (a)]. The thickness of the wall of the tube is to be 0.25 in.

(a) What is the minimum required outer diameter d_{min} if the allowable tensile stress is 12,000 psi?
(b) Repeat part (a) if the tube will have a hole of diameter $d/10$ at mid-length [see figure parts (b) and (c)].



Solution 1.8-1

NUMERICAL DATA

$P = 33$ kips $\quad t = 0.25$ in. $\quad \sigma_a = 12$ ksi

(a) MIN. DIAMETER OF TUBE (NO HOLES)

$$A_1 = \frac{\pi}{4}\left[d^2 - (d-2t)^2\right] \qquad A_2 = \frac{P}{\sigma_a}$$

$A_2 = 2.75\ \text{in}^2$

equating A_1 & A_2 and solving for d:

$$d = \frac{P}{\pi\sigma_a t} + t \qquad d = 3.75\ \text{in.} \quad \leftarrow$$

(b) MIN. DIAMETER OF TUBE (WITH HOLES)

$$A_1 = \left[\frac{\pi}{4}\left[d^2 - (d-2t)^2\right] - 2\left(\frac{d}{10}\right)t\right]$$

$$A_1 = d\left(\pi t - \frac{t}{5}\right) - \pi t^2$$

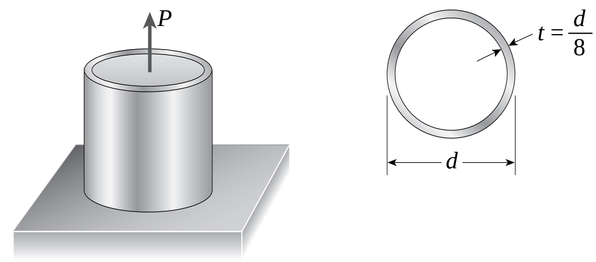
equating A_1 & A_2 and solving for d:

$$d = \frac{\frac{P}{\sigma_a} + \pi t^2}{\pi t - \frac{t}{5}} \qquad d = 4.01\ \text{in.} \quad \leftarrow$$

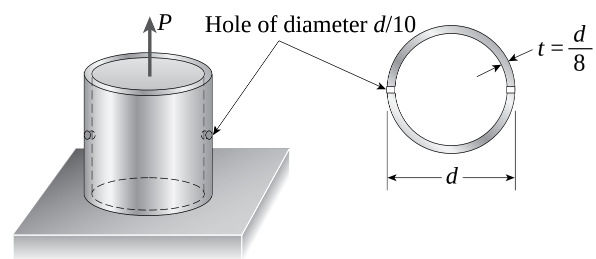
Problem 1.8-2 A copper alloy pipe having yield stress $\sigma_Y = 290$ MPa is to carry an axial tensile load $P = 1500$ kN [see figure part (a)]. A factor of safety of 1.8 against yielding is to be used.

(a) If the thickness t of the pipe is to be one-eighth of its outer diameter, what is the minimum required outer diameter d_{min}?

(b) Repeat part (a) if the tube has a hole of diameter $d/10$ drilled through the entire tube as shown in the figure [part (b)].



(a)



(b)

Solution 1.8-2

NUMERICAL DATA

$\sigma_Y = 290$ MPa

$P = 1500$ kN

$FS_y = 1.8$

(a) MIN. DIAMETER (NO HOLES)

$$A_1 = \frac{\pi}{4}\left[d^2 - \left(d - \frac{d}{8}\right)^2\right]$$

$$A_1 = \frac{\pi}{4}\left(\frac{15}{64}d^2\right) \qquad A_1 = \frac{15}{256}\pi d^2$$

$$A_2 = \frac{P}{\frac{\sigma_Y}{FS_y}} \qquad A_2 = 9.31 \times 10^3 \text{ mm}^2$$

equate A_1 & A_2 and solve for d:

$$d^2 = \frac{256}{15\pi}\left(\frac{P}{\frac{\sigma_Y}{FS_y}}\right)$$

$$d_{min} = \sqrt{\frac{256}{15\pi}\left(\frac{P}{\frac{\sigma_Y}{FS_y}}\right)}$$

$$d_{min} = 225\text{mm} \quad \leftarrow$$

(b) MIN. DIAMETER (WITH HOLES)

Redefine A_1 - subtract area for two holes - then equate to A_2

$$A_1 = \left[\frac{\pi}{4}\left[d^2 - \left(d - \frac{d}{8}\right)^2\right] - 2\left(\frac{d}{10}\right)\left(\frac{d}{8}\right)\right]$$

$$A_1 = \frac{15}{256}\pi d^2 - \frac{1}{40}d^2$$

$$A_1 = d^2\left(\frac{15}{256}\pi - \frac{1}{40}\right) \qquad \frac{15}{256}\pi - \frac{1}{40} = 0.159$$

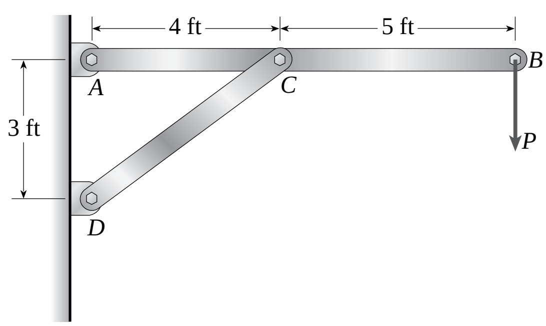
$$d^2 = \frac{\left(\dfrac{P}{\dfrac{\sigma y}{FS_y}}\right)}{\left(\dfrac{15}{256}\pi - \dfrac{1}{40}\right)}$$

$$d_{min} = \sqrt{\frac{\left(\dfrac{P}{\dfrac{\sigma_y}{FS_y}}\right)}{\left(\dfrac{15}{256}\pi - \dfrac{1}{40}\right)}} \qquad d_{min} = 242 \text{ mm} \quad \longleftarrow$$

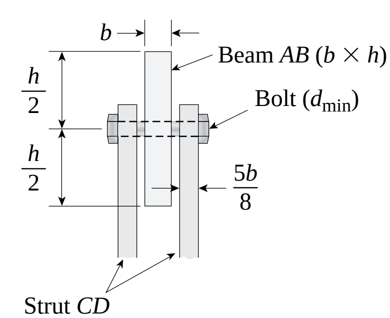
Problem 1.8-3 A horizontal beam *AB* with cross-sectional dimensions ($b = 0.75$ in.) $\times$ ($h = 8.0$ in.) is supported by an inclined strut *CD* and carries a load $P = 2700$ lb at joint *B* [see figure part (a)]. The strut, which consists of two bars each of thickness $5b/8$, is connected to the beam by a bolt passing through the three bars meeting at joint *C* [see figure part (b)].

(a) If the allowable shear stress in the bolt is 13,000 psi, what is the minimum required diameter d_{min} of the bolt at *C*?

(b) If the allowable bearing stress in the bolt is 19,000 psi, what is the minimum required diameter d_{min} of the bolt at *C*?



(a)



(b)

Solution 1.8-3

NUMERICAL DATA

$P = 2.7$ kips $\quad b = 0.75$ in. $\quad h = 8$ in.

$\tau_a = 13$ ksi $\quad \sigma_{ba} = 19$ ksi

(a) d_{min} BASED ON ALLOWABLE SHEAR - DOUBLE SHEAR IN STRUT

$$\tau_a = \frac{F_{DC}}{A_s} \qquad F_{DC} = \frac{15}{4}P$$

$$A_s = 2\left(\frac{\pi}{4}d^2\right)$$

$$d_{min} = \sqrt{\frac{\frac{15}{4}P}{\tau_a\left(\frac{\pi}{2}\right)}} \qquad d_{min} = 0.704 \text{ inches} \quad \leftarrow$$

(b) d_{min} BASED ON ALLOWABLE BEARING AT JT C

Bearing from beam ACB $\quad \sigma_b = \dfrac{15\ P/4}{b\,d}$

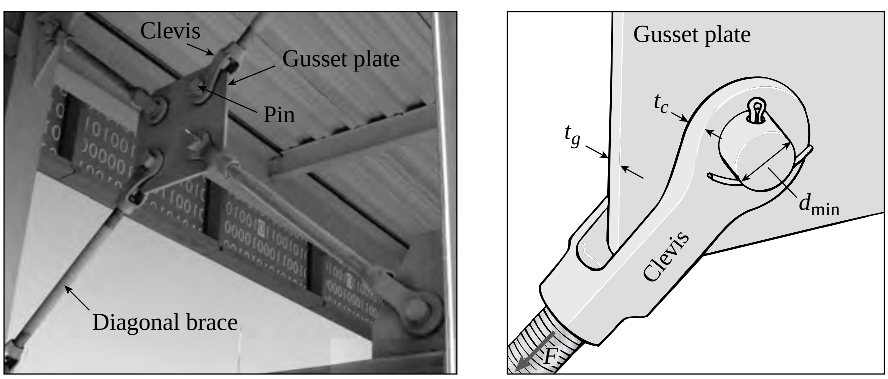
$$d_{min} = \frac{15\ P/4}{b\,\sigma_{ba}} \qquad d_{min} = 0.711 \text{ inches} \quad \leftarrow$$

Bearing from strut DC $\quad \sigma_b = \dfrac{\frac{15}{4}P}{2\frac{5}{8}b\,d}$

$$\sigma_b = 3\frac{P}{b\,d} \qquad \text{(lower than ACB)}$$

Problem 1.8-4 Lateral bracing for an elevated pedestrian walkway is shown in the figure part (a). The thickness of the clevis plate t_c = 16 mm and the thickness of the gusset plate t_g = 20 mm [see figure part (b)]. The maximum force in the diagonal bracing is expected to be F = 190 kN.

If the allowable shear stress in the pin is 90 MPa and the allowable bearing stress between the pin and both the clevis and gusset plates is 150 MPa, what is the minimum required diameter d_{min} of the pin?



Solution 1.8-4

NUMERICAL DATA

$F = 190 \text{ kN}$ $\tau_a = 90 \text{ MPa}$ $\sigma_{ba} = 150 \text{ MPa}$

$t_g = 20 \text{ mm}$ $t_c = 16 \text{ mm}$

(1) d_{min} BASED ON ALLOW SHEAR - DOUBLE SHEAR IN STRUT

$$\tau = \frac{F}{A_s} \qquad A_s = 2\left(\frac{\pi}{4}d^2\right)$$

$$d_{min} = \sqrt{\frac{F}{\tau_a\left(\frac{\pi}{2}\right)}} \qquad d_{min} = 36.7 \text{ mm}$$

(2) d_{min} BASED ON ALLOW BEARING IN GUSSET & CLEVIS PLATES

Bearing on gusset plate

$$\sigma_b = \frac{F}{A_b} \qquad A_b = t_g d \qquad d_{min} = \frac{F}{t_g \sigma_{ba}}$$

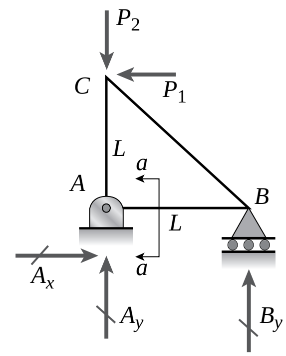
$$d_{min} = 63.3 \text{ mm} \qquad < \text{controls} \qquad \leftarrow$$

Bearing on clevis $A_b = d(2t_c)$

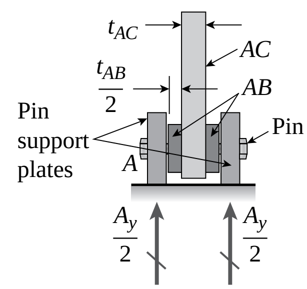
$$d_{min} = \frac{F}{2t_c \sigma_{ba}} \qquad d_{min} = 39.6 \text{ mm}$$

Problem 1.8-5 Forces $P_1 = 1500$ lb and $P_2 = 2500$ lb are applied at joint C of plane truss ABC shown in the figure part (a). Member AC has thickness $t_{AC} = 5/16$ in. and member AB is composed of two bars each having thickness $t_{AB}/2 = 3/16$ in. [see figure part (b)]. Ignore the effect of the two plates which make up the pin support at A.

If the allowable shear stress in the pin is 12,000 psi and the allowable bearing stress in the pin is 20,000 psi, what is the minimum required diameter d_{min} of the pin?



(a)



Section a–a

(b)

Solution 1.8-5

NUMERICAL DATA

$P_1 = 1.5$ kips $\qquad P_2 = 2.5$ kips

$t_{AC} = \frac{5}{16}$ in. $\qquad t_{AB} = 2\left(\frac{3}{16}\right)$ in.

$\tau_a = 12$ ksi $\qquad \sigma_{ba} = 20$ ksi

(1) d_{min} BASED ON ALLOWABLE SHEAR - DOUBLE SHEAR IN STRUT; FIRST CHECK AB (SINGLE SHEAR IN EACH BAR HALF)

Force in each bar of AB is $P_1/2$

$$\tau = \frac{\frac{P_1}{2}}{A_S} \qquad A_s = \left(\frac{\pi}{4}d^2\right)$$

$$d_{min} = \sqrt{\frac{\frac{P_1}{2}}{\tau_2\left(\frac{\pi}{4}\right)}} \qquad d_{min} = 0.282 \text{ in.}$$

Next check double shear to AC; force in AC is $(P_1 + P_2)/2$

$$d_{min} = \sqrt{\frac{(P_1 + P_2)/2}{\tau_a\left(\frac{\pi}{4}\right)}} \qquad d_{min} = 0.461 \text{ inches} \qquad \leftarrow$$

Finally check RESULTANT force on pin at A

$$R = \sqrt{\left(\frac{P_1}{2}\right)^2 + \left(\frac{P_1 + P_2}{2}\right)^2} \qquad R = 2.136 \text{ kips}$$

$$d_{min} = \sqrt{\frac{\frac{R}{2}}{\tau_a\left(\frac{\pi}{4}\right)}} \qquad d_{min} = 0.476 \text{ in.}$$

(2) d_{min} BASED ON ALLOWABLE BEARING ON PIN

member AB bearing on pin $\quad \sigma_b = \dfrac{P_1}{A_b} \qquad A_b = t_{AB}d$

$$d_{min} = \frac{P_1}{t_{AB}\sigma_{ba}} \qquad d_{min} = 0.2 \text{ in.}$$

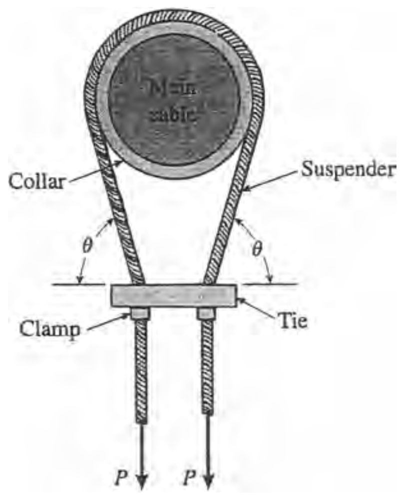
member AC bearing on pin $\qquad A_b = d(t_{AC})$

$$d_{min} = \frac{P_1 + P_2}{t_{AC}\sigma_{ba}} \qquad d_{min} = 0.64 \text{ in.} \qquad \text{controls} \qquad \leftarrow$$

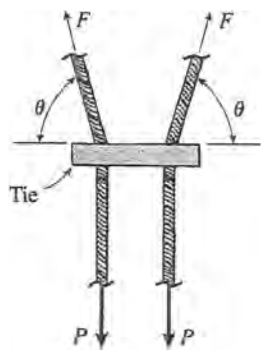
Problem 1.8-6 A suspender on a suspension bridge consists of a cable that passes over the main cable (see figure) and supports the bridge deck, which is far below. The suspender is held in position by a metal tie that is prevented from sliding downward by clamps around the suspender cable.

Let P represent the load in each part of the suspender cable, and let θ represent the angle of the suspender cable just above the tie. Finally, let σ_{allow} represent the allowable tensile stress in the metal tie.

(a) Obtain a formula for the minimum required cross-sectional area of the tie.
(b) Calculate the minimum area if $P = 130$ kN, $\theta = 75°$, and $\sigma_{allow} = 80$ MPa.



Solution 1.8-6 Suspender tie on a suspension bridge



F = tensile force in cable above tie

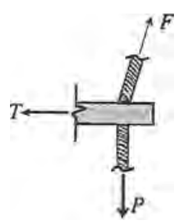
P = tensile force in cable below tie

σ_{allow} = allowable tensile stress in the tie

FREE-BODY DIAGRAM OF HALF THE TIE

Note: Include a small amount of the cable in the free-body diagram

T = tensile force in the tie



FORCE TRIANGLE

$$\cot\theta = \frac{T}{P}$$

$$T = P \cot \theta$$

(a) MINIMUM REQUIRED AREA OF TIE

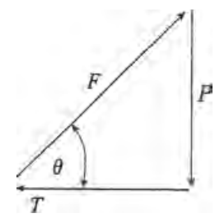
$$A_{min} = \frac{T}{\sigma_{allow}} = \frac{P\cot\theta}{\sigma_{allow}} \quad \leftarrow$$

(b) SUBSTITUTE NUMERICAL VALUES:

$P = 130$ kN $\qquad \theta = 75°$

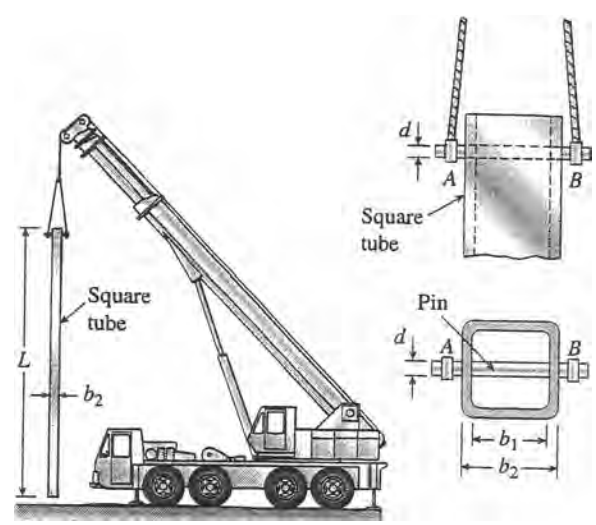
$\sigma_{allow} = 80$ MPa

$A_{min} = 435\ \text{mm}^2 \quad \leftarrow$

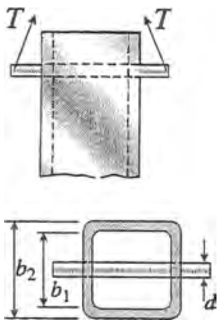


Problem 1.8-7 A square steel tube of length $L = 20$ ft and width $b_2 = 10.0$ in. is hoisted by a crane (see figure). The tube hangs from a pin of diameter d that is held by the cables at points A and B. The cross section is a hollow square with inner dimension $b_1 = 8.5$ in. and outer dimension $b_2 = 10.0$ in. The allowable shear stress in the pin is 8,700 psi, and the allowable bearing stress between the pin and the tube is 13,000 psi.

Determine the minimum diameter of the pin in order to support the weight of the tube. (*Note*: Disregard the rounded corners of the tube when calculating its weight.)



Solution 1.8-7 Tube hoisted by a crane



T = tensile force in cable

W = weight of steel tube

d = diameter of pin

b_1 = inner dimension of tube
= 8.5 in.

b_2 = outer dimension of tube
= 10.0 in.

L = length of tube = 20 ft

τ_{allow} = 8,700 psi

σ_b = 13,000 psi

WEIGHT OF TUBE

γ_s = weight density of steel
= 490 lb/ft^3

A = area of tube

$$= b_2^2 - b_1^2 = (10.0 \text{ in.})^2 - (8.5 \text{ in.})^2$$
$$= 27.75 \text{ in.}$$

$$W = \gamma_s AL$$
$$= (490 \text{ lb/ft}^3)(27.75 \text{ in.}^2)\left(\frac{1 \text{ ft}^2}{144 \text{ in.}}\right)(20 \text{ ft})$$
$$= 1{,}889 \text{ lb}$$

DIAMETER OF PIN BASED UPON SHEAR

Double shear. $2\tau_{allow} A_{pin} = W$

$$2(8{,}700 \text{ psi})\left(\frac{\pi d^2}{4}\right) = 1889 \text{ lb}$$

$d^2 = 0.1382$ in.^{2} $\quad d_1 = 0.372$ in.

DIAMETER OF PIN BASED UPON BEARING

$\sigma_b(b_2 - b_1)d = W$

(13,000 psi)(10.0 in. − 8.5 in.) d = 1,889 lb

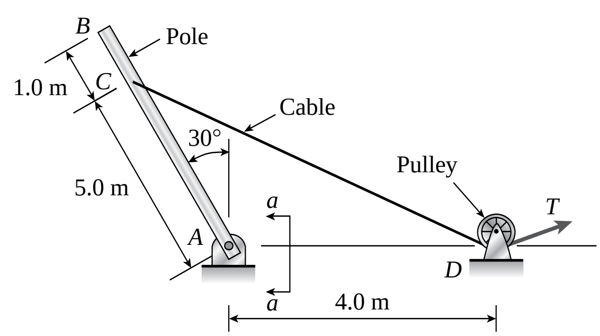
$d_2 = 0.097$ in.

MINIMUM DIAMETER OF PIN

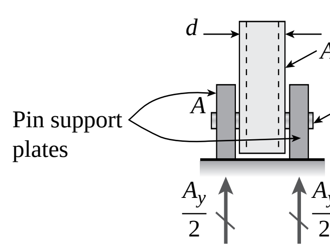
Shear governs. $\quad d_{min} = 0.372$ in.

Problem 1.8-8 A cable and pulley system at D is used to bring a 230-kg pole (ACB) to a vertical position as shown in the figure part (a). The cable has tensile force T and is attached at C. The length L of the pole is 6.0 m, the outer diameter is d = 140 mm, and the wall thickness t = 12 mm. The pole pivots about a pin at A in figure part (b). The allowable shear stress in the pin is 60 MPa and the allowable bearing stress is 90 MPa.

Find the minimum diameter of the pin at A in order to support the weight of the pole in the position shown in the figure part (a).



(a)



(b)

Solution 1.8-8

ALLOWABLE SHEAR & BEARING STRESSES

$\tau_a = 60\text{ MPa} \qquad \sigma_{ba} = 90\text{ MPa}$

FIND INCLINATION OF & FORCE IN CABLE, T

let α = angle between pole & cable at C; use Law of Cosines

$$DC = \sqrt{5^2 + 4^2 - 2(5)(4)\cos\left(120\frac{\pi}{180}\right)}$$

$$DC = 7.81\text{ m} \qquad \alpha = \operatorname{acos}\left[\frac{5^2 + DC^2 - 4^2}{2DC(5)}\right]$$

$$\alpha\frac{180}{\pi} = 26.33\text{ degrees} \qquad \theta = 60\left(\frac{\pi}{180}\right) - \alpha$$

$$\theta\frac{180}{\pi} = 33.67 \quad \text{<ange between cable \& horiz. at D}$$

$$W = 230\text{ kg}(9.81\text{ m/s}^2) \qquad W = 2.256 \times 10^3\text{ N}$$

STATICS TO FIND CABLE FORCE T

$$\sum M_A = 0 \qquad W(3\sin(30\text{ deg})) - T_X(5\cos(30\text{ deg})) + T_y(5\sin(30\text{ deg})) = 0$$

substitute for T_x & T_y in terms of T & solve for T:

$$T = \frac{\frac{3}{2}W}{\frac{-5}{2}\sin(\theta) + \frac{5\sqrt{3}}{2}\cos(\theta)}$$

$$T = 1.53 \times 10^3\text{ N} \qquad T_x = T\cos(\theta)$$

$$T_y = T\sin(\theta) \qquad T_x = 1.27 \times 10^3\text{ N} \qquad T_y = 846.11\text{ N}$$

(1) d_{min} BASED ON ALLOWABLE SHEAR - DOUBLE SHEAR AT A

$$A_x = -T_x \qquad A_y = T_y + W$$

CHECK SHEAR DUE TO RESULTANT FORCE ON PIN AT A

$$R_A = \sqrt{A_x^2 + A_y^2} \qquad R_A = 3.35 \times 10^3\text{ N}$$

$$d_{min} = \sqrt{\frac{\frac{R_A}{2}}{\tau_a\left(\frac{\pi}{4}\right)}}$$

$$d_{min} = 5.96\text{ mm} \quad \text{<controls} \quad \leftarrow$$

(2) d_{min} BASED ON ALLOWABLE BEARING ON PIN

$$d_{pole} = 140\text{ mm} \qquad t_{pole} = 12\text{ mm} \qquad L_{pole} = 6000\text{ mm}$$

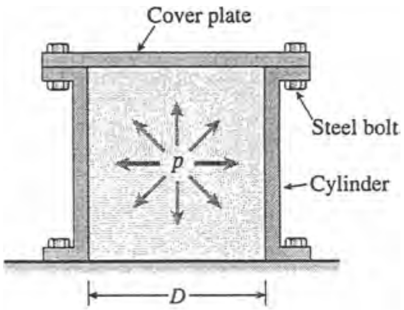
member AB BEARING ON PIN

$$\sigma_b = \frac{R_A}{A_b} \qquad A_b = 2t_{pole}d$$

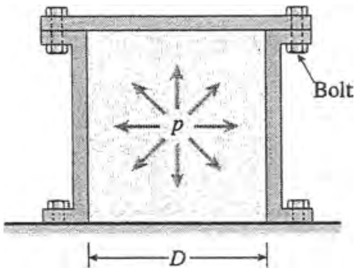
$$d_{min} = \frac{R_A}{2t_{pole}\sigma_{ba}} \qquad d_{min} = 1.55\text{ mm}$$

Problem 1.8-9 A pressurized circular cylinder has a sealed cover plate fastened with steel bolts (see figure). The pressure p of the gas in the cylinder is 290 psi, the inside diameter D of the cylinder is 10.0 in., and the diameter d_B of the bolts is 0.50 in.

If the allowable tensile stress in the bolts is 10,000 psi, find the number n of bolts needed to fasten the cover.



Solution 1.8-9 Pressurized cylinder



$p = 290$ psi $\quad D = 10.0$ in. $\quad d_b = 0.50$ in.

$\sigma_{\text{allow}} = 10{,}000$ psi $\quad n =$ number of bolts

$F =$ total force acting on the cover plate from the internal pressure

$$F = p\left(\frac{\pi D^2}{4}\right)$$

Number of bolts

$P =$ tensile force in one bolt

$$P = \frac{F}{n} = \frac{\pi p D^2}{4n}$$

$$A_b = \text{area of one bolt} = \frac{\pi}{4} d_b^2$$

$$P = \sigma_{\text{allow}} A_b$$

$$\sigma_{\text{allow}} = \frac{P}{A_b} = \frac{\pi p D^2}{(4n)(\frac{\pi}{4})d_b^2} = \frac{pD^2}{n d_b^2}$$

$$n = \frac{pD^2}{d_b^2 \sigma_{\text{allow}}}$$

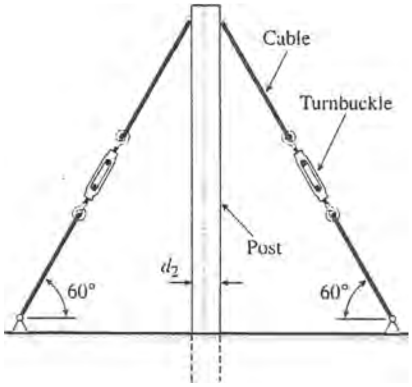
Substitute numerical values:

$$n = \frac{(290\text{ psi})(10\text{ in.})^2}{(0.5\text{ in.})^2(10{,}000\text{ psi})} = 11.6$$

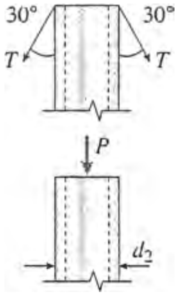
Use 12 bolts $\leftarrow$

Problem 1.8-10 A tubular post of outer diameter d_2 is guyed by two cables fitted with turnbuckles (see figure). The cables are tightened by rotating the turnbuckles, thus producing tension in the cables and compression in the post. Both cables are tightened to a tensile force of 110 kN. Also, the angle between the cables and the ground is 60°, and the allowable compressive stress in the post is $\sigma_c = 35$ MPa.

If the wall thickness of the post is 15 mm, what is the minimum permissible value of the outer diameter d_2?



Solution 1.8-10 Tubular post with guy cables



d_2 = outer diameter

d_1 = inner diameter

t = wall thickness
= 15 mm

T = tensile force in a cable
= 110 kN

σ_{allow} = 35 MPa

P = compressive force in post
= $2T\cos 30°$

REQUIRED AREA OF POST

$$A = \frac{P}{\sigma_{allow}} = \frac{2T\cos 30°}{\sigma_{allow}}$$

AREA OF POST

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = \frac{\pi}{4}[d_2^2 - (d_2 - 2t)^2]$$

$$= \pi t(d_2 - t)$$

EQUATE AREAS AND SOLVE FOR d_2:

$$\frac{2T\cos 30°}{\sigma_{allow}} = \pi t(d_2 - t)$$

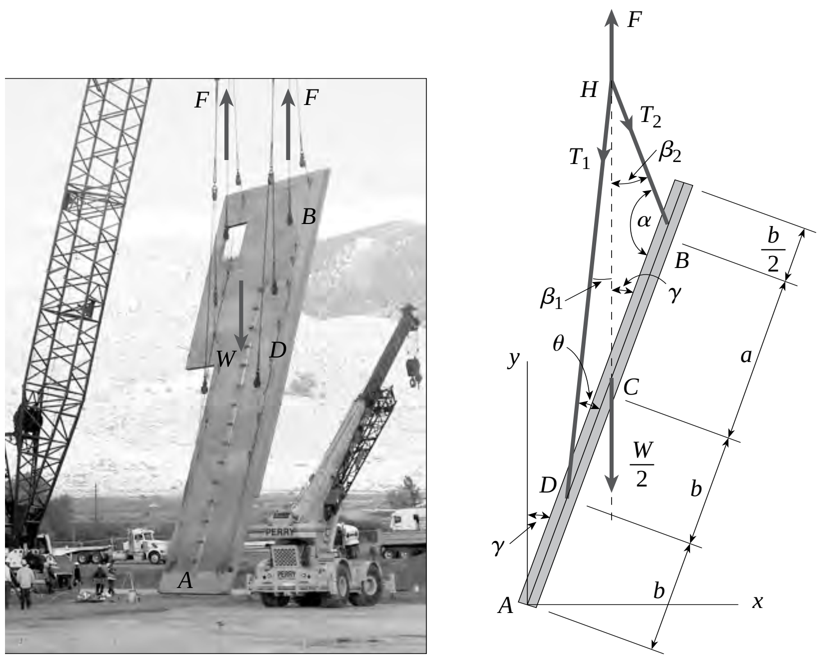
$$d_2 = \frac{2T\cos 30°}{\pi t \sigma_{allow}} + t \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$(d_2)_{min} = 131$ mm $\leftarrow$

Problem 1.8-11 A large precast concrete panel for a warehouse is being raised to a vertical position using two sets of cables at two lift lines as shown in the figure part (a). Cable 1 has length $L_1 = 22$ ft and distances along the panel (see figure part (b)) are $a = L_1/2$ and $b = L_1/4$. The cables are attached at lift points B and D and the panel is rotated about its base at A. However, as a worst case, assume that the panel is momentarily lifted off the ground and its total weight must be supported by the cables. Assuming the cable lift forces F at each lift line are about equal, use the simplified model of one half of the panel in figure part (b) to perform your analysis for the lift position shown. The total weight of the panel is $W = 85$ kips. The orientation of the panel is defined by the following angles: $\gamma = 20°$ and $\theta = 10°$.

Find the required cross-sectional area A_C of the cable if its breaking stress is 91 ksi and a factor of safety of 4 with respect to failure is desired.



(a) (b)

Solution 1.8-11

GEOMETRY

$L_1 = 22$ ft $\qquad a = \frac{1}{2}L_1 \qquad b = \frac{1}{4}L_1$

$\theta = 10$ deg $\qquad a + 2.5b = 24.75$ ft

$\gamma = 20$ deg

Using Law of cosines

$$L_2 = \sqrt{(a + b)^2 + L_1^2 - 2(a + b)L_1\cos(\theta)}$$

$L_2 = 6.425$ ft

$$\beta = \operatorname{acos}\left[\frac{L_1^2 + L_2^2 - (a + b)^2}{2L_1L_2}\right]$$

$\beta = 26.484$ degrees

$\beta_1 = \pi - (\theta + \pi - \gamma) \qquad \beta_1 = 10$ deg

$\beta_2 = \beta - \beta_1 \qquad \beta_2 = 16.484$ deg

SOLUTION APPROACH: FIND T THEN $A_C = T/(\sigma_U/FS)$

STATICS at point H

$$\sum_{H} F_x = 0 \qquad T_1 \sin(\beta_1) = T_2 \sin(\beta_2)$$

SO $\quad T_2 = T_1 \dfrac{\sin(\beta_1)}{\sin(\beta_2)}$

$$\sum_{H} F_Y = 0 \qquad T_1 \cos(\beta_1) + T_2 \cos(\beta_2) = F$$

and $\quad F = W/2, \qquad W = 85 \text{ kips}$

SO $\quad T_1\left(\cos(\beta_1) + \dfrac{\sin(\beta_1)}{\sin(\beta_2)}\cos(\beta_2)\right) = F$

$$T_1 = \frac{\dfrac{W}{2}}{\left(\cos(\beta_1) + \dfrac{\sin(\beta_1)}{\sin(\beta_2)}\cos(\beta_2)\right)}$$

$T_1 = 27.042 \text{ kips}$

$$T_2 = T_1 \frac{\sin(\beta_1)}{\sin(\beta_2)} \qquad T_2 = 16.549 \text{ kips}$$

COMPUTE REQUIRED CROSS-SECTIONAL AREA

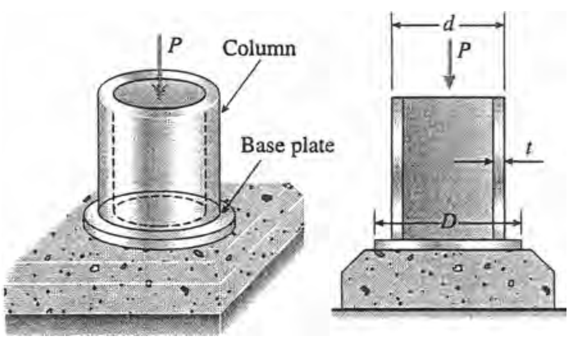
$$\sigma_u = 91 \text{ ksi} \qquad FS = 4 \qquad \frac{\sigma_u}{FS} = 22.75 \text{ ksi}$$

$$A_c = \frac{T_1}{\dfrac{\sigma_u}{FS}} \qquad A_c = 1.189 \text{ in}^2 \quad \leftarrow$$

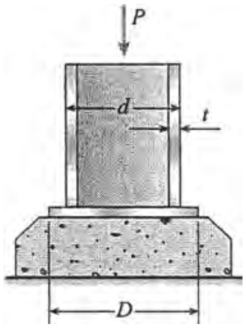
Problem 1.8-12 A steel column of hollow circular cross section is supported on a circular steel base plate and a concrete pedestal (see figure). The column has outside diameter $d = 250$ mm and supports a load $P = 750$ kN.

(a) If the allowable stress in the column is 55 MPa, what is the minimum required thickness t? Based upon your result, select a thickness for the column. (Select a thickness that is an even integer, such as 10, 12, 14, . . . , in units of millimeters.)

(b) If the allowable bearing stress on the concrete pedestal is 11.5 MPa, what is the minimum required diameter D of the base plate if it is designed for the allowable load P_{allow} that the column with the selected thickness can support?



Solution 1.8-12 Hollow circular column



$d = 250$ mm $\qquad P = 750$ kN

$\sigma_{allow} = 55$ MPa (compression in column)

t = thickness of column

D = diameter of base plate

$\sigma_b = 11.5$ MPa (allowable pressure on concrete)

(a) THICKNESS t OF THE COLUMN

$$A = \frac{P}{\sigma_{allow}} \qquad A = \frac{\pi d^2}{4} - \frac{\pi}{4}(d - 2t)^2$$

$$= \frac{\pi}{4}(4t)(d - t) = \pi t(d - t)$$

$$\pi t(d - t) = \frac{P}{\sigma_{allow}}$$

$$\pi t^2 - \pi t d + \frac{P}{\sigma_{allow}} = 0$$

$$t^2 - td + \frac{P}{\pi \sigma_{allow}} = 0 \qquad \text{(Eq. 1)}$$

SUBSTITUTE NUMERICAL VALUES IN EQ. (1):

$$t^2 - 250\,t + \frac{(750 \times 10^3\ \text{N})}{\pi(55\ \text{N/mm}^2)} = 0$$

(Note: In this eq., t has units of mm.)

$t^2 - 250t + 4{,}340.6 = 0$

Solve the quadratic eq. for t:

$t = 18.77$ mm $\qquad t_{min} = 18.8$ mm $\quad \leftarrow$

Use $t = 20$ mm $\quad \leftarrow$

(b) DIAMETER D OF THE BASE PLATE

For the column, $\qquad P_{allow} = \sigma_{allow} A$

where A is the area of the column with $t = 20$ mm.

$A = \pi t(d - t)\ P_{allow} = \sigma_{allow}\ \pi t(d - t)$

$$\text{Area of base plate} = \frac{\pi D^2}{4} = \frac{P_{allow}}{\sigma_b}$$

$$\frac{\pi D^2}{4} = \frac{\sigma_{allow} \pi t(d - t)}{\sigma_b}$$

$$D^2 = \frac{4\sigma_{allow} t(d - t)}{\sigma_b}$$

$$= \frac{4(55\ \text{MPa})(20\ \text{mm})(230\ \text{mm})}{11.5\ \text{MPa}}$$

$D^2 = 88{,}000\ \text{mm}^2 \qquad D = 296.6$ mm

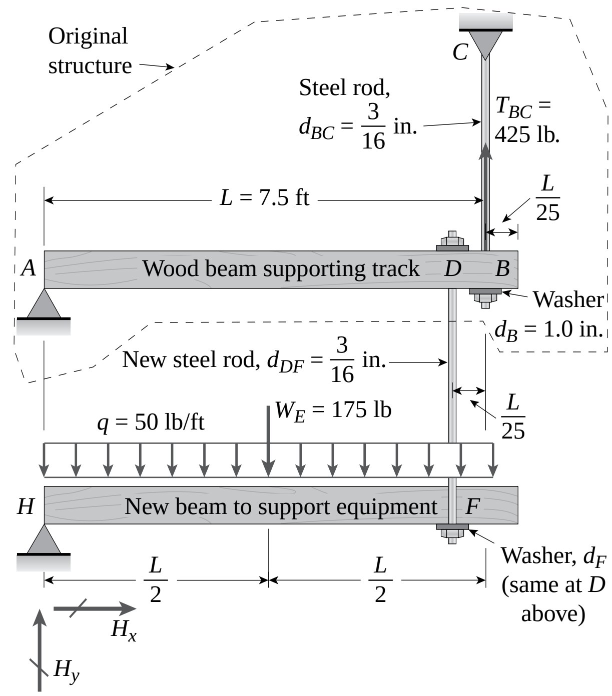
$D_{min} = 297$ mm $\quad \leftarrow$

Problem 1.8-13 An elevated jogging track is supported at intervals by a wood beam AB ($L = 7.5$ ft) which is pinned at A and supported by steel rod BC and a steel washer at B. Both the rod ($d_{BC} = 3/16$ in.) and the washer ($d_B = 1.0$ in.) were designed using a rod tension force of $T_{BC} = 425$ lb. The rod was sized using a factor of safety of 3 against reaching the ultimate stress $\sigma_u = 60$ ksi. An allowable bearing stress $\sigma_{ba} = 565$ psi was used to size the washer at B.

Now, a small platform HF is to be suspended below a section of the elevated track to support some mechanical and electrical equipment. The equipment load is uniform load $q = 50$ lb/ft and concentrated load $W_E = 175$ lb at mid-span of beam HF. The plan is to drill a hole through beam AB at D and install the same rod (d_{BC}) and washer (d_B) at both D and F to support beam HF.

(a) Use σ_u and σ_{ba} to check the proposed design for rod DF and washer d_F; are they acceptable?

(b) Also re-check the normal tensile stress in rod BC and bearing stress at B; if either is inadequate under the additional load from platform HF, redesign them to meet the original design criteria.



Solution 1.8-13

NUMERICAL DATA

$L = 7.5(12)$ $\quad L = 90$ in. $\quad T_{BC} = 425$ lb

$\sigma_u = 60$ ksi $\quad FS_u = 3$ $\quad \sigma_{ba} = 0.565$ ksi

$$q = \frac{50}{12} \qquad q = 4.167\frac{\text{lb}}{\text{in}} \qquad W_E = 175 \text{ lb}$$

$$d_{BC} = \frac{3}{16} \text{ in.} \qquad d_B = 1.0 \text{ in}$$

(a) FIND FORCE IN ROD DF AND FORCE ON WASHER AT F

$$\Sigma M_H = 0 \qquad T_{DF} = \frac{W_E\frac{L}{2} + qL\frac{L}{2}}{\left(L - \frac{L}{25}\right)}$$

$$T_{DF} = 286.458 \text{ lb}$$

NORMAL STRESS IN ROD DF:

$$\sigma_{DF} = \frac{T_{DF}}{\frac{\pi}{4}d_{BC}^2}$$

$\sigma_{DF} = 10.38$ ksi OK - less than σ_a; rod is acceptable ←

$$\sigma_a = \frac{\sigma_u}{FS_u} \qquad \sigma_a = 20 \text{ ksi}$$

BEARING STRESS ON WASHER AT F:

$$\sigma_{bF} = \frac{T_{DF}}{\frac{\pi}{4}(d_B^2 - d_{BC}^2)}$$

$\sigma_{bF} = 378$ psi OK - less than σ_{ba}; washer is acceptable ←

(b) FIND NEW FORCE IN ROD BC - SUM MOMENT ABOUT A FOR UPPER FBD - THEN CHECK NORMAL STRESS IN BC & BEARING STRESS AT B

$$\Sigma M_A = 0$$

$$T_{BC2} = \frac{T_{BC}L + T_{DF}\left(L - \frac{L}{25}\right)}{L}$$

$$T_{BC2} = 700 \text{ lb}$$

REVISED NORMAL STRESS IN ROD BC:

$$\sigma_{BC2} = \frac{T_{BC2}}{\left(\frac{\pi}{4} d_{BC}{}^2\right)}$$

$\sigma_{BC2} = 25.352$ ksi exceeds $\sigma_a = 20$ ksi

SO RE-DESIGN ROD BC:

$$d_{BCreqd} = \sqrt{\frac{T_{BC2}}{\frac{\pi}{4}\sigma_a}}$$

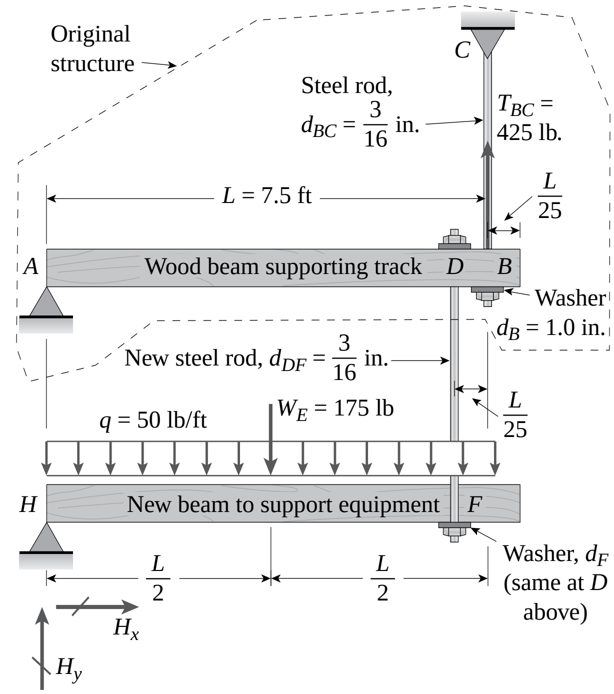
$d_{BCreqd} = 0.211$ in. $d_{BCreqd} \cdot 16 = 3.38$ in.

^say $4/16 = 1/4$ in. $d_{BC2} = \frac{1}{4}$ in.

RE-CHECK BEARING STRESS IN WASHER AT B:

$$\sigma_{bB2} = \frac{T_{BC2}}{\left[\frac{\pi}{4}\left(d_B{}^2 - d_{BC}{}^2\right)\right]}$$

$\sigma_{bB2} = 924$ psi
^ exceeds
$\sigma_{ba} = 565$ psi



SO RE-DESIGN WASHER AT B:

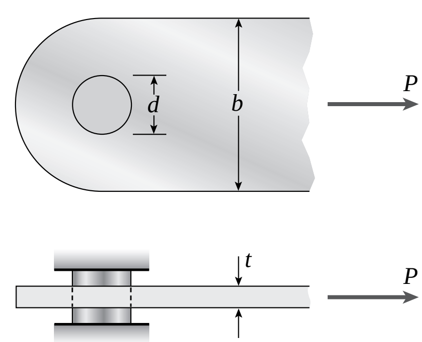
$$d_{Breqd} = \sqrt{\frac{T_{BC2}}{\frac{\pi}{4}\sigma_{ba}} + d_{BC}{}^2}$$

$d_{Breqd} = 1.281$ in.

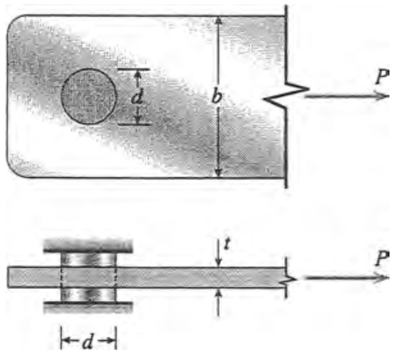
use $1 - 5/16$ in washer at B: $1 + 5/16 = 1.312$ in. ←

Problem 1.8-14 A flat bar of width $b = 60$ mm and thickness $t = 10$ mm is loaded in tension by a force P (see figure). The bar is attached to a support by a pin of diameter d that passes through a hole of the same size in the bar. The allowable tensile stress on the net cross section of the bar is $\sigma_T = 140$ MPa, the allowable shear stress in the pin is $\tau_S = 80$ MPa, and the allowable bearing stress between the pin and the bar is $\sigma_B = 200$ MPa.

(a) Determine the pin diameter d_m for which the load P will be a maximum.
(b) Determine the corresponding value P_{max} of the load.



Solution 1.8-14 Bar with a pin connection



$b = 60$ mm

$t = 10$ mm

d = diameter of hole and pin

$\sigma_T = 140$ MPa

$\tau_S = 80$ MPa

$\sigma_B = 200$ MPa

UNITS USED IN THE FOLLOWING CALCULATIONS:

P is in kN

σ and τ are in N/mm^2 (same as MPa)

b, t, and d are in mm

TENSION IN THE BAR

$$P_T = \sigma_T(\text{Net area}) = \sigma_t(t)(b - d)$$

$$= (140 \text{ MPa})(10 \text{ mm})(60 \text{ mm} - d)\left(\frac{1}{1000}\right)$$

$$= 1.40(60 - d) \qquad \text{(Eq. 1)}$$

SHEAR IN THE PIN

$$P_S = 2\tau_S A_{\text{pin}} = 2\tau_S\left(\frac{\pi d^2}{4}\right)$$

$$= 2(80 \text{ MPa})\left(\frac{\pi}{4}\right)(d^2)\left(\frac{1}{1000}\right)$$

$$= 0.040\,\pi d^2 = 0.12566 d^2 \qquad \text{(Eq. 2)}$$

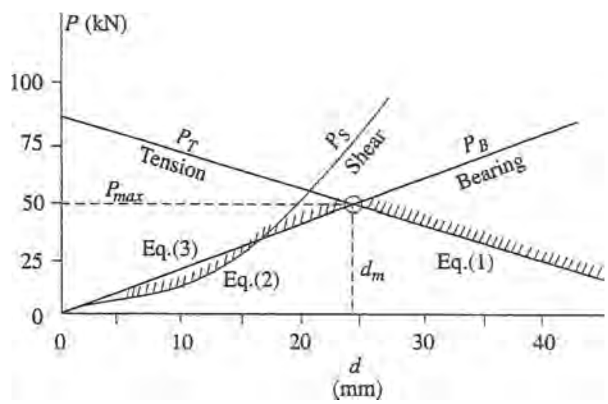
BEARING BETWEEN PIN AND BAR

$$P_B = \sigma_B\, td$$

$$= (200 \text{ MPa})(10 \text{ mm})(d)\left(\frac{1}{1000}\right)$$

$$= 2.0\, d \qquad \text{(Eq. 3)}$$

GRAPH OF EQS. (1), (2), AND (3)



(a) PIN DIAMETER d_m

$P_T = P_B$ or $1.40(60 - d) = 2.0\, d$

Solving, $d_m = \dfrac{84.0}{3.4}$ mm $= 24.7$ mm $\leftarrow$

(b) LOAD $P_{\max}$

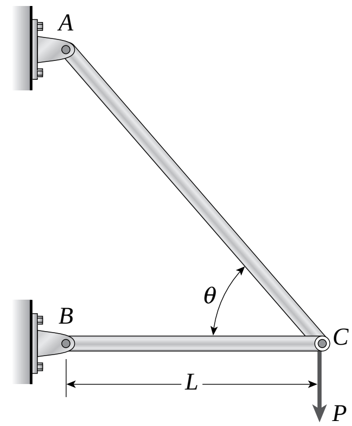
Substitute d_m into Eq. (1) or Eq. (3):

$P_{\max} = 49.4$ kN $\leftarrow$

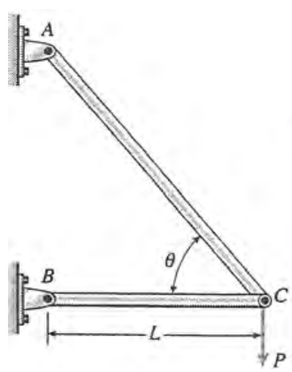
Problem 1.8-15 Two bars *AC* and *BC* of the same material support a vertical load *P* (see figure). The length *L* of the horizontal bar is fixed, but the angle θ can be varied by moving support *A* vertically and changing the length of bar *AC* to correspond with the new position of support *A*. The allowable stresses in the bars are the same in tension and compression.

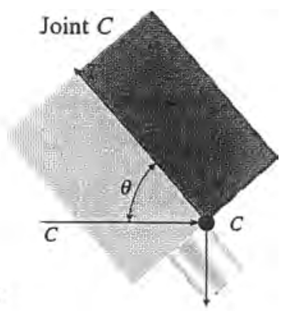
We observe that when the angle θ is reduced, bar *AC* becomes shorter but the cross-sectional areas of both bars increase (because the axial forces are larger). The opposite effects occur if the angle θ is increased. Thus, we see that the weight of the structure (which is proportional to the volume) depends upon the angle θ.

Determine the angle θ so that the structure has minimum weight without exceeding the allowable stresses in the bars. (*Note*: The weights of the bars are very small compared to the force *P* and may be disregarded.)



Solution 1.8-15 Two bars supporting a load *P*





T = tensile force in bar *AC*

C = compressive force in bar *BC*

$$\sum F_{\text{vert}} = 0 \qquad T = \frac{P}{\sin\theta}$$

$$\sum F_{\text{horiz}} = 0 \qquad C = \frac{P}{\tan\theta}$$

AREAS OF BARS

$$A_{AC} = \frac{T}{\sigma_{\text{allow}}} = \frac{P}{\sigma_{\text{allow}} \sin\theta}$$

$$A_{BC} = \frac{C}{\sigma_{\text{allow}}} = \frac{P}{\sigma_{\text{allow}} \tan\theta}$$

LENGTHS OF BARS

$$L_{AC} = \frac{L}{\cos\theta} \qquad L_{BC} = L$$

WEIGHT OF TRUSS

γ = weight density of material

$$W = \gamma(A_{AC}L_{AC} + A_{BC}L_{BC})$$

$$= \frac{\gamma PL}{\sigma_{\text{allow}}}\left(\frac{1}{\sin\theta\cos\theta} + \frac{1}{\tan\theta}\right)$$

$$= \frac{\gamma PL}{\sigma_{\text{allow}}}\left(\frac{1 + \cos^2\theta}{\sin\theta\cos\theta}\right) \qquad \text{Eq. (1)}$$

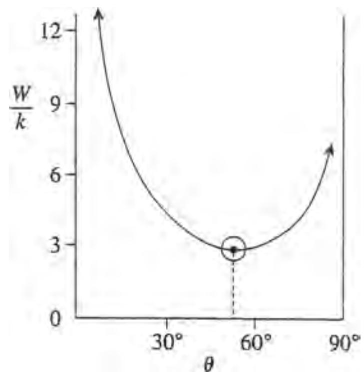
γ, P, L, and σ_{allow} are constants

W varies only with θ

Let $k = \dfrac{\gamma PL}{\sigma_{\text{allow}}}$ (k has unis of force)

$$\frac{W}{k} = \frac{1 + \cos^2\theta}{\sin\theta\cos\theta} \text{ (Nondimensional)} \qquad \text{Eq. (2)}$$

GRAPH OF EQ. (2):



ANGLE θ THAT MAKES W A MINIMUM

Use Eq. (2)

Let $f = \dfrac{1 + \cos^2\theta}{\sin\theta\cos\theta}$

$$\frac{df}{d\theta} = 0$$

$$\frac{df}{d\theta} = \frac{(\sin\theta\cos\theta)(2)(\cos\theta)(-\sin\theta) - (1 + \cos^2\theta)(-\sin^2\theta + \cos^2\theta)}{\sin^2\theta\cos^2\theta}$$

$$= \frac{-\sin^2\theta\cos^2\theta + \sin^2\theta - \cos^2\theta - \cos^4\theta}{\sin^2\theta\cos^2\theta}$$

SET THE NUMERATOR = 0 AND SOLVE FOR θ:

$$-\sin^2\theta\cos^2\theta + \sin^2\theta - \cos^2\theta - \cos^4\theta = 0$$

Replace $\sin^2\theta$ by $1 - \cos^2\theta$:

$$-(1 - \cos^2\theta)(\cos^2\theta) + 1 - \cos^2\theta - \cos^2\theta - \cos^4\theta = 0$$

Combine terms to simplify the equation:

$$1 - 3\cos^2\theta = 0 \qquad \cos\theta = \frac{1}{\sqrt{3}}$$

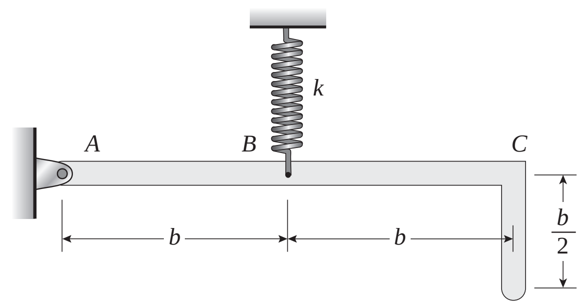
$$\theta = 54.7° \qquad \leftarrow$$

2

Axially Loaded Members

Changes in Lengths of Axially Loaded Members

Problem 2.2-1 The L-shaped arm *ABC* shown in the figure lies in a vertical plane and pivots about a horizontal pin at *A*. The arm has constant cross-sectional area and total weight *W*. A vertical spring of stiffness *k* supports the arm at point *B*. Obtain a formula for the elongation of the spring due to the weight of the arm.



Solution 2.2-1

Take first moments about A to find c.g.

$$x = \frac{\left(\dfrac{2b}{\frac{5}{2}b}\right)W(b) + \left[\dfrac{\frac{b}{2}}{\left(\frac{5}{2}b\right)}\right]W(2b)}{W}$$

$$x = \frac{6}{5}b$$

Find force in spring due to weight of arm

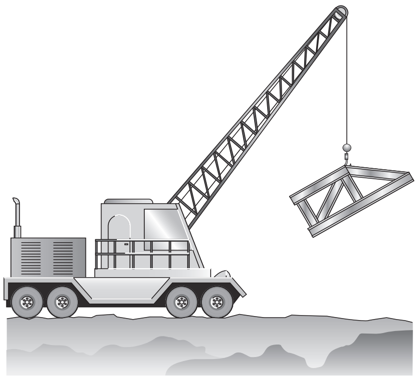
$$\sum M_A = 0 \qquad F_k = \frac{W\left(\frac{6}{5}b\right)}{b} \qquad F_k = \frac{6}{5}W$$

Find elongation of spring due to weight of arm

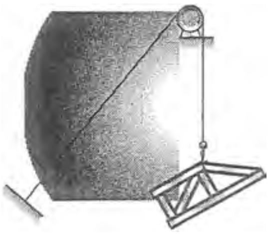
$$\delta = \frac{F_k}{k} \qquad \delta = \frac{6W}{5k} \quad \longleftarrow$$

Problem 2.2-2 A steel cable with nominal diameter 25 mm (see Table 2-1) is used in a construction yard to lift a bridge section weighing 38 kN, as shown in the figure. The cable has an effective modulus of elasticity $E = 140$ GPa.

(a) If the cable is 14 m long, how much will it stretch when the load is picked up?
(b) If the cable is rated for a maximum load of 70 kN, what is the factor of safety with respect to failure of the cable?



Solution 2.2-2 Bridge section lifted by a cable



$A = 304\ \text{mm}^2$ (from Table 2-1)

$W = 38$ kN

$E = 140$ GPa

$L = 14$ m

(a) STRETCH OF CABLE

$$\delta = \frac{WL}{EA} = \frac{(38\ \text{kN})(14\ \text{m})}{(140\ \text{GPa})(304\ \text{mm}^2)}$$

$$= 12.5\ \text{mm} \quad \leftarrow$$

(b) FACTOR OF SAFETY

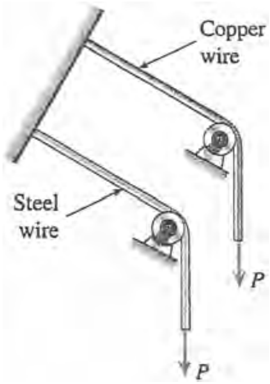
$P_{ULT} = 406$ kN (from Table 2-1)

$P_{\max} = 70$ kN

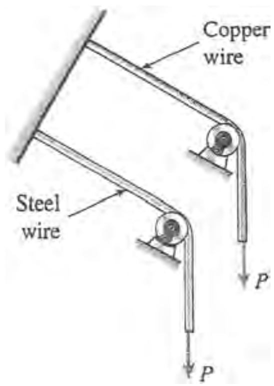
$$n = \frac{P_{ULT}}{P_{\max}} = \frac{406\ \text{kN}}{70\ \text{kN}} = 5.8 \quad \leftarrow$$

Problem 2.2-3 A steel wire and a copper wire have equal lengths and support equal loads P (see figure). The moduli of elasticity for the steel and copper are $E_s = 30{,}000$ ksi and $E_c = 18{,}000$ ksi, respectively.

(a) If the wires have the same diameters, what is the ratio of the elongation of the copper wire to the elongation of the steel wire?
(b) If the wires stretch the same amount, what is the ratio of the diameter of the copper wire to the diameter of the steel wire?



Solution 2.2-3 Steel wire and copper wire



Equal lengths and equal loads

Steel: $E_s = 30{,}000$ ksi

Copper: $E_c = 18{,}000$ ksi

(a) Ratio of elongations (equal diameters)

$$\delta_c = \frac{PL}{E_c A} \qquad \delta_s = \frac{PL}{E_s A}$$

$$\frac{\delta_c}{\delta_s} = \frac{E_s}{E_c} = \frac{30}{18} = 1.67 \quad \leftarrow$$

(b) Ratio of diameters (equal elongations)

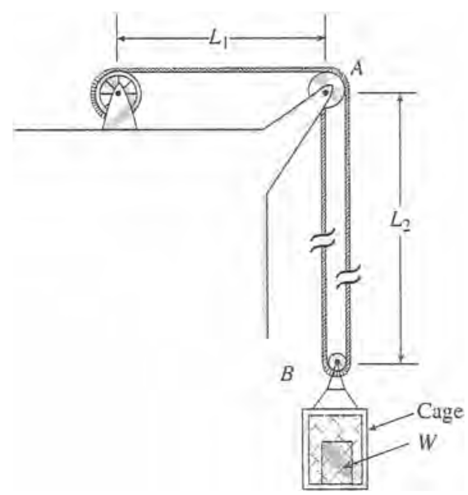
$$\delta_c = \delta_s \quad \frac{PL}{E_c A_c} = \frac{PL}{E_s A_s} \text{ or } E_c A_c = E_s A_s$$

$$E_c\left(\frac{\pi}{4}\right)d_c^2 = E_s\left(\frac{\pi}{4}\right)d_s^2$$

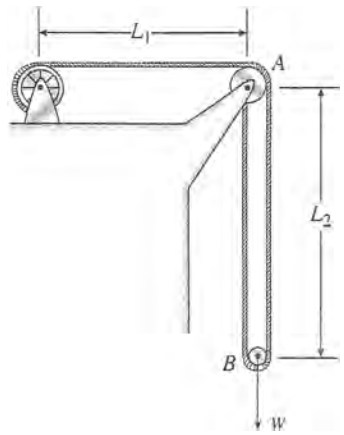
$$\frac{d_c^2}{d_s^2} = \frac{E_s}{E_c} \quad \frac{d_c}{d_s} = \sqrt{\frac{E_s}{E_c}} = \sqrt{\frac{30}{18}} = \; = 1.29 \quad \leftarrow$$

Problem 2.2-4 By what distance h does the cage shown in the figure move downward when the weight W is placed inside it?

Consider only the effects of the stretching of the cable, which has axial rigidity $EA = 10{,}700$ kN. The pulley at A has diameter $d_A = 300$ mm and the pulley at B has diameter $d_B = 150$ mm. Also, the distance $L_1 = 4.6$ m, the distance $L_2 = 10.5$ m, and the weight $W = 22$ kN. (*Note*: When calculating the length of the cable, include the parts of the cable that go around the pulleys at A and B.)



Solution 2.2-4 Cage supported by a cable



$d_A = 300$ mm
$d_B = 150$ mm
$L_1 = 4.6$ m
$L_2 = 10.5$ m
$EA = 10{,}700$ kN
$W = 22$ kN

TENSILE FORCE IN CABLE

$$T = \frac{W}{2} = 11 \text{ kN}$$

LENGTH OF CABLE

$$L = L_1 + 2L_2 + \frac{1}{4}(\pi d_A) + \frac{1}{2}(\pi d_B)$$

$$= 4{,}600 \text{ mm} + 21{,}000 \text{ mm} + 236 \text{ mm} + 236 \text{ mm}$$

$$= 26{,}072 \text{ mm}$$

ELONGATION OF CABLE

$$\delta = \frac{TL}{EA} = \frac{(11 \text{ kN})(26{,}072 \text{ mm})}{(10{,}700 \text{ kN})} = 26.8 \text{ mm}$$

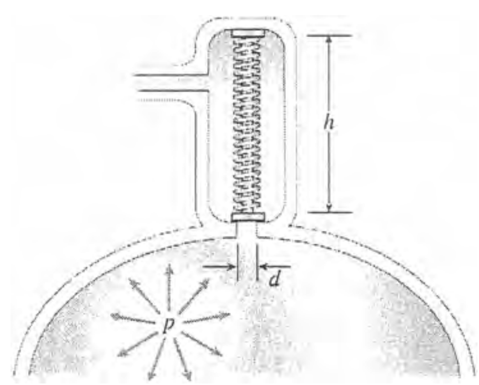
LOWERING OF THE CAGE

h = distance the cage moves downward

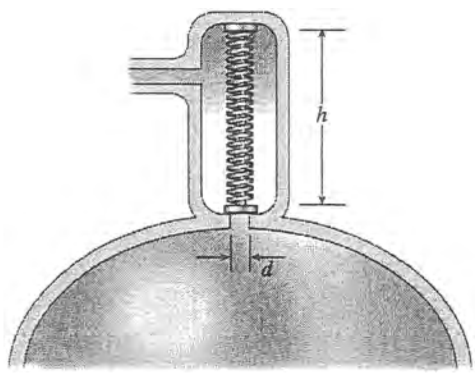
$$h = \frac{1}{2}\delta = 13.4 \text{ mm} \quad \leftarrow$$

Problem 2.2-5 A safety valve on the top of a tank containing steam under pressure p has a discharge hole of diameter d (see figure). The valve is designed to release the steam when the pressure reaches the value p_{max}.

If the natural length of the spring is L and its stiffness is k, what should be the dimension h of the valve? (Express your result as a formula for h.)



Solution 2.2-5 Safety valve



h = height of valve (compressed length of the spring)

d = diameter of discharge hole

p = pressure in tank

p_{max} = pressure when valve opens

L = natural length of spring ($L > h$)

k = stiffness of spring

FORCE IN COMPRESSED SPRING

$F = k(L - h)$ (From Eq. 2-1a)

PRESSURE FORCE ON SPRING

$$P = p_{max}\left(\frac{\pi d^2}{4}\right)$$

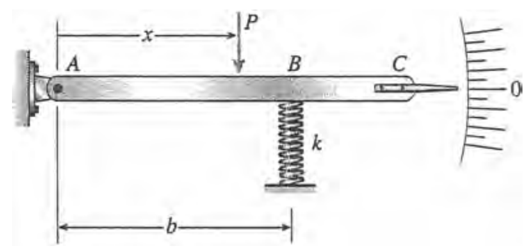
EQUATE FORCES AND SOLVE FOR h:

$$F = P \quad k(L - h) = \frac{\pi p_{max} d^2}{4}$$

$$h = L - \frac{\pi p_{max}\, d^2}{4\,k} \quad \leftarrow$$

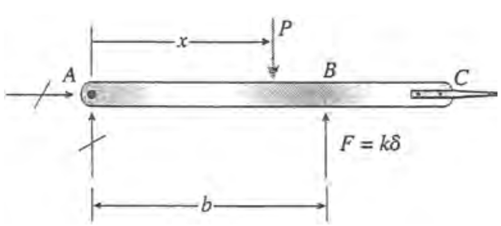
Problem 2.2-6 The device shown in the figure consists of a pointer ABC supported by a spring of stiffness $k = 800$ N/m. The spring is positioned at distance $b = 150$ mm from the pinned end A of the pointer. The device is adjusted so that when there is no load P, the pointer reads zero on the angular scale.

If the load $P = 8$ N, at what distance x should the load be placed so that the pointer will read 3° on the scale?



Solution 2.2-6 Pointer supported by a spring

FREE-BODY DIAGRAM OF POINTER



$P = 8$ N

$k = 800$ N/m

$b = 150$ mm

δ = displacement of spring

F = force in spring

$= k\delta$

$$\Sigma M_A = 0$$

$$-Px + (k\delta)b = 0 \quad \text{or} \quad \delta = \frac{Px}{kb}$$

Let α = angle of rotation of pointer

$$\tan\alpha = \frac{\delta}{b} = \frac{Px}{kb^2} \qquad x = \frac{kb^2}{P}\tan\alpha \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

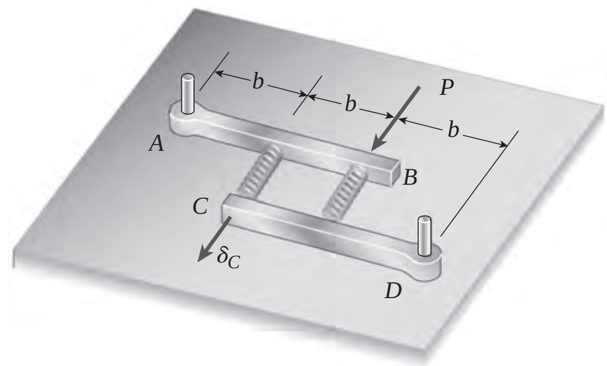
$\alpha = 3°$

$$x = \frac{(800\text{ N/m})(150\text{ mm})^2}{8\text{ N}}\tan 3°$$

$$= 118\text{ mm} \quad \leftarrow$$

Problem 2.2-7 Two rigid bars, *AB* and *CD*, rest on a smooth horizontal surface (see figure). Bar *AB* is pivoted end *A*, and bar *CD* is pivoted at end *D*. The bars are connected to each other by two linearly elastic springs of stiffness *k*. Before the load *P* is applied, the lengths of the springs are such that the bars are parallel and the springs are without stress.

Derive a formula for the displacement δ_C at point *C* when the load *P* is acting near point *B* as shown. (Assume that the bars rotate through very small angles under the action of the load *P*.)



Solution 2.2-7

(1) first sum moments about A for the entire structure to get R_D then sum vertical forces to get R_A

$$\sum M_A = 0 \qquad R_D = \frac{1}{3b}[\,P(2b)]$$

$$R_D = \frac{2}{3}P$$

$$\sum F_V = 0 \qquad R_A = P - R_D \qquad R_A = \frac{P}{3}$$

(2) next, cut through both springs & consider equilibrium of upper free body (UFBD) to find forces in springs (assume initially that both springs are in tension)

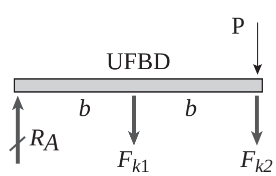
$$\sum M_{k1} = 0 \qquad (P + F_{k2})b = -R_A b$$

UFBD

$$F_{k2} = -R_A - P$$

$$F_{k2} = \frac{-4}{3}P$$

^ spring 2 is in compression



$$\sum F_V = 0 \qquad F_{k1} = R_A - (P + F_{k2})$$

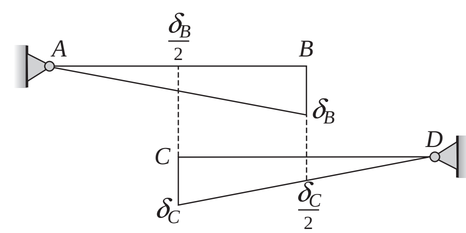
UFBD

$$F_{k1} = \left(\frac{P}{3} - P + \frac{4}{3}P\right) \qquad F_{k1} = \frac{2}{3}P$$

^ spring 1 is in tension

(3) solve displacement equations to find δ_C

DISPLACEMENT DIAGRAMS



$$\text{elongation of spring 1} = \delta_C - \frac{\delta_B}{2} = \frac{F_{k1}}{k} = \frac{2}{3}\frac{P}{k}$$

$$\text{elongation of spring 2} = \frac{\delta_C}{2} - \delta_B = \frac{F_{k2}}{k} = \frac{-4}{3}\frac{P}{k}$$

multiply 2nd equation above by $(-1/2)$ and add to first equation

$$\frac{3}{4}\delta_C = \frac{4}{3}\frac{P}{k} \qquad \delta_C = \frac{16}{9}\frac{P}{k} \qquad \leftarrow \qquad \frac{16}{9} = 1.778$$

(4) substitute δ_C into either equation to find δ_B (not a required part of this problem)

$$\text{1st equ} > \qquad \delta_B = 2\delta_C - \frac{4}{3}\frac{P}{k}$$

$$\delta_B = \left[2\left(\frac{16}{9}\frac{P}{k}\right) - \frac{4}{3}\frac{P}{k}\right]$$

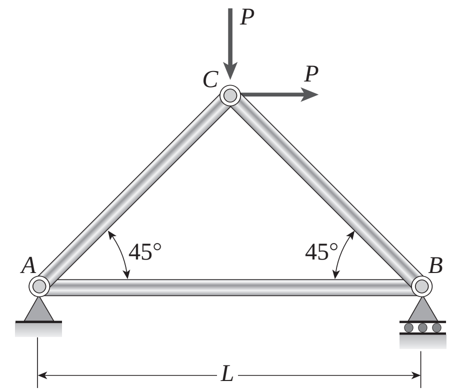
$$\delta_B = \frac{20}{9}\frac{P}{k} \qquad \frac{20}{9} = 2.222$$

$$\text{2nd equ} > \qquad \delta_B = \frac{\delta_C}{2} + \frac{4}{3}\frac{P}{k}$$

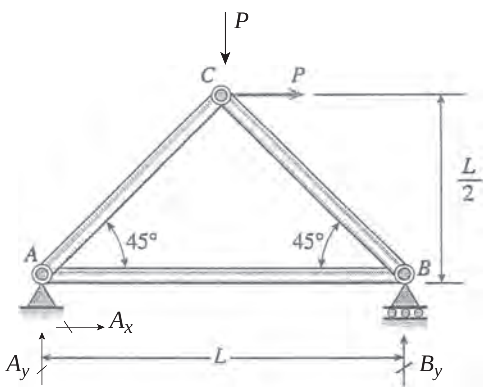
$$\delta_B = \left[\frac{1}{2}\left(\frac{16}{9}\frac{P}{k}\right) + \frac{4}{3}\frac{P}{k}\right] \qquad \delta_B = \frac{20}{9}\frac{P}{k}$$

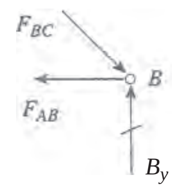
Problem 2.2-8 The three-bar truss ABC shown in the figure has a span $L = 3$ m and is constructed of steel pipes having cross-sectional area $A = 3900\ \text{mm}^2$ and modulus of elasticity $E = 200$ GPa. Identical loads P act both vertically and horizontally at joint C, as shown.

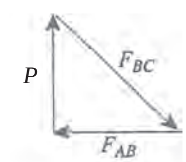
(a) If $P = 650$ kN, what is the horizontal displacement of joint B?
(b) What is the maximum permissible load value P_{max} if the displacement of joint B is limited to 1.5 mm?



Solution 2.2-8







NUMERICAL DATA

$A = 3900\ \text{mm}^2$ $\qquad$ $E = 200$ GPa

$P = 650$ kN $\qquad$ $L = 3000$ mm

$\delta_{Bmax} = 1.5$ mm

(a) FIND HORIZ. DISPL. OF JOINT B

$$\sum M_A = 0 \qquad B_y = \frac{1}{L}\left(2P\frac{L}{2}\right)$$

$$B_y = P$$

$$\sum F_H = 0 \quad A_x = -P$$

$$\sum F_V = 0 \quad A_y = P - B_y \quad A_y = 0$$

Method of Joints: $\quad F_{AC_V} = A_y \qquad F_{AC_V} = 0$
$\qquad F_{AC} = 0$

$F_{AB} = A_x$ $\qquad$ force in AB is P (tension) so elongation of AB = horiz. displ. of jt B

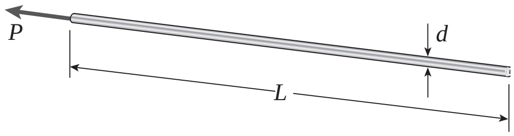
$$\delta_B = \frac{F_{AB}L}{EA} \qquad \delta_B = \frac{PL}{EA} \qquad \delta_B = 2.5\ \text{mm} \qquad \leftarrow$$

(b) FIND P_{max} IF DISPL. OF JOINT $B = \delta_{Bmax} = 1.5$ mm

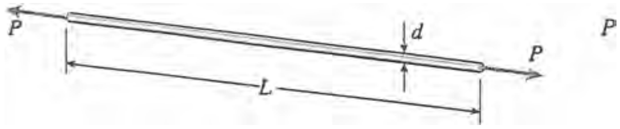
$$P_{max} = \frac{EA}{L}\delta_{Bmax} \qquad P_{max} = 390\ \text{kN} \qquad \leftarrow$$

Problem 2.2-9 An aluminum wire having a diameter $d = 1/10$ in. and length $L = 12$ ft is subjected to a tensile load P (see figure). The aluminum has modulus of elasticity $E = 10{,}600$ ksi

If the maximum permissible elongation of the wire is 1/8 in. and the allowable stress in tension is 10 ksi, what is the allowable load P_{max}?



Solution 2.2-9



$$d = \frac{1}{10}\text{ in} \quad L = 12(12)\text{ in} \quad E = 10600 \times (10^3)\text{ psi}$$

$$\delta_a = \frac{1}{8}\text{ in} \quad \sigma_a = 10 \times (10^3)\text{ psi}$$

$$A = \frac{\pi d^2}{4} \quad A = 7.854 \times 10^{-3}\text{ in}^2$$

$$EA = 8.325 \times 10^4\text{ lb}$$

Max. load based on elongation

$$P_{max1} = \frac{EA}{L}\delta_a \quad P_{max1} = 72.3\text{ lb} \quad \leftarrow \text{controls}$$

Max. load based on stress

$$P_{max2} = \sigma_a A \quad P_{max2} = 78.5\text{ lb}$$

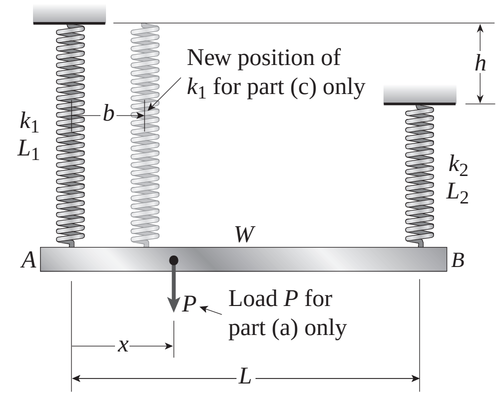
Problem 2.2-10 A uniform bar *AB* of weight $W = 25$ N is supported by two springs, as shown in the figure. The spring on the left has stiffness $k_1 = 300$ N/m and natural length $L_1 = 250$ mm. The corresponding quantities for the spring on the right are $k_2 = 400$ N/m and $L_2 = 200$ mm. The distance between the springs is $L = 350$ mm, and the spring on the right is suspended from a support that is distance $h = 80$ mm below the point of support for the spring on the left. Neglect the weight of the springs.

(a) At what distance *x* from the left-hand spring (figure part a) should a load $P = 18$ N be placed in order to bring the bar to a horizontal position?

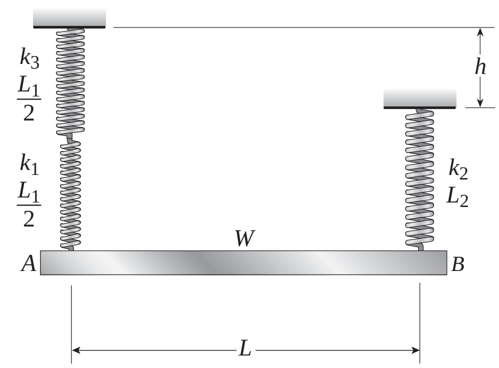
(b) If *P* is now removed, what new value of k_1 is required so that the bar (figure part a) will hang in a horizontal position under weight *W*?

(c) If *P* is removed and $k_1 = 300$ N/m, what distance *b* should spring k_1 be moved to the right so that the bar (figure part a) will hang in a horizontal position under weight *W*?

(d) If the spring on the left is now replaced by two springs in series ($k_1 = 300$N/m, k_3) with overall natural length $L_1 = 250$ mm (see figure part b), what value of k_3 is required so that the bar will hang in a horizontal position under weight *W*?



(a)



(b)

Solution 2.2-10

NUMERICAL DATA

$$W = 25\ \text{N} \qquad k_1 = 0.300\,\frac{\text{N}}{\text{mm}} \qquad L_1 = 250\ \text{mm}$$

$$k_2 = 0.400\,\frac{\text{N}}{\text{mm}} \qquad L_2 = 200\ \text{mm}$$

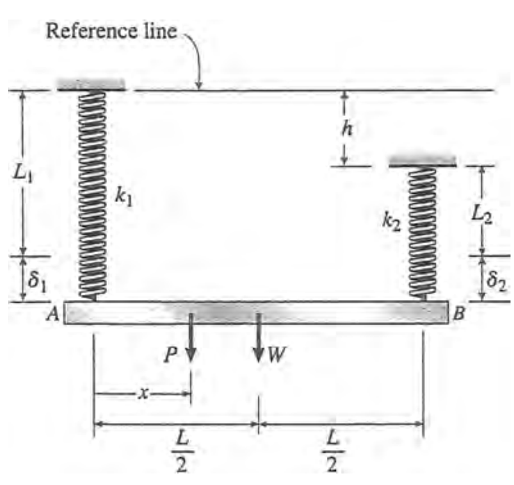
$$L = 350\ \text{mm} \qquad h = 80\ \text{mm} \qquad P = 18\ \text{N}$$

(a) LOCATION OF LOAD P TO BRING BAR TO HORIZ. POSITION

use statics to get forces in both springs

$$\sum M_A = 0 \qquad F_2 = \frac{1}{L}\left(W\frac{L}{2} + Px\right)$$

$$F_2 = \frac{W}{2} + P\frac{x}{L}$$



$$\sum F_V = 0 \qquad F_1 = W + P - F_2$$

$$F_1 = \frac{W}{2} + P\left(1 - \frac{x}{L}\right)$$

use constraint equation to define horiz. position,
then solve for location x

$$L_1 + \frac{F_1}{k_1} = L_2 + h + \frac{F_2}{k_2}$$

substitute expressions for F_1 & F_2 above into constraint equ. & solve for x

$$x = \frac{-2L_1 L k_1 k_2 - k_2 WL - 2k_2 P L + 2L_2 L k_1 k_2 + 2 h L k_1 k_2 + k_1 W L}{-2P(k_1 + k_2)}$$

$$x = 134.7 \text{ mm} \quad \leftarrow$$

(b) Next remove P and find new value of spring constant k_1 so that bar is horiz. under weight W

$$\text{Now,} \quad F_1 = \frac{W}{2} \qquad F_2 = \frac{W}{2} \quad \text{since } P = 0$$

same constraint equation as above but now $P = 0$:

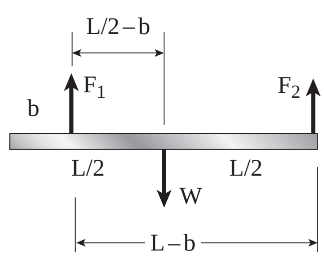
$$L_1 + \frac{\frac{W}{2}}{k_1} - (L_2 + h) - \frac{\left(\frac{W}{2}\right)}{k_2} = 0$$

solve for k_1

$$k_1 = \frac{-Wk_2}{[2k_2[L_1 - (L_2 + h)]] - W}$$

$$k_1 = 0.204 \frac{\text{N}}{\text{mm}} \quad \leftarrow$$

(c) Use $k_1 = 0.300$ N/mm but relocate spring k_1 ($x = b$) so that bar ends up in horiz. position under weight W



FBD

Part (c) - continued

statics

$$\sum M_{k_1} = 0 \qquad F_2 = \frac{w\left(\frac{L}{2} - b\right)}{L - b}$$

$$\sum F_V = 0$$

$$F_1 = W - F_2$$

$$F_1 = W - \frac{W\left(\frac{L}{2} - b\right)}{L - b}$$

$$F_1 = \frac{WL}{2(L - b)}$$

constraint equation - substitute above expressions for F_1 & F_2 and solve for b

$$L_1 + \frac{F_1}{k_1} - (L_2 + h) - \frac{F_2}{k_2} = 0$$

use the following data

$$k_1 = 0.300 \frac{\text{N}}{\text{mm}} \qquad k_2 = 0.4 \frac{\text{N}}{\text{mm}} \qquad L_1 = 250 \text{ mm}$$

$$L_2 = 200 \text{ mm} \quad L = 350 \text{ mm}$$

$$b = \frac{2L_1k_1k_2L + WLk_2 - 2L_2k_1k_2L - 2hk_1k_2L - Wk_1L}{(2L_1k_1k_2) - 2L_2k_1k_2 - 2hk_1k_2 - 2Wk_1} \qquad b = 74.1 \text{ mm} \quad \leftarrow$$

(d) REPLACE SPRING K_1 WITH SPRINGS IN SERIES: $K_1 = 0.3$N/mm, $L_1/2$ AND K_3, $L_1/2$ - FIND K_3 SO THAT BAR HANGS IN HORIZ. POSITION

statics $\quad F_1 = \frac{W}{2} \quad F_2 = \frac{W}{2}$

new constraint equation; solve for k_3

$$L_1 + \frac{F_1}{k_1} + \frac{F_1}{k_3} - (L_2 + h) - \frac{F_2}{k_2} = 0$$

$$L_1 + \frac{\frac{W}{2}}{k_1} + \frac{\frac{W}{2}}{k_3} - (L_2 + h) - \frac{\frac{W}{2}}{k_2} = 0$$

$$k_3 = \frac{Wk_1k_2}{-2L_1k_1k_2 - Wk_2 + 2L_2k_1k_2 + 2hk_1k_2 + Wk_1} \qquad k_3 = 0.638\,\frac{N}{mm} \quad \leftarrow$$

NOTE - equivalent spring constant for series springs

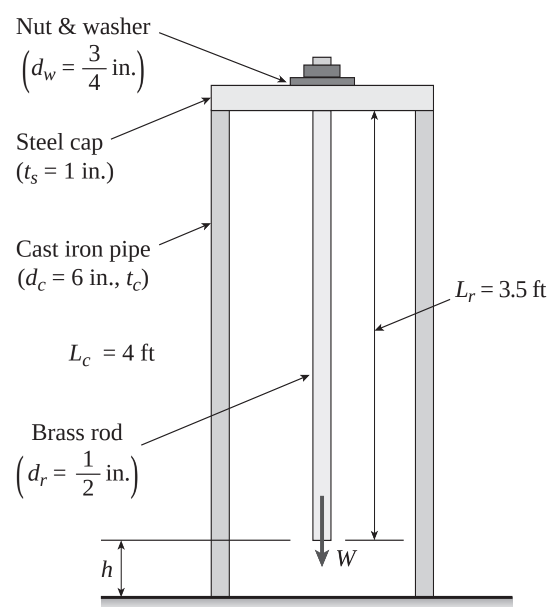
$$k_e = \frac{k_1k_3}{k_1 + k_3} \qquad k_e = 0.204\,\frac{N}{mm} \quad \leftarrow \text{ checks - same as (b) above}$$

Problem 2.2-11 A hollow, circular, cast-iron pipe (E_c = 12,000 ksi) supports a brass rod (E_b = 14,000 ksi) and weight W = 2 kips, as shown. The outside diameter of the pipe is d_c = 6 in.

(a) If the allowable compressive stress in the pipe is 5000 psi and the allowable shortening of the pipe is 0.02 in., what is the minimum required wall thickness $t_{c,min}$? (Include the weights of the rod and steel cap in your calculations.)

(b) What is the elongation of the brass rod δ_r due to both load W and its own weight?

(c) What is the minimum required clearance h?



Solution 2.2-11

The figure shows a section cut through the pipe, cap and rod.

NUMERICAL DATA

$E_c = 12000$ ksi $\quad E_b = 14000$ ksi

$W = 2$ kips $\quad d_c = 6$ in $\quad d_r = \frac{1}{2}$ in.

$\sigma_a = 5$ ksi $\quad \delta_a = 0.02$ in.

unit weights (see Table H-1) $\quad \gamma_s = 2.836 \times 10^{-4}\dfrac{\text{kips}}{\text{in}^3}$

$\gamma_b = 3.009 \times 10^{-4}\dfrac{\text{kips}}{\text{in}^3}$

$L_c = 48$ in $\quad L_r = 42$ in

$t_s = 1$ in.

(a) MIN. REQ'D WALL THICKNESS OF CI PIPE, t_{cmin}

first check allowable stress then allowable shortening

$$W_{cap} = \gamma_s\left(\frac{\pi}{4}d_c^2 t_s\right)$$

$W_{cap} = 8.018 \times 10^{-3}$ kips

$$W_{rod} = \gamma_b\left(\frac{\pi}{4}d_r^2 L_r\right)$$

$W_{rod} = 2.482 \times 10^{-3}$ kips

$W_t = W + W_{cap} + W_{rod} \qquad W_t = 2.01$ kips

$A_{min} = \dfrac{W_t}{\sigma_a} \qquad A_{min} = 0.402\ \text{in}^2$

$$A_{pipe} = \frac{\pi}{4}[d_c^2 - (d_c - 2t_c)^2]$$

$A_{pipe} = \pi t_c(d_c - t_c)$

$$t_c(d_c - t_c) = \frac{W_t}{\pi\sigma_a}$$

LET $\alpha = \dfrac{W_t}{\pi\sigma_a} \qquad \alpha = 0.128$

$t_c^2 - d_c t_c + \alpha = 0$

$t_c = \dfrac{d_c - \sqrt{d_c^2 - 4\alpha}}{2} \qquad t_c = 0.021$ in

^ min. based on σ_a

now check allowable shortening requirement

$\delta_{pipe} = \dfrac{W_t L_c}{E_c A_{min}} \qquad A_{min} = \dfrac{W_t L_c}{E_c \delta_a}$

$A_{min} = 0.447\ \text{in}^2$ < larger than value based on σ_a above

$$\pi t_c(d_c - t_c) = \frac{W_t L_c}{E_c \delta_a}$$

$t_c^2 - d_c t_c + \beta = 0 \qquad \beta = \dfrac{W_t L_c}{\pi E_c \delta_a}$

$\beta = 0.142$

$$t_c = \frac{d_c - \sqrt{d_c^2 - 4\beta}}{2}$$

$t_c = 0.021$ in. $\quad \leftarrow$ min. based on δ_a and σ_a controls

(b) ELONGATION OF ROD DUE TO SELF WEIGHT & ALSO WEIGHT W

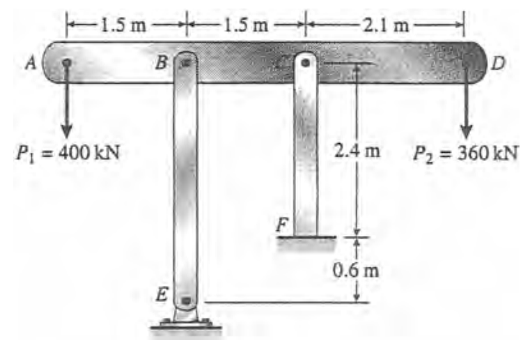
$\delta_r = \dfrac{\left(W + \dfrac{W_{rod}}{2}\right)L_r}{E_b\left(\dfrac{\pi}{4}d_r^2\right)} \qquad \delta_r = 0.031$ in $\quad \leftarrow$

(c) MIN. CLEARANCE h

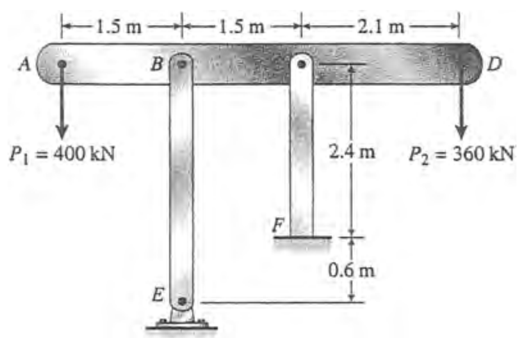
$h_{min} = \delta_a + \delta_r \qquad h_{min} = 0.051$ in. $\quad \leftarrow$

Problem 2.2-12 The horizontal rigid beam *ABCD* is supported by vertical bars *BE* and *CF* and is loaded by vertical forces $P_1 = 400$ kN and $P_2 = 360$ kN acting at points *A* and *D*, respectively (see figure). Bars *BE* and *CF* are made of steel ($E = 200$ GPa) and have cross-sectional areas $A_{BE} = 11{,}100$ mm^2 and $A_{CF} = 9{,}280$ mm^2. The distances between various points on the bars are shown in the figure.

Determine the vertical displacements δ_A and δ_D of points *A* and *D*, respectively.



Solution 2.2-12 Rigid beam supported by vertical bars



$A_{BE} = 11{,}100$ mm^2

$A_{CF} = 9{,}280$ mm^2

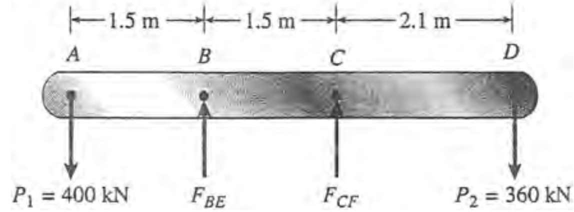
$E = 200$ GPa

$L_{BE} = 3.0$ m

$L_{CF} = 2.4$ m

$P_1 = 400$ kN; $P_2 = 360$ kN

FREE-BODY DIAGRAM OF BAR *ABCD*



$\Sigma M_B = 0$

$(400 \text{ kN})(1.5 \text{ m}) + F_{CF}(1.5 \text{ m}) - (360 \text{ kN})(3.6 \text{ m}) = 0$

$F_{CF} = 464$ kN

$\Sigma M_C = 0$

$(400 \text{ kN})(3.0 \text{ m}) - F_{BE}(1.5 \text{ m}) - (360 \text{ kN})(2.1 \text{ m}) = 0$

$F_{BE} = 296$ kN

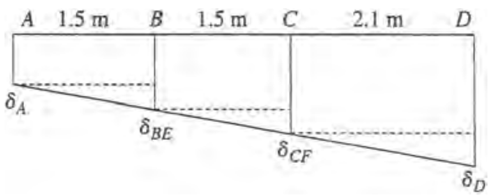
SHORTENING OF BAR *BE*

$$\delta_{BE} = \frac{F_{BE}L_{BE}}{EA_{BE}} = \frac{(296 \text{ kN})(3.0 \text{ m})}{(200 \text{ GPa})(11{,}100 \text{ mm}^2)}$$
$$= 0.400 \text{ mm}$$

SHORTENING OF BAR *CF*

$$\delta_{CF} = \frac{F_{CF}L_{CF}}{EA_{CF}} = \frac{(464 \text{ kN})(2.4 \text{ m})}{(200 \text{ GPa})(9{,}280 \text{ mm}^2)} = 0.600 \text{ mm}$$

DISPLACEMENT DIAGRAM



$$\delta_{BE} - \delta_A = \delta_{CF} - \delta_{BE} \text{ or } \delta_A = 2\delta_{BE} - \delta_{CF}$$

$$\delta_A = 2(0.400 \text{ mm}) - 0.600 \text{ m}$$

$$= 0.200 \text{ mm} \quad \leftarrow$$

(Downward)

$$\delta_D - \delta_{CF} = \frac{2.1}{1.5}(\delta_{CF} - \delta_{BE})$$

or $$\delta_D = \frac{12}{5}\delta_{CF} - \frac{7}{5}\delta_{BE}$$

$$= \frac{12}{5}(0.600 \text{ mm}) - \frac{7}{5}(0.400 \text{ mm})$$

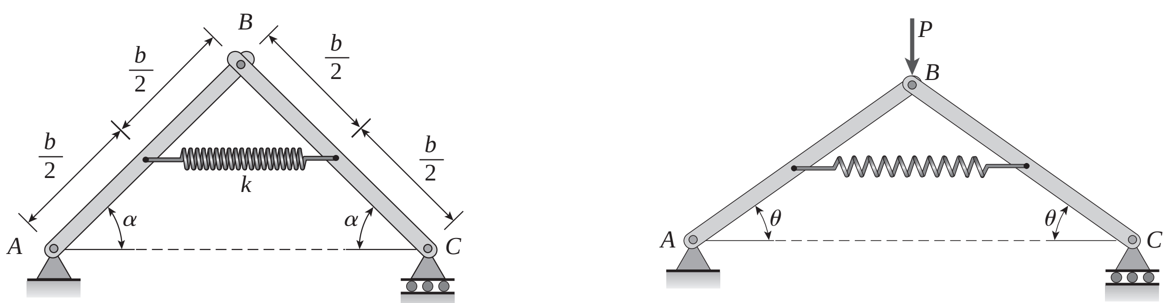
$$= 0.880 \text{ mm} \quad \leftarrow$$

(Downward)

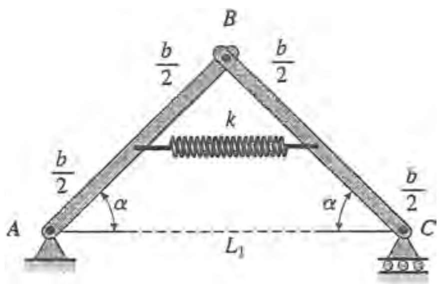
Problem 2.2-13 A framework *ABC* consists of two rigid bars *AB* and *BC*, each having length *b* (see the first part of the figure). The bars have pin connections at *A*, *B*, and *C* and are joined by a spring of stiffness *k*. The spring is attached at the midpoints of the bars. The framework has a pin support at *A* and a roller support at *C*, and the bars are at an angle α to the hoizontal.

When a vertical load *P* is applied at joint *B* (see the second part of the figure) the roller support *C* moves to the right, the spring is stretched, and the angle of the bars decreases from α to the angle θ.

Determine the angle θ and the increase δ in the distance between points *A* and *C*. (Use the following data; $b = 8.0$ in., $k = 16$ lb/in., $\alpha = 45°$, and $P = 10$ lb.)



Solution 2.2-13 Framework with rigid bars and a spring



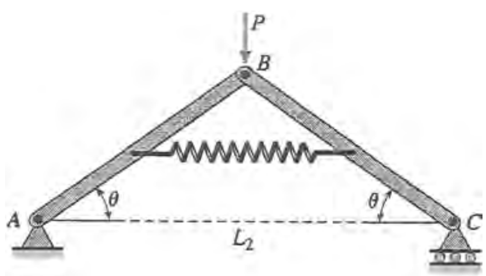
WITH NO LOAD

L_2 = span from A to C

$= 2b \cos \theta$

S_1 = length of spring

$$= \frac{L_1}{2} = b\cos\alpha$$



WITH LOAD P

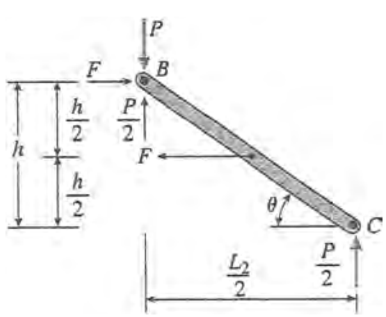
L_1 = span from A to C

$= 2b \cos \alpha$

S_2 = length of spring

$$= \frac{L_2}{2} = b\cos\theta$$

FREE-BODY DIAGRAM OF BC



h = height from C to $B = b \sin \theta$

$$\frac{L_2}{2} = b\cos\theta$$

F = force in spring due to load P

$\Sigma M_B = 0$

$$\frac{P}{2}\left(\frac{L_2}{2}\right) - F\left(\frac{h}{2}\right) = 0 \text{ or } P\cos\theta = F\sin\theta \qquad \text{(Eq. 1)}$$

DETERMINE THE ANGLE θ

ΔS = elongation of spring

$= S_2 - S_1 = b(\cos \theta - \cos \alpha)$

For the spring: $F = k(\Delta S)$

$F = bk(\cos \theta - \cos \alpha)$

Substitute F into Eq. (1):

$P \cos \theta = bk(\cos \theta - \cos \alpha)(\sin \theta)$

$$\text{or} \quad \frac{P}{bk}\cot\theta - \cos\theta + \cos\alpha = 0 \quad \leftarrow \qquad \text{(Eq. 2)}$$

This equation must be solved numerically for the angle θ.

DETERMINE THE DISTANCE δ

$\delta = L_2 - L_1 = 2b \cos \theta - 2b \cos \alpha$

$= 2b(\cos \theta - \cos \alpha)$

From Eq. (2): $\cos\alpha = \cos\theta - \dfrac{P\cot\theta}{bk}$

Therefore,

$$\delta = 2b\left(\cos\theta - \cos\theta + \frac{P\cot\theta}{bk}\right)$$

$$= \frac{2P}{k}\cot\theta \quad \leftarrow \qquad \text{(Eq. 3)}$$

NUMERICAL RESULTS

$b = 8.0$ in. $\quad k = 16$ lb/in. $\quad \alpha = 45°$ $\quad P = 10$ lb

Substitute into Eq. (2):

$$0.078125 \cot \theta - \cos \theta + 0.707107 = 0 \qquad \text{(Eq. 4)}$$

Solve Eq. (4) numerically:

$\theta = 35.1° \quad \leftarrow$

Substitute into Eq. (3):

$\delta = 1.78$ in. $\quad \leftarrow$

Problem 2.2-14 Solve the preceding problem for the following data: $b = 200$ mm, $k = 3.2$ kN/m, $\alpha = 45°$, and $P = 50$ N.

Solution 2.2-14 Framework with rigid bars and a spring

See the solution to the preceding problem.

Eq. (2): $$\frac{P}{bk}\cot\theta - \cos\theta + \cos\alpha = 0$$

Eq. (3): $$\delta = \frac{2P}{k}\cot\theta$$

NUMERICAL RESULTS

$b = 200$ mm $\quad k = 3.2$ kN/m $\quad \alpha = 45° \quad P = 50$ N

Substitute into Eq. (2):

$$0.078125 \cot\theta - \cos\theta + 0.707107 = 0 \qquad \text{(Eq. 4)}$$

Solve Eq. (4) numerically:

$\theta = 35.1° \quad \leftarrow$

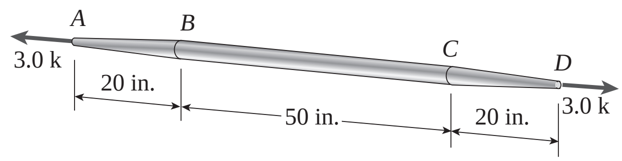
Substitute into Eq. (3):

$\delta = 44.5$ mm $\quad \leftarrow$

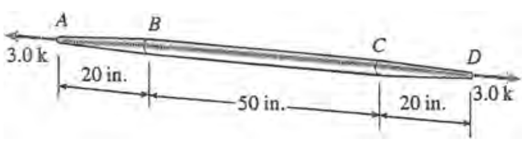
Changes in Lengths under Nonuniform Conditions

Problem 2.3-1 Calculate the elongation of a copper bar of solid circular cross section with tapered ends when it is stretched by axial loads of magnitude 3.0 k (see figure).

The length of the end segments is 20 in. and the length of the prismatic middle segment is 50 in. Also, the diameters at cross sections *A*, *B*, *C*, and *D* are 0.5, 1.0, 1.0, and 0.5 in., respectively, and the modulus of elasticity is 18,000 ksi. (*Hint*: Use the result of Example 2-4.)



Solution 2.3-1 Bar with tapered ends



$d_A = d_D = 0.5$ in. $\quad P = 3.0$ k

$d_B = d_C = 1.0$ in. $\quad E = 18{,}000$ ksi

End segment ($L = 20$ in.)

From Example 2-4:

$$\delta = \frac{4PL}{\pi E\, d_A\, d_B}$$

$$\delta_1 = \frac{4(3.0\text{ k})(20\text{ in.})}{\pi(18{,}000\text{ ksi})(0.5\text{ in.})(1.0\text{ in.})} = 0.008488\text{ in.}$$

Middle segment ($L = 50$ in.)

$$\delta_2 = \frac{PL}{EA} = \frac{(3.0\text{ k})(50\text{ in.})}{(18{,}000\text{ ksi})\left(\frac{\pi}{4}\right)(1.0\text{ in.})^2}$$

$$= 0.0106\text{ in.}$$

Elongation of bar

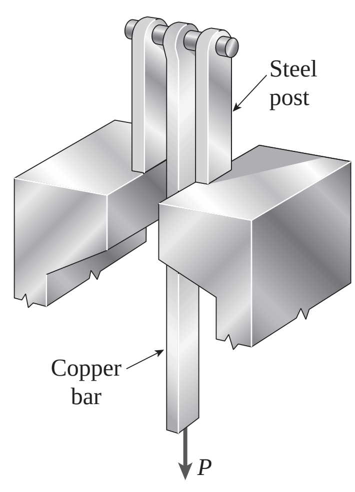
$$\delta = \Sigma\frac{NL}{EA} = 2\delta_1 + \delta_2$$

$$= 2(0.008488\text{ in.}) + (0.01061\text{ in.})$$

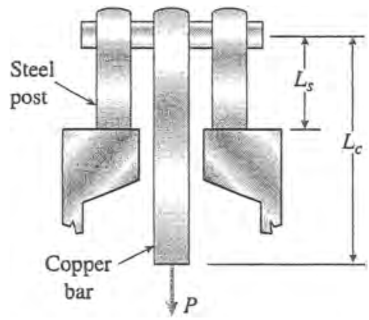
$$= 0.0276\text{ in.} \quad \longleftarrow$$

Problem 2.3-2 A long, rectangular copper bar under a tensile load *P* hangs from a pin that is supported by two steel posts (see figure). The copper bar has a length of 2.0 m, a cross-sectional area of 4800 mm^2, and a modulus of elasticity $E_c = 120$ GPa. Each steel post has a height of 0.5 m, a cross-sectional area of 4500 mm^2, and a modulus of elasticity $E_s = 200$ GPa.

(a) Determine the downward displacement δ of the lower end of the copper bar due to a load $P = 180$ kN.
(b) What is the maximum permissible load P_{max} if the displacement δ is limited to 1.0 mm?



Solution 2.3-2 Copper bar with a tensile load



$L_c = 2.0$ m

$A_c = 4800$ mm^2

$E_c = 120$ GPa

$L_s = 0.5$ m

$A_s = 4500$ mm^2

$E_s = 200$ GPa

(a) DOWNWARD DISPLACEMENT δ ($P = 180$ kN)

$$\delta_c = \frac{PL_c}{E_c A_c} = \frac{(180 \text{ kN})(2.0 \text{ m})}{(120 \text{ GPa})(4800 \text{ mm}^2)}$$
$$= 0.625 \text{ mm}$$
$$\delta_s = \frac{(P/2)L_s}{E_s A_s} = \frac{(90 \text{ kN})(0.5 \text{ m})}{(200 \text{ GPa})(4500 \text{ mm}^2)}$$
$$= 0.050 \text{ mm}$$
$$\delta = \delta_c + \delta_s = 0.625 \text{ mm} + 0.050 \text{ mm}$$
$$= 0.675 \text{ mm} \quad \longleftarrow$$

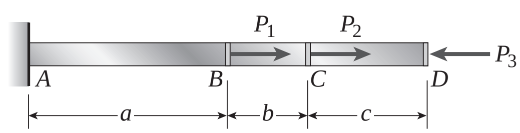
(b) MAXIMUM LOAD P_{max} ($\delta_{max} = 1.0$ mm)

$$\frac{P_{max}}{P} = \frac{\delta_{max}}{\delta} \qquad P_{max} = P\left(\frac{\delta_{max}}{\delta}\right)$$
$$P_{max} = (180 \text{ kN})\left(\frac{1.0 \text{ mm}}{0.675 \text{ mm}}\right) = 267 \text{ kN} \qquad \longleftarrow$$

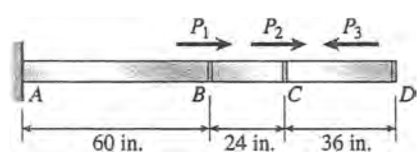
Problem 2.3-3 A steel bar *AD* (see figure) has a cross-sectional area of 0.40 in.^{2} and is loaded by forces $P_1 = 2700$ lb, $P_2 = 1800$ lb, and $P_3 = 1300$ lb. The lengths of the segments of the bar are $a = 60$ in., $b = 24$ in., and $c = 36$ in.

(a) Assuming that the modulus of elasticity $E = 30 \times 10^6$ psi, calculate the change in length δ of the bar. Does the bar elongate or shorten?

(b) By what amount P should the load P_3 be increased so that the bar does not change in length when the three loads are applied?



Solution 2.3-3 Steel bar loaded by three forces



$A = 0.40$ in.^{2} $\quad P_1 = 2700$ lb $\quad P_2 = 1800$ lb

$P_3 = 1300$ lb $\quad E = 30 \times 10^6$ psi

AXIAL FORCES

$N_{AB} = P_1 + P_2 - P_3 = 3200$ lb

$N_{BC} = P_2 - P_3 = 500$ lb

$N_{CD} = -P_3 = -1300$ lb

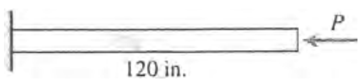
(a) CHANGE IN LENGTH

$$\delta = \Sigma \frac{N_i L_i}{E_i A_i}$$
$$= \frac{1}{EA}(N_{AB}L_{AB} + N_{BC}L_{BC} + N_{CD}L_{CD})$$

$$= \frac{1}{(30 \times 10^6 \text{psi})(0.40 \text{ in.}^2)} [(3200\ lb)(60 \text{ in.}) + (500\ lb)(24 \text{ in.}) - (1300\ lb)(36 \text{ in.})]$$

$$= 0.0131 \text{ in. (elongation)} \quad \longleftarrow$$

(b) INCREASE IN P_3 FOR NO CHANGE IN LENGTH



P = increase in force P_3

The force P must produce a shortening equal to 0.0131 in. in order to have no change in length.

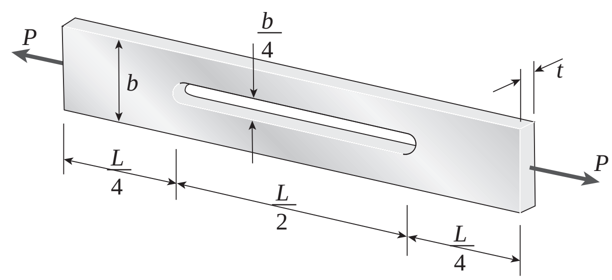
$$\therefore\ 0.0131 \text{ in.} = \delta = \frac{PL}{EA}$$

$$= \frac{P(120 \text{ in.})}{(30 \times 10^6 \text{ psi})(0.40 \text{ in.}^2)}$$

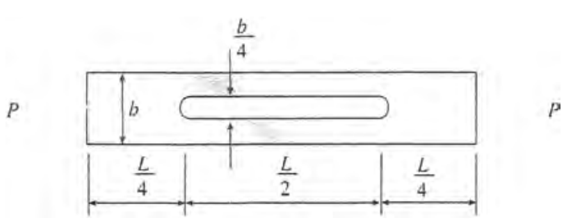
$$P = 1310 \text{ lb} \quad \longleftarrow$$

Problem 2.3-4 A rectangular bar of length L has a slot in the middle half of its length (see figure). The bar has width b, thickness t, and modulus of elasticity E. The slot has width $b/4$.

(a) Obtain a formula for the elongation δ of the bar due to the axial loads P.
(b) Calculate the elongation of the bar if the material is high-strength steel, the axial stress in the middle region is 160 MPa, the length is 750 mm, and the modulus of elasticity is 210 GPa.



Solution 2.3-4 Bar with a slot



t = thickness $\quad L$ = length of bar

(a) ELONGATION OF BAR

$$\delta = \Sigma \frac{N_i L_i}{EA_i} = \frac{P(L/4)}{E(bt)} + \frac{P(L/2)}{E\left(\frac{3}{4}bt\right)} + \frac{P(L/4)}{E(bt)}$$

$$= \frac{PL}{Ebt}\left(\frac{1}{4} + \frac{4}{6} + \frac{1}{4}\right) = \frac{7PL}{6Ebt} \quad \longleftarrow$$

STRESS IN MIDDLE REGION

$$\sigma = \frac{P}{A} = \frac{P}{\left(\frac{3}{4}bt\right)} = \frac{4P}{3bt} \quad \text{or} \quad \frac{P}{bt} = \frac{3\sigma}{4}$$

Substitute into the equation for δ:

$$\delta = \frac{7PL}{6Ebt} = \frac{7L}{6E}\left(\frac{P}{bt}\right) = \frac{7L}{6E}\left(\frac{3\sigma}{4}\right)$$

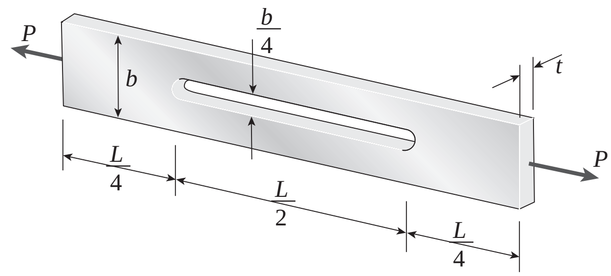
$$= \frac{7\sigma L}{8E}$$

(b) SUBSTITUTE NUMERICAL VALUES:

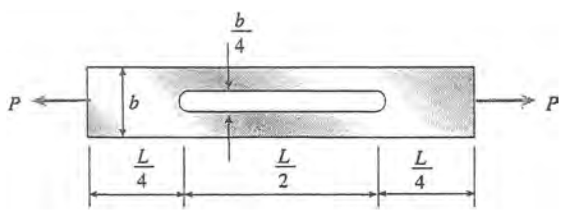
$\sigma = 160$ MPa $\quad L = 750$ mm $\quad E = 210$ GPa

$$\delta = \frac{7(160 \text{ MPa})(750 \text{ mm})}{8(210 \text{ GPa})} = 0.500 \text{ mm} \quad \longleftarrow$$

Problem 2.3-5 Solve the preceding problem if the axial stress in the middle region is 24,000 psi, the length is 30 in., and the modulus of elasticity is 30×10^6 psi.



Solution 2.3-5 Bar with a slot



t = thickness $\qquad L$ = length of bar

(a) Elongation of bar

$$\delta = \Sigma\frac{N_i L_i}{EA_i} = \frac{P(L/4)}{E(bt)} + \frac{P(L/2)}{E(\frac{3}{4}bt)} + \frac{P(L/4)}{E(bt)}$$

$$= \frac{PL}{Ebt}\left(\frac{1}{4} + \frac{4}{6} + \frac{1}{4}\right) = \frac{7PL}{6Ebt} \quad \leftarrow$$

Stress in middle region

$$\sigma = \frac{P}{A} = \frac{P}{\left(\frac{3}{4}bt\right)} = \frac{4P}{3bt} \quad \text{or} \quad \frac{P}{bt} = \frac{3\sigma}{4}$$

Substitute into the equation for δ:

$$\delta = \frac{7PL}{6Ebt} = \frac{7L}{6E}\left(\frac{P}{bt}\right) = \frac{7L}{6E}\left(\frac{3\sigma}{4}\right)$$

$$= \frac{7\sigma L}{8E}$$

(b) Substitute numerical values:

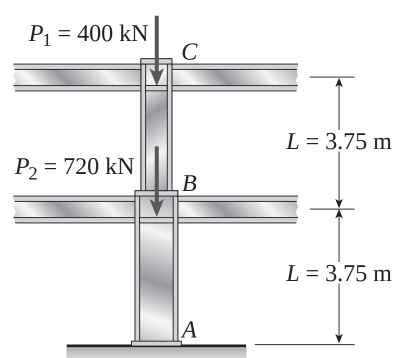
$\sigma = 24{,}000$ psi $\quad L = 30$ in.

$E = 30 \times 10^6$ psi

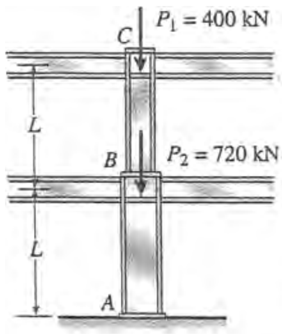
$$\delta = \frac{7(24{,}000 \text{ psi})(30 \text{ in.})}{8(30 \times 10^6 \text{ psi})} = 0.0210 \text{ in.} \quad \leftarrow$$

Problem 2.3-6 A two-story building has steel columns AB in the first floor and BC in the second floor, as shown in the figure. The roof load P_1 equals 400 kN and the second-floor load P_2 equals 720 kN. Each column has length $L = 3.75$ m. The cross-sectional areas of the first- and second-floor columns are 11,000 mm^2 and 3,900 mm^2, respectively.

(a) Assuming that $E = 206$ GPa, determine the total shortening δ_{AC} of the two columns due to the combined action of the loads P_1 and P_2.
(b) How much additional load P_0 can be placed at the top of the column (point C) if the total shortening δ_{AC} is not to exceed 4.0 mm?



Solution 2.3-6 Steel columns in a building



L = length of each column = 3.75 m

E = 206 GPa

$A_{AB} = 11{,}000\ \text{mm}^2$

$A_{BC} = 3{,}900\ \text{mm}^2$

(a) Shortening δ_{AC} of the two columns

$$\delta_{AC} = \Sigma\frac{N_iL_i}{E_iA_i} = \frac{N_{AB}L}{EA_{AB}} + \frac{N_{BC}L}{EA_{BC}}$$

$$= \frac{(1120\ \text{kN})(3.75\ \text{m})}{(206\ \text{GPa})(11{,}000\ \text{mm}^2)}$$

$$+ \frac{(400\ \text{kN})(3.75\ \text{m})}{(206\ \text{GPa})(3{,}900\ \text{mm}^2)}$$

$$= 1.8535\ \text{mm} + 1.8671\ \text{mm} = 3.7206\ \text{mm}$$

$$\delta_{AC} = 3.72\ \text{mm} \quad \leftarrow$$

(b) Additional load P_0 at point C

$(\delta_{AC})_{\max} = 4.0$ mm

δ_0 = additional shortening of the two columns due to the load P_0

$$\delta_0 = (\delta_{AC})_{\max} - \delta_{AC} = 4.0\ \text{mm} - 3.7206\ \text{mm}$$

$$= 0.2794\ \text{mm}$$

Also, $\delta_0 = \dfrac{P_0L}{EA_{AB}} + \dfrac{P_0L}{EA_{BC}} = \dfrac{P_0L}{E}\left(\dfrac{1}{A_{AB}} + \dfrac{1}{A_{BC}}\right)$

Solve for P_0:

$$P_0 = \frac{E\delta_0}{L}\left(\frac{A_{AB}A_{BC}}{A_{AB} + A_{BC}}\right)$$

Substitute numerical values:

$E = 206 \times 10^9\ \text{N/m}^2 \quad \delta_0 = 0.2794 \times 10^{-3}$ m

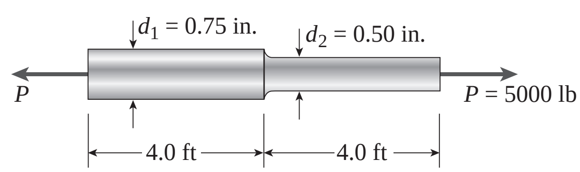
$L = 3.75$ m $\quad A_{AB} = 11{,}000 \times 10^{-6}\ \text{m}^2$

$A_{BC} = 3{,}900 \times 10^{-6}\ \text{m}^2$

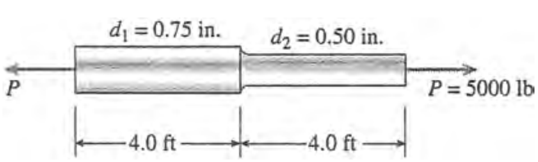
$$P_0 = 44{,}200\ \text{N} = 44.2\ \text{kN} \quad \leftarrow$$

Problem 2.3-7 A steel bar 8.0 ft long has a circular cross section of diameter $d_1 = 0.75$ in. over one-half of its length and diameter $d_2 = 0.5$ in. over the other half (see figure). The modulus of elasticity $E = 30 \times 10^6$ psi.

(a) How much will the bar elongate under a tensile load $P = 5000$ lb?
(b) If the same volume of material is made into a bar of constant diameter d and length 8.0 ft, what will be the elongation under the same load P?



Solution 2.3-7 Bar in tension



$P = 5000$ lb

$E = 30 \times 10^6$ psi

$L = 4$ ft $= 48$ in.

(a) Elongation of nonprismatic bar

$$\delta = \Sigma\frac{N_iL_i}{E_iA_i} = \frac{PL}{E}\Sigma\frac{1}{A_i}$$

$$\delta = \frac{(5000\ \text{lb})(48\ \text{in.})}{30 \times 10^6\ \text{psi}}$$

$$\times \left[\frac{1}{\frac{\pi}{4}(0.75 \text{ in})^2} + \frac{1}{\frac{\pi}{4}(0.50 \text{ in.})^2} \right]$$

$$= 0.0589 \text{ in.} \quad \leftarrow$$

(b) ELONGATION OF PRISMATIC BAR OF SAME VOLUME

Original bar: $V_o = A_1L + A_2L = L(A_1 + A_2)$

Prismatic bar: $V_p = A_p(2L)$

Equate volumes and solve for A_p:

$V_o = V_p \quad L(A_1 + A_2) = A_p(2L)$

$$A_p = \frac{A_1 + A_2}{2} = \frac{1}{2}\left(\frac{\pi}{4}\right)(d_1^2 + d_2^2)$$

$$= \frac{\pi}{8}[(0.75 \text{ in.})^2 + (0.50 \text{ in.})^2] = 0.3191 \text{ in.}^2$$

$$\delta = \frac{P(2L)}{EA_p} = \frac{(5000 \text{ lb})(2)(48 \text{ in.})}{(30 \times 10^6 \text{ psi})(0.3191 \text{ in.}^2)}$$

$$= 0.0501 \text{ in.} \quad \leftarrow$$

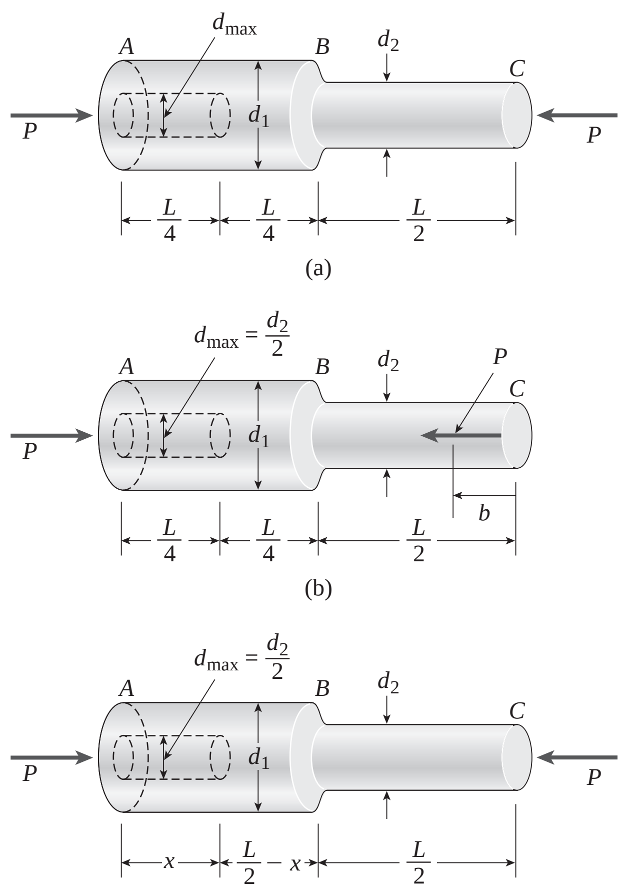
NOTE: A prismatic bar of the same volume will *always* have a smaller change in length than will a nonprismatic bar, provided the constant axial load P, modulus E, and total length L are the same.

Problem 2.3-8 A bar ABC of length L consists of two parts of equal lengths but different diameters. Segment AB has diameter $d_1 = 100$ mm, and segment BC has diameter $d_2 = 60$ mm. Both segments have length $L/2 = 0.6$ m. A longitudinal hole of diameter d is drilled through segment AB for one-half of its length (distance $L/4 = 0.3$ m). The bar is made of plastic having modulus of elasticity $E = 4.0$ GPa. Compressive loads $P = 110$ kN act at the ends of the bar.

(a) If the shortening of the bar is limited to 8.0 mm, what is the maximum allowable diameter d_{max} of the hole? (See figure part a.)

(b) Now, if d_{max} is instead set at $d_2/2$, at what distance b from end C should load P be applied to limit the bar shortening to 8.0 mm? (See figure part b.)

(c) Finally, if loads P are applied at the ends and $d_{max} = d_2/2$, what is the permissible length x of the hole if shortening is to be limited to 8.0 mm? (See figure part c.)



(a) (b) (c)

Solution 2.3-8

NUMERICAL DATA

$d_1 = 100$ mm $\quad d_2 = 60$ mm

$L = 1200$ mm $\quad E = 4.0$ GPa $\quad P = 110$ kN

$\delta_a = 8.0$ mm

(a) find d_{max} if shortening is limited to δ_a

$$A_1 = \frac{\pi}{4} d_1^{\,2} \qquad A_2 = \frac{\pi}{4} d_2^{\,2}$$

$$\delta = \frac{P}{E}\left[\frac{\frac{L}{4}}{\frac{\pi}{4}\left(d_1^{\,2} - d_{max}^{\,2}\right)} + \frac{\frac{L}{4}}{A_1} + \frac{\frac{L}{2}}{A_2}\right]$$

set δ to δ_a and solve for d_{max}

$$d_{max} = d_1\sqrt{\frac{E\delta_a \pi d_1^{\,2} d_2^{\,2} - 2PLd_2^{\,2} - 2PLd_1^{\,2}}{E\delta_a \pi d_1^{\,2} d_2^{\,2} - PLd_2^{\,2} - 2PLd_1^{\,2}}}$$

$d_{max} = 23.9$ mm $\quad \leftarrow$

(b) Now, if d_{max} is instead set at $d_2/2$, at what distance b from end C should load P be applied to limit the bar shortening to $\delta_a = 8.0$ mm?

$$A_0 = \frac{\pi}{4}\left[d_1^2 - \left(\frac{d_2}{2}\right)^2\right]$$

$$A_1 = \frac{\pi}{4} d_1^{\,2} \qquad A_2 = \frac{\pi}{4} d_2^{\,2}$$

$$\delta = \frac{P}{E}\left[\frac{L}{4A_0} + \frac{L}{4A_1} + \frac{\left(\frac{L}{2} - b\right)}{A_2}\right]$$

no axial force in segment at end of length b; set $\delta = \delta_a$ & solve for b

$$b = \left[\frac{L}{2} - A_2\left[\frac{E\delta_a}{P} - \left(\frac{L}{4A_0} + \frac{L}{4A_1}\right)\right]\right]$$

$b = 4.16$ mm $\quad \leftarrow$

(c) Finally if loads P are applied at the ends and $d_{max} = d_2/2$, what is the permissible length x of the hole if shortening is to be limited to $\delta_a = 8.0$ mm?

$$\delta = \frac{P}{E}\left[\frac{x}{A_0} + \frac{\left(\frac{L}{2} - x\right)}{A_1} + \frac{\left(\frac{L}{2}\right)}{A_2}\right]$$

set $\delta = \delta_a$ & solve for x

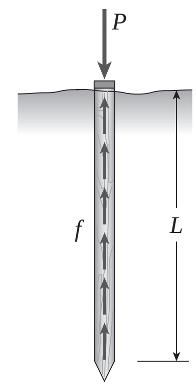
$$x = \frac{\left[A_0 A_1\left(\frac{E\delta_a}{P} - \frac{L}{2A_2}\right)\right] - \frac{1}{2}A_0 L}{A_1 - A_0}$$

$x = 183.3$ mm $\quad \leftarrow$

Problem 2.3-9 A wood pile, driven into the earth, supports a load P entirely by friction along its sides (see figure). The friction force f per unit length of pile is assumed to be uniformly distributed over the surface of the pile. The pile has length L, cross-sectional area A, and modulus of elasticity E.

(a) Derive a formula for the shortening δ of the pile in terms of P, L, E, and A.

(b) Draw a diagram showing how the compressive stress σ_c varies throughout the length of the pile.



Solution 2.3-9 Wood pile with friction

FROM FREE-BODY DIAGRAM OF PILE:

$\Sigma F_{\text{vert}} = 0$ *uarr*$_+$ *darr*$^-$ $fL - P = 0$ $f = \dfrac{P}{L}$ (Eq. 1)

(a) SHORTENING δ OF PILE:

At distance y from the base:

$N(y) =$ axial force $N(y) = fy$ (Eq. 2)

$$d\delta = \frac{N(y)dy}{EA} = \frac{fy\,dy}{EA}$$

$$\delta = \int_0^L d\delta = \frac{f}{EA}\int_0^L y\,dy = \frac{fL^2}{2EA} = \frac{PL}{2EA}$$

$$\delta = \frac{PL}{2EA} \quad \leftarrow$$

(b) COMPRESSIVE STRESS σ_c IN PILE

$$\sigma_c = \frac{N(y)}{A} = \frac{fy}{A} = \frac{Py}{AL} \quad \leftarrow$$

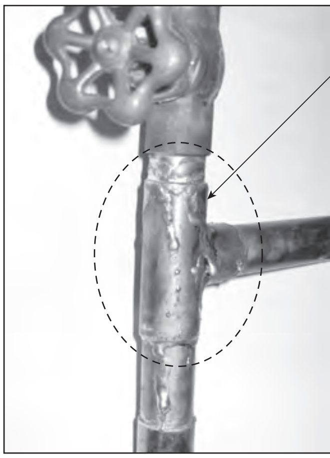
At the base ($y = 0$): $\sigma_c = 0$

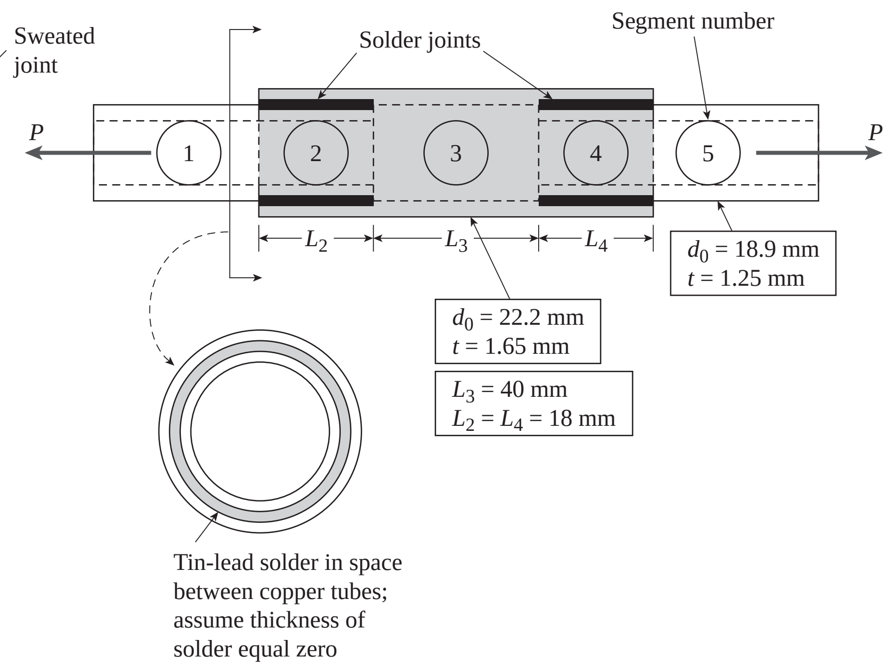
At the top($y = L$): $\sigma_c = \dfrac{P}{A}$

See the diagram above.

Problem 2.3-10 Consider the copper tubes joined below using a "sweated" joint. Use the properties and dimensions given.

(a) Find the total elongation of segment 2-3-4 ($\delta_{2\text{-}4}$) for an applied tensile force of $P = 5$ kN. Use $E_c = 120$ GPa.
(b) If the yield strength in shear of the tin-lead solder is $\tau_y = 30$ MPa and the tensile yield strength of the copper is $\sigma_y = 200$ MPa, what is the maximum load P_{max} that can be applied to the joint if the desired factor of safety in shear is $FS_\tau = 2$ and in tension is $FS_\sigma = 1.7$?
(c) Find the value of L_2 at which tube and solder capacities are equal.





Solution 2.3-10

NUMERICAL DATA

$P = 5\text{ kN} \qquad E_c = 120\text{ GPa}$

$L_2 = 18\text{ mm} \qquad L_4 = L_2$

$L_3 = 40\text{ mm}$

$d_{o3} = 22.2\text{ mm} \qquad t_3 = 1.65\text{ mm}$

$d_{o5} = 18.9\text{ mm} \qquad t_5 = 1.25\text{ mm}$

$\tau_Y = 30\text{ MPa} \qquad \sigma_Y = 200\text{ MPa}$

$FS_\tau = 2 \qquad FS_\sigma = 1.7$

$$\tau_a = \frac{\tau_Y}{FS_\tau} \qquad \tau_a = 15\text{ MPa}$$

$$\sigma_a = \frac{\sigma_Y}{FS_\sigma} \qquad \sigma_a = 117.6\text{ MPa}$$

(a) ELONGATION OF SEGMENT 2-3-4

$$A_2 = \frac{\pi}{4}[d_{o3}^2 - (d_{o5} - 2t_5)^2]$$

$$A_3 = \frac{\pi}{4}[d_{o3}^2 - (d_{o3} - 2t_3)^2]$$

$$A_2 = 175.835\text{ mm}^2 \qquad A_3 = 106.524\text{ mm}^2$$

$$\delta_{24} = \frac{P}{E_c}\left(\frac{L_2 + L_4}{A_2} + \frac{L_3}{A_3}\right)$$

$$\delta_{24} = 0.024\text{ mm} \quad \leftarrow$$

(b) MAXIMUM LOAD P_{max} THAT CAN BE APPLIED TO THE JOINT

FIRST CHECK NORMAL STRESS

$$A_1 = \frac{\pi}{4}[d_{o5}^2 - (d_{o5} - 2t_5)^2]$$

$A_1 = 69.311\ \text{mm}^2$ < smallest cross-sectional area controls normal stress

$P_{max\sigma} = \sigma_a A_1 \quad P_{max\sigma} = 8.15\ \text{kN}$ ← smaller than P_{max} based on shear below so normal stress controls

next check shear stress in solder joint

$A_{sh} = \pi d_{o5} L_2 \qquad A_{sh} = 1.069 \times 10^3\ \text{mm}^2$

$P_{max\tau} = \tau_a A_{sh} \quad P_{max\tau} = 16.03\ \text{kN}$

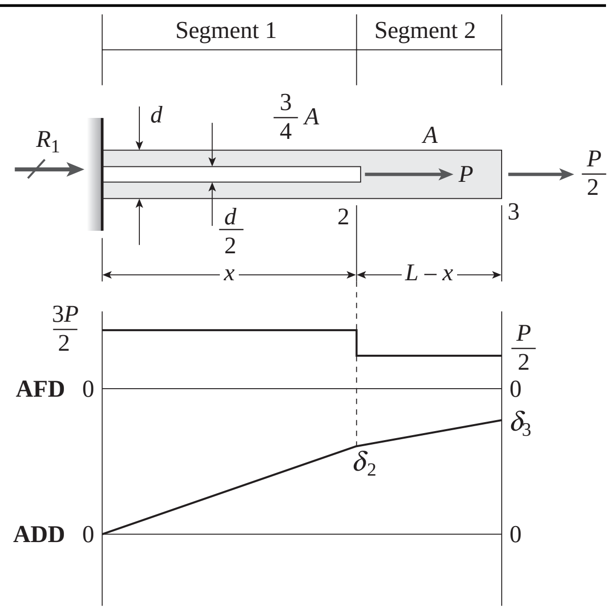
(c) FIND THE VALUE OF L_2 AT WHICH TUBE AND SOLDER CAPACITIES ARE EQUAL

set P_{max} based on shear strength equal to P_{max} based on tensile strength & solve for L_2

$$L_2 = \frac{\sigma_a A_1}{\tau_a(\pi d_{o5})} \qquad L_2 = 9.16\ \text{mm} \quad \leftarrow$$

Problem 2.3-11 The nonprismatic cantilever circular bar shown has an internal cylindrical hole of diameter $d/2$ from 0 to x, so the net area of the cross section for Segment 1 is $(3/4)A$. Load P is applied at x, and load $P/2$ is applied at $x = L$. Assume that E is constant.

(a) Find reaction force R_1.
(b) Find internal axial forces N_i in segments 1 and 2.
(c) Find x required to obtain axial displacement at joint 3 of $\delta_3 = PL/EA$.
(d) In (c), what is the displacement at joint 2, δ_2?
(e) If P acts at $x = 2L/3$ and $P/2$ at joint 3 is replaced by βP, find β so that $\delta_3 = PL/EA$.
(f) Draw the *axial force* (AFD: $N(x)$, $0 \le x \le L$) and *axial displacement* (ADD: $\delta(x)$, $0 \le x \le L$) *diagrams* using results from (b) through (d) above.



Solution 2.3-11

(a) STATICS $\sum F_H = 0 \quad R_1 = -P - \frac{P}{2}$

$$R_1 = \frac{-3}{2}P \quad \leftarrow$$

(b) DRAW FBD'S CUTTING THROUGH SEGMENT 1 & AGAIN THROUGH SEGMENT 2

$N_1 = \frac{3P}{2}$ < tension $\qquad N_2 = \frac{P}{2}$ < tension

(c) FIND X REQUIRED TO OBTAIN AXIAL DISPLACEMENT AT JOINT 3 OF $\delta_3 = PL/EA$

add axial deformations of segments 1 & 2 then set to δ_3; solve for x

$$\frac{N_1 x}{E\frac{3}{4}A} + \frac{N_2(L - x)}{EA} = \frac{PL}{EA}$$

$$\frac{\frac{3P}{2}x}{E\frac{3}{4}A} + \frac{\frac{P}{2}(L - x)}{EA} = \frac{PL}{EA}$$

$$\frac{3}{2}x = \frac{L}{2} \qquad x = \frac{L}{3} \quad \leftarrow$$

(d) WHAT IS THE DISPLACEMENT AT JOINT 2, δ_2?

$$\delta_2 = \frac{N_1 x}{E\frac{3}{4}A} \qquad \delta_2 = \frac{\left(\frac{3P}{2}\right)\frac{L}{3}}{E\frac{3}{4}A}$$

$$\delta_2 = \frac{2}{3}\frac{PL}{EA}$$

(e) IF $x = 2L/3$ AND $P/2$ AT JOINT 3 IS REPLACED BY βP, FIND β SO THAT $\delta_3 = PL/EA$

$$N_1 = (1 + \beta)P \quad N_2 = \beta P \quad x = \frac{2L}{3}$$

substitute in axial deformation expression above & solve for β

$$\frac{[(1 + \beta)P]\dfrac{2L}{3}}{E\dfrac{3}{4}A} + \frac{\beta P\left(L - \dfrac{2L}{3}\right)}{EA} = \frac{PL}{EA}$$

$$\frac{1}{9}PL\frac{8 + 11\beta}{EA} = \frac{PL}{EA}$$

$$(8 + 11\beta) = 9$$

$$\beta = \frac{1}{11} \quad \leftarrow$$

$$\beta = 0.091$$

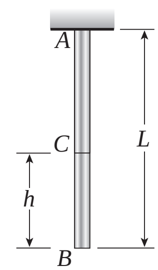
(f) Draw AFD, ADD - see plots above for $x = \dfrac{L}{3}$

Problem 2.3-12 A prismatic bar AB of length L, cross-sectional area A, modulus of elasticity E, and weight W hangs vertically under its own weight (see figure).

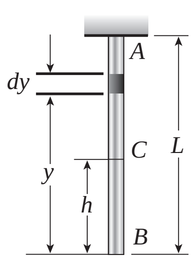
(a) Derive a formula for the downward displacement δ_C of point C, located at distance h from the lower end of the bar.

(b) What is the elongation δ_B of the entire bar?

(c) What is the ratio β of the elongation of the upper half of the bar to the elongation of the lower half of the bar?



Solution 2.3-12 Prismatic bar hanging vertically



W = Weight of bar

(a) DOWNWARD DISPLACEMENT δ_C

Consider an element at distance y from the lower end.

$$N(y) = \frac{Wy}{L} \quad d\delta = \frac{N(y)dy}{EA} = \frac{Wydy}{EAL}$$

$$\delta_C = \int_h^L d\delta = \int_h^L \frac{Wydy}{EAL} = \frac{W}{2EAL}(L^2 - h^2)$$

$$\delta_C = \frac{W}{2EAL}(L^2 - h^2) \quad \leftarrow$$

(b) ELONGATION OF BAR ($h = 0$)

$$\delta_B = \frac{WL}{2EA} \quad \leftarrow$$

(c) RATIO OF ELONGATIONS

Elongation of upper half of bar $\left(h = \dfrac{L}{2}\right)$:

$$\delta_{\text{upper}} = \frac{3WL}{8EA}$$

Elongation of lower half of bar:

$$\delta_{\text{lower}} = \delta_B - \delta_{\text{upper}} = \frac{WL}{2EA} - \frac{3WL}{8EA} = \frac{WL}{8EA}$$

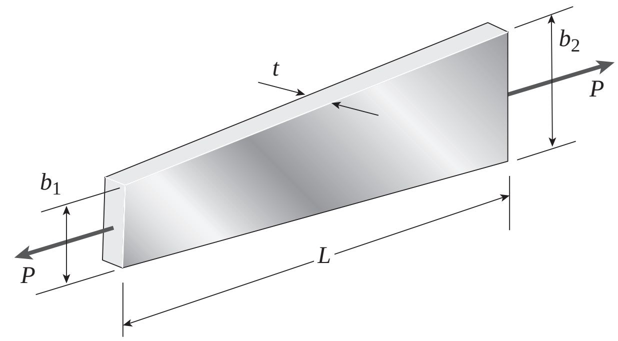
$$\beta = \frac{\delta_{\text{upper}}}{\delta_{\text{lower}}} = \frac{3/8}{1/8} = 3 \quad \leftarrow$$

Problem 2.3-13 A flat bar of rectangular cross section, length L, and constant thickness t is subjected to tension by forces P (see figure). The width of the bar varies linearly from b_1 at the smaller end to b_2 at the larger end. Assume that the angle of taper is small.

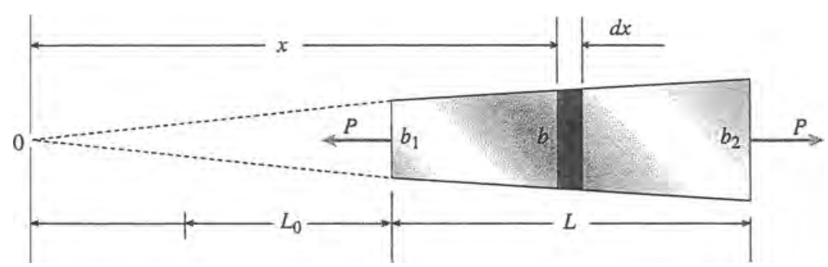
(a) Derive the following formula for the elongation of the bar:

$$\delta = \frac{PL}{Et(b_2 - b_1)} \ln \frac{b_2}{b_1}$$

(b) Calculate the elongation, assuming $L = 5$ ft, $t = 1.0$ in., $P = 25$ k, $b_1 = 4.0$ in., $b_2 = 6.0$ in., and $E = 30 \times 10^6$ psi.



Solution 2.3-13 Tapered bar (rectangular cross section)



t = thickness (constant)

$$b = b_1\left(\frac{x}{L_0}\right) \quad b_2 = b_1\left(\frac{L_0 + L}{L_0}\right) \qquad \text{(Eq. 1)}$$

$$A(x) = bt = b_1\, t\left(\frac{x}{L_0}\right)$$

(a) Elongation of the bar

$$d\delta = \frac{P dx}{EA(x)} = \frac{PL_0\, dx}{Eb_1\, tx}$$

$$\delta = \int_{L0}^{L_0+L} d\delta = \frac{PL_0}{Eb_1\, t}\int_{L_0}^{L_0+L}\frac{dx}{x}$$

$$= \frac{PL_0}{Eb_1\, t} \ln x \Big|_{L_0}^{L_0+L} = \frac{PL_0}{Eb_1\, t} \ln \frac{L_0 + L}{L_0} \qquad \text{(Eq. 2)}$$

From Eq. (1): $\dfrac{L_0 + L}{L_0} = \dfrac{b_2}{b_1}$ (Eq. 3)

Solve Eq. (3) for L_0: $L_0 = L\left(\dfrac{b_1}{b_2 - b_1}\right)$ (Eq. 4)

Substitute Eqs. (3) and (4) into Eq. (2):

$$\delta = \frac{PL}{Et\,(b_2 - b_1)} \ln \frac{b_2}{b_1} \qquad \text{(Eq. 5)}$$

(b) Substitute numerical values:

$L = 5$ ft $= 60$ in. $\quad t = 10$ in.

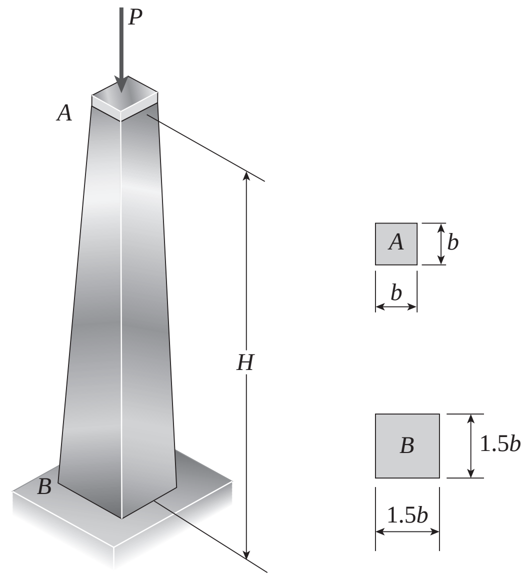
$P = 25$ k $\quad b_1 = 4.0$ in.

$b_2 = 6.0$ in. $\quad E = 30 \times 10^6$ psi

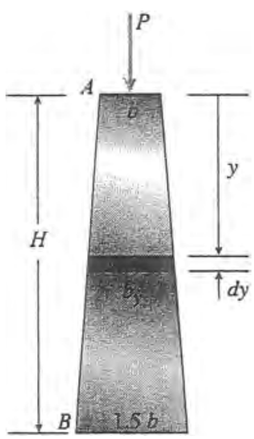
From Eq. (5): $\delta = 0.010$ in. $\leftarrow$

Problem 2.3-14 A post *AB* supporting equipment in a laboratory is tapered uniformly throughout its height *H* (see figure). The cross sections of the post are square, with dimensions $b \times b$ at the top and $1.5b \times 1.5b$ at the base.

Derive a formula for the shortening δ of the post due to the compressive load *P* acting at the top. (Assume that the angle of taper is small and disregard the weight of the post itself.)



Solution 2.3-14 Tapered post



Square cross sections

b = width at *A*

$1.5b$ = width at *B*

b_y = width at distance y

$$= b + (1.5b - b)\frac{y}{H}$$

$$= \frac{b}{H}(H + 0.5y)$$

A_y = cross sectional area at distance y

$$= (b_y)^2 = \frac{b^2}{H^2}(H + 0.5y)^2$$

Shortening of element dy

$$d\delta = \frac{P\,dy}{EA_y} = \frac{P\,dy}{E\left(\frac{b^2}{H^2}\right)(H + 0.5y)^2}$$

Shortening of entire post

$$\delta = \int d\delta = \frac{PH^2}{Eb^2}\int_0^H \frac{dy}{(H + 0.5y)^2}$$

From Appendix C: $\displaystyle\int \frac{dx}{(a + bx)^2} = -\frac{1}{b(a + bx)}$

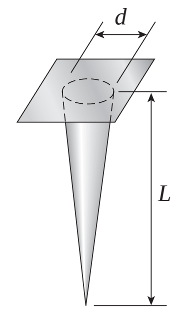
$$\delta = \frac{PH^2}{Eb^2}\left[-\frac{1}{(0.5)(H + 0.5y)}\right]_0^H$$

$$= \frac{PH^2}{Eb^2}\left[-\frac{1}{(0.5)(1.5H)} + \frac{1}{0.5H}\right]$$

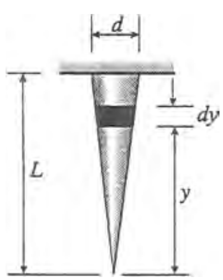
$$= \frac{2PH}{3Eb^2} \quad \leftarrow$$

Problem 2.3-15 A long, slender bar in the shape of a right circular cone with length L and base diameter d hangs vertically under the action of its own weight (see figure). The weight of the cone is W and the modulus of elasticity of the material is E.

Derive a formula for the increase δ in the length of the bar due to its own weight. (Assume that the angle of taper of the cone is small.)



Solution 2.3-15 Conical bar hanging vertically



TERMINOLOGY

N_y = axial force acting on element dy

A_y = cross-sectional area at element dy

A_B = cross-sectional area at base of cone

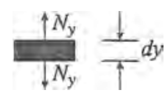
$$= \frac{\pi d^2}{4} \quad V = \text{volume of cone}$$

$$= \frac{1}{3}A_B L \quad V_y = \text{volume of cone below element } dy$$

$$= \frac{1}{3}A_y\, y \quad W_y = \text{weight of cone below element } dy$$

$$= \frac{V_y}{V}(W) = \frac{A_y\, yW}{A_B\, L} \quad N_y = W_y$$

ELEMENT OF BAR



W = weight of cone

ELONGATION OF ELEMENT dy

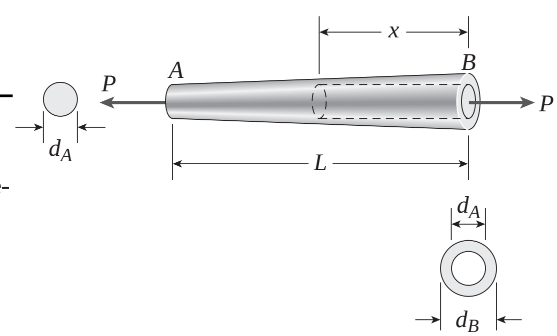
$$d\delta = \frac{N_y\, dy}{E\, A_y} = \frac{Wy\, dy}{E\, A_B L} = \frac{4W}{\pi d^2\, EL}\, y\, dy$$

ELONGATION OF CONICAL BAR

$$\delta = \int d\delta = \frac{4W}{\pi d^2\, EL}\int_0^L y\, dy = \frac{2WL}{\pi d^2\, E} \quad \leftarrow$$

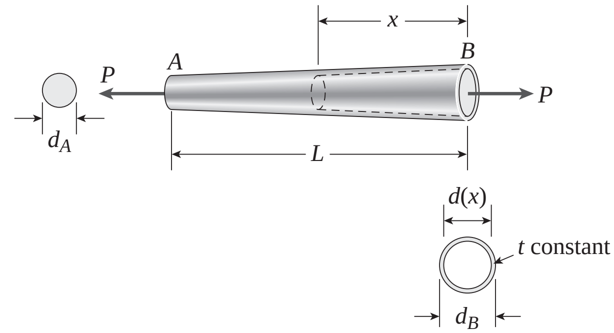
Problem 2.3-16 A uniformly tapered plastic tube AB of circular cross section and length L is shown in the figure. The average diameters at the ends are d_A and $d_B = 2d_A$. Assume E is constant. Find the elongation δ of the tube when it is subjected to loads P acting at the ends. Use the following numerial data: $d_A = 35$ mm, $L = 300$ mm, E = 2.1 GPa, P = 25 kN. Consider two cases as follows:

(a) A hole of *constant* diameter d_A is drilled from B toward A to form a hollow section of length $x = L/2$ (see figure part a).



(a)

(b) A hole of *variable* diameter $d(x)$ is drilled from B toward A to form a hollow section of length $x = L/2$ and constant thickness t (see figure part b). (Assume that $t = d_A/20$.)



(b)

Solution 2.3-16

(a) ELONGATION δ FOR CASE OF CONSTANT DIAMETER HOLE

$$d(\zeta) = d_A\left(1 + \frac{\zeta}{L}\right) \qquad A(\zeta) = \frac{\pi}{4}d(\zeta)^2 \quad < \text{solid portion of length L-x}$$

$$A(\zeta) = \frac{\pi}{4}\left(d(\zeta)^2 - d_A^{\,2}\right) \quad < \text{hollow portion of length x}$$

$$\delta = \frac{P}{E}\left(\int \frac{1}{A(\zeta)}\,d\zeta\right) \qquad \delta = \frac{P}{E}\left[\int_0^{L-x} \frac{4}{\pi d(\zeta)^2}\,d\zeta + \int_{L-x}^{L} \frac{4}{\pi\left(d(\zeta)^2 - d_A^{\,2}\right)}\,d\zeta\right]$$

$$\delta = \frac{P}{E}\left[\int_0^{L-x} \frac{1}{\left[\frac{\pi}{4}\left[d_A\left(1 + \frac{\zeta}{L}\right)\right]^2\right]}\,d\zeta + \int_{L-x}^{L} \frac{1}{\left[\frac{\pi}{4}\left[\left[d_A\left(1 + \frac{\zeta}{L}\right)\right]^2 - d_A^{\,2}\right]\right]}\,d\zeta\right]$$

$$\delta = \frac{P}{E}\left[4\frac{L^2}{(-2 + x)\pi d_A^{\,2}} + \left[\left[4\frac{L}{\pi d_A^2} + \int_{L-x}^{L} \frac{1}{\left[\frac{\pi}{4}\left[\left[d_A\left(1 + \frac{\zeta}{L}\right)\right]^2 - d_A^{\,2}\right]\right]}\,d\zeta\right]\right]\right]$$

$$\delta = \frac{P}{E}\left[4\frac{L^2}{(-2 + x)\pi d_A^{\,2}} + \left(4\frac{L}{\pi d_A^{\,2}} - 2L\frac{\ln(3)}{\pi d_A^{\,2}} + 2L\frac{-\ln(L - x) + \ln(3L - x)}{\pi d_A^{\,2}}\right)\right]$$

$$\text{if } x = L/2 \quad \delta = \frac{P}{E}\left(\frac{4}{3}\frac{L}{\pi d_A^2} - 2L\frac{\ln(3)}{\pi d_A^2} + 2L\frac{-\ln\left(\frac{1}{2}L\right) + \ln\left(\frac{5}{2}L\right)}{\pi d_A^2}\right)$$

Substitute numerical data

$\delta = 2.18$ mm $\leftarrow$

(b) ELONGATION δ FOR CASE OF VARIABLE DIAMETER HOLE BUT CONSTANT WALL THICKNESS $t = d_A/20$ OVER SEGMENT x

$$d(\zeta) = d_A\left(1 + \frac{\zeta}{L}\right) \qquad A(\zeta) = \frac{\pi}{4}d(\zeta)^2 \quad < \text{solid portion of length L-x}$$

$$A(\zeta) = \frac{\pi}{4}\left[d(\zeta)^2 - \left(d(\zeta) - 2\frac{d_A}{20}\right)^2\right] \quad < \text{hollow portion of length x}$$

$$\delta = \frac{P}{E}\left(\int \frac{1}{A(\zeta)}\,d\zeta\right) \qquad \delta = \frac{P}{E}\left[\int_0^{L-x} \frac{4}{\pi d(\zeta)^2}\,d\zeta + \int_{L-x}^{L} \frac{4}{\pi\left[d(\zeta)^2 - \left(d(\zeta) - 2\frac{d_A}{20}\right)^2\right]}\,d\zeta\right]$$

$$\delta = \frac{P}{E}\left[\int_0^{L-x} \frac{4}{\pi\left[d_A\left(1 + \frac{\zeta}{L}\right)\right]}\,d\zeta + \int_{L-x}^{L} \frac{4}{\pi\left[\left[d_A\left(1 + \frac{\zeta}{L}\right)\right]^2 - \left[d_A\left(1 + \frac{\zeta}{L}\right) - 2\frac{d_A}{20}\right]^2\right]}\,d\zeta\right]$$

$$\delta = \frac{P}{E}\left[4\frac{L^2}{(-2L + x)\pi d_A^{\,2}} + 4\frac{L}{\pi d_A^{\,2}} + 20L\frac{\ln(3) + \ln(13) + 2\ln(d_A) + \ln(L)}{\pi d_A^{\,2}} - 20L\frac{2\ln(d_A) + \ln(39L - 20x)}{\pi d_A^{\,2}}\right]$$

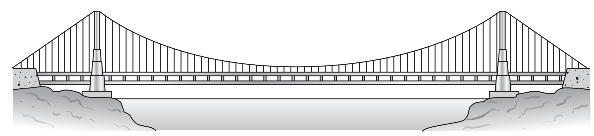
if $x = L/2$

$$\delta = \frac{P}{E}\left(\frac{4}{3}\frac{L}{\pi d_A^{\,2}} + 20L\frac{\ln(3) + \ln(13) + 2\ln(d_A) + \ln(L)}{\pi d_A^{\,2}} - 20L\frac{2\ln(d_A) + \ln(29L)}{\pi d_A^{\,2}}\right)$$

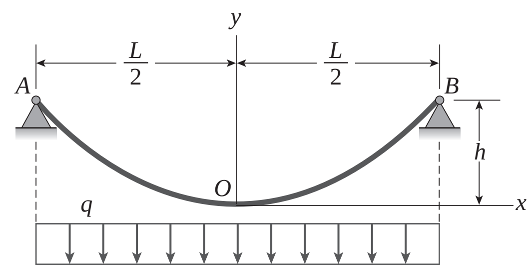
Substitute numerical data

$\delta = 6.74$ mm $\quad\leftarrow$

Problem 2.3-17 The main cables of a suspension bridge [see part (a) of the figure] follow a curve that is nearly parabolic because the primary load on the cables is the weight of the bridge deck, which is uniform in intensity along the horizontal. Therefore, let us represent the central region *AOB* of one of the main cables [see part (b) of the figure] as a parabolic cable supported at points *A* and *B* and carrying a uniform load of intensity *q* along the horizontal. The span of the cable is *L*, the sag is *h*, the axial rigidity is *EA*, and the origin of coordinates is at midspan.



(a)



(b)

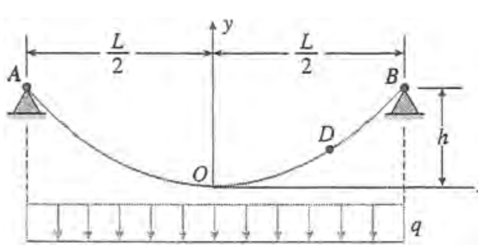
(a) Derive the following formula for the elongation of cable *AOB* shown in part (b) of the figure:

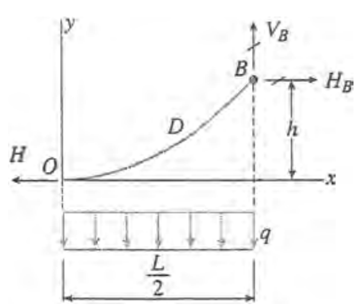
$$\delta = \frac{qL^3}{8hEA}\left(1 + \frac{16h^2}{3L^2}\right)$$

(b) Calculate the elongation δ of the central span of one of the main cables of the Golden Gate Bridge, for which the dimensions and properties are $L = 4200$ ft, $h = 470$ ft, $q = 12{,}700$ lb/ft, and $E = 28{,}800{,}000$ psi. The cable consists of 27,572 parallel wires of diameter 0.196 in.

Hint: Determine the tensile force *T* at any point in the cable from a free-body diagram of part of the cable; then determine the elongation of an element of the cable of length *ds*; finally, integrate along the curve of the cable to obtain an equation for the elongation δ.

Solution 2.3-17 Cable of a suspension bridge





Equation of parabolic curve:

$$y = \frac{4hx^2}{L^2}$$

$$\frac{dy}{dx} = \frac{8hx}{L^2}$$

FREE-BODY DIAGRAM OF HALF OF CABLE

$\Sigma M_B = 0$

$$-Hh + \frac{qL}{2}\left(\frac{L}{4}\right) = 0$$

$$H = \frac{qL^2}{8h}$$

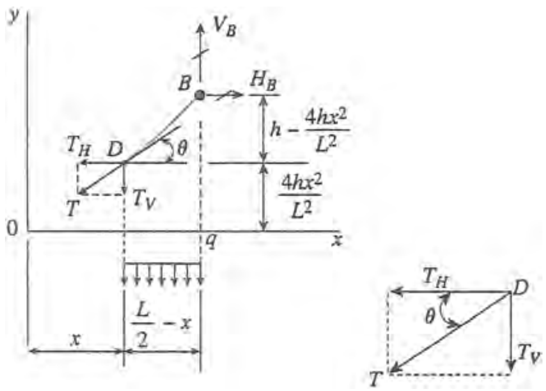
$\Sigma F_{\text{horizontal}} = 0$

$$H_B = H = \frac{qL^2}{8h} \qquad \text{(Eq. 1)}$$

$\Sigma F_{\text{vertical}} = 0$

$$V_B = \frac{qL}{2} \qquad \text{(Eq. 2)}$$

FREE-BODY DIAGRAM OF SEGMENT DB OF CABLE



$$\Sigma F_{\text{horiz}} = 0 \qquad T_H = H_B = \frac{qL^2}{8h} \qquad \text{(Eq. 3)}$$

$$\Sigma F_{\text{vert}} = 0 \quad V_B - T_v - q\left(\frac{L}{2} - x\right) = 0$$

$$T_v = V_B - q\left(\frac{L}{2} - x\right) = \frac{qL}{2} - \frac{qL}{2} + qx$$

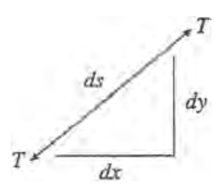
$$= qx \qquad \text{(Eq. 4)}$$

TENSILE FORCE T IN CABLE

$$T = \sqrt{T_H^2 + T_v^2} = \sqrt{\left(\frac{qL^2}{8h}\right)^2 + (qx)^2}$$

$$= \frac{qL^2}{8h}\sqrt{1 + \frac{64h^2x^2}{L^4}} \qquad \text{(Eq. 5)}$$

ELONGATION $d\delta$ OF AN ELEMENT OF LENGTH ds



$$d\delta = \frac{Tds}{EA}$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= dx\sqrt{1 + \left(\frac{8hx}{L^2}\right)^2}$$

$$= dx\sqrt{1 + \frac{64h^2x^2}{L^4}} \qquad \text{(Eq. 6)}$$

(a) ELONGATION δ OF CABLE AOB

$$\delta = \int d\delta = \int \frac{T\,ds}{EA}$$

Substitute for T from Eq. (5) and for ds from Eq. (6):

$$\delta = \frac{1}{EA}\int \frac{qL^2}{8h}\left(1 + \frac{64h^2x^2}{L^4}\right)dx$$

For both halves of cable:

$$\delta = \frac{2}{EA}\int_0^{L/2} \frac{qL^2}{8h}\left(1 + \frac{64h^2x^2}{L^4}\right)dx$$

$$\delta = \frac{qL^3}{8hEA}\left(1 + \frac{16h^2}{3L^4}\right) \quad \leftarrow \qquad \text{(Eq. 7)}$$

(b) GOLDEN GATE BRIDGE CABLE

$L = 4200$ ft $\qquad h = 470$ ft

$q = 12{,}700$ lb/ft $\quad E = 28{,}800{,}000$ psi

27,572 wires of diameter $d = 0.196$ in.

$$A = (27{,}572)\left(\frac{\pi}{4}\right)(0.196 \text{ in.})^2 = 831.90 \text{ in.}^2$$

Substitute into Eq. (7):

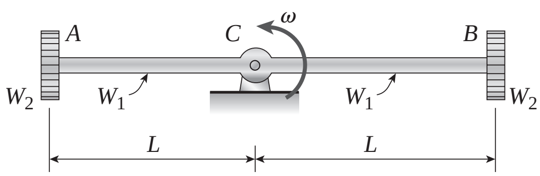
$\delta = 133.7$ in $= 11.14$ ft $\quad \leftarrow$

Problem 2.3-18 A bar *ABC* revolves in a horizontal plane about a vertical axis at the midpoint *C* (see figure). The bar, which has length $2L$ and cross-sectional area *A*, revolves at constant angular speed ω. Each half of the bar (*AC* and *BC*) has weight W_1 and supports a weight W_2 at its end.

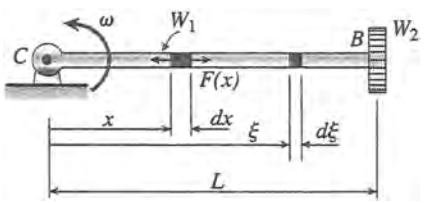
Derive the following formula for the elongation of one-half of the bar (that is, the elongation of either *AC* or *BC*):

$$\delta = \frac{L^2\omega^2}{3gEA}(W_1 + 3W_2)$$

in which *E* is the modulus of elasticity of the material of the bar and *g* is the acceleration of gravity.



Solution 2.3-18 Rotating bar



ω = angular speed

A = cross-sectional area

E = modulus of elasticity

g = acceleration of gravity

$F(x)$ = axial force in bar at distance x from point C

Consider an element of length dx at distance x from point C.

To find the force $F(x)$ acting on this element, we must find the inertia force of the part of the bar from distance x to distance L, plus the inertia force of the weight W_2.

Since the inertia force varies with distance from point C, we now must consider an element of length $d\xi$ at distance ξ, where ξ varies from x to L.

Mass of element $d\xi = \dfrac{d\xi}{L}\left(\dfrac{W_1}{g}\right)$

Acceleration of element $= \xi\omega^2$

Centrifugal force produced by element

$$= (\text{ mass})(\text{ acceleration}) = \frac{W_1\omega^2}{gL}\,\xi d\xi$$

Centrifugal force produced by weight W_2

$$= \left(\frac{W_2}{g}\right)(L\omega^2)$$

Axial force $F(x)$

$$F(x) = \int_{\xi=x}^{\xi=L}\frac{W_1\omega^2}{gL}\,\xi d\xi + \frac{W_2L\omega^2}{g}$$

$$= \frac{W_1\omega^2}{2gL}(L^2 - x^2) + \frac{W_2L\omega^2}{g}$$

Elongation of bar *BC*

$$\delta = \int_0^L \frac{F(x)\,dx}{EA}$$

$$= \int_0^L \frac{W_1\omega^2}{2gL}(L^2 - x^2)dx + \int_0^L \frac{W_2L\omega^2 dx}{gEA}$$

$$= \frac{W1L\omega^2}{2gLEA}\left[\int_0^L L^2\,dx - \int_0^L x^2\,dx\right] + \frac{W_2L\omega^2 dx}{gEA}\int_0^L dx$$

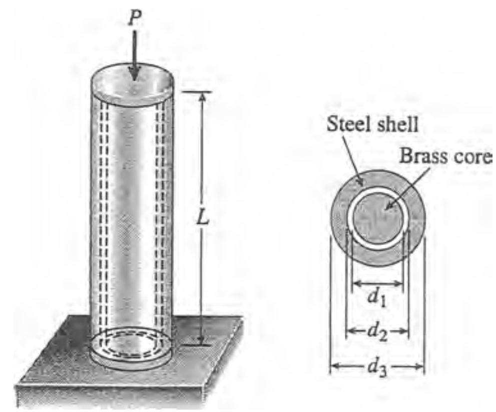
$$= \frac{W_1L^2\omega^2}{3gEA} + \frac{W_2L^2\omega^2}{gEA}$$

$$= \frac{L^2\omega^2}{3gEA} + (W_1 + 3W_2) \quad \leftarrow$$

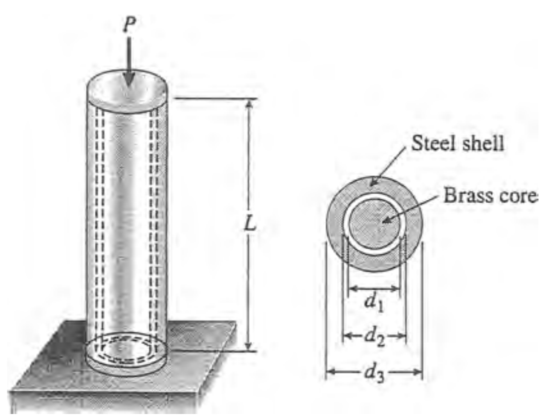
Statically Indeterminate Structures

Problem 2.4-1 The assembly shown in the figure consists of a brass core (diameter $d_1 = 0.25$ in.) surrounded by a steel shell (inner diameter $d_2 = 0.28$ in., outer diameter $d_3 = 0.35$ in.). A load P compresses the core and shell, which have length $L = 4.0$ in. The moduli of elasticity of the brass and steel are $E_b = 15 \times 10^6$ psi and $E_s = 30 \times 10^6$ psi, respectively.

(a) What load P will compress the assembly by 0.003 in.?
(b) If the allowable stress in the steel is 22 ksi and the allowable stress in the brass is 16 ksi, what is the allowable compressive load P_{allow}? (*Suggestion:* Use the equations derived in Example 2-5.)



Solution 2.4-1 Cylindrical assembly in compression



$d_1 = 0.25$ in. $\quad E_b = 15 \times 10^6$ psi

$d_2 = 0.28$ in. $\quad E_s = 30 \times 10^6$ psi

$d_3 = 0.35$ in. $\quad A_s = \dfrac{\pi}{4}(d_3^2 - d_2^2) = 0.03464 \text{ in.}^2$

$L = 4.0$ in. $\quad A_b = \dfrac{\pi}{4}d_1^2 = 0.04909 \text{ in.}^2$

(a) Decrease in length ($\delta = 0.003$ in.)

Use Eq. (2-13) of Example 2-5.

$$\delta = \frac{PL}{E_s A_s + E_b A_b} \quad \text{or}$$

$$P = (E_s A_s + E_s A_b)\left(\frac{\delta}{L}\right)$$

Substitute numerical values:

$$E_s A_s + E_b A_b = (30 \times 10^6 \text{ psi})(0.03464 \text{ in.}^2)$$

$$+ (15 \times 10^6 \text{ psi})(0.04909 \text{ in.}^2)$$

$$= 1.776 \times 10^6 \text{ lb}$$

$$P = (1.776 \times 10^6 \text{ lb})\left(\frac{0.003 \text{ in.}}{4.0 \text{ in.}}\right)$$

$$= 1330 \text{ lb} \quad \leftarrow$$

(b) Allowable load

$\sigma_s = 22$ ksi $\quad \sigma_b = 16$ ksi

Use Eqs. (2-12a and b) of Example 2-5.
For steel:

$$\sigma_s = \frac{PE_s}{E_s A_s + E_b A_b} \qquad P_s = (E_s A_s + E_b A_b)\frac{\sigma_s}{E_s}$$

$$P_s = (1.776 \times 10^6 \text{ lb})\left(\frac{22 \text{ ksi}}{30 \times 10^6 \text{ psi}}\right) = 1300 \text{ lb}$$

For brass:

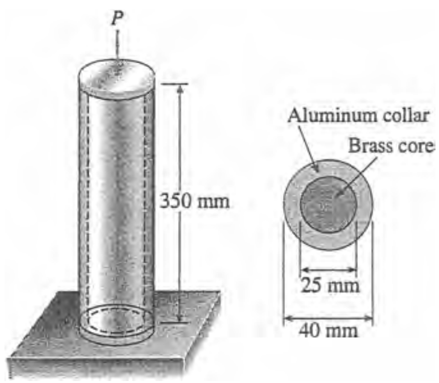
$$\sigma_b = \frac{PE_b}{E_s A_s + E_b A_b} \qquad P_s = (E_s A_s + E_b A_b)\frac{\sigma_b}{E_b}$$

$$P_s = (1.776 \times 10^6 \text{ lb})\left(\frac{16 \text{ ksi}}{15 \times 10^6 \text{ psi}}\right) = 1890 \text{ lb}$$

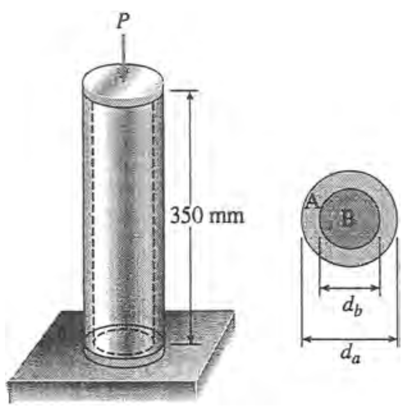
Steel governs. $\quad P_{allow} = 1300$ lb $\quad \leftarrow$

Problem 2.4-2 A cylindrical assembly consisting of a brass core and an aluminum collar is compressed by a load P (see figure). The length of the aluminum collar and brass core is 350 mm, the diameter of the core is 25 mm, and the outside diameter of the collar is 40 mm. Also, the moduli of elasticity of the aluminum and brass are 72 GPa and 100 GPa, respectively.

(a) If the length of the assembly decreases by 0.1% when the load P is applied, what is the magnitude of the load?
(b) What is the maximum permissible load P_{max} if the allowable stresses in the aluminum and brass are 80 MPa and 120 MPa, respectively? (*Suggestion:* Use the equations derived in Example 2-5.)



Solution 2.4-2 Cylindrical assembly in compression



A = aluminum

B = brass

$L = 350$ mm

$d_a = 40$ mm

$d_b = 25$ mm

$$A_a = \frac{\pi}{4}(d_a^2 - d_b^2)$$

$$= 765.8 \text{ mm}^2$$

$E_a = 72$ GPa $\quad E_b = 100$ GPa $\quad A_b = \dfrac{\pi}{4}d_b^2$

$= 490.9 \text{ mm}^2$

(a) DECREASE IN LENGTH

($\delta = 0.1\%$ of $L = 0.350$ mm)

Use Eq. (2-13) of Example 2-5.

$$\delta = \frac{PL}{E_a A_a + E_b A_b} \quad \text{or}$$

$$P = (E_a A_a + E_b A_b)\left(\frac{\delta}{L}\right)$$

Substitute numerical values:

$$E_a A_a + E_b A_b = (72 \text{ GPa})(765.8 \text{ mm}^2)$$
$$+(100 \text{ GPa})(490.9 \text{ mm}^2)$$
$$= 55.135 \text{ MN} + 49.090 \text{ MN}$$
$$= 104.23 \text{ MN}$$

$$P = (104.23 \text{ MN})\left(\frac{0.350 \text{ mm}}{350 \text{ mm}}\right)$$
$$= 104.2 \text{ kN} \quad \longleftarrow$$

(b) ALLOWABLE LOAD

$\sigma_a = 80$ MPa $\quad \sigma_b = 120$ MPa

Use Eqs. (2-12a and b) of Example 2-5.

For aluminum:

$$\sigma_a = \frac{PE_a}{E_a A_a + E_b A_b} \quad P_a = (E_a A_a + E_b A_b)\left(\frac{\sigma_a}{E_a}\right)$$

$$P_a = (104.23 \text{ MN})\left(\frac{80 \text{ MPa}}{72 \text{ GPa}}\right) = 115.8 \text{ kN}$$

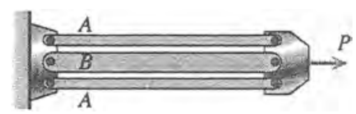
For brass:

$$\sigma_b = \frac{PE_b}{E_a A_a + E_b A_b} \quad P_b = (E_a A_a + E_b A_b)\left(\frac{\sigma_b}{E_b}\right)$$

$$P_b = (104.23 \text{ MN})\left(\frac{120 \text{ MPa}}{100 \text{ GPa}}\right) = 125.1 \text{ kN}$$

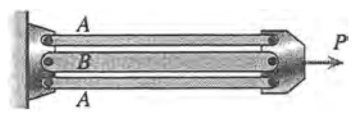
Aluminum governs. $P_{max} = 116$ kN $\quad \longleftarrow$

Problem 2.4-3 Three prismatic bars, two of material A and one of material B, transmit a tensile load P (see figure). The two outer bars (material A) are identical. The cross-sectional area of the middle bar (material B) is 50% larger than the cross-sectional area of one of the outer bars. Also, the modulus of elasticity of material A is twice that of material B.

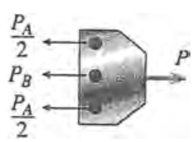


(a) What fraction of the load P is transmitted by the middle bar?
(b) What is the ratio of the stress in the middle bar to the stress in the outer bars?
(c) What is the ratio of the strain in the middle bar to the strain in the outer bars?

Solution 2.4-3 Prismatic bars in tension



Free-body diagram of end plate



Equation of equilibrium

$$\Sigma F_{\text{horiz}} = 0 \quad P_A + P_B - P = 0 \tag{1}$$

Equation of compatibility

$$\delta_A = \delta_B \tag{2}$$

Force-displacement relations

A_A = total area of both outer bars

$$\delta_A = \frac{P_A L}{E_A A_k} \quad \delta_B = \frac{P_B L}{E_B A_B} \tag{3}$$

Substitute into Eq. (2):

$$\frac{P_A L}{E_A A_A} = \frac{P_B L}{E_B A_B} \tag{4}$$

Solution of the equations

Solve simultaneously Eqs. (1) and (4):

$$P_A = \frac{E_A A_A P}{E_A A_A + E_B A_B} \quad P_B = \frac{E_B A_B P}{E_A A_A + E_B A_B} \tag{5}$$

Substitute into Eq. (3):

$$\delta = \delta_A = \delta_B = \frac{PL}{E_A A_A + E_B A_B} \tag{6}$$

Stresses:

$$\sigma_A = \frac{P_A}{A_A} = \frac{E_A P}{E_A A_A + E_B A_B}$$

$$\sigma_B = \frac{P_B}{A_B} = \frac{E_B P}{E_A A_A + E_B A_B} \tag{7}$$

(a) Load in middle bar

$$\frac{P_B}{P} = \frac{E_B A_B}{E_A A_A + E_B A_B} = \frac{1}{\frac{E_A A_A}{E_B A_B} + 1}$$

$$\text{Given: } \frac{E_A}{E_B} = 2 \quad \frac{A_A}{A_B} = \frac{1+1}{1.5} = \frac{4}{3}$$

$$\therefore \frac{P_B}{P} = \frac{1}{\left(\frac{E_A}{E_B}\right)\left(\frac{A_A}{A_B}\right) + 1} = \frac{1}{\frac{8}{3} + 1} = \frac{3}{11} \quad \leftarrow$$

(b) Ratio of stresses

$$\frac{\sigma_B}{\sigma_A} = \frac{E_B}{E_A} = \frac{1}{2} \quad \leftarrow$$

(c) Ratio of strains

All bars have the same strain

Ratio = 1 $\leftarrow$

Problem 2.4-4 A circular bar *ACB* of diameter d having a cylindrical hole of length x and diameter $d/2$ from A to C is held between rigid supports at A and B. A load P acts at $L/2$ from ends A and B. Assume E is constant.

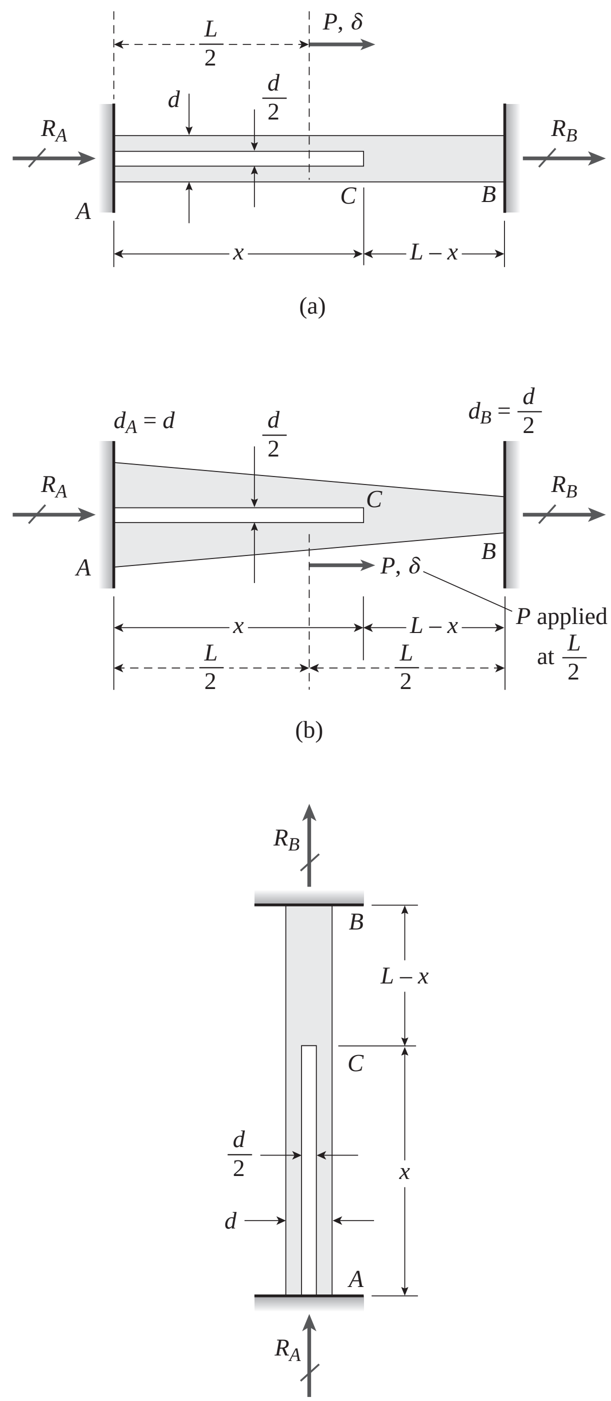
(a) Obtain formulas for the reactions R_A and R_B at supports A and B, respectively, due to the load P (see figure part a).

(b) Obtain a formula for the displacement δ at the point of load application (see figure part a).

(c) For what value of x is $R_B = (6/5)\,R_A$? (See figure part a.)

(d) Repeat (a) if the bar is now tapered linearly from A to B as shown in figure part b and $x = L/2$.

(e) Repeat (a) if the bar is now rotated to a vertical position, load P is removed, and the bar is hanging under its own weight (assume mass density $= \rho$). (See figure part c.) Assume that $x = L/2$



Solution 2.4-4

(a) **reactions at A & B due to load P at L/2**

$$A_{AC} = \frac{\pi}{4}\left[d^2 - \left(\frac{d}{2}\right)^2\right] \qquad A_{AC} = \frac{3}{16}\pi d^2$$

$$A_{CB} = \frac{\pi}{4}d^2$$

select R_B as the redundant; use superposition and a compatibility equation at B

$$\text{if } x \le L/2 \qquad \delta_{B1a} = \frac{Px}{EA_{AC}} + \frac{P\left(\frac{L}{2} - x\right)}{A_{CB}} \qquad \delta_{B1a} = \frac{P}{E}\left(\frac{x}{\frac{3}{16}\pi d^2} + \frac{\frac{L}{2} - x}{\frac{\pi}{4}d^2}\right)$$

$$\delta_{B1a} = \frac{2}{3}P\frac{2x + 3L}{E\pi d^2}$$

$$\text{if } x \ge L/2 \qquad \delta_{B1b} = \frac{P\frac{L}{2}}{EA_{AC}} \qquad \delta_{B1b} = \frac{P\frac{L}{2}}{E\left(\frac{3}{16}\pi d^2\right)} \qquad \delta_{B1b} = \frac{8}{3}\frac{PL}{E\pi d^2}$$

the following expression for δ_{B2} is good for all x

$$\delta_{B2} = \frac{R_B}{E}\left(\frac{x}{A_{AC}} + \frac{L - x}{A_{CB}}\right) \qquad \delta_{B2} = \frac{R_B}{E}\left(\frac{x}{\frac{3}{16}\pi d^2} + \frac{L - x}{\frac{\pi}{4}d^2}\right)$$

$$\delta_{B2} = \frac{R_B}{E}\left(\frac{16}{3}\frac{x}{\pi d^2} + 4\frac{L - x}{\pi d^2}\right)$$

(a.1) **solve for R_B and R_A assuming that $x \le L/2$**

$$\text{compatibility:} \qquad \delta_{B1a} + \delta_{B2} = 0 \qquad R_{Ba} = \frac{-\left(\frac{2}{3}P\frac{2x + 3L}{\pi d^2}\right)}{\left(\frac{16}{3}\frac{x}{\pi d^2} + 4\frac{L - x}{\pi d^2}\right)} \qquad R_{Ba} = \frac{-1}{2}P\frac{2x + 3L}{x + 3L} \quad \longleftarrow$$

^ check − if $x = 0$, $R_B = -P/2$

$$\text{statics:} \qquad R_{Aa} = -P - R_{Ba} \qquad R_{Aa} = -P - \frac{-1}{2}P\frac{2x + 3L}{x + 3L} \qquad R_{Aa} = \frac{-3}{2}P\frac{L}{x + 3L} \quad \longleftarrow$$

^ check − if $x = 0$, $R_{A_a} = -P/2$

(a.2) **solve for R_B and R_A assuming that $x \geq L/2$**

compatibility: $\delta_{B1b} + \delta_{B2} = 0$ $\quad R_{Bb} = \dfrac{\dfrac{-8}{3}\dfrac{PL}{\pi d^2}}{\left(\dfrac{16}{3}\dfrac{x}{\pi d^2} + 4\dfrac{L - x}{\pi d^2}\right)}$ $\quad R_{Bb} = \dfrac{-2PL}{x + 3L} \leftarrow$

^ check − if $x = L$, $R_B = -P/2$

statics: $R_{Ab} = -P - R_{Bb}$ $\quad R_{Ab} = -P - \left(\dfrac{-2PL}{x + 3L}\right)$ $\quad R_{Ab} = -P\dfrac{x + L}{x + 3L} \leftarrow$

(b) find δ at point of load application; axial force for segment 0 to $L/2 = -R_A$ & δ = elongation of this segment

(b.1) **assume that $x \leq L/2$**

$$\delta_a = \frac{-R_{Aa}}{E}\left(\frac{x}{A_{AC}} + \frac{\frac{L}{2} - x}{A_{CB}}\right) \qquad \delta_a = \frac{-\left(\frac{-3}{2}P\frac{L}{x + 3L}\right)}{E}\left(\frac{x}{\frac{3}{16}\pi d^2} + \frac{\frac{L}{2} - x}{\frac{\pi}{4}d^2}\right)$$

$$\delta_a = PL\frac{2x + 3L}{(x + 3L)E\pi d^2} \qquad \text{for } x = L/2 \qquad \delta_a = \frac{8}{7}L\frac{P}{E\pi d^2} \leftarrow$$

(b.2) **assume that $x \geq L/2$**

$$\delta_b = \frac{(-R_{Ab})\frac{L}{2}}{EA_{AC}} \qquad \delta_b = \frac{\left(P\frac{x + L}{x + 3L}\right)\frac{L}{2}}{E\left(\frac{3}{16}\pi d^2\right)} \qquad \delta_b = \frac{8}{3}P\left(\frac{x + L}{x + 3L}\right)\frac{L}{E\pi d^2} \leftarrow$$

for $x = L/2$ $\quad \delta_b = \dfrac{8}{7}P\dfrac{L}{E\pi d^2}$ $\quad$ < same as δ_a above (OK)

(c) **For what value of x is $R_B = (6/5)\, R_A$**? Guess that $x < L/2$ here & use R_{Ba} expression above to find x

$$\frac{-1}{2}P\frac{2x + 3L}{x + 3L} - \frac{6}{5}\left(\frac{-3}{2}P\frac{L}{x + 3L}\right) = 0 \qquad \frac{-1}{10}P\frac{10x - 3L}{x + 3L} = 0 \qquad x = \frac{3L}{10} \leftarrow$$

Now try $R_{Bb} = (6/5)R_{Ab}$ assuming that $x > L/2$

$$\frac{-2PL}{x + 3L} - \frac{6}{5}\left(-P\frac{x + L}{x + 3L}\right) = 0 \qquad \frac{2}{5}P\frac{-2L + 3x}{x + 3L} = 0 \qquad x = \frac{2}{3}L \leftarrow$$

So, there are two solutions for x.

(d) **repeat (a) above for tapered bar & x = L/2**

outer diameter

$$d(x) = d\left(1 - \frac{x}{2L}\right)$$

$$A_{AC} = \frac{\pi}{4}\left[d(x)^2 - \left(\frac{d}{2}\right)^2\right] \qquad 0 \le x \le \frac{L}{2} \qquad A_{AC} = \frac{\pi}{4}\left[\left[d\left(1 - \frac{x}{2L}\right)\right]^2 - \left(\frac{d}{2}\right)^2\right]$$

$$A_{AC} = \frac{1}{16}\pi\left(\frac{d}{L}\right)^2(3L^2 - 4Lx + x^2)$$

$$A_{CB} = \frac{\pi}{4}d(x)^2 \qquad \frac{L}{2} \le x \le L \qquad A_{CB} = \frac{\pi}{4}\left[d\left(1 - \frac{x}{2L}\right)\right]^2$$

$$A_{CB} = \frac{1}{16}\pi\left(\frac{d}{L}\right)^2(4L^2 - 4Lx + x^2)$$

As in (a), use superposition and compatibility to find redundant R_B & then R_A

$$\delta_{B1} = \frac{P}{E}\int_0^{\frac{L}{2}}\frac{1}{A_{AC}}d\zeta \qquad \delta_{B1} = \frac{P}{E}\int_0^{\frac{L}{2}}\frac{1}{\left[\frac{1}{16}\pi\left(\frac{d}{L}\right)^2(3L^2 - 4L\zeta + \zeta^2)\right]}d\zeta$$

$$\delta_{B1} = \frac{-8PL}{E\pi d^2}(-\ln(5) + \ln(3)) \qquad \delta_{B1} = 1.301\frac{PL}{Ed^2}$$

$$\delta_{B2} = \frac{R_B}{E}\left(\int_0^{\frac{L}{2}}\frac{1}{A_{AC}}d\zeta + \int_{\frac{L}{2}}^{L}\frac{1}{A_{CB}}d\zeta\right)$$

$$\delta_{B2} = \frac{R_B}{E}\left[\int_0^{\frac{L}{2}}\frac{1}{\frac{1}{16}\pi\left(\frac{d}{L}\right)^2(3L^2 - 4L\zeta + \zeta^2)}d\zeta + \int_{\frac{L}{2}}^{L}\frac{1}{\frac{1}{16}\pi\left(\frac{d}{L}\right)^2(4L^2 - 4L\zeta + \zeta^2)}d\zeta\right]$$

$$\delta_{B2} = \frac{-8L}{3(\pi d^2)}\frac{R_B}{E}(-3\ln(5) + 3\ln(3) - 2) \qquad \delta_{B2} = 2.998\frac{R_BL}{Ed^2}$$

compatibility: $\delta_{B1} + \delta_{B2} = 0 \qquad R_B = \dfrac{-\left(1.301\frac{PL}{Ed^2}\right)}{\left(2.998\frac{L}{Ed^2}\right)} \qquad R_B = -0.434P \quad \leftarrow$

statics: $R_A = -P - R_B \qquad R_A = (-P - -0.434P) \qquad R_A = -0.566P \quad \leftarrow$

(e) **Find reactions if the bar is now rotated to a vertical position, load P is removed, and the bar is hanging under its own weight (assume mass density = ρ). Assume that x = L/2.**

$$A_{AC} = \frac{3}{16}\pi d^2 \qquad A_{CB} = \frac{\pi}{4} d^2$$

select $\mathbf{R_B}$ **as the redundant**; use superposition and a compatibility equation at B

from (a) above: compatibility: $\delta_{B1} + \delta_{B2} = 0$

$$\delta_{B2} = \frac{R_B}{E}\left(\frac{x}{A_{AC}} + \frac{L - x}{A_{CB}}\right) \qquad \text{for } x = L/2,\ \delta_{B2} = \frac{R_B}{E}\left(\frac{14}{3}\frac{L}{\pi d^2}\right)$$

$$\delta_{B1} = \int_0^{\frac{L}{2}} \frac{N_{AC}}{EA_{AC}} d\zeta + \int_{\frac{L}{2}}^{L} \frac{N_{CB}}{EA_{CB}} d\zeta$$

where axial forces in bar due to self weight are: $W_{AC} = \rho g A_{AC}\dfrac{L}{2} \qquad W_{CB} = \rho g A_{CB}\dfrac{L}{2}$
(assume ζ is measured upward from A)

$$N_{AC} = -\left[\rho g A_{CB}\frac{L}{2} + \rho g A_{AC}\left(\frac{L}{2} - \zeta\right)\right] \qquad A_{AC} = \frac{3}{16}\pi d^2 \qquad A_{CB} = \frac{\pi}{4} d^2$$

$$N_{CB} = -[\rho\, g A_{CB}(L - \zeta)]$$

$$N_{AC} = \frac{-1}{8}\rho g\pi\, d^2\, L - \frac{3}{16}\rho g\pi\, d^2\left(\frac{1}{2}L - \zeta\right) \qquad N_{CB} = -\left[\frac{1}{4}\rho g\pi\, d^2(L - \zeta)\right]$$

$$\delta_{B1} = \int_0^{\frac{L}{2}} \frac{\frac{-1}{8}\rho g\pi d^2 L - \frac{3}{16}\rho g\pi d^2\left(\frac{1}{2}L - \zeta\right)}{E\left(\frac{3}{16}\pi d^2\right)} d\zeta + \int_{\frac{L}{2}}^{L} \frac{-\left[\frac{1}{4}\rho g\pi d^2(L - \zeta)\right]}{E\left(\frac{\pi}{4}d^2\right)} d\zeta$$

$$\delta_{B1} = \left(\frac{-11}{24}\rho g\frac{L^2}{E} + \frac{-1}{8}\rho g\frac{L^2}{E}\right) \qquad \delta_{B1} = \frac{-7}{12}\rho\, g\frac{L^2}{E} \qquad \frac{7}{12} = 0.583$$

compatibility: $\delta_{B1} + \delta_{B2} = 0$

$$R_B = \frac{-\left(\frac{-7}{12}\rho g\frac{L^2}{E}\right)}{\left(\frac{14}{3}\frac{L}{E\pi d^2}\right)} \qquad R_B = \frac{1}{8}\rho g\pi d^2 L \qquad \leftarrow$$

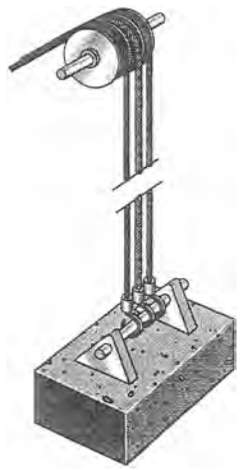
statics: $R_A = (W_{AC} + W_{CB}) - R_B$

$$R_A = \left[\left[\rho g\left(\frac{3}{16}\pi d^2\right)\frac{L}{2} + \rho g\left(\frac{\pi}{4}d^2\right)\frac{L}{2}\right] - \frac{1}{8}\rho g\pi d^2 L\right]$$

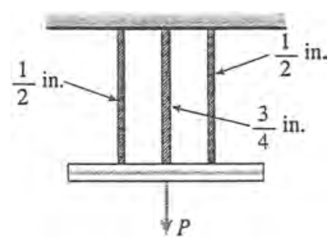
$$R_A = \frac{3}{32}\rho g\pi\, d^2\, L \qquad \leftarrow$$

Problem 2.4-5 Three steel cables jointly support a load of 12 k (see figure). The diameter of the middle cable is 3/4 in. and the diameter of each outer cable is 1/2 in. The tensions in the cables are adjusted so that each cable carries one-third of the load (i.e., 4 k). Later, the load is increased by 9 k to a total load of 21 k.

(a) What percent of the total load is now carried by the middle cable?
(b) What are the stresses σ_M and σ_O in the middle and outer cables, respectively? (**NOTE:** See Table 2-1 in Section 2.2 for properties of cables.)



Solution 2.4-5 Three cables in tension



AREAS OF CABLES (from Table 2-1)

Middle cable: $A_M = 0.268 \text{ in.}^2$

Outer cables: $A_O = 0.119 \text{ in.}^2$

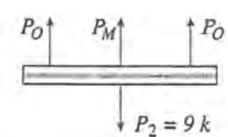
(for each cable)

FIRST LOADING

$$P_1 = 12\text{ k}\left(\text{Each cable carries } \frac{P_1}{3} \text{ or } 4\text{ k.}\right)$$

SECOND LOADING

$P_2 = 9$ k (additional load)



EQUATION OF EQUILIBRIUM

$$\Sigma F_{vert} = 0 \qquad 2P_O + P_M - P_2 = 0 \tag{1}$$

EQUATION OF COMPATIBILITY

$$\delta_M = \delta_O \tag{2}$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_M = \frac{P_M L}{EA_M} \quad \delta_O = \frac{P_o L}{EA_o} \tag{3, 4}$$

SUBSTITUTE INTO COMPATIBILITY EQUATION:

$$\frac{P_M L}{EA_M} = \frac{P_O L}{EA_O} \quad \frac{P_M}{A_M} = \frac{P_O}{A_O} \tag{5}$$

SOLVE SIMULTANEOUSLY EQS. (1) AND (5):

$$P_M = P_2\left(\frac{A_M}{A_M + 2A_O}\right) = (9\text{ k})\left(\frac{0.268\text{ in.}^2}{0.506\text{ in.}^2}\right)$$

$$= 4.767\text{ k}$$

$$P_o = P_2\left(\frac{A_o}{A_M + 2A_O}\right) = (9\text{ k})\left(\frac{0.119\text{ in.}^2}{0.506\text{ in.}^2}\right)$$

$$= 2.117\text{ k}$$

FORCES IN CABLES

Middle cable: Force $= 4\text{ k} + 4.767\text{ k} = 8.767\text{ k}$

Outer cables: Force $= 4\text{ k} + 2.117\text{ k} = 6.117\text{ k}$

(for each cable)

(a) PERCENT OF TOTAL LOAD CARRIED BY MIDDLE CABLE

$$\text{Percent} = \frac{8.767\text{ k}}{21\text{ k}}(100\%) = 41.7\% \quad \leftarrow$$

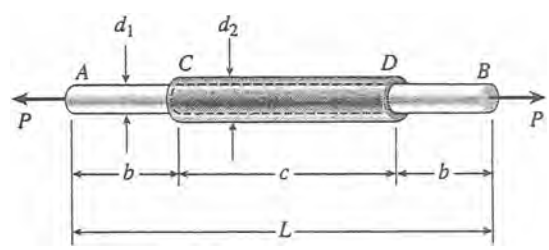
(b) STRESSES IN CABLES ($\sigma = P/A$)

$$\text{Middle cable: } \sigma_M = \frac{8.767\text{ k}}{0.268\text{ in.}^2} = 32.7\text{ ksi} \quad \leftarrow$$

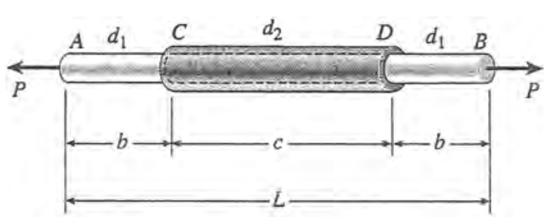
$$\text{Outer cables: } \sigma_O = \frac{6.117\text{ k}}{0.119\text{ in.}^2} = 51.4\text{ ksi} \quad \leftarrow$$

Problem 2.4-6 A plastic rod AB of length $L = 0.5$ m has a diameter $d_1 = 30$ mm (see figure). A plastic sleeve CD of length $c = 0.3$ m and outer diameter $d_2 = 45$ mm is securely bonded to the rod so that no slippage can occur between the rod and the sleeve. The rod is made of an acrylic with modulus of elasticity $E_1 = 3.1$ GPa and the sleeve is made of a polyamide with $E_2 = 2.5$ GPa.

(a) Calculate the elongation δ of the rod when it is pulled by axial forces $P = 12$ kN.
(b) If the sleeve is extended for the full length of the rod, what is the elongation?
(c) If the sleeve is removed, what is the elongation?



Solution 2.4-6 Plastic rod with sleeve



$P = 12$ kN $\quad d_1 = 30$ mm $\quad b = 100$ mm

$L = 500$ mm $\quad d_2 = 45$ mm $\quad c = 300$ mm

Rod: $E_1 = 3.1$ GPa

Sleeve: $E_2 = 2.5$ GPa

$$\text{Rod: } A_1 = \frac{\pi d_1^2}{4} = 706.86\text{ mm}^2$$

$$\text{Sleeve: } A_2 = \frac{\pi}{4}(d_2^2 - d_1^2) = 883.57\text{ mm}^2$$

$$E_1A_1 + E_2A_2 = 4.400\text{ MN}$$

(a) ELONGATION OF ROD

Part AC: $\delta_{AC} = \dfrac{Pb}{E_1A_1} = 0.5476 \text{ mm}$

Part CD: $\delta_{CD} = \dfrac{P_C}{E_1A_1E_2A_2}$

$= 0.81815 \text{ mm}$

(From Eq. 2-13 of Example 2-5)

$\delta = 2\delta_{AC} + \delta_{CD} = 1.91 \text{ mm}$ ←

(b) SLEEVE AT FULL LENGTH

$$\delta = \delta_{CD}\left(\frac{L}{c}\right) = (0.81815 \text{ mm})\left(\frac{500 \text{ mm}}{300 \text{ mm}}\right)$$

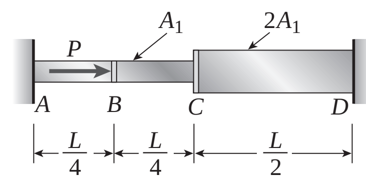
$= 1.36 \text{ mm}$ ←

(c) SLEEVE REMOVED

$\delta = \dfrac{PL}{E_1A_1} = 2.74 \text{ mm}$ ←

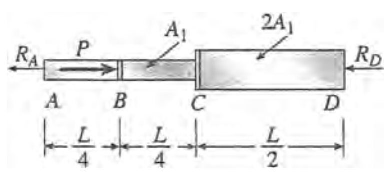
Problem 2.4-7 The axially loaded bar $ABCD$ shown in the figure is held between rigid supports. The bar has cross-sectional area A_1 from A to C and $2A_1$ from C to D.

(a) Derive formulas for the reactions R_A and R_D at the ends of the bar.
(b) Determine the displacements δ_B and δ_C at points B and C, respectively.
(c) Draw a diagram in which the abscissa is the distance from the left-hand support to any point in the bar and the ordinate is the horizontal displacement δ at that point.



Solution 2.4-7 Bar with fixed ends

FREE-BODY DIAGRAM OF BAR



EQUATION OF EQUILIBRIUM

$\Sigma F_{\text{horiz}} = 0 \qquad R_A + R_D = P$ (Eq. 1)

EQUATION OF COMPATIBILITY

$\delta_{AB} + \delta_{BC} + \delta_{CD} = 0$ (Eq. 2)

Positive means elongation.

FORCE-DISPLACEMENT EQUATIONS

$$\delta_{AB} = \frac{R_A(L/4)}{EA_1} \qquad \delta_{BC} = \frac{(R_A - P)(L/4)}{EA_1} \qquad \text{(Eqs. 3, 4)}$$

$$\delta_{CD} = -\frac{R_D(L/2)}{E(2A_1)} \qquad \text{(Eq. 5)}$$

SOLUTION OF EQUATIONS

Substitute Eqs. (3), (4), and (5) into Eq. (2):

$$\frac{R_AL}{4EA_1} + \frac{(R_A - P)(L)}{4EA_1} - \frac{R_DL}{4EA_1} = 0 \qquad \text{(Eq. 6)}$$

(a) REACTIONS

Solve simultaneously Eqs. (1) and (6):

$R_A = \dfrac{2P}{3} \qquad R_D = \dfrac{P}{3}$ ←

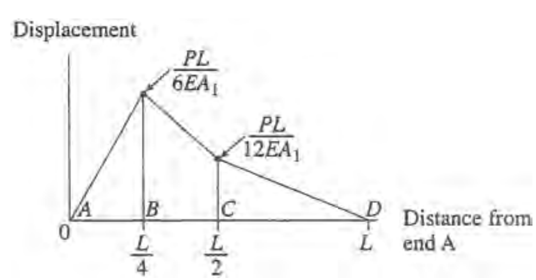
(b) DISPLACEMENTS AT POINTS B AND C

$\delta_B = \delta_{AB} = \dfrac{R_AL}{4EA_1} = \dfrac{PL}{6EA_1}$ (To the right) ←

$\delta_C = |\delta_{CD}| = \dfrac{R_DL}{4EA_1}$

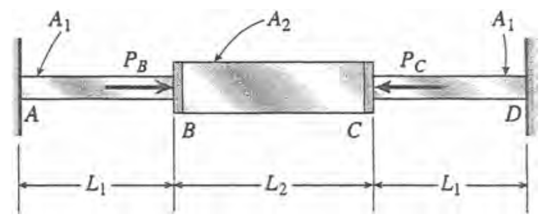
$= \dfrac{PL}{12EA_1}$ (To the right) ←

(c) AXIAL DISPLACEMENT DIAGRAM (ADD)

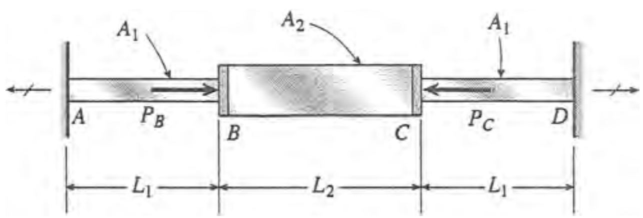


Problem 2.4-8 The fixed-end bar *ABCD* consists of three prismatic segments, as shown in the figure. The end segments have cross-sectional area $A_1 = 840\ \text{mm}^2$ and length $L_1 = 200$ mm. The middle segment has cross-sectional area $A_2 = 1260\ \text{mm}^2$ and length $L_2 = 250$ mm. Loads P_B and P_C are equal to 25.5 kN and 17.0 kN, respectively.

(a) Determine the reactions R_A and R_D at the fixed supports.
(b) Determine the compressive axial force F_{BC} in the middle segment of the bar.



Solution 2.4-8 Bar with three segments



FREE-BODY DIAGRAM

EQUATION OF EQUILIBRIUM

$\Sigma F_{\text{horiz}} = 0 \quad \overset{+}{\rightarrow} \quad \leftarrow$

$P_B + R_D - P_C - R_A = 0$ or

$R_A - R_D = P_B - P_C = 8.5$ kN (Eq. 1)

EQUATION OF COMPATIBILITY

δ_{AD} = elongation of entire bar

$\delta_{AD} = \delta_{AB} + \delta_{BC} + \delta_{CD} = 0$ (Eq. 2)

FORCE-DISPLACEMENT RELATIONS

$$\delta_{AB} = \frac{R_A L_1}{EA_1} = \frac{R_A}{E}\left(238.05\frac{1}{\text{m}}\right) \quad \text{(Eq. 3)}$$

$$\delta_{BC} = \frac{(R_A - P_B)L_2}{EA_2}$$

$$= \frac{R_A}{E}\left(198.413\frac{1}{\text{m}}\right) - \frac{P_B}{E}\left(198.413\frac{1}{\text{m}}\right) \quad \text{(Eq. 4)}$$

$$\delta_{CD} = \frac{R_D L_1}{EA_1} = \frac{R_D}{E}\left(238.095\frac{1}{\text{m}}\right) \quad \text{(Eq. 5)}$$

$P_B = 25.5$ kN $\quad P_C = 17.0$ kN

$L_1 = 200$ mm $\quad L_2 = 250$ mm

$A_1 = 840\ \text{mm}^2 \quad A_2 = 1260\ \text{mm}^2$

m = meter

SOLUTION OF EQUATIONS

Substitute Eqs. (3), (4), and (5) into Eq. (2):

$$\frac{R_A}{E}\left(238.095\frac{1}{\text{m}}\right) + \frac{R_A}{E}\left(198.413\frac{1}{\text{m}}\right)$$

$$-\frac{P_B}{E}\left(198.413\frac{1}{\text{m}}\right) + \frac{R_D}{E}\left(238.095\frac{1}{\text{m}}\right) = 0$$

Simplify and substitute $P_B = 25.5$ kN:

$$R_A\left(436.508\frac{1}{\text{m}}\right) + R_D\left(238.095\frac{1}{\text{m}}\right)$$

$$= 5{,}059.53\frac{\text{kN}}{\text{m}} \quad \text{(Eq. 6)}$$

(a) REACTIONS R_A AND R_D

Solve simultaneously Eqs. (1) and (6).

From (1): $R_D = R_A - 8.5$ kN

Substitute into (6) and solve for R_A:

$$R_A\left(674.603\frac{1}{\text{m}}\right) = 7083.34\frac{\text{kN}}{\text{m}}$$

$R_A = 10.5$ kN $\quad \leftarrow$

$R_D = R_A - 8.5\text{ kN} = 2.0$ kN $\quad \leftarrow$

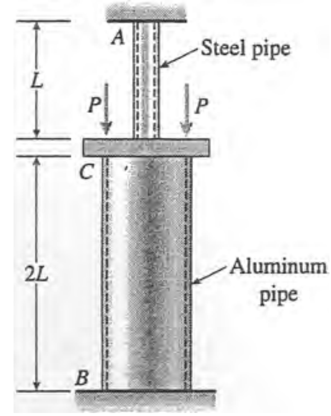
(b) COMPRESSIVE AXIAL FORCE F_{BC}

$F_{BC} = P_B - R_A = P_C - R_D = 15.0$ kN $\quad \leftarrow$

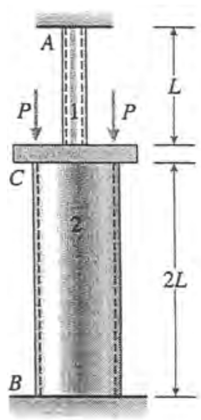
Problem 2.4-9 The aluminum and steel pipes shown in the figure are fastened to rigid supports at ends A and B and to a rigid plate C at their junction. The aluminum pipe is twice as long as the steel pipe. Two equal and symmetrically placed loads P act on the plate at C.

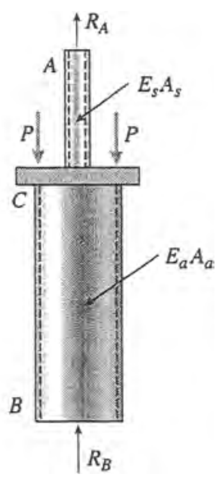
(a) Obtain formulas for the axial stresses σ_a and σ_s in the aluminum and steel pipes, respectively.

(b) Calculate the stresses for the following data: $P = 12$ k, cross-sectional area of aluminum pipe $A_a = 8.92$ in.^{2}, cross-sectional area of steel pipe $A_s = 1.03$ in.^{2}, modulus of elasticity of aluminum $E_a = 10 \times 10^6$ psi, and modulus of elasticity of steel $E_s = 29 \times 10^6$ psi.



Solution 2.4-9 Pipes with intermediate loads





Pipe 1 is steel.
Pipe 2 is aluminum.

EQUATION OF EQUILIBRIUM

$\Sigma F_{vert} = 0 \qquad R_A + R_B = 2P$ (Eq. 1)

EQUATION OF COMPATIBILITY

$\delta_{AB} = \delta_{AC} + \delta_{CB} = 0$ (Eq. 2)

(A positive value of δ means elongation.)

FORCE-DISPLACEMENT RELATIONS

$$\delta_{AC} = \frac{R_A L}{E_s A_s} \qquad \delta_{BC} = -\frac{R_B(2L)}{E_a A_a} \qquad \text{(Eqs. 3, 4))}$$

SOLUTION OF EQUATIONS

Substitute Eqs. (3) and (4) into Eq. (2):

$$\frac{R_A L}{E_s A_s} - \frac{R_B(2L)}{E_a A_a} = 0 \qquad \text{(Eq. 5)}$$

Solve simultaneously Eqs. (1) and (5):

$$R_A = \frac{4E_s A_s P}{E_a A_a + 2E_s A_s} \qquad R_B = \frac{2E_a A_a P}{E_a A_a + 2E_s A_s}$$

(Eqs. 6, 7)

(a) AXIAL STRESSES

$$\text{Aluminum: } \sigma_a = \frac{R_B}{A_a} = \frac{2E_a P}{E_a A_a + 2E_s A_s} \qquad \leftarrow$$

(Eq. 8)

(compression)

$$\text{Steel: } \sigma_s = \frac{R_A}{A_s} = \frac{4E_s P}{E_a A_a + 2E_s A_s} \qquad \leftarrow \qquad \text{(Eq. 9)}$$

(tension)

(b) NUMERICAL RESULTS

$P = 12$ k $\qquad A_a = 8.92$ in.^{2} $\qquad A_s = 1.03$ in.^{2}

$E_a = 10 \times 10^6$ psi $\qquad E_s = 29 \times 10^6$ psi

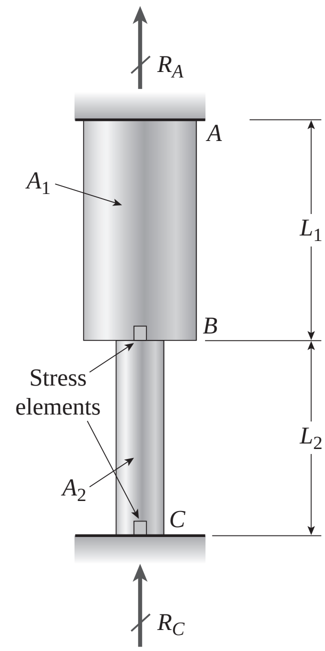
Substitute into Eqs. (8) and (9):

$\sigma_a = 1{,}610$ psi (compression) $\leftarrow$

$\sigma_s = 9{,}350$ psi (tension) $\leftarrow$

Problem 2.4-10 A nonprismatic bar *ABC* is composed of two segments: *AB* of length L_1 and cross-sectional area A_1; and *BC* of length L_2 and cross-sectional area A_2. The modulus of elasticity E, mass density ρ, and acceleration of gravity g are constants. Initially, bar *ABC* is horizontal and then is restrained at *A* and *C* and rotated to a vertical position. The bar then hangs vertically under its own weight (see figure). Let $A_1 = 2A_2 = A$ and $L_1 = \frac{3}{5}L$, $L_2 = \frac{2}{5}L$.

(a) Obtain formulas for the reactions R_A and R_C at supports *A* and *C*, respectively, due to gravity.
(b) Derive a formula for the downward displacement δ_B of point *B*.
(c) Find expressions for the axial stresses a small distance above points *B* and *C*, respectively.



Solution 2.4-10

(a) find reactions in 1-degree statically indeterminate structure

use superposition; select R_A as the redundant

compatibility: $\delta_{A1} + \delta_{A2} = 0$

segment weights: $W_{AB} = \rho g A_1 L_1$

$W_{BC} = \rho g A_2 L_2$

find axial forces in each segment;

use variable ζ measured from C toward A

$N_{AB} = -\rho g A_1(L_1 + L_2 - \zeta) \qquad L_2 \le \zeta \le L_1 + L_2$

$N_{BC} = -[W_{AB} + \rho g A_2(L_2 - \zeta)] \qquad 0 \le \zeta \le L_2$

displacement at A in released structure due to self weight

$$\delta_{A1} = \int_0^{L_2} \frac{N_{BC}}{EA_2}\,d\zeta + \int_{L_2}^{L_1+L_2} \frac{N_{AB}}{EA_1}\,d\zeta$$

$$\delta_{A1} = \int_0^{L_2} \frac{-[\rho g A_1 L_1 + \rho g A_2(L_2 - \zeta)]}{EA_2}\,d\zeta + \int_{L_2}^{L_1+L_2} \frac{-\rho g A_1(L_1 + L_2 - \zeta)}{EA_1}\,d\zeta$$

$$\delta_{A1} = \left[\frac{-1}{2}\rho g L_2 \frac{2A_1L_1 + A_2L_2}{EA_2} + \left(\frac{-1}{2}\rho g \frac{L_1^2 + 2L_1L_2 + L_2^2}{E} + \frac{1}{2}\rho g L_2 \frac{2L_1 + L_2}{E}\right)\right]$$

$$\delta_{A1} = \frac{-\rho g}{2(EA_2)}(2L_2A_1L_1 + A_2L_2^2 + A_2L_1^2)$$

Next, displacement at A in released structure due to redundant R_A

$$\delta_{A2} = R_A(f_{AB} + f_{BC}) \qquad \delta_{A2} = R_A\left(\frac{L_1}{EA_1} + \frac{L_2}{EA_2}\right)$$

enforce compatibility: $\delta_{A1} + \delta_{A2} = 0$ solve for R_A

$$R_A = \frac{\frac{\rho g}{2(EA_2)}(2L_2A_1L_1 + A_2L_2^2 + A_2L_1^2)}{f_{AB} + f_B}$$

$$R_A = \frac{1}{2}\rho g A_1 \frac{A_2L_1^2 + 2A_1L_1L_2 + A_2L_2^2}{L_1A_2 + L_2A_1} \quad \leftarrow$$

statics: $R_C = W_{AB} + W_{BC} - R_A$

$$R_C = \left[\rho g A_1L_1 + \rho g A_2L_2 - \frac{1}{2}\rho g(2L_2A_1L_1 + A_2L_2^2 + A_2L_1^2)\frac{A_1}{L_1A_2 + L_2A_1}\right]$$

$$R_C = \frac{1}{2}\rho g A_2 \frac{A_1L_1^2 + 2A_2L_1L_2 + A_1L_2^2}{L_1A_2 + L_2A_1} \quad \leftarrow$$

For $A_1 = A \qquad A_2 = \frac{A}{2} \qquad L_1 = \frac{3L}{5} \qquad L_2 = \frac{2L}{5}$

$$R_A = \frac{1}{2}\rho g A \frac{\frac{A}{2}\left(\frac{3L}{5}\right)^2 + 2A\frac{3L}{5}\frac{2L}{5} + \frac{A}{2}\left(\frac{2L}{5}\right)^2}{\frac{3L}{5}\frac{A}{2} + \frac{2L}{5}A} \qquad R_A = \frac{37}{70}\rho g AL \quad \leftarrow \qquad \frac{37}{70} = 0.529$$

$$R_C = \frac{1}{2}\rho g \frac{A}{2}\frac{A\left(\frac{3L}{5}\right)^2 + 2\frac{A}{2}\frac{3L}{5}\frac{2L}{5} + A\left(\frac{2L}{5}\right)^2}{\frac{3L}{5}\frac{A}{2} + \frac{2L}{5}A} \qquad R_C = \frac{19}{70}\rho g LA \quad \leftarrow \qquad \frac{19}{70} = 0.271$$

(b) use superposition to find displacement at point B $\delta_B = \delta_{B1} + \delta_{B2}$ where δ_{B1} is due to gravity and δ_{B2} is due to R_A

$$\delta_{B1} = \int_0^{L_2} \frac{N_{BC}}{EA_2}\,d\zeta \qquad < \text{due to shortening of BC}$$

$$\delta_{B1} = \frac{-\rho g L_2}{2(EA_2)}(2A_1L_1 + A_2L_2)$$

$$\delta_{B2} = R_A(f_{BC}) \qquad \delta_{B2} = R_A\left(\frac{L_2}{EA_2}\right)$$

$$\delta_B = \left[\frac{-\rho g L_2}{2(EA_2)}(2A_1L_1 + A_2L_2) + \frac{1}{2}\rho g A_1 \frac{A_2L_1^2 + 2A_1L_1L_2 + A_2L_2^2}{L_1A_2 + L_2A_1}\left(\frac{L_2}{EA_2}\right)\right]$$

$$\delta_B = \frac{-1}{2}\rho g L_2 L_1 \frac{A_1L_1 + A_2L_2}{(L_1A_2 + L_2A_1)E}$$

For $\quad A_1 = A \qquad A_2 = \frac{A}{2} \qquad L_1 = \frac{3L}{5} \qquad L_2 = \frac{2L}{5}$

$$\delta_B = \frac{-1}{2}\rho g \frac{2L}{5}\frac{3L}{5}\frac{A\frac{3L}{5} + \frac{A}{2}\frac{2L}{5}}{\left(\frac{3L}{5}\frac{A}{2} + \frac{2L}{5}A\right)E} \qquad \delta_B = \frac{-24}{175}\rho g\frac{L^2}{E} \qquad \leftarrow$$

(c) expressions for the *average* axial stresses a small distance above points B and C

N_B = axial force near B = $R_A - W_{AB}$

$$N_B = \left(\frac{1}{2}\rho g A_1 \frac{A_2L_1^2 + 2A_1L_1L_2 + A_2L_2^2}{L_1A_2 + L_2A_1}\right) - \rho g A_1 L_1$$

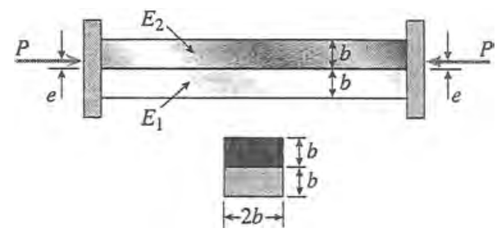
$$N_B = \frac{37}{70}\rho g A L - \rho g A \frac{3L}{5} \qquad N_B = \frac{-1}{14}\rho g A L$$

$$\sigma_B = \frac{N_B}{A} \qquad \sigma_B = \frac{-1}{14}\rho g L \qquad \leftarrow$$

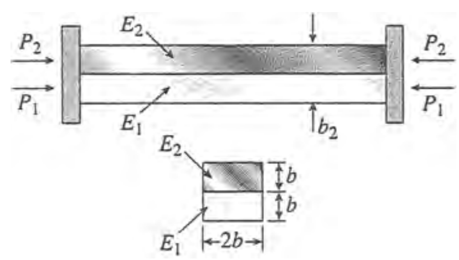
$$N_C = -R_C \qquad \sigma_C = \frac{-\left(\frac{19}{70}\rho g L A\right)}{\frac{A}{2}} \qquad \sigma_C = \frac{-19}{35}\rho g L \qquad \leftarrow$$

Problem 2.4-11 A *bimetallic* bar (or composite bar) of square cross section with dimensions $2b \times 2b$ is constructed of two different metals having moduli of elasticity E_1 and E_2 (see figure). The two parts of the bar have the same cross-sectional dimensions. The bar is compressed by forces P acting through rigid end plates. The line of action of the loads has an eccentricity e of such magnitude that each part of the bar is stressed uniformly in compression.

(a) Determine the axial forces P_1 and P_2 in the two parts of the bar.
(b) Determine the eccentricity e of the loads.
(c) Determine the ratio σ_1/σ_2 of the stresses in the two parts of the bar.

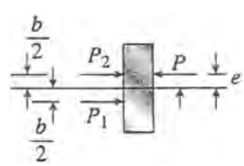


Solution 2.4-11 Bimetallic bar in compression



FREE-BODY DIAGRAM

(Plate at right-hand end)



EQUATIONS OF EQUILIBRIUM

$\Sigma F = 0 \quad P_1 + P_2 = P$ (Eq. 1)

$$\Sigma M = 0 \quad Pe + P_1\left(\frac{b}{2}\right) - P_2\left(\frac{b}{2}\right) = 0 \quad \text{(Eq. 2)}$$

EQUATION OF COMPATIBILITY

$\delta_2 = \delta_1$

$$\frac{P_2 L}{E_2 A} = \frac{P_1 L}{E_1 A} \quad \text{or} \quad \frac{P_2}{E_2} = \frac{P_1}{E_1} \quad \text{(Eq. 3)}$$

(a) AXIAL FORCES

Solve simultaneously Eqs. (1) and (3):

$$P_1 = \frac{PE_1}{E_1 + E_2} \quad P_2 = \frac{PE_2}{E_1 + E_2} \quad \leftarrow$$

(b ECCENTRICITY OF LOAD P

Substitute P_1 and P_2 into Eq. (2) and solve for e:

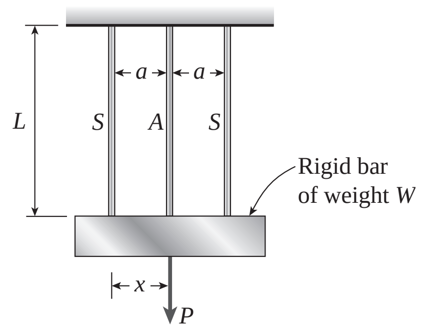
$$e = \frac{b(E_2 - E_1)}{2(E_2 + E_1)} \quad \leftarrow$$

(c) RATIO OF STRESSES

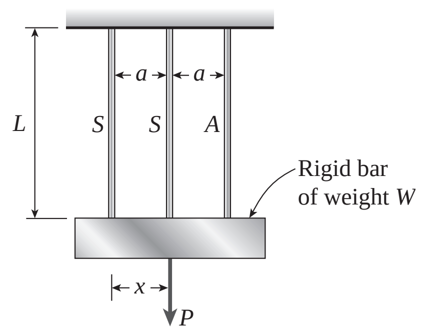
$$\sigma_1 = \frac{P_1}{A} \quad \sigma_2 = \frac{P_2}{A} \quad \frac{\sigma_1}{\sigma_2} = \frac{P_1}{P_2} = \frac{E_1}{E_2} \quad \leftarrow$$

Problem 2.4-12 A rigid bar of weight $W = 800$ N hangs from three equally spaced vertical wires (length $L = 150$ mm, spacing $a = 50$ mm): two of steel and one of aluminum. The wires also support a load P acting on the bar. The diameter of the steel wires is $d_s = 2$ mm, and the diameter of the aluminum wire is $d_a = 4$ mm. Assume $E_s = 210$ GPa and $E_a = 70$ GPa.

(a) What load P_{allow} can be supported *at the midpoint of the bar* ($x = a$) if the allowable stress in the steel wires is 220 MPa and in the aluminum wire is 80 MPa? (See figure part a.)
(b) What is P_{allow} if the load is positioned at $x = a/2$? (See figure part a.)
(c) Repeat (b) above if the second and third wires are *switched* as shown in figure part b.



(a)



(b)

Solution 2.4-12

numerical data

$W = 800$ N $\qquad L = 150$ mm

$a = 50$ mm $\qquad d_S = 2$ mm

$d_A = 4$ mm $\qquad E_S = 210$ GPa

$E_A = 70$ GPa

$\sigma_{Sa} = 220$ MPa $\qquad \sigma_{Aa} = 80$ MPa

$$A_A = \frac{\pi}{4} d_A^2 \qquad A_S = \frac{\pi}{4} d_S^2$$

$A_A = 13\ \text{mm}^2 \qquad A_S = 3\ \text{mm}^2$

(a) P_{allow} at center of bar

1-degree stat-indet - use reaction (R_A) at top of aluminum bar as the redundant

compatibility: $\delta_1 - \delta_2 = 0$ $\qquad$ statics: $\qquad 2R_S + R_A = P + W$

$$\delta_1 = \frac{P + W}{2}\left(\frac{L}{E_S A_S}\right)$$

< downward displacement due to elongation of each steel wire under P + W if alum. wire is cut at top

$$\delta_2 = R_A\left(\frac{L}{2E_SA_S} + \frac{L}{E_AA_A}\right)$$ $<$ upward displ. due to shortening of steel wires & elongation of alum. wire under redundant R_A

enforce compatibility & then solve for R_A

$$\delta_1 = \delta_2 \qquad \text{so} \qquad R_A = \frac{\frac{P+W}{2}\left(\frac{L}{E_SA_S}\right)}{\frac{L}{2E_SA_S} + \frac{L}{E_AA_A}} \qquad R_A = (P+W)\frac{E_AA_A}{E_AA_A + 2E_SA_S} \qquad \text{and} \qquad \sigma_{Aa} = \frac{R_A}{A_A}$$

now use statics to find R_S

$$R_S = \frac{P + W - R_A}{2} \qquad R_S = \frac{P + W - (P+W)\frac{E_AA_A}{E_AA_A + 2E_SA_S}}{2} \qquad R_S = (P+W)\frac{E_SA_S}{E_AA_A + 2E_SA_S}$$

$$\text{and} \qquad \sigma_{Sa} = \frac{R_S}{A_S}$$

compute stresses & apply allowable stress values

$$\sigma_{Aa} = (P+W)\frac{E_A}{E_AA_A + 2E_SA_S} \qquad \sigma_{Sa} = (P+W)\frac{E_S}{E_AA_A + 2E_SA_S}$$

solve for allowable load P

$$P_{Aa} = \sigma_{Aa}\left(\frac{E_AA_A + 2E_SA_S}{E_A}\right) - W \qquad P_{Sa} = \sigma_{Sa}\left(\frac{E_AA_A + 2E_SA_S}{E_S}\right) - W$$ lower value of P controls

$P_{Aa} = 1713\text{ N}$ $\qquad$ $P_{Sa} = 1504\text{ N}$ $\leftarrow$ P_{allow} is controlled by steel wires

(b) P_{allow} if load P at $x = a/2$

again, cut aluminum wire at top, then compute elongations of left & right steel wires

$$\delta_{1L} = \left(\frac{3P}{4} + \frac{W}{2}\right)\left(\frac{L}{E_SA_S}\right) \qquad \delta_{1R} = \left(\frac{P}{4} + \frac{W}{2}\right)\left(\frac{L}{E_SA_S}\right)$$

$$\delta_1 = \frac{\delta_{1L} + \delta_{1R}}{2} \qquad \delta_1 = \frac{P+W}{2}\left(\frac{L}{E_SA_S}\right)$$ where δ_1 = displ. at $x = a$

Use δ_2 from (a) above

$$\delta_2 = R_A\left(\frac{L}{2E_SA_S} + \frac{L}{E_AA_A}\right)$$ so equating δ_1 & δ_2, solve for R_A $\quad R_A = (P+W)\dfrac{E_AA_A}{E_AA_A + 2E_SA_S}$

same as in (a)

$$R_{SL} = \frac{3P}{4} + \frac{W}{2} - \frac{R_A}{2}$$ $<$ stress in left steel wire exceeds that in right steel wire

$$R_{SL} = \frac{3P}{4} + \frac{W}{2} - \frac{(P+W)\frac{E_AA_A}{E_AA_A + 2E_SA_S}}{2}$$

$$R_{SL} = \frac{PE_AA_A + 6PE_SA_S + 4WE_SA_S}{4E_AA_A + 8E_SA_S} \qquad \sigma_{Sa} = \frac{PE_AA_A + 6PE_SA_S + 4WE_SA_S}{4E_AA_A + 8E_SA_S}\left(\frac{1}{A_S}\right)$$

solve for P_{allow} based on allowable stresses in steel & alum.

$$P_{Sa} = \frac{\sigma_{Sa}(4A_SE_AA_A + 8E_SA_S{}^2) - (4WE_SA_S)}{E_AA_A + 6E_SA_S} \qquad P_{Aa} = 1713\text{ N} \qquad < \text{same as in (a)}$$

$P_{Sa} = 820$ N $\leftarrow$ steel controls

(c) P_{allow} if wires are switched as shown & $x = a/2$

select R_A as the redundant

statics on the two released structures

(1) cut alum. wire - apply P & W, compute forces in left & right steel wires, then compute displacements at each steel wire

$$R_{SL} = \frac{P}{2} \qquad R_{SR} = \frac{P}{2} + W$$

$$\delta_{1L} = \frac{P}{2}\left(\frac{L}{E_SA_S}\right) \qquad \delta_{1R} = \left(\frac{P}{2} + W\right)\left(\frac{L}{E_SA_S}\right)$$

by geometry, δ at alum. wire location at far right is $\delta_1 = \left(\frac{P}{2} + 2W\right)\left(\frac{L}{E_SA_S}\right)$

(2) next apply redundant R_A at right wire, compute wire force & displ. at alum. wire

$$R_{SL} = -R_A \qquad R_{SR} = 2R_A \qquad \delta_2 = R_A\left(\frac{5L}{E_SA_S} + \frac{L}{E_AA_A}\right)$$

(3) compatibility equate δ_1, δ_2 and solve for R_A then P_{allow} for alum. wire

$$R_A = \frac{\left(\frac{P}{2} + 2W\right)\left(\frac{L}{E_SA_S}\right)}{\frac{5L}{E_SA_S} + \frac{L}{E_AA_A}} \qquad R_A = \frac{E_AA_AP + 4E_AA_AW}{10E_AA_A + 2E_SA_S} \qquad \sigma_{Aa} = \frac{R_A}{A_A}$$

$$\sigma_{Aa} = \frac{E_AP + 4E_AW}{10E_AA_A + 2E_SA_S}$$

$$P_{Aa} = \frac{\sigma_{Aa}(10E_AA_A + 2E_SA_S) - 4E_AW}{E_A} \qquad P_{Aa} = 1713\text{ N}$$

(4) statics or superposition - find forces in steel wires then P_{allow} for steel wires

$$R_{SL} = \frac{P}{2} + R_A \qquad R_{SL} = \frac{P}{2} + \frac{E_AA_AP + 4E_AA_AW}{10E_AA_A + 2E_SA_S}$$

$$R_{SL} = \frac{6E_AA_AP + PE_SA_S + 4E_AA_AW}{10E_AA_A + 2E_SA_S}$$

$<$ larger than R_{SR} below so use in allow. stress calcs

$$R_{SR} = \frac{P}{2} + W - 2R_A \qquad R_{SR} = \frac{P}{2} + W - \frac{E_AA_AP + 4E_AA_AW}{5E_AA_A + E_SA_S}$$

$$R_{SR} = \frac{3E_AA_AP + PE_SA_S + 2E_AA_AW + 2WE_SA_S}{10E_AA_A + 2E_SA_S}$$

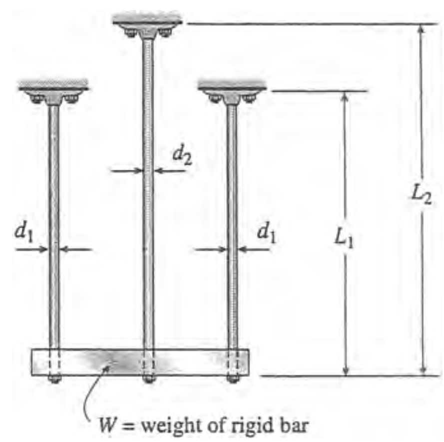
$$\sigma_{Sa} = \frac{R_{SL}}{A_S} \qquad P_{Sa} = \sigma_{Sa}A_S\left(\frac{10E_AA_A + 2E_SA_S}{6E_AA_A + E_SA_S}\right) - \frac{4E_AA_AW}{6E_AA_A + E_SA_S}$$

$$P_{Sa} = \frac{10\sigma_{Sa}A_SE_AA_A + 2\sigma_{Sa}A_S^2E_S - 4E_AA_AW}{6E_AA_A + E_SA_S}$$

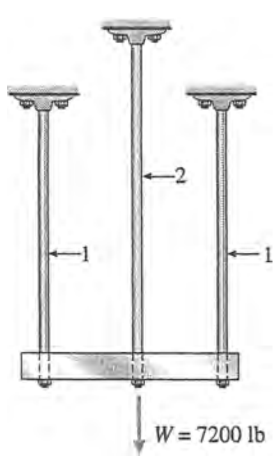
$P_{Sa} = 703$ N $\leftarrow$
^ steel controls

Problem 2.4-13 A horizontal rigid bar of weight $W = 7200$ lb is supported by three slender circular rods that are equally spaced (see figure). The two outer rods are made of aluminum ($E_1 = 10 \times 10^6$ psi) with diameter $d_1 = 0.4$ in. and length $L_1 = 40$ in. The inner rod is magnesium ($E_2 = 6.5 \times 10^6$ psi) with diameter d_2 and length L_2. The allowable stresses in the aluminum and magnesium are 24,000 psi and 13,000 psi, respectively.

If it is desired to have all three rods loaded to their maximum allowable values, what should be the diameter d_2 and length L_2 of the middle rod?



Solution 2.4-13 Bar supported by three rods



BAR 1 ALUMINUM

$E_1 = 10 \times 10^6$ psi

$d_1 = 0.4$ in.

$L_1 = 40$ in.

$\sigma_1 = 24{,}000$ psi

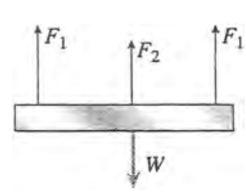
BAR 2 MAGNESIUM

$E_2 = 6.5 \times 10^6$ psi

$d_2 = ?$ $\qquad L_2 = ?$

$\sigma_2 = 13{,}000$ psi

FREE-BODY DIAGRAM OF RIGID BAR
EQUATION OF EQUILIBRIUM



$\Sigma F_{vert} = 0$

$2F_1 + F_2 - W = 0 \qquad$ (Eq. 1)

FULLY STRESSED RODS

$F_1 = \sigma_1 A_1 \qquad F_2 = \sigma_2 A_2$

$$A_1 = \frac{\pi d_1^2}{4} \qquad A_2 = \frac{\pi d_2^2}{4}$$

Substitute into Eq. (1):

$$2\sigma_1\left(\frac{\pi d_1^2}{4}\right) + \sigma_2\left(\frac{\pi d_2^2}{4}\right) = W$$

Diameter d_1 is known; solve for d_2:

$$d_2^2 = \frac{4W}{\pi\sigma_2} - \frac{2\sigma_1 d_1^2}{\sigma_2} \quad \longleftarrow \qquad \text{(Eq. 2)}$$

SUBSTITUTE NUMERICAL VALUES:

$$d_2^2 = \frac{4(7200\text{ lb})}{\pi(13{,}000\text{ psi})} - \frac{2(24{,}000\text{ psi})(0.4\text{ in.})^2}{13{,}000\text{ psi}}$$

$$= 0.70518\text{ in}^2. - 0.59077\text{ in.}^2 = 0.11441\text{ in.}^2$$

$$d_2 = 0.338\text{ in.} \quad \longleftarrow$$

EQUATION OF COMPATIBILITY

$$\delta_1 = \delta_2 \qquad \text{(Eq. 3)}$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_1 = \frac{F_1 L_1}{E_1 A_1} = \sigma_1\left(\frac{L_1}{E_1}\right) \qquad \text{(Eq. 4)}$$

$$\delta_2 = \frac{F_2 L_2}{E_2 A_2} = \sigma_2\left(\frac{L_2}{E_2}\right) \qquad \text{(Eq. 5)}$$

Substitute (4) and (5) into Eq. (3):

$$\sigma_1\left(\frac{L_1}{E_1}\right) = \sigma_2\left(\frac{L_2}{E_2}\right)$$

Length L_1 is known; solve for L_2:

$$L_2 = L_1\left(\frac{\sigma_1 E_2}{\sigma_2 E_1}\right) \quad \longleftarrow \qquad \text{(Eq. 6)}$$

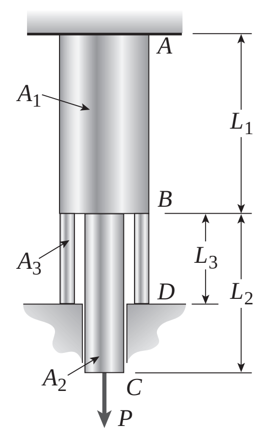
SUBSTITUTE NUMERICAL VALUES:

$$L_2 = (40\text{ in.})\left(\frac{24{,}000\text{ psi}}{13{,}000\text{ psi}}\right)\left(\frac{6.5 \times 10^6\text{ psi}}{10 \times 10^6\text{ psi}}\right)$$

$$= 48.0\text{ in.}$$

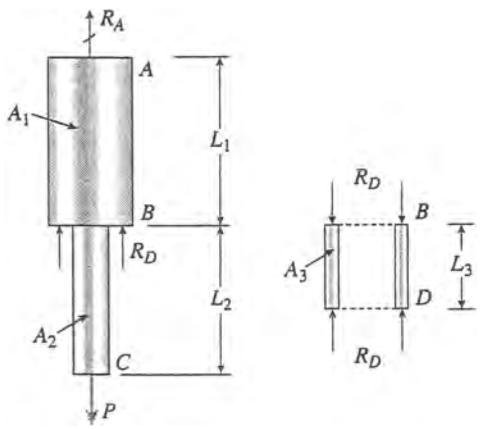
Problem 2.4-14 A circular steel bar ABC ($E = 200$ GPa) has cross-sectional area A_1 from A to B and cross-sectional area A_2 from B to C (see figure). The bar is supported rigidly at end A and is subjected to a load P equal to 40 kN at end C. A circular steel collar BD having cross-sectional area A_3 supports the bar at B. The collar fits snugly at B and D when there is no load.

Determine the elongation δ_{AC} of the bar due to the load P. (Assume $L_1 = 2L_3 = 250$ mm, $L_2 = 225$ mm, $A_1 = 2A_3 = 960\text{ mm}^2$, and $A_2 = 300\text{ mm}^2$.)



Solution 2.4-14 Bar supported by a collar

FREE-BODY DIAGRAM OF BAR *ABC* AND COLLAR *BD*



EQUILIBRIUM OF BAR *ABC*

$$\Sigma F_{\text{vert}} = 0 \qquad R_A + R_D - P = 0 \qquad \text{(Eq. 1)}$$

COMPATIBILITY (distance *AD* does not change)

$$\delta_{AB}(\text{bar}) + \delta_{BD}(\text{collar}) = 0 \qquad \text{(Eq. 2)}$$

(Elongation is positive.)

FORCE-DISPLACEMENT RELATIONS

$$\delta_{AB} = \frac{R_A L_1}{EA_1} \qquad \delta_{BD} = -\frac{R_D L_3}{EA_3}$$

Substitute into Eq. (2):

$$\frac{R_A L_1}{EA_1} - \frac{R_D L_3}{EA_3} = 0 \qquad \text{(Eq. 3)}$$

SOLVE SIMULTANEOUSLY EQS. (1) AND (3):

$$R_A = \frac{PL_3A_1}{L_1A_3 + L_3A_1} \qquad R_D = \frac{PL_1A_3}{L_1A_3 + L_3A_1}$$

CHANGES IN LENGTHS (Elongation is positive)

$$\delta_{AB} = \frac{R_A L_1}{EA_1} = \frac{PL_1L_3}{E(L_1A_3 + L_3A_1)} \qquad \delta_{BC} = \frac{PL_2}{EA_2}$$

ELONGATION OF BAR *ABC*

$$\delta_{AC} = \delta_{AB} + \delta_{AC}$$

SUBSTITUTE NUMERICAL VALUES:

$P = 40$ kN $\qquad E = 200$ GPa

$L_1 = 250$ mm

$L_2 = 225$ mm

$L_3 = 125$ mm

$A_1 = 960\ \text{mm}^2$

$A_2 = 300\ \text{mm}^2$

$A_3 = 480\ \text{mm}^2$

RESULTS:

$R_A = R_D = 20$ kN

$\delta_{AB} = 0.02604$ mm

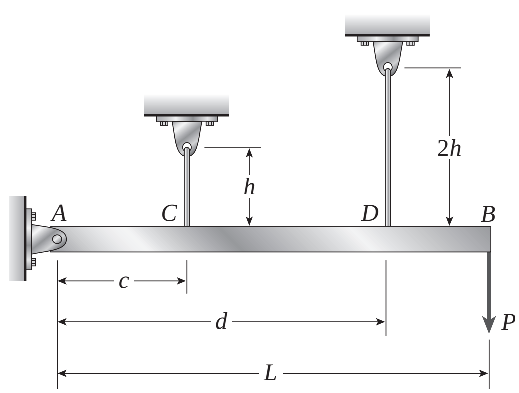
$\delta_{BC} = 0.15000$ mm

$\delta_{AC} = \delta_{AB} + \delta_{AC} = 0.176$ mm $\leftarrow$

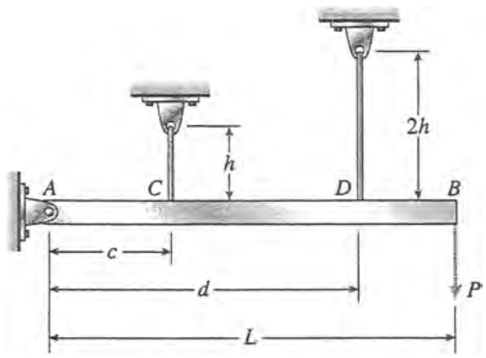
Problem 2.4-15 A rigid bar *AB* of length $L = 66$ in. is hinged to a support at *A* and supported by two vertical wires attached at points *C* and *D* (see figure). Both wires have the same cross-sectional area ($A = 0.0272\ \text{in.}^2$) and are made of the same material (modulus $E = 30 \times 10^6$ psi). The wire at *C* has length $h = 18$ in. and the wire at *D* has length twice that amount. The horizontal distances are $c = 20$ in. and $d = 50$ in.

(a) Determine the tensile stresses σ_C and σ_D in the wires due to the load $P = 340$ lb acting at end *B* of the bar.

(b) Find the downward displacement δ_B at end *B* of the bar.



Solution 2.4-15 Bar supported by two wires



$h = 18$ in.

$2h = 36$ in.

$c = 20$ in.

$d = 50$ in.

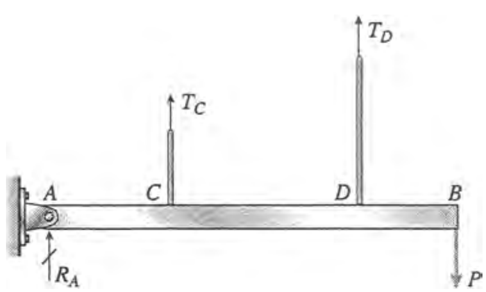
$L = 66$ in.

$E = 30 \times 10^6$ psi

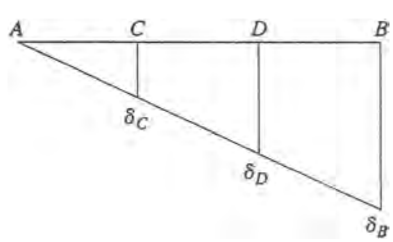
$A = 0.0272$ in.^{2}

$P = 340$ lb

FREE-BODY DIAGRAM



DISPLACEMENT DIAGRAM



EQUATION OF EQUILIBRIUM

$\Sigma M_A = 0 \curvearrowleft\!\!+ \quad T_C(c) + T_D(d) = PL$ (Eq. 1)

EQUATION OF COMPATIBILITY

$$\frac{\delta_C}{c} = \frac{\delta_D}{d} \qquad \text{(Eq. 2)}$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_C = \frac{T_C h}{EA} \qquad \delta_D = \frac{T_D(2h)}{EA} \qquad \text{(Eqs. 3, 4)}$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{T_C h}{cEA} = \frac{T_D(2h)}{dEA} \quad \text{or} \quad \frac{T_C}{c} = \frac{2T_D}{d} \qquad \text{(Eq. 5)}$$

TENSILE FORCES IN THE WIRES

Solve simultaneously Eqs. (1) and (5):

$$T_C = \frac{2cPL}{2c^2 + d^2} \qquad T_D = \frac{dPL}{2c^2 + d^2} \qquad \text{(Eqs. 6, 7)}$$

TENSILE STRESSES IN THE WIRES

$$\sigma_C = \frac{T_C}{A} = \frac{2cPL}{A(2c^2 + d^2)} \qquad \text{(Eq. 8)}$$

$$\sigma_D = \frac{T_D}{A} = \frac{dPL}{A(2c^2 + d^2)} \qquad \text{(Eq. 9)}$$

DISPLACEMENT AT END OF BAR

$$\delta_B = \delta_D\left(\frac{L}{d}\right) = \frac{2hT_D}{EA}\left(\frac{L}{d}\right) = \frac{2hPL^2}{EA(2c^2 + d^2)} \qquad \text{(Eq. 10)}$$

SUBSTITUTE NUMERICAL VALUES

$$2c^2 + d^2 = 2(20 \text{ in.})^2 + (50 \text{ in.})^2 = 3300 \text{ in.}^2$$

(a) $$\sigma_C = \frac{2cPL}{A(2c^2 + d^2)} = \frac{2(20 \text{ in.})(340 \text{ lb})(66 \text{ in.})}{(0.0272 \text{ in.}^2)(3300 \text{ in.}^2)}$$

$$= 10{,}000 \text{ psi} \quad \leftarrow$$

$$\sigma_D = \frac{dPL}{A(2c^2 + d^2)} = \frac{(50 \text{ in.})(340 \text{ lb})(66 \text{ in.})}{(0.0272 \text{ in.}^2)(3300 \text{ in.}^2)}$$

$$= 12{,}500 \text{ psi} \quad \leftarrow$$

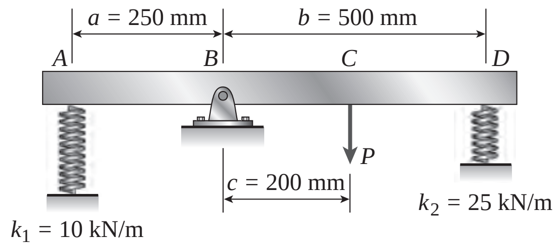
(b) $$\delta_B = \frac{2hPL^2}{EA(2c^2 + d^2)}$$

$$= \frac{2(18 \text{ in.})(340 \text{ lb})(66 \text{ in.})^2}{(30 \times 10^6 \text{ psi})(0.0272 \text{ in.}^2)(3300 \text{ in.}^2)}$$

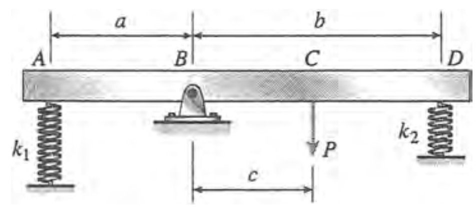
$$= 0.0198 \text{ in.} \quad \leftarrow$$

Problem 2.4-16 A rigid bar $ABCD$ is pinned at point B and supported by springs at A and D (see figure). The springs at A and D have stiffnesses $k_1 = 10$ kN/m and $k_2 = 25$ kN/m, respectively, and the dimensions a, b, and c are 250 mm, 500 mm, and 200 mm, respectively. A load P acts at point C.

If the angle of rotation of the bar due to the action of the load P is limited to 3°, what is the maximum permissible load P_{max}?



Solution 2.4-16 Rigid bar supported by springs



NUMERICAL DATA

$a = 250$ mm

$b = 500$ mm

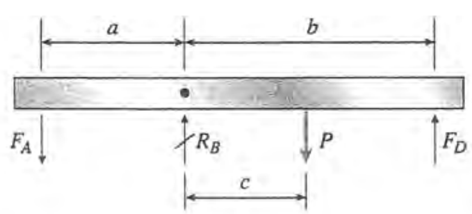
$c = 200$ mm

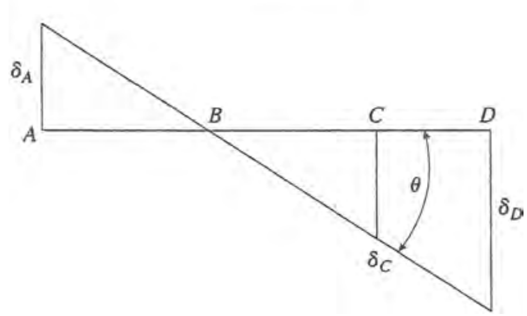
$k_1 = 10$ kN/m

$k_2 = 25$ kN/m

$$\theta_{max} = 3° = \frac{\pi}{60} \text{ rad}$$

FREE-BODY DIAGRAM AND DISPLACEMENT DIAGRAM





EQUATION OF EQUILIBRIUM

$$\Sigma M_B = 0 \quad \curvearrowleft + \quad -F_A(a) - P(c) + F_D(b) = 0 \qquad \text{(Eq. 1)}$$

EQUATION OF COMPATIBILITY

$$\frac{\delta_A}{a} = \frac{\delta_D}{b} \qquad \text{(Eq. 2)}$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_A = \frac{F_A}{k_1} \qquad \delta_D = \frac{F_D}{k_2} \qquad \text{(Eqs. 3, 4)}$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{F_A}{ak_1} = \frac{F_D}{bk_2} \qquad \text{(Eq. 5)}$$

SOLVE SIMULTANEOUSLY EQS. (1) AND (5):

$$F_A = \frac{ack_1P}{a^2k_1 + b^2k_2} \qquad F_D = \frac{bck_2P}{a^2k_1 + b^2k_2}$$

ANGLE OF ROTATION

$$\delta_D = \frac{F_D}{k_2} = \frac{bcP}{a^2k_1 + b^2k_2} \qquad \theta = \frac{\delta_D}{b} = \frac{cP}{a^2k_1 + b^2k_2}$$

MAXIMUM LOAD

$$P = \frac{\theta}{c}(a^2k_1 + b^2k_2)$$

$$P_{max} = \frac{\theta_{max}}{c}(a^2k_1 + b^2k_2) \quad \leftarrow$$

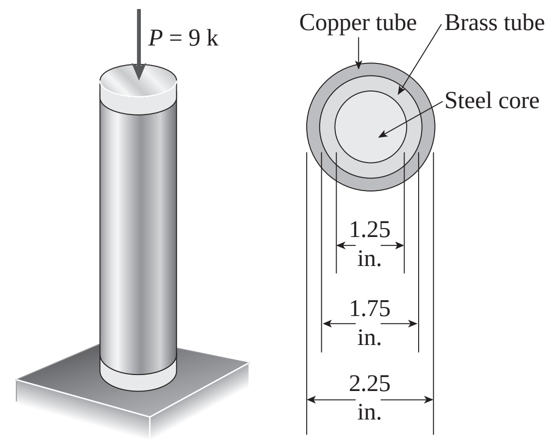
SUBSTITUTE NUMERICAL VALUES:

$$P_{max} = \frac{\pi/60 \text{ rad}}{200 \text{ mm}}[(250 \text{ mm})^2(10 \text{ kN/m}) + (500 \text{ mm})^2(25 \text{ kN/m})]$$

$$= 1800 \text{ N} \quad \leftarrow$$

Problem 2.4-17 A trimetallic bar is uniformly compressed by an axial force $P = 9$ kips applied through a rigid end plate (see figure). The bar consists of a circular steel core surrounded by brass and copper tubes. The steel core has diameter 1.25 in., the brass tube has outer diameter 1.75 in., and the copper tube has outer diameter 2.25 in. The corresponding moduli of elasticity are $E_s = 30{,}0000$ ksi, $E_b = 16{,}000$ ksi, and $E_c = 18{,}000$ ksi.

Calculate the compressive stresses σ_s, σ_b, and σ_c in the steel, brass, and copper, respectively, due to the force P.



Solution 2.4-17

numerical properties (kips, inches)

$d_c = 2.25$ in. $\quad d_b = 1.75$ in. $\quad d_s = 1.25$ in. $\qquad A_s = \frac{\pi}{4} d_s^2$

$E_c = 18000$ ksi $\quad E_b = 16000$ ksi $\qquad A_b = \frac{\pi}{4}(d_b^2 - d_s^2)$

$E_s = 30000$ ksi $\qquad A_c = \frac{\pi}{4}(d_c^2 - d_b^2)$

$P = 9$ kips

EQUATION OF EQUILIBRIUM

$\Sigma F_{vert} = 0 \qquad P_s + P_b + P_c = P$ (Eq. 1)

EQUATIONS OF COMPATIBILITY

$\delta_s = \delta_b \qquad \delta_c = \delta_s$ (Eqs. 2)

FORCE-DISPLACEMENT RELATIONS

$$\delta_s = \frac{P_s L}{E_s A_s} \quad \delta_b = \frac{P_b L}{E_b A_b} \quad \delta_c = \frac{P_c L}{E_c A_c} \qquad \text{(Eqs. 3, 4, 5)}$$

SOLUTION OF EQUATIONS

Substitute (3), (4), and (5) into Eqs. (2):

$$P_b = P_s \frac{E_b A_b}{E_s A_s} \quad P_c = P_s \frac{E_c A_c}{E_s A_s} \qquad \text{(Eqs. 6, 7)}$$

SOLVE SIMULTANEOUSLY EQS. (1), (6), AND (7):

$$P_s = P\frac{E_s A_s}{E_s A_s + E_b A_b + E_c A_c} = 4 \text{ kips}$$

$$P_b = P\frac{E_b A_b}{E_s A_s + E_b A_b + E_c A_c} = 2 \text{ kips}$$

$$P_c = P\frac{E_c A_c}{E_s A_s + E_b A_b + E_c A_c} = 3 \text{ kips}$$

$\mathrm{P_s + P_b + P_c = 9}$ statics check

COMPRESSIVE STRESSES

Let $\Sigma EA = E_s A_s + E_b A_b + E_c A_c$

$$\sigma_s = \frac{P_s}{A_s} = \frac{PE_s}{\Sigma EA} \qquad \sigma_b = \frac{P_b}{A_b} = \frac{PE_b}{\Sigma EA}$$

$$\sigma_c = \frac{P_c}{A_c} = \frac{PE_c}{\Sigma EA}$$

compressive stresses $\sigma_s = \frac{P_s}{A_s}$ $\sigma_s = 3$ ksi $\leftarrow$

$\sigma_b = \frac{P_b}{A_b}$ $\sigma_b = 2$ ksi $\leftarrow$

$\sigma_c = \frac{P_c}{A_c}$ $\sigma_c = 2$ ksi $\leftarrow$

Thermal Effects

Problem 2.5-1 The rails of a railroad track are welded together at their ends (to form continuous rails and thus eliminate the clacking sound of the wheels) when the temperature is 60°F.

What compressive stress σ is produced in the rails when they are heated by the sun to 120°F if the coefficient of thermal expansion $\alpha = 6.5 \times 10^{-6}$/°F and the modulus of elasticity $E = 30 \times 10^6$ psi?

Solution 2.5-1 Expansion of railroad rails

The rails are prevented from expanding because of their great length and lack of expansion joints.

Therefore, each rail is in the same condition as a bar with fixed ends (see Example 2-7).

The compressive stress in the rails may be calculated from Eq. (2-18).

$$\Delta T = 120°\text{F} - 60°\text{F} = 60°\text{F}$$

$$\sigma = E\alpha(\Delta T)$$

$$= (30 \times 10^6 \text{ psi})(6.5 \times 10^{-6}/°\text{F})(60°\text{F})$$

$$\sigma = 11{,}700 \text{ psi} \quad \leftarrow$$

Problem 2.5-2 An aluminum pipe has a length of 60 m at a temperature of 10°C. An adjacent steel pipe at the same temperature is 5 mm longer than the aluminum pipe.

At what temperature (degrees Celsius) will the aluminum pipe be 15 mm longer than the steel pipe? (Assume that the coefficients of thermal expansion of aluminum and steel are $\alpha_a = 23 \times 10^{-6}$/°C and $\alpha_s = 12 \times 10^{-6}$/°C, respectively.)

Solution 2.5-2 Aluminum and steel pipes

INITIAL CONDITIONS

$L_a = 60$ m $\qquad T_0 = 10°$C

$L_s = 60.005$ m $\qquad T_0 = 10°$C

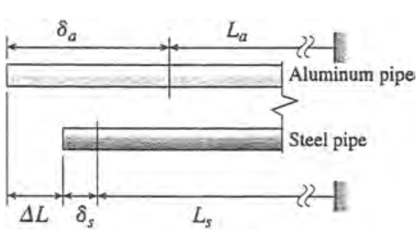
$\alpha_a = 23 \times 10^{-6}$/°C $\qquad \alpha_s = 12 \times 10^{-6}$/°C

FINAL CONDITIONS

Aluminum pipe is longer than the steel pipe by the amount $\Delta L = 15$ mm.

ΔT = increase in temperature

$$\delta_a = \alpha_a(\Delta T)L_a \qquad \delta_s = \alpha_s(\Delta T)L_s$$



From the figure above:

$$\delta_a + L_a = \Delta L + \delta_s + L_s$$

or, $\alpha_a(\Delta T)L_a + L_a = \Delta L + \alpha_s(\Delta T)L_s + L_s$

Solve for ΔT:

$$\Delta T = \frac{\Delta L + (L_s - L_a)}{\alpha_a L_a - \alpha_s L_s} \quad \leftarrow$$

Substitute numerical values:

$$\alpha_a L_a - \alpha_s L_s = 659.9 \times 10^{-6} \text{ m/°C}$$

$$\Delta T = \frac{15 \text{ mm} + 5 \text{ mm}}{659.9 \times 10^{-6} \text{ m/°C}} = 30.31°\text{C}$$

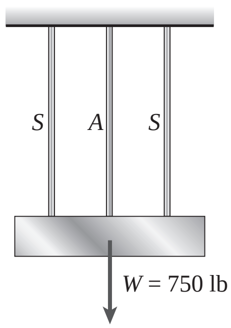
$$T = T_0 + \Delta T = 10°\text{C} + 30.31°\text{C}$$

$$= 40.3°\text{C} \quad \leftarrow$$

Problem 2.5-3 A rigid bar of weight $W = 750$ lb hangs from three equally spaced wires, two of steel and one of aluminum (see figure). The diameter of the wires is $^1/_8$ in . Before they were loaded, all three wires had the same length.

What temperature increase ΔT in all three wires will result in the entire load being carried by the steel wires? (Assume $E_s = 30 \times 10^6$ psi, $\alpha_s = 6.5 \times 10^{-6}$/°F, and $\alpha_a = 12 \times 10^{-6}$/°F.)

Solution 2.5-3 Bar supported by three wires



S = steel $\quad$ A = aluminum

$W = 750$ lb

$$d = \frac{1}{8}\text{ in.}$$

$$A_s = \frac{\pi d^2}{4} = 0.012272\text{ in.}^2$$

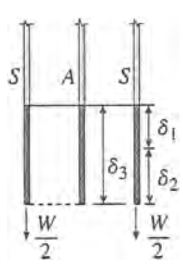
$E_s = 30 \times 10^6$ psi

$E_sA_s = 368{,}155$ lb

$\alpha_s = 6.5 \times 10^{-6}$/°F

$\alpha_a = 12 \times 10^{-6}$/°F

L = Initial length of wires



δ_1 = increase in length of a steel wire due to temperature increase ΔT

$= \alpha_s (\Delta T)L$

δ_2 = increase in length of a steel wire due to load $W/2$

$$= \frac{WL}{2E_sA_s}$$

δ_3 = increase in length of aluminum wire due to temperature increase ΔT

$= \alpha_a(\Delta T)L$

For no load in the aluminum wire:

$\delta_1 + \delta_2 = \delta_3$

$$\alpha_s(\Delta T)L + \frac{WL}{2E_sA_s} = \alpha_a(\Delta T)L$$

or

$$\Delta T = \frac{W}{2E_sA_s(\alpha_a - \alpha_s)} \quad \leftarrow$$

Substitute numerical values:

$$\Delta T = \frac{750\text{ lb}}{(2)(368{,}155\text{ lb})(5.5 \times 10^{-6}/°\text{ F})}$$

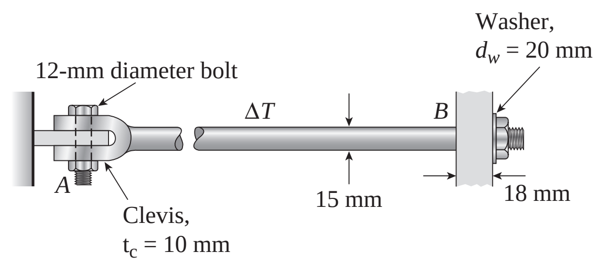
$$= 185°F \quad \leftarrow$$

NOTE: If the temperature increase is larger than ΔT, the aluminum wire would be in compression, which is not possible. Therefore, the steel wires continue to carry all of the load. If the temperature increase is less than ΔT, the aluminum wire will be in tension and carry part of the load.

Problem 2.5-4 A steel rod of 15-mm diameter is held snugly (but without any initial stresses) between rigid walls by the arrangement shown in the figure. (For the steel rod, use $\alpha = 12 \times 10^{-6}$/°C and $E = 200$ GPa.)

(a) Calculate the temperature drop ΔT (degrees Celsius) at which the average shear stress in the 12-mm diameter bolt becomes 45 MPa.

(b) What are the average bearing stresses in the bolt and clevis at A and the washer ($d_w = 20$ mm) and wall ($t_{wall} = 18$mm) at B?



Solution 2.5-4

numerical properties

$$d_r = 15 \text{ mm} \qquad d_b = 12 \text{ mm}$$
$$d_w = 20 \text{ mm} \qquad t_c = 10 \text{ mm}$$
$$t_{wall} = 18 \text{ mm} \qquad \tau_b = 45 \text{ MPa}$$
$$\alpha = 12 \times (10^{-6}) \qquad E = 200 \text{ GPa}$$

(a) Temperature drop resulting in bolt shear stress

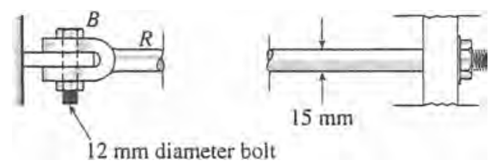
$$\varepsilon = \alpha \Delta T \qquad \sigma = E\alpha\Delta T$$

rod force $= P = (E\alpha\Delta T)\frac{\pi}{4}d_r^2$ and bolt in double

shear with shear stress $\tau = \dfrac{\frac{P}{2}}{A_S} \qquad \tau = \dfrac{P}{2\frac{\pi}{4}d_b^{\,2}}$

$$\text{so} \quad \tau_b = \frac{2}{\pi d_b^{\,2}}\left[(E\alpha\Delta T)\frac{\pi}{4}d_r^{\,2}\right]$$

$$\tau_b = \frac{E\alpha\Delta T}{2}\left(\frac{d_r}{d_b}\right)^2$$



solve for ΔT

$$\Delta T = \frac{2\tau_b}{E\alpha}\left(\frac{d_b}{d_r}\right)^2$$

$$\Delta T = 24°\text{C} \quad \leftarrow$$

$$P = (E\alpha\Delta T)\frac{\pi}{4}d_r^2 \qquad P = 10.18 \text{ kN}$$

(b) Bearing stresses

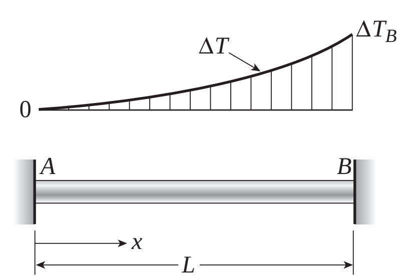
bolt and clevis $\sigma_{bc} = \dfrac{\frac{P}{2}}{d_b t_c} \qquad \sigma_{bc} = 42.4$ MPa $\leftarrow$

washer at wall $\sigma_{bw} = \dfrac{P}{\frac{\pi}{4}(d_w^{\,2} - d_r^{\,2})}$

$\sigma_{bw} = 74.1$ MPa $\leftarrow$

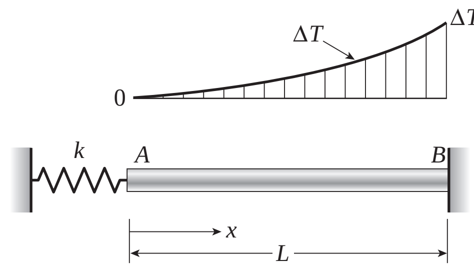
Problem 2.5-5 A bar AB of length L is held between rigid supports and heated nonuniformly in such a manner that the temperature increase ΔT at distance x from end A is given by the expression $\Delta T = \Delta T_B x^3/L^3$, where ΔT_B is the increase in temperature at end B of the bar (see figure part a).

(a) Derive a formula for the compressive stress σ_c in the bar. (Assume that the material has modulus of elasticity E and coefficient of thermal expansion α).



(a)

(b) Now modify the formula in (a) if the rigid support at A is replaced by an elastic support at A having a spring constant k (see figure part b). Assume that only bar AB is subject to the temperature increase.

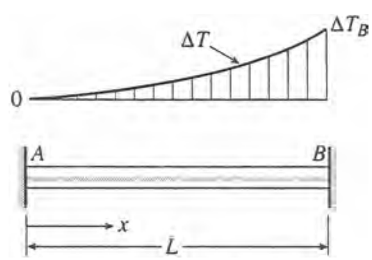


(b)

Solution 2.5-5

(a) one degree statically indeterminate - use superposition select reaction R_B as the redundant; follow procedure below

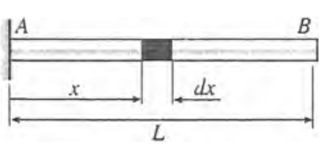
Bar with nonuniform temperature change



At distance x:

$$\Delta T = \Delta T_B \left(\frac{x^3}{L^3}\right)$$

REMOVE THE SUPPORT AT THE END B OF THE BAR:



Consider an element dx at a distance x from end A.

$d\delta$ = Elongation of element dx

$$d\delta = \alpha(\Delta T)dx = \alpha(\Delta T_B)\left(\frac{x^3}{L^3}\right)dx$$

$d\delta$ = elongation of bar

$$\delta = \int_0^L d\delta = \int_0^L \alpha(\Delta T_B)\left(\frac{x^3}{L^3}\right)dx = \frac{1}{4}\alpha(\Delta T_B)L$$

COMPRESSIVE FORCE P REQUIRED TO SHORTEN THE BAR BY THE AMOUNT δ

$$P = \frac{EA\delta}{L} = \frac{1}{4}EA\alpha(\Delta T_B)$$

COMPRESSIVE STRESS IN THE BAR

$$\alpha_c = \frac{P}{A} = \frac{E\alpha(\Delta T_B)}{4} \quad \longleftarrow$$

(b) one degree statically indeterminate - use superposition

select reaction R_B as the redundant then compute bar elongations due to ΔT & due to R_B

$$\delta_{B1} = \alpha \Delta T_B \frac{L}{4} \quad \text{due to temp. from above}$$

$$\delta_{B2} = R_B\left(\frac{1}{k} + \frac{L}{EA}\right)$$

compatibility: solve for R_B $\quad \delta_{B1} + \delta_{B2} = 0$

$$R_B = \frac{-\left(\alpha \Delta T_B \frac{L}{4}\right)}{\left(\frac{1}{k} + \frac{L}{EA}\right)}$$

$$R_B = -\alpha \Delta T_B \left[\frac{EA}{4\left(\frac{EA}{kL} + 1\right)}\right]$$

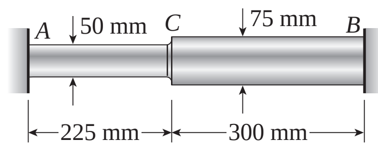
so compressive stress in bar is:

$$\sigma_c = \frac{R_B}{A} \qquad \sigma_c = \frac{E\alpha(\Delta T_B)}{4\left(\frac{EA}{kL} + 1\right)} \quad \longleftarrow$$

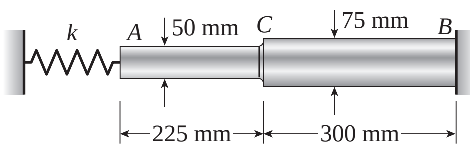
NOTE: σ_c in (b) is the same as in (a) if spring const. k goes to infinity.

Problem 2.5-6 A plastic bar *ACB* having two different solid circular cross sections is held between rigid supports as shown in the figure. The diameters in the left- and right-hand parts are 50 mm and 75 mm, respectively. The corresponding lengths are 225 mm and 300 mm. Also, the modulus of elasticity *E* is 6.0 GPa, and the coefficient of thermal expansion α is 100×10^{-6}/°C. The bar is subjected to a uniform temperature increase of 30°C.

(a) Calculate the following quantities: (1) the compressive force *N* in the bar; (2) the maximum compressive stress σ_c; and (3) the displacement δ_C of point *C*.
(b) Repeat (a) if the rigid support at *A* is replaced by an elastic support having spring constant $k = 50$ MN/m (see figure part b; assume that only the bar *ACB* is subject to the temperature increase).



(a)



(b)

Solution

NUMERICAL DATA

$d_1 = 50$ mm $\quad d_2 = 75$ mm

$L_1 = 225$ mm $\quad L_2 = 300$ mm

$E = 6.0$ GPa $\quad \alpha = 100 \times (10^{-6}/°C$

$\Delta T = 30°C \quad k = 50$ MN/m

(a) COMPRESSIVE FORCE N, MAX. COMPRESSIVE STRESS & DISPL. OF PT. C

$$A_1 = \frac{\pi}{4} d_1^2 \qquad A_2 = \frac{\pi}{4} d_2^2$$

one-degree stat-indet - use R_B as redundant

$$\delta_{B1} = \alpha \Delta T (L_1 + L_2)$$

$$\delta_{B2} = R_B \left(\frac{L_1}{EA_1} + \frac{L_2}{EA_2} \right)$$

compatibility: $\delta_{B1} = \delta_{B2}$, solve for R_B

$$R_B = \frac{\alpha \Delta T (L_1 + L_2)}{\frac{L_1}{EA_1} + \frac{L_2}{EA_2}} \qquad N = R_B$$

$N = 51.8$ kN $\leftarrow$

max. compressive stress in AC since it has the smaller area ($A_1 < A_2$)

$$\sigma_{cmax} = \frac{N}{A_1} \qquad \sigma_{cmax} = 26.4 \text{ MPa}$$

displacement δ_C of point C = superposition of displacements in two released structures at C

$$\delta_C = \alpha \Delta T (L_1) - R_B \frac{L_1}{EA_1}$$

$\delta_C = -0.314$ mm $\leftarrow$ (−) sign means jt C moves left

(b) COMPRESSIVE FORCE N, MAX. COMPRESSIVE STRESS & DISPL. OF PT. C FOR ELASTIC SUPPORT CASE

Use R_B as redundant as in (a)

$$\delta_{B1} = \alpha \Delta T (L_1 + L_2)$$

$$\delta_{B_2} = R_B \left(\frac{L_1}{EA_1} + \frac{L_2}{EA_2} + \frac{1}{k} \right)$$

^ now add effect of elastic support; equate δ_{B1} and δ_{B2} then solve for R_B

$$R_B = \frac{\alpha \Delta T (L_1 + L_2)}{\frac{L_1}{EA_1} + \frac{L_2}{EA_2} + \frac{1}{k}} \qquad N = R_B$$

$N = 31.2$ kN $\leftarrow$

$$\sigma_{cmax} = \frac{N}{A_1} \qquad \sigma_{cmax} = 15.91 \text{ MPa} \quad \leftarrow$$

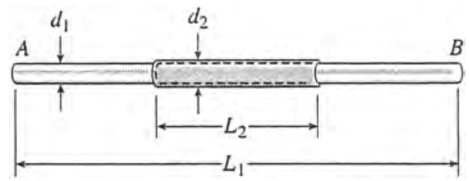
super position

$$\delta_C = \alpha \Delta T (L_1) - R_B \left(\frac{L_1}{EA_1} + \frac{1}{k} \right)$$

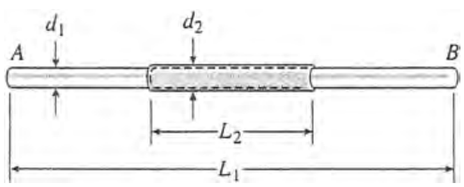
$\delta_C = -0.546$ mm $\leftarrow$ (−) sign means jt C moves left

Problem 2.5-7 A circular steel rod AB (diameter $d_1 = 1.0$ in., length $L_1 =$ 3.0 ft) has a bronze sleeve (outer diameter $d_2 = 1.25$ in., length $L_2 = 1.0$ ft) shrunk onto it so that the two parts are securely bonded (see figure).

Calculate the total elongation δ of the steel bar due to a temperature rise $\Delta T = 500°F$. (Material properties are as follows: for steel, $E_s = 30 \times 10^6$ psi and $\alpha_s = 6.5 \times 10^{-6}/°F$; for bronze, $E_b = 15 \times 10^6$ psi and $\alpha_b = 11 \times 10^{-6}/°F$.)



Solution 2.5-7 Steel rod with bronze sleeve



$L_1 = 36$ in. $\quad L_2 = 12$ in.

ELONGATION OF THE TWO OUTER PARTS OF THE BAR

$$\delta_1 = \alpha_s(\Delta T)(L_1 - L_2)$$
$$= (6.5 \times 10^{-6}/°F)(500°F)(36 \text{ in.} - 12 \text{ in.})$$
$$= 0.07800 \text{ in.}$$

ELONGATION OF THE MIDDLE PART OF THE BAR

The steel rod and bronze sleeve lengthen the same amount, so they are in the same condition as the bolt and sleeve of Example 2-8. Thus, we can calculate the elongation from Eq. (2-21):

$$\delta_2 = \frac{(\alpha_s E_s A_s + \alpha_b E_b A_b)(\Delta T)L_2}{E_s A_s + E_b A_b}$$

SUBSTITUTE NUMERICAL VALUES:

$\alpha_s = 6.5 \times 10^{-6}/°F \quad \alpha_b = 11 \times 10^{-6}/°F$

$E_s = 30 \times 10^6$ psi $\quad E_b = 15 \times 10^6$ psi

$d_1 = 1.0$ in.

$$A_s = \frac{\pi}{4} d_1^2 = 0.78540 \text{ in.}^2$$

$d_2 = 1.25$ in.

$$A_b = \frac{\pi}{4}(d_2^2 - d_1^2) = 0.44179 \text{ in.}^2$$

$\Delta T = 500°F \quad L_2 = 12.0$ in.

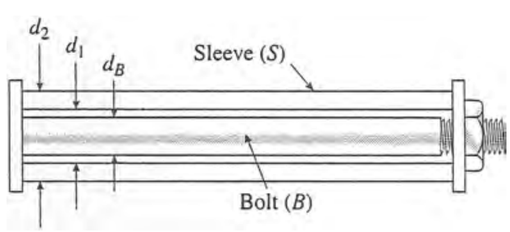
$\delta_2 = 0.04493$ in.

TOTAL ELONGATION

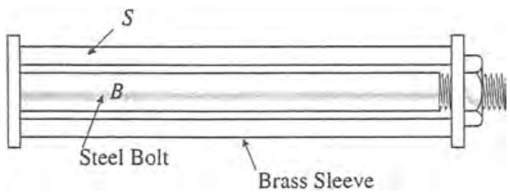
$\delta = \delta_1 + \delta_2 = 0.123$ in. $\quad \leftarrow$

Problem 2.5-8 A brass sleeve S is fitted over a steel bolt B (see figure), and the nut is tightened until it is just snug. The bolt has a diameter $d_B = 25$ mm, and the sleeve has inside and outside diameters $d_1 = 26$ mm and $d_2 = 36$ mm, respectively.

Calculate the temperature rise ΔT that is required to produce a compressive stress of 25 MPa in the sleeve. (Use material properties as follows: for the sleeve, $\alpha_S = 21 \times 10^{-6}/°C$ and $E_S = 100$ GPa; for the bolt, $\alpha_B = 10 \times 10^{-6}/°C$ and $E_B = 200$ GPa.)
(*Suggestion*: Use the results of Example 2-8.)



Solution 2.5-8 Brass sleeve fitted over a Steel bolt



Subscript S means "sleeve".

Subscript B means "bolt".

Use the results of Example 2-8.

σ_S = compressive force in sleeve

EQUATION (2-20A):

$$\sigma_S = \frac{(\alpha_S - \alpha_B)(\Delta T)E_S E_B A_B}{E_S A_S + E_B A_B} \text{ (Compression)}$$

SOLVE FOR ΔT:

$$\Delta T = \frac{\sigma_S(E_S A_S + E_B A_B)}{(\alpha_S - \alpha_B)E_S E_B A_B}$$

or

$$\Delta T = \frac{\sigma_S}{E_S(\alpha_S - \alpha_B)}\left(1 + \frac{E_S A_S}{E_B A_B}\right) \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$\sigma_S = 25$ MPa

$d_2 = 36$ mm $\quad d_1 = 26$ mm $\quad d_B = 25$ mm

$E_S = 100$ GPa $\quad E_B = 200$ GPa

$\alpha_S = 21 \times 10^{-6}/°C \quad \alpha_B = 10 \times 10^{-6}/°C$

$$A_S = \frac{\pi}{4}(d_2^2 - d_1^2) = \frac{\pi}{4}(620 \text{ mm}^2)$$

$$A_B = \frac{\pi}{4}(d_B)^2 = \frac{\pi}{4}(625 \text{ mm}^2) \; 1 + \frac{E_S A_S}{E_B A_B} = 1.496$$

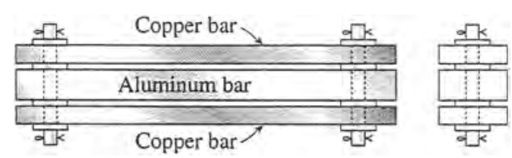
$$\Delta T = \frac{25 \text{ MPa } (1.496)}{(100 \text{ GPa})(11 \times 10^{-6}/°C)}$$

$\Delta T = 34°C \quad \leftarrow$

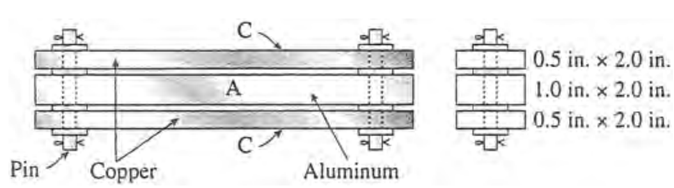
(Increase in temperature)

Problem 2.5-9 Rectangular bars of copper and aluminum are held by pins at their ends, as shown in the figure. Thin spacers provide a separation between the bars. The copper bars have cross-sectional dimensions 0.5 in. × 2.0 in., and the aluminum bar has dimensions 1.0 in. × 2.0 in.

Determine the shear stress in the 7/16 in. diameter pins if the temperature is raised by 100°F. (For copper, $E_c = 18{,}000$ ksi and $\alpha_c = 9.5 \times 10^{-6}/°F$; for aluminum, $E_a = 10{,}000$ ksi and $\alpha_a = 13 \times 10^{-6}/°F$.) *Suggestion*: Use the results of Example 2-8.



Solution 2.5-9 Rectangular bars held by pins



Diameter of pin: $d_P = \frac{7}{16}$ in. $= 0.4375$ in.

Area of pin: $A_P = \frac{\pi}{4} d_P^2 = 0.15033$ in.^{2}

Area of two copper bars: $A_c = 2.0$ in.^{2}

Area of aluminum bar: $A_a = 2.0$ in.^{2}

$\Delta T = 100°F$

Copper: $E_c = 18{,}000$ ksi $\quad \alpha_c = 9.5 \times 10^{-6}/°\text{F}$

Aluminum: $E_a = 10{,}000$ ksi

$\alpha_a = 13 \times 10^{-6}/°\text{F}$

Use the results of Example 2-8.

Find the forces P_a and P_c in the aluminum bar and copper bar, respectively, from Eq. (2-19).

Replace the subscript "S" in that equation by "a" (for aluminum) and replace the subscript "B" by "c" (for copper), because α for aluminum is larger than α for copper.

$$P_a = P_c = \frac{(\alpha_a - \alpha_c)(\Delta T)E_a A_a E_c A_c}{E_a A_a + E_c A_c}$$

Note that P_a is the compressive force in the aluminum bar and P_c is the combined tensile force in the two copper bars.

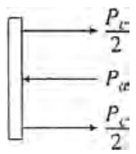
$$P_a = P_c = \frac{(\alpha_a - \alpha_c)(\Delta T)E_c A_c}{1 + \dfrac{E_c A_c}{E_a A_a}}$$

SUBSTITUTE NUMERICAL VALUES:

$$P_a = P_c = \frac{(3.5 \times 10^{-6}/° \text{ F})(100°\text{F})(18{,}000 \text{ ksi})(2 \text{ in.}^2)}{1 + \left(\dfrac{18}{10}\right)\left(\dfrac{2.0}{2.0}\right)}$$

$= 4{,}500$ lb

FREE-BODY DIAGRAM OF PIN AT THE LEFT END



V = shear force in pin

$= P_c/2$

$= 2{,}250$ lb

τ = average shear stress on cross section of pin

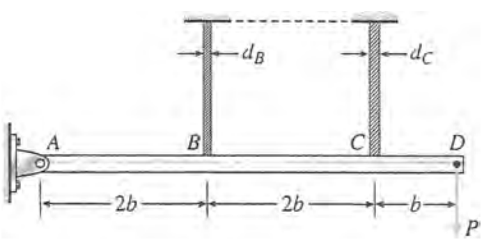
$$\tau = \frac{V}{A_P} = \frac{2{,}250 \text{ lb}}{0.15033 \text{ in.}^2}$$

$\tau = 15.0$ ksi $\quad\longleftarrow$

Problem 2.5-10 A rigid bar $ABCD$ is pinned at end A and supported by two cables at points B and C (see figure). The cable at B has nominal diameter $d_B = 12$ mm and the cable at C has nominal diameter $d_C = 20$ mm. A load P acts at end D of the bar.

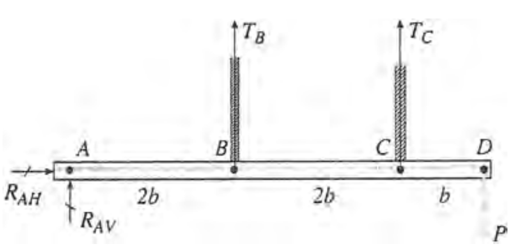
What is the allowable load P if the temperature rises by 60°C and each cable is required to have a factor of safety of at least 5 against its ultimate load?

(*Note:* The cables have effective modulus of elasticity $E = 140$ GPa and coefficient of thermal expansion $\alpha = 12 \times 10^{-6}/°\text{C}$. Other properties of the cables can be found in Table 2-1, Section 2.2.)



Solution 2.5-10 Rigid bar supported by two cables

FREE-BODY DIAGRAM OF BAR *ABCD*



T_B = force in cable B $\quad T_C$ = force in cable C

$d_B = 12$ mm $\quad d_C = 20$ mm

From Table 2-1:

$A_B = 76.7 \text{ mm}^2 \quad E = 140$ GPa

$\Delta T = 60°\text{C} \quad A_C = 173 \text{ mm}^2$

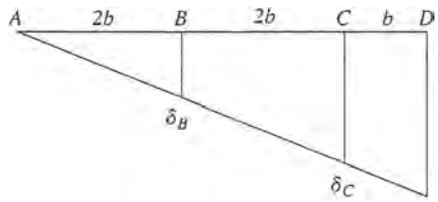
$\alpha = 12 \times 10^{-6}/°\text{C}$

EQUATION OF EQUILIBRIUM

$\Sigma M_A = 0 \quad T_B(2b) + T_C(4b) - P(5b) = 0$

or $2T_B + 4T_C = 5P$ (Eq. 1)

DISPLACEMENT DIAGRAM



COMPATIBILITY:

$\delta_C = 2\delta_B$ (Eq. 2)

FORCE-DISPLACEMENT AND TEMPERATURE-DISPLACEMENT RELATIONS

$$\delta_B = \frac{T_B L}{EA_B} + \alpha(\Delta T)L \qquad \text{(Eq. 3)}$$

$$\delta_C = \frac{T_C L}{EA_C} + \alpha(\Delta T)L \qquad \text{(Eq. 4)}$$

SUBSTITUTE EQS. (3) AND (4) INTO EQ. (2):

$$\frac{T_C L}{EA_C} + \alpha(\Delta T)L = \frac{2T_B L}{EA_B} + 2\alpha(\Delta T)L$$

or

$$2T_B A_C - T_C A_B = -E\alpha(\Delta T)A_B A_C \qquad \text{(Eq. 5)}$$

SUBSTITUTE NUMERICAL VALUES INTO EQ. (5):

$T_B(346) - T_C(76.7) = -1{,}338{,}000$ (Eq. 6)

in which T_B and T_C have units of newtons.

SOLVE SIMULTANEOUSLY EQS. (1) AND (6):

$T_B = 0.2494\,P - 3{,}480$ (Eq. 7)

$T_C = 1.1253\,P + 1{,}740$ (Eq. 8)

in which P has units of newtons.

SOLVE EQS. (7) AND (8) FOR THE LOAD P:

$P_B = 4.0096\,T_B + 13{,}953$ (Eq. 9)

$P_C = 0.8887\,T_C - 1{,}546$ (Eq. 10)

ALLOWABLE LOADS

From Table 2-1:

$(T_B)_{ULT} = 102{,}000$ N $\quad (T_C)_{ULT} = 231{,}000$ N

Factor of safety = 5

$(T_B)_{allow} = 20{,}400$ N $\quad (T_C)_{allow} = 46{,}200$ N

From Eq. (9): $P_B = (4.0096)(20{,}400\text{ N}) + 13{,}953\text{ N}$
$= 95{,}700$ N

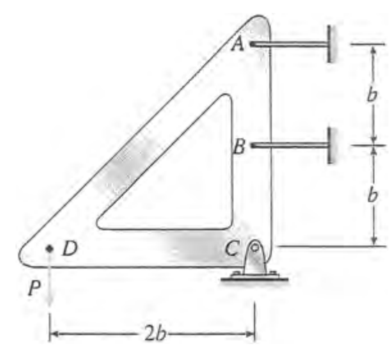
From Eq. (10): $P_C = (0.8887)(46{,}200\text{ N}) - 1546\text{ N}$
$= 39{,}500$ N

Cable C governs.

$P_{allow} = 39.5$ kN $\leftarrow$

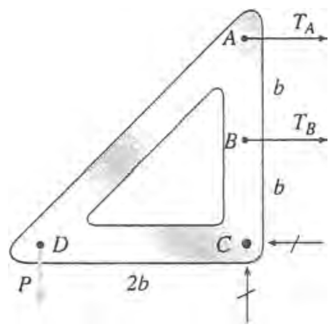
Problem 2.5-11 A rigid triangular frame is pivoted at C and held by two identical horizontal wires at points A and B (see figure). Each wire has axial rigidity $EA = 120$ k and coefficient of thermal expansion $\alpha = 12.5 \times 10^{-6}/°F$.

(a) If a vertical load $P = 500$ lb acts at point D, what are the tensile forces T_A and T_B in the wires at A and B, respectively?
(b) If, while the load P is acting, both wires have their temperatures raised by 180°F, what are the forces T_A and T_B?
(c) What further increase in temperature will cause the wire at B to become slack?



Solution 2.5-11 Triangular frame held by two wires

FREE-BODY DIAGRAM OF FRAME

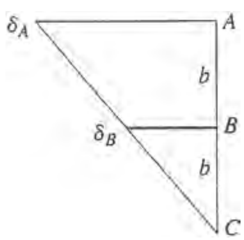


EQUATION OF EQUILIBRIUM

$\Sigma M_C = 0$

$P(2b) - T_A(2b) - T_B(b) = 0$ or $2T_A + T_B = 2P$ (Eq. 1)

DISPLACEMENT DIAGRAM



EQUATION OF COMPATIBILITY

$\delta_A = 2\delta_B$ (Eq. 2)

(a) LOAD P ONLY

Force-displacement relations:

$$\delta_A = \frac{T_A L}{EA} \quad \delta_B = \frac{T_B L}{EA} \qquad \text{(Eq. 3, 4)}$$

(L = length of wires at A and B.)
Substitute (3) and (4) into Eq. (2):

$$\frac{T_A L}{EA} = \frac{2T_B L}{EA}$$

or $T_A = 2T_B$ (Eq. 5)

Solve simultaneously Eqs. (1) and (5):

$$T_A = \frac{4P}{5} \quad T_B = \frac{2P}{5} \qquad \text{(Eqs. 6, 7)}$$

Numerical values:

$P = 500$ lb

$\therefore T_A = 400$ lb $\quad T_B = 200$ lb $\leftarrow$

(b) LOAD P AND TEMPERATURE INCREASE ΔT

Force-displacement and temperature-displacement relations:

$$\delta_A = \frac{T_A L}{EA} + \alpha(\Delta T)L \qquad \text{(Eq. 8)}$$

$$\delta_B = \frac{T_B L}{EA} + \alpha(\Delta T)L \qquad \text{(Eq. 9)}$$

Substitute (8) and (9) into Eq. (2):

$$\frac{T_A L}{EA} + \alpha(\Delta T)L = \frac{2T_B L}{EA} + 2\alpha(\Delta T)L$$

or $T_A - 2T_B = EA\alpha(\Delta T)$ (Eq. 10)

Solve simultaneously Eqs. (1) and (10):

$$T_A = \frac{1}{5}[4P + EA\alpha(\Delta T)] \qquad \text{(Eq. 11)}$$

$$T_B = \frac{2}{5}[P - EA\alpha(\Delta T)] \qquad \text{(Eq. 12)}$$

Substitute numerical values:

$P = 500$ lb $\quad EA = 120{,}000$ lb

$\Delta T = 180°F$

$\alpha = 12.5 \times 10^{-6}/°F$

$$T_A = \frac{1}{5}(2000 \text{ lb} + 270 \text{ lb}) = 454 \text{ lb} \quad \leftarrow$$

$$T_B = \frac{2}{5}(500 \text{ lb} - 270 \text{ lb}) = 92 \text{ lb} \quad \leftarrow$$

(c) WIRE B BECOMES SLACK

Set $T_B = 0$ in Eq. (12):

$$P = EA\alpha(\Delta T)$$

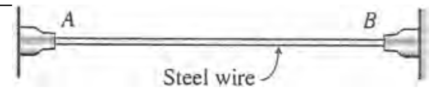
or

$$\Delta T = \frac{P}{EA\alpha} = \frac{500 \text{ lb}}{(120{,}000 \text{ lb})(12.5 \times 10^{-6}/°F)}$$
$$= 333.3°F$$

Further increase in temperature:

$$\Delta T = 333.3°F - 180°F$$
$$= 153°F \quad \leftarrow$$

Misfits and Prestrains



Problem 2.5-12 A steel wire *AB* is stretched between rigid supports (see figure). The initial prestress in the wire is 42 MPa when the temperature is 20°C.

(a) What is the stress σ in the wire when the temperature drops to 0°C?
(b) At what temperature T will the stress in the wire become zero?
(Assume $\alpha = 14 \times 10^{-6}$/°C and $E = 200$ GPa.)

Solution 2.5-12 Steel wire with initial prestress

Initial prestress: $\sigma_1 = 42$ MPa

Initial temperature: $T_1 = 20$°C

$E = 200$ GPa

$\alpha = 14 \times 10^{-6}$/°C

(a) Stress σ when temperature drops to 0°C

$T_2 = 0°\text{C} \quad \Delta T = 20°\text{C}$

NOTE: *Positive* ΔT means a *decrease* in temperature and an *increase* in the stress in the wire.

Negative ΔT means an *increase* in temperature and a *decrease* in the stress.

Stress σ equals the initial stress σ_1 plus the additional stress σ_2 due to the temperature drop.

From Eq. (2-18): $\sigma_2 = E\alpha(\Delta T)$

$$\sigma = \sigma_1 + \sigma_2 = \sigma_1 + E\alpha(\Delta T)$$

$$= 42 \text{ MPa} + (200 \text{ GPa})(14 \times 10^{-6}/°\text{C})(20°\text{C})$$

$$= 42 \text{ MPa} + 56 \text{ MPa} = 98 \text{ MPa} \quad \leftarrow$$

(b) Temperature when stress equals zero

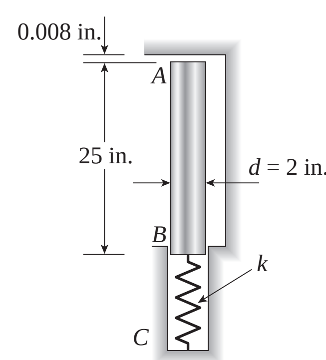
$$\sigma = \sigma_1 + \sigma_2 = 0 \quad \sigma_1 + E\alpha(\Delta T) = 0$$

$$\Delta T = -\frac{\sigma_1}{E\alpha}$$

(Negative means increase in temp.)

$$\Delta T = -\frac{42 \text{ MPa}}{(200 \text{ GPa})(14 \times 10^{-6}/°\text{C}} = -15°\text{C}$$

$$T = 20°\text{C} + 15°\text{C} = 35°\text{C} \quad \leftarrow$$



Problem 2.5-13 A copper bar *AB* of length 25 in. and diameter 2 in. is placed in position at room temperature with a gap of 0.008 in. between end *A* and a rigid restraint (see figure). The bar is supported at end B by an elastic spring with spring constant $k = 1.2 \times 10^6$ lb/in.

(a) Calculate the axial compressive stress σ_c in the bar if the temperature rises 50°F. (For copper, use $\alpha = 9.6 \times 10^{-6}$/°F and $E = 16 \times 10^6$ psi.)
(b) What is the force in the spring? (Neglect gravity effects.)
(C) Repeat (a) if $k \rightarrow \infty$

Solution 2.5-13

numerical data

$L = 25$ in. $d = 2$ in. $\delta = 0.008$ in.

$k = 1.2 \times (10^6)$ lb/in. $E = 16 \times (10^6)$ psi

$\alpha = 9.6 \times (10^{-6})/°F$ $\Delta T = 50°F$

$A = \dfrac{\pi}{4}d^2$ $A = 3.14159$ in^2

(a) one-degree stat.-indet. if gap closes

$\Delta = \alpha \Delta T L$ $\Delta = 0.012$ in. <exceeds gap

select R_A as redundant & do superposition analysis

$$\delta_{A1} = \Delta \quad \delta_{A2} = R_A\left(\frac{L}{EA} + \frac{1}{k}\right)$$

compatibility $\delta_{A1} + \delta_{A2} = \delta$ $\delta_{A2} = \delta - \delta_{A1}$

$$R_A = \frac{\delta - \Delta}{\dfrac{L}{EA} + \dfrac{1}{k}} \qquad R_A = -3006 \text{ lb}$$

compressive stress in bar

$$\sigma = \frac{R_A}{A} \qquad \sigma = -957 \text{ psi}$$

(b) force in spring $F_k = R_C$
statics $R_A + R_C = 0$
$R_C = -R_A$
$R_C = 3006$ lb ←

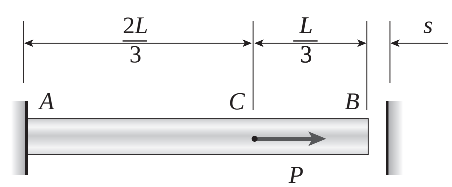
(c) find compressive stress in bar if k goes to infinity
from expression for R_A above, 1/k goes to zero, so

$$R_A = \frac{\delta - \Delta}{\dfrac{L}{EA}} \qquad R_A = -8042 \text{ lb} \qquad \sigma = \frac{R_A}{A}$$

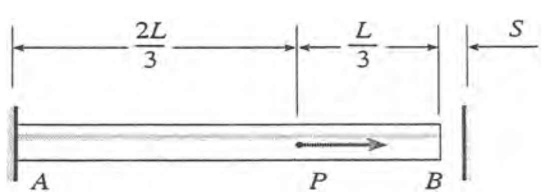
$\sigma = -2560$ psi ←

Problem 2.5-14 A bar *AB* having length *L* and axial rigidity *EA* is fixed at end *A* (see figure). At the other end a small gap of dimension *s* exists between the end of the bar and a rigid surface. A load *P* acts on the bar at point *C*, which is two-thirds of the length from the fixed end.

If the support reactions produced by the load *P* are to be equal in magnitude, what should be the size *s* of the gap?



Solution 2.5-14 Bar with a gap (load *P*)



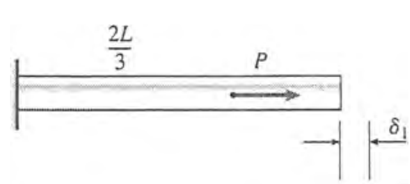
L = length of bar

S = size of gap

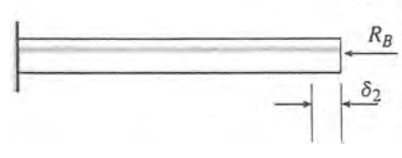
EA = axial rigidity

Reactions must be equal; find S.

FORCE-DISPLACEMENT RELATIONS



$$\delta_1 = \frac{P\left(\frac{2L}{3}\right)}{EA}$$



$$\delta_2 = \frac{R_B L}{EA}$$

COMPATIBILITY EQUATION

$\delta_1 - \delta_2 = S$ or

$$\frac{2PL}{3EA} - \frac{R_B L}{EA} = S \qquad \text{(Eq. 1)}$$

EQUILIBRIUM EQUATION

R_A = reaction at end A (to the left)

R_B = reaction at end B (to the left)

$P = R_A + R_B$

Reactions must be equal.

$$\therefore R_A = R_B \quad P = 2R_B \quad R_B = \frac{P}{2}$$

Substitute for R_B in Eq. (1):

$$\frac{2PL}{3EA} - \frac{PL}{2EA} = S \quad \text{or} \quad S = \frac{PL}{6EA} \quad \leftarrow$$

NOTE: The gap closes when the load reaches the value $P/4$. When the load reaches the value P, equal to $6EAs/L$, the reactions are equal ($R_A = R_B = P/2$). When the load is between $P/4$ and P, R_A is greater than R_B. If the load exceeds P, R_B is greater than R_A.

Problem 2.5-15 Pipe 2 has been inserted snugly into Pipe 1, but the holes for a connecting pin do not line up: there is a gap s. The user decides to apply *either* force P_1 to Pipe 1 *or* force P_2 to Pipe 2, whichever is smaller. Determine the following using the numerical properties in the box.

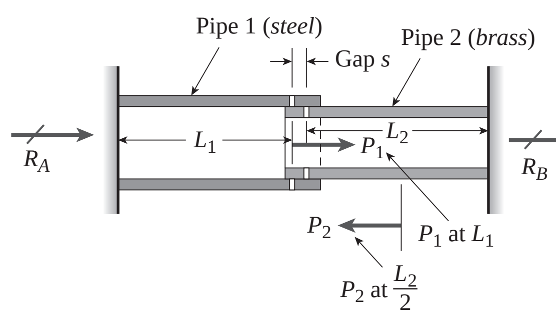
(a) If only P_1 is applied, find P_1 (kips) required to close gap s; if a pin is then inserted and P_1 removed, what are reaction forces R_A and R_B for this load case?

(b) If only P_2 is applied, find P_2 (kips) required to close gap s; if a pin is inserted and P_2 removed, what are reaction forces R_A and R_B for this load case?

(c) What is the maximum *shear* stress in the pipes, for the loads in (a) and (b)?

(d) If a temperature increase ΔT is to be applied to the entire structure to close gap s (*instead of applying forces P_1 and P_2*), find the ΔT required to close the gap. If a pin is inserted after the gap has closed, what are reaction forces R_A and R_B for this case?

(e) Finally, if the structure (with pin inserted) then cools to the *original* ambient temperature, what are reaction forces R_A and R_B?



Numerical properties
$E_1 = 30{,}000$ ksi, $E_2 = 14{,}000$ ksi
$\alpha_1 = 6.5 \times 10^{-6}/°F$, $\alpha_2 = 11 \times 10^{-6}/°F$
Gap $s = 0.05$ in.
$L_1 = 56$ in., $d_1 = 6$ in., $t_1 = 0.5$ in., $A_1 = 8.64$ in.^{2}
$L_2 = 36$ in., $d_2 = 5$ in., $t_2 = 0.25$ in., $A_2 = 3.73$ in.^{2}

Solution 2.5-15

(a) find reactions at A & B for applied force P_1: first compute P_1, required to close gap

$$P_1 = \frac{E_1 A_1}{L_1} s \qquad P_1 = 231.4 \text{ kips} \quad \leftarrow$$

stat-indet analysis with R_B as the redundant

$$\delta_{B1} = -s \qquad \delta_{B2} = R_B\left(\frac{L_1}{E_1 A_1} + \frac{L_2}{E_2 A_2}\right)$$

compatibility: $\delta_{B1} + \delta_{B2} = 0$

$$R_B = \frac{s}{\left(\frac{L_1}{E_1 A_1} + \frac{L_2}{E_2 A_2}\right)} \qquad R_B = 55.2 \text{ k} \quad \leftarrow$$

$$R_A = -R_B \quad \leftarrow$$

(b) find reactions at A & B for applied force P_2

$$P_2 = \frac{E_2 A_2}{\frac{L_2}{2}} s \qquad P_2 = 145.1 \text{ kips} \quad \leftarrow$$

analysis after removing P_2 is same as in (a) so reaction forces are the same

(c) max. shear stress in pipe 1 or 2 when either P_1 or P_2 is applied

$$\tau_{maxa} = \frac{\frac{P_1}{A_1}}{2} \qquad \tau_{maxa} = 13.39 \text{ ksi} \qquad \longleftarrow$$

$$\tau_{maxb} = \frac{\frac{P_2}{A_2}}{2} \qquad \tau_{maxb} = 19.44 \text{ ksi} \qquad \longleftarrow$$

(d) required ΔT and reactions at A & B

$$\Delta T_{reqd} = \frac{s}{\alpha_1 L_1 + \alpha_2 L_2} \qquad \Delta T_{reqd} = 65.8°\text{F} \qquad \longleftarrow$$

if pin is inserted but temperature remains at ΔT above ambient temp., reactions are zero

(e) if temp. returns to original ambient temperature, find reactions at A & B
stat-indet analysis with R_B as the redundant
compatibility: $\delta_{B1} + \delta_{B2} = 0$
analysis is the same as in (a) & (b) above since gap s is the same, so reactions are the same

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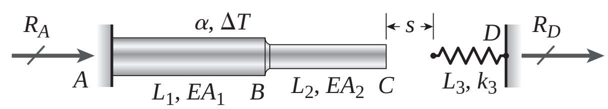
Problem 2.5-16 A nonprismatic bar *ABC* made up of segments *AB* (length L_1, cross-sectional area A_1) and *BC* (length L_2, cross-sectional area A_2) is fixed at end *A* and free at end *C* (see figure). The modulus of elasticity of the bar is *E*. A small gap of dimension *s* exists between the end of the bar and an elastic spring of length L_3 and spring constant k_3. If bar *ABC* only (*not the spring*) is subjected to temperature increase ΔT determine the following.



(a) Write an expression for reaction forces R_A and R_D if the elongation of *ABC* exceeds gap length *s*.
(b) Find expressions for the displacements of points *B* and *C* if the elongation of *ABC* exceeds gap length *s*.

Solution 2.5-16

With gap s closed due to ΔT, structure is one-degree statically-indeterminate; select internal force (Q) at juncture of bar & spring as the redundant; use superposition of two released structures in the solution

δ_{rel1} = relative displ. between end of bar at C & end of spring due to ΔT

$\delta_{rel1} = \alpha\Delta T \cdot (L_1 + L_2)$ $\qquad \delta_{rel1}$ is greater than gap length s

δ_{rel2} = relative displ. between ends of bar & spring due to pair of forces Q, one on end of bar at C & the other on end of spring

$$\delta_{rel2} = Q\left(\frac{L_1}{EA_1} + \frac{L_2}{EA_2}\right) + \frac{Q}{k_3}$$

$$\delta_{rel2} = Q\left(\frac{L_1}{EA_1} + \frac{L_2}{EA_2} + \frac{1}{k_3}\right)$$

compatibility: $\delta_{rel1} + \delta_{rel2} = s \qquad \delta_{rel2} = s - \delta_{rel1}$

$$\delta_{rel2} = s - \alpha\Delta T(L_1 + L_2)$$

$$Q = \frac{s - \alpha\Delta T(L_1 + L_2)}{\frac{L_1}{EA_1} + \frac{L_2}{EA_2} + \frac{1}{k_3}}$$

$$Q = \frac{EA_1A_2k_3}{L_1A_2k_3 + L_2A_1k_3 + EA_1A_2}$$
$$[s - \alpha\Delta T(L_1 + L_2)]$$

(a) Reactions at A & D

statics: $R_A = -Q \quad R_D = Q$

$$R_A = \frac{-s + \alpha\Delta T(L_1 + L_2)}{\frac{L_1}{EA_1} + \frac{L_2}{EA_2} + \frac{1}{k_3}} \qquad \longleftarrow$$

$R_D = -R_A \quad \longleftarrow$

(b) Displacements at B & C
use superposition of displacements in the two released structures

$$\delta_B = \alpha\Delta T(L_1) - R_A\left(\frac{L_1}{EA_1}\right) \quad \leftarrow$$

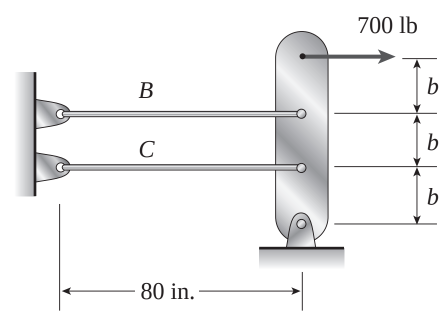
$$\delta_B = \alpha\Delta T(L_1) - \frac{[-s + \alpha\Delta T(L_1 + L_2)]}{\frac{L_1}{EA_1} + \frac{L_2}{EA_2} + \frac{1}{k_3}}\left(\frac{L_1}{EA_1}\right)$$

$$\delta_C = \alpha\Delta T(L_1 + L_2) - R_A\left(\frac{L_1}{EA_1} + \frac{L_2}{EA_2}\right) \quad \leftarrow$$

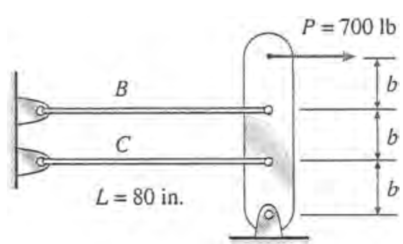
$$\delta_C = \alpha\Delta T(L_1 + L_2) - \frac{[-s + \alpha\Delta T(L_1 + L_2)]}{\frac{L_1}{EA_1} + \frac{L_2}{EA_2} + \frac{1}{k_3}}\left(\frac{L_1}{EA_1} + \frac{L_2}{EA_2}\right)$$

Problem 2.5–17 Wires B and C are attached to a support at the left-hand end and to a pin-supported rigid bar at the right-hand end (see figure). Each wire has cross-sectional area $A = 0.03$ in.^{2} and modulus of elasticity $E = 30 \times 10^6$ psi. When the bar is in a vertical position, the length of each wire is $L = 80$ in. However, before being attached to the bar, the length of wire B was 79.98 in. and of wire C was 79.95 in.

Find the tensile forces T_B and T_C in the wires under the action of a force $P = 700$ lb acting at the upper end of the bar.



Solution 2.5–17 Wires B and C attached to a bar



$P = 700$ lb

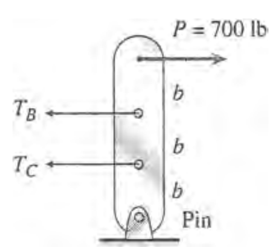
$A = 0.03$ in.^{2}

$E = 30 \times 10^6$ psi

$L_B = 79.98$ in.

$L_C = 79.95$ in.

Equilibrium Equation



$$\Sigma M_{pin} = 0 \quad \leftarrow$$

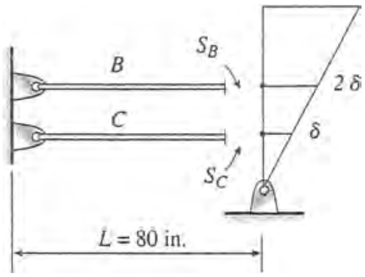
$$T_C(b) + T_B(2b) = P(3b)$$

$$2T_B + T_C = 3P \qquad \text{(Eq. 1)}$$

DISPLACEMENT DIAGRAM

$S_B = 80 \text{ in.} - L_B = 0.02 \text{ in.}$

$S_C = 80 \text{ in.} - L_C = 0.05 \text{ in.}$



Elongation of wires:

$$\delta_B = S_B + 2\delta \qquad \text{(Eq. 2)}$$

$$\delta_C = S_C + \delta \qquad \text{(Eq. 3)}$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_B = \frac{T_B L}{EA} \quad \delta_C = \frac{T_C L}{EA} \qquad \text{(Eqs. 4, 5)}$$

SOLUTION OF EQUATIONS

Combine Eqs. (2) and (4):

$$\frac{T_B L}{EA} = S_B + 2\delta \qquad \text{(Eq. 6)}$$

Combine Eqs. (3) and (5):

$$\frac{T_C L}{EA} = S_C + \delta \qquad \text{(Eq. 7)}$$

Eliminate δ between Eqs. (6) and (7):

$$T_B - 2T_C = \frac{EAS_B}{L} - \frac{2EAS_C}{L} \qquad \text{(Eq. 8)}$$

Solve simultaneously Eqs. (1) and (8):

$$T_B = \frac{6P}{5} + \frac{EAS_B}{5L} - \frac{2EAS_C}{5L} \quad \longleftarrow$$

$$T_C = \frac{3P}{5} - \frac{2EAS_B}{5L} + \frac{4EAS_C}{5L} \quad \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$$\frac{EA}{5L} = 2250 \text{ lb/in.}$$

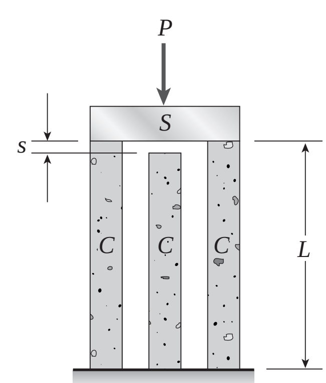
$$T_B = 840 \text{ lb} + 45 \text{ lb} - 225 \text{ lb} = 660 \text{ lb} \quad \longleftarrow$$

$$T_C = 420 \text{ lb} - 90 \text{ lb} + 450 \text{ lb} = 780 \text{ lb} \quad \longleftarrow$$

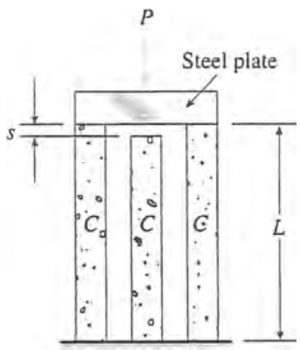
(Both forces are positive, which means tension, as required for wires.)

Problem 2.5-18 A rigid steel plate is supported by three posts of high-strength concrete each having an effective cross-sectional area $A = 40{,}000 \text{ mm}^2$ and length $L = 2$ m (see figure). Before the load P is applied, the middle post is shorter than the others by an amount $s = 1.0$ mm.

Determine the maximum allowable load P_{allow} if the allowable compressive stress in the concrete is $\sigma_{\text{allow}} = 20$ MPa. (Use $E = 30$ GPa for concrete.)



Solution 2.5-18 Plate supported by three posts



s = size of gap = 1.0 mm

L = length of posts = 2.0 m

$A = 40{,}000\ \text{mm}^2$

$\sigma_{\text{allow}} = 20$ MPa

$E = 30$ GPa

C = concrete post

DOES THE GAP CLOSE?

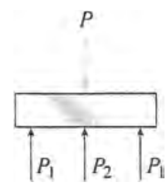
Stress in the two outer posts when the gap is just closed:

$$\sigma = E\varepsilon = E\left(\frac{s}{L}\right) = (30\ \text{GPa})\left(\frac{1.0\ \text{mm}}{2.0\ \text{m}}\right)$$

$$= 15\ \text{MPa}$$

Since this stress is less than the allowable stress, the allowable force P will close the gap.

EQUILIBRIUM EQUATION



$$2P_1 + P_2 = P \qquad \text{(Eq. 1)}$$

COMPATIBILITY EQUATION

δ_1 = shortening of outer posts

δ_2 = shortening of inner post

$$\delta_1 = \delta_2 + s \qquad \text{(Eq. 2)}$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_1 = \frac{P_1 L}{EA} \quad \delta_2 = \frac{P_2 L}{EA} \qquad \text{(Eqs. 3, 4)}$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{P_1 L}{EA} = \frac{P_2 L}{EA} + s \quad \text{or} \quad P_1 - P_2 = \frac{EAs}{L} \qquad \text{(Eq. 5)}$$

Solve simultaneously Eqs. (1) and (5):

$$P = 3P_1 - \frac{EAs}{L}$$

By inspection, we know that P_1 is larger than P_2. Therefore, P_1 will control and will be equal to $\sigma_{\text{allow}} A$.

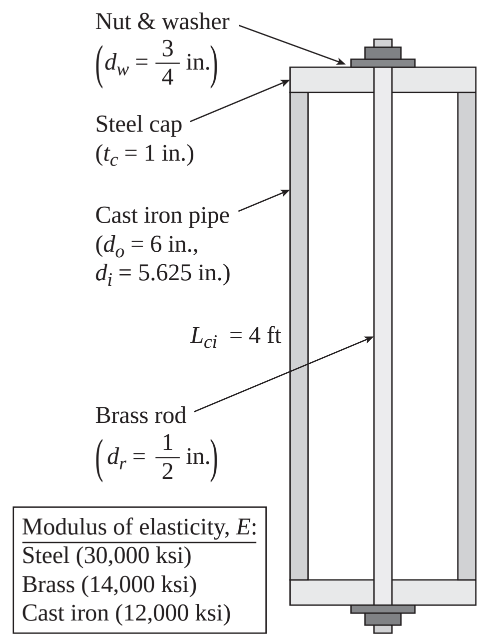
$$P_{\text{allow}} = 3\sigma_{\text{allow}} A - \frac{EAs}{L}$$

$$= 2400\ \text{kN} - 600\ \text{kN} = 1800\ \text{kN}$$

$$= 1.8\ \text{MN} \quad \longleftarrow$$

Problem 2.5-19 A capped cast-iron pipe is compressed by a brass rod, as shown. The nut is turned until it is just snug, then add an additional quarter turn to pre-compress the CI pipe. The pitch of the threads of the bolt is $p = 52$ mils (a mil is one-thousandth of an inch). Use the numerical properties provided.

(a) What stresses σ_p and σ_r will be produced in the cast-iron pipe and brass rod, respectively, by the additional quarter turn of the nut?
(b) Find the bearing stress σ_b beneath the washer and the shear stress τ_c in the steel cap.



Solution 2.5-19

The figure shows a section through the pipe, cap and rod

NUMERICAL PROPERTIES

$L_{ci} = 48$ in. $\quad E_s = 30000$ ksi $\quad E_b = 14000$ ksi

$E_c = 12000$ ksi $\quad t_c = 1$ in. $\quad p = 52 \times (10^{-3})$ in. $\quad n = \frac{1}{4}$

$d_w = \frac{3}{4}$ in. $\quad d_r = \frac{1}{2}$ in. $\quad d_o = 6$ in. $\quad d_i = 5.625$ in.

(a) FORCES & STRESSES IN PIPE & ROD

one degree stat-indet - cut rod at cap & use force in rod (Q) as the redundant

δ_{rel1} = relative displ. between cut ends of rod due to 1/4 turn of nut

$\delta_{rel1} = -np$ ends of rod move apart, not together, so this is (−)

δ_{rel2} = relative displ. between cut ends of rod due pair of forces Q

$$\delta_{rel2} = Q\left(\frac{L + 2t_c}{E_b A_{rod}} + \frac{L_{ci}}{E_c A_{pipe}}\right)$$

$$A_{rod} = \frac{\pi}{4} d_r^2 \qquad A_{pipe} = \frac{\pi}{4}(d_o^2 - d_i^2)$$

$A_{rod} = 0.196 \text{ in}^2 \qquad A_{pipe} = 3.424 \text{ in}^2$

compatibility equation $\quad \delta_{rel1} + \delta_{rel2} = 0$

$$Q = \frac{np}{\dfrac{L_{ci} + 2t_c}{E_b A_{rod}} + \dfrac{L_{ci}}{E_c A_{pipe}}}$$

$Q = 0.672$ kips $\qquad F_{rod} = Q$

statics $\quad F_{pipe} = -Q$

stresses $\quad \sigma_c = \dfrac{F_{pipe}}{A_{pipe}} \qquad \sigma_c = -0.196 \text{ ksi} \quad \leftarrow$

$\sigma_b = \dfrac{F_{rod}}{A_{rod}} \qquad \sigma_b = 3.42 \text{ ksi} \quad \leftarrow$

(b) BEARING AND SHEAR STRESSES IN STEEL CAP

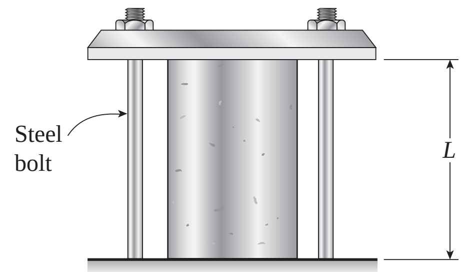
$$\sigma_b = \frac{F_{rod}}{\frac{\pi}{4}(d_w^2 - d_r^2)} \qquad \sigma_b = 2.74 \text{ ksi} \quad \leftarrow$$

$$\tau_c = \frac{F_{rod}}{\pi d_w t_c} \qquad \tau_c = 0.285 \text{ ksi} \quad \leftarrow$$

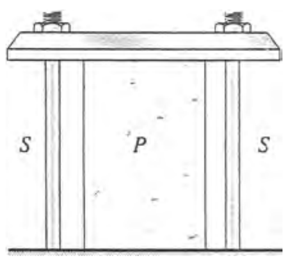
Problem 2.5-20 A plastic cylinder is held snugly between a rigid plate and a foundation by two steel bolts (see figure).

Determine the compressive stress σ_p in the plastic when the nuts on the steel bolts are tightened by one complete turn.

Data for the assembly are as follows: length $L = 200$ mm, pitch of the bolt threads $p = 1.0$ mm, modulus of elasticity for steel $E_s = 200$ GPa, modulus of elasticity for the plastic $E_p = 7.5$ GPa, cross-sectional area of one bolt $A_s = 36.0\ \text{mm}^2$, and cross-sectional area of the plastic cylinder $A_p = 960\ \text{mm}^2$.



Solution 2.5-20 Plastic cylinder and two steel bolts



$L = 200$ mm

$P = 1.0$ mm

$E_s = 200$ GPa

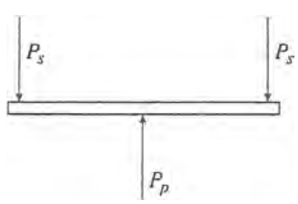
$A_s = 36.0\ \text{mm}^2$ (for one bolt)

$E_p = 7.5$ GPa

$A_p = 960\ \text{mm}^2$

$n = 1$ (See Eq. 2-22)

EQUILIBRIUM EQUATION

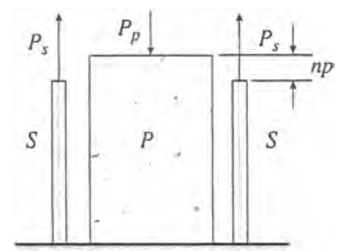


P_s = tensile force in one steel bolt

P_p = compressive force in plastic cylinder

$$P_p = 2P_s \qquad \text{(Eq. 1)}$$

COMPATIBILITY EQUATION



δ_s = elongation of steel bolt

δ_p = shortening of plastic cylinder

$$\delta_s + \delta_p = np \qquad \text{(Eq. 2)}$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_s = \frac{P_s L}{E_s A_s} \quad \delta_p = \frac{P_p L}{E_p A_p} \qquad \text{(Eq. 3, Eq. 4)}$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{P_s L}{E_s A_s} + \frac{P_p L}{E_p A_p} = np \qquad \text{(Eq. 5)}$$

Solve simultaneously Eqs. (1) and (5):

$$P_p = \frac{2npE_s A_s E_p A_p}{L(E_p A_p + 2E_s A_s)}$$

Stress in the plastic cylinder

$$\sigma_p = \frac{P_p}{A_p} = \frac{2np\,E_s\,A_s\,E_p}{L(E_pA_p + 2E_sA_s)}$$

Substitute numerical values:

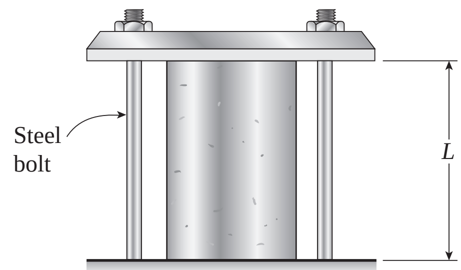
$N = E_s\,A_s\,E_p = 54.0 \times 10^{15}\ \mathrm{N^2/m^2}$

$D = E_pA_p + 2E_sA_s = 21.6 \times 10^6\ \mathrm{N}$

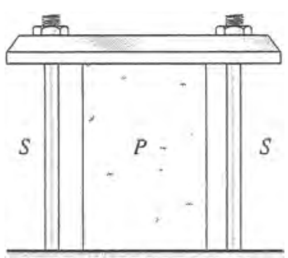
$$\sigma_p = \frac{2np}{L}\left(\frac{N}{D}\right) = \frac{2(1)(1.0\ \mathrm{mm})}{200\ \mathrm{mm}}\left(\frac{N}{D}\right)$$

$= 25.0\ \mathrm{MPa}$ ←

Problem 2.5-21 Solve the preceding problem if the data for the assembly are as follows: length $L = 10$ in., pitch of the bolt threads $p = 0.058$ in., modulus of elasticity for steel $E_s = 30 \times 10^6$ psi, modulus of elasticity for the plastic $E_p = 500$ ksi, cross-sectional area of one bolt $A_s = 0.06$ in.^{2}, and cross-sectional area of the plastic cylinder $A_p = 1.5$ in.^{2}



Solution 2.5-21 Plastic cylinder and two steel bolts



$L = 10$ in.

$p = 0.058$ in.

$E_s = 30 \times 10^6$ psi

$A_s = 0.06$ in.^{2} (for one bolt)

$E_p = 500$ ksi

$A_p = 1.5$ in.^{2}

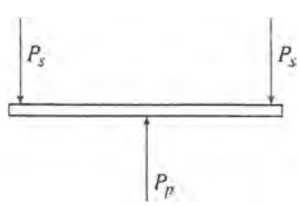
$n = 1$ (see Eq. 2-22)

Equilibrium equation

P_s = tensile force in one steel bolt

P_p = compressive force in plastic cylinder

$P_p = 2P_s$ (Eq. 1)

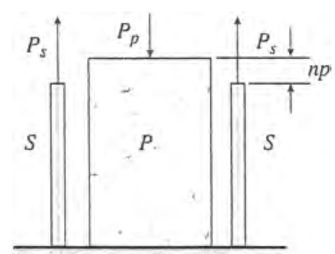


Compatibility equation

δ_s = elongation of steel bolt

δ_p = shortening of plastic cylinder

$\delta_s + \delta_p = np$ (Eq. 2)



Force-displacement relations

$$\delta_s = \frac{P_s\,L}{E_s\,A_s} \qquad \delta_p = \frac{P_p\,L}{E_p\,A_p} \qquad \text{(Eq. 3, Eq. 4)}$$

Solution of equations

Substitute (3) and (4) into Eq. (2):

$$\frac{P_s\,L}{E_s\,A_s} + \frac{P_p\,L}{E_p\,A_p} = np \qquad \text{(Eq. 5)}$$

Solve simultaneously Eqs. (1) and (5):

$$P_p = \frac{2\,np\,E_s\,A_s\,E_p\,A_p}{L(E_p\,A_p + 2E_s\,A_s)}$$

Stress in the plastic cylinder

$$\sigma_p = \frac{P_p}{A_p} = \frac{2\,np\,E_s\,A_s\,E_p}{L(E_p\,A_p + 2E_s\,A_s)} \quad \leftarrow$$

Substitute numerical values:

$$N = E_s A_s E_p = 900 \times 10^9\ \text{lb}^2/\text{in.}^2$$

$$D = E_p A_p + 2E_s A_s = 4350 \times 10^3\ \text{lb}$$

$$\sigma_p = \frac{2np}{L}\left(\frac{N}{D}\right) = \frac{2(1)(0.058\text{in.})}{10\text{ in.}}\left(\frac{N}{D}\right)$$

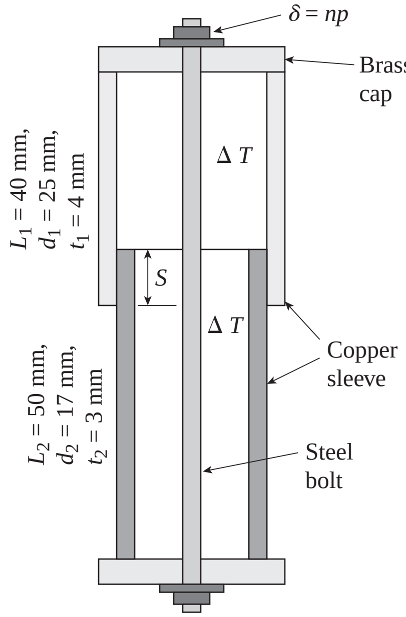
$$= 2400\ \text{psi} \quad \leftarrow$$

Problem 2.5-22 Consider the sleeve made from two copper tubes joined by tin-lead solder over distance s. The sleeve has brass caps at both ends, which are held in place by a steel bolt and washer with the nut turned just snug at the outset. Then, two "loadings" are applied: $n = 1/2$ turn applied to the nut; at the same time the internal temperature is raised by $\Delta T = 30°\text{C}$.

(a) Find the forces in the sleeve and bolt, P_s and P_B, due to both the prestress in the bolt and the temperature increase. For copper, use $E_c = 120$ GPa and $\alpha_c = 17 \times 10^{-6}/°\text{C}$; for steel, use $E_s = 200$ GPa and $\alpha_s = 12 \times 10^{-6}/°\text{C}$. The pitch of the bolt threads is $p = 1.0$ mm. Assume $s = 26$ mm and bolt diameter $d_b = 5$ mm.

(b) Find the required length of the solder joint, s, if shear stress in the sweated joint cannot exceed the allowable shear stress $\tau_{aj} = 18.5$ MPa.

(c) What is the final elongation of the entire assemblage due to both temperature change ΔT and the initial prestress in the bolt?



Solution 2.5-22

The figure shows a section through the sleeve, cap and bolt

Numerical properties (SI units)

$n = \frac{1}{2}$ $\quad p = 1.0$ mm $\quad \Delta T = 30°\text{C}$

$E_c = 120$ GPa $\quad \alpha_c = 17 \times (10^{-6})/°\text{C}$

$E_s = 200$ GPa $\quad \alpha_s = 12 \times (10^{-6})/°\text{C}$

$\tau_{aj} = 18.5$ MPa $\quad s = 26$ mm $\quad d_b = 5$ mm

$L_1 = 40$ mm $\quad t_1 = 4$ mm $\quad L_2 = 50$ mm $\quad t_2 = 3$ mm

$d_1 = 25$ mm $\quad d_1 - 2t_1 = 17$ mm $\quad d_2 = 17$ mm

$$A_b = \frac{\pi}{4}d_b^2 \qquad A_1 = \frac{\pi}{4}[d_1^2 - (d_1 - 2t_1)^2]$$

$$A_b = 19.635\ \text{mm}^2 \qquad A_1 = 263.894\ \text{mm}^2$$

$$A_2 = \frac{\pi}{4}[d_2^2 - (d_2 - 2t_2)^2] \qquad A_2 = 131.947\ \text{mm}^2$$

(a) Forces in sleeve & bolt

one-degree stat-indet - cut bolt & use force in bolt (P_B) as redundant (see sketches below)

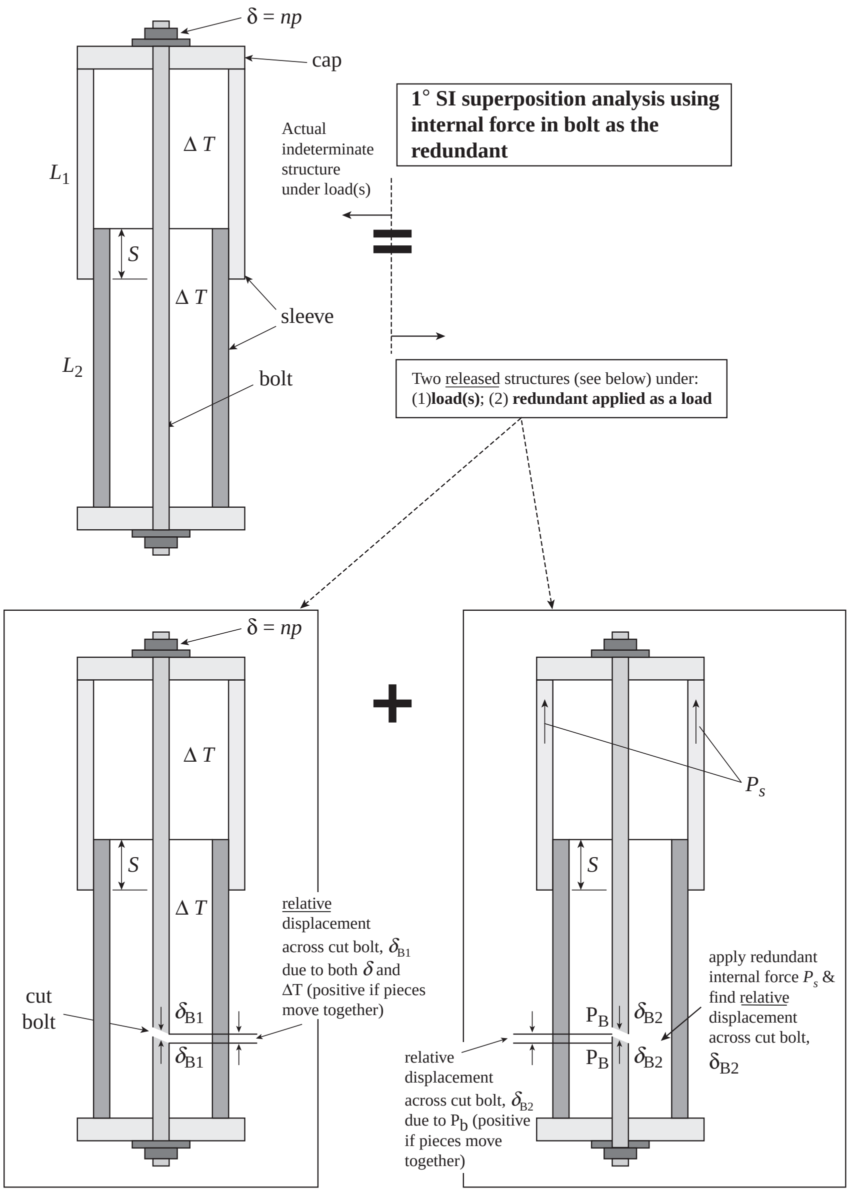
$$\delta_{B1} = -np + \alpha_s \Delta T(L_1 + L_2 - s)$$

$$\delta_{B2} = P_B\left[\frac{L_1 + L_2 - s}{E_s A_b} + \frac{L_1 - s}{E_c A_1} + \frac{L_2 - s}{E_c A_2} + \frac{s}{E_c(A_1 + A_2)}\right]$$

compatibility $\qquad \delta_{B1} + \delta_{B2} = 0$

$$P_B = \frac{-[-np + \alpha_s \Delta T(L_1 + L_2 - s)]}{\left[\frac{L_1 + L_2 - s}{E_s A_b} + \frac{L_1 - s}{E_c A_1} + \frac{L_2 - s}{E_c A_2} + \frac{s}{E_c(A_1 + A_2)}\right]} \qquad P_B = 25.4\text{ kN} \longleftarrow \qquad P_s = -P_B \longleftarrow$$

Sketches illustrating superposition procedure for statically-indeterminate analysis



(b) REQUIRED LENGTH OF SOLDER JOINT≈

$\tau = \frac{P}{A_s} \qquad A_s = \pi d_2 s$

$s_{reqd} = \frac{P_B}{\pi d_2 \tau_{aj}} \qquad s_{reqd} = 25.7 \text{ mm}$

(c) FINAL ELONGATION

δ_f = net of elongation of bolt (δ_b) & shortening of sleeve (δ_s)

$\delta_b = P_B\left(\frac{L_1 + L_2 - s}{E_s A_b}\right) \qquad \delta_b = 0.413 \text{ mm}$

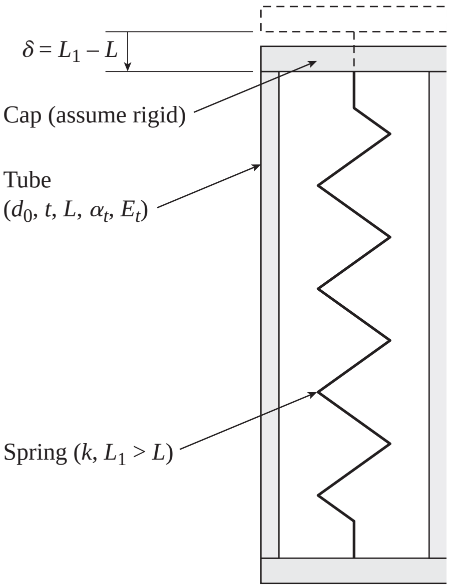
$\delta_s = P_s\left[\frac{L_1 - s}{E_c A_1} + \frac{L_2 - s}{E_c A_2} + \frac{s}{E_c(A_1 + A_2)}\right]$

$\delta_s = -0.064 \text{ mm}$

$\delta_f = \delta_b + \delta_s \qquad \delta_f = 0.35 \text{ mm} \quad \longleftarrow$

Problem 2.5-23 A polyethylene tube (length L) has a cap which when installed compresses a spring (with undeformed length $L_1 > L$) by amount $\delta = (L_1 - L)$. Ignore deformations of the cap and base. Use the force at the base of the spring as the redundant. Use numerical properties in the boxes given.

(a) What is the resulting force in the spring, F_k?
(b) What is the resulting force in the tube, F_t?
(c) What is the final length of the tube, L_f?
(d) What temperature change ΔT inside the tube will result in zero force in the spring?



Modulus of elasticity
Polyethylene tube (E_t = 100 ksi)

Coefficients of thermal expansion
$\alpha_t = 80 \times 10^{-6}/°F$, $\alpha_k = 6.5 \times 10^{-6}/°F$

Properties and dimensions
$d_0 = 6$ in. $t = \frac{1}{8}$ in.
$L_1 = 12.125$ in. $> L = 12$ in. $k = 1.5 \frac{\text{kip}}{\text{in.}}$

Solution 2.5-23

The figure shows a section through the tube, cap and spring

Properties & dimensions

$d_o = 6\text{ in.} \qquad t = \dfrac{1}{8}\text{ in.} \qquad E_t = 100\text{ ksi}$

$A_t = \dfrac{\pi}{4}\left[d_o^2 - (d_o - 2t)^2\right] \qquad A_t = 2.307\text{ in}^2$

$L_1 = 12.125\text{ in.} > L = 12\text{ in.} \qquad k = 1.5\dfrac{\text{kip}}{\text{in}}$

spring is 1/8 in. longer than tube $\qquad \delta = L_1 - L \qquad \delta = 0.125\text{ in.}$

$\alpha_k = 6.5(10^{-6})/°F \quad < \quad \alpha_t = 80 \times (10^{-6})/°F$

$\Delta T = 0$ < note that Q result below is for zero temp. (until part(d))

(a) Force in spring F_k = redundant Q

Flexibilities $\qquad f = \dfrac{1}{k} \qquad f_t = \dfrac{L}{E_t A_t}$

δ_2 = rel. displ. across cut spring due to redundant $= Q(f + f_t)$

δ_1 = rel. displ. across cut spring due to precompression and $\Delta T = \delta + \alpha_k \Delta T L_1 - \alpha_t \Delta T L$

compatibility: $\delta_1 + \delta_2 = 0$

solve for redundant Q

$$Q = \frac{-\delta + \Delta T(-\alpha_k L_1 + \alpha_t L)}{f + f_t} = F_k$$

$F_k = -0.174\text{ kips}$ ← compressive force in spring (F_k) & also tensile force in tube

(b) F_t = force in tube $= -Q$ ←

NOTE: if tube is rigid, $F_k = -k\delta = -0.1875\text{ kips}$

(c) Final length of tube

$L_f = L + \delta_{c1} + \delta_{c2}$ < i.e., add displacements for the two released structures to initial tube length L

$L_f = L - Qf_t + \alpha_t(\Delta T)L \qquad L_f = 12.01\text{ in.}$ ←

(d) Set $Q = 0$ to find ΔT required to reduce spring force to zero

$$\Delta T_{reqd} = \frac{\delta}{(-\alpha_k L_1 + \alpha_t L)}$$

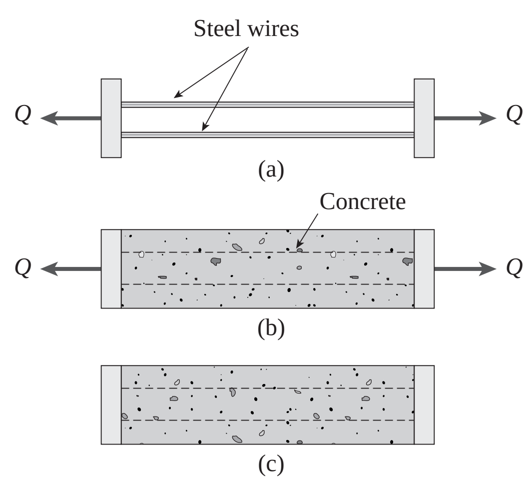
$\Delta T_{reqd} = 141.9°F$ since $\alpha_t > \alpha_k$, a temp. increase is req'd to expand tube so that spring force goes to zero

---

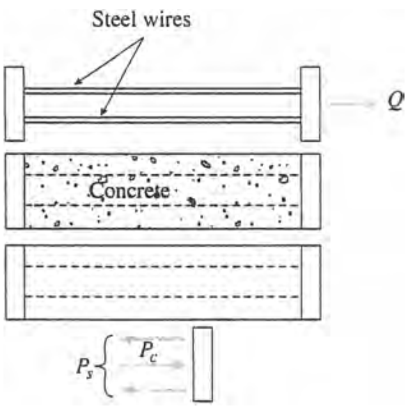
Problem 2.5-24 Prestressed concrete beams are sometimes manufactured in the following manner. High-strength steel wires are stretched by a jacking mechanism that applies a force *Q*, as represented schematically in part (a) of the figure. Concrete is then poured around the wires to form a beam, as shown in part (b).

After the concrete sets properly, the jacks are released and the force *Q* is removed [see part (c) of the figure]. Thus, the beam is left in a prestressed condition, with the wires in tension and the concrete in compression.

Let us assume that the prestressing force *Q* produces in the steel wires an initial stress $\sigma_0 = 620$ MPa. If the moduli of elasticity of the steel and concrete are in the ratio 12:1 and the cross-sectional areas are in the ratio 1:50, what are the final stresses σ_s and σ_c in the two materials?



Solution 2.5-24 Prestressed concrete beam



EQUILIBRIUM EQUATION

$P_s = P_c$ (Eq. 1)

COMPATIBILITY EQUATION AND FORCE-DISPLACEMENT RELATIONS

δ_1 = initial elongation of steel wires

$$= \frac{QL}{E_s A_s} = \frac{\sigma_0 L}{E_s}$$

δ_2 = final elongation of steel wires

$$= \frac{P_s L}{E_s A_s}$$

δ_3 = shortening of concrete

$$= \frac{P_c L}{E_c A_c}$$

$\delta_1 - \delta_2 = \delta_3$ or

$$\frac{\sigma_0 L}{E_s} - \frac{P_s L}{E_s A_s} = \frac{P_c L}{E_c A_c} \qquad \text{(Eq. 2, Eq. 3)}$$

Solve simultaneously Eqs. (1) and (3):

$$P_s = P_c = \frac{\sigma_0 A_s}{1 + \dfrac{E_s A_s}{E_c A_c}}$$

L = length

σ_0 = initial stress in wires

$$= \frac{Q}{A_s} = 620 \text{ MPa}$$

A_s = total area of steel wires

A_c = area of concrete

$= 50\, A_s$

$E_s = 12\, E_c$

P_s = final tensile force in steel wires

P_c = final compressive force in concrete

STRESSES

$$\sigma_s = \frac{P_s}{A_s} = \frac{\sigma_0}{1 + \dfrac{E_s A_s}{E_c A_c}} \qquad \longleftarrow$$

$$\sigma_c = \frac{P_c}{A_c} = \frac{\sigma_0}{\dfrac{A_c}{A_s} + \dfrac{E_s}{E_c}} \qquad \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

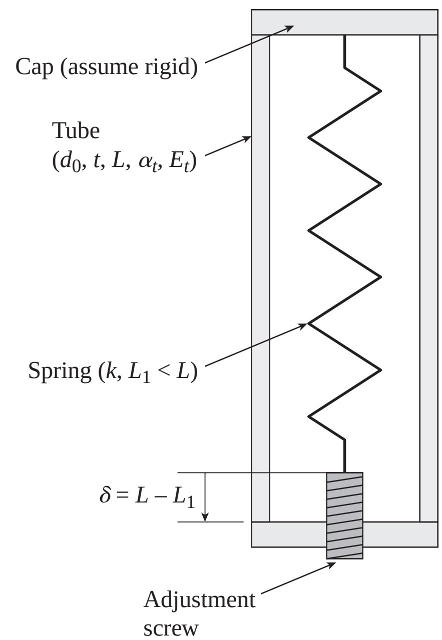
$$\sigma_0 = 620 \text{ MPa} \qquad \frac{E_s}{E_c} = 12 \qquad \frac{A_s}{A_c} = \frac{1}{50}$$

$$\sigma_s = \frac{620 \text{ MPa}}{1 + \dfrac{12}{50}} = 500 \text{ MPa (Tension)} \qquad \longleftarrow$$

$$\sigma_c = \frac{620 \text{ MPa}}{50 + 12} = 10 \text{ MPa (Compression)} \qquad \longleftarrow$$

Problem 2.5-25 A polyethylene tube (length L) has a cap which is held in place by a spring (with undeformed length $L_1 < L$). After installing the cap, the spring is post-tensioned by turning an adjustment screw by amount δ. Ignore deformations of the cap and base. Use the force at the base of the spring as the redundant. Use numerical properties in the boxes below.

(a) What is the resulting force in the spring, F_k?
(b) What is the resulting force in the tube, F_t?
(c) What is the final length of the tube, L_f?
(d) What temperature change ΔT inside the tube will result in zero force in the spring?



Modulus of elasticity
Polyethylene tube (E_t = 100 ksi)

Coefficients of thermal expansion
$\alpha_t = 80 \times 10^{-6}/°F$, $\alpha_k = 6.5 \times 10^{-6}/°F$

Properties and dimensions
$d_0 = 6$ in. $t = \frac{1}{8}$ in.
$L = 12$ in. $L_1 = 11.875$ in. $k = 1.5 \frac{\text{kip}}{\text{in.}}$

Solution 2.5-25

The figure shows a section through the tube, cap and spring

Properties & dimensions

$$d_o = 6 \text{ in.} \qquad t = \frac{1}{8} \text{ in.} \qquad E_t = 100 \text{ ksi}$$

$$L = 12 \text{ in.} > L_1 = 11.875 \text{ in.} \qquad k = 1.5 \frac{\text{kip}}{\text{in}}$$

$$\alpha_k = 6.5(10^{-6}) < \alpha_t = 80 \times (10^{-6})$$

$$A_t = \frac{\pi}{4}[d_o^2 - (d_o - 2t)^2]$$

$$A_t = 2.307 \text{ in}^2$$

Pretension & temperature — spring is 1/8 in. shorter than tube

$$\delta = L - L_1 \qquad \delta = 0.125 \text{ in.} \qquad \Delta T = 0$$

note that Q result below is for zero temp. (until part (d))

Flexibilities $\qquad f = \frac{1}{k} \qquad f_t = \frac{L}{E_t A_t}$

(a) Force in spring (F_k) = redundant (Q)

follow solution procedure outlined in Prob. 2.5-23 solution

$$Q = \frac{\delta + \Delta T(-\alpha_k L_1 + \alpha_t L)}{f + f_t} = F_k$$

$F_k = 0.174$ kips $\longleftarrow$ also the compressive force in the tube

(b) force in tube $F_t = -Q = -\ 0.174$ k $\longleftarrow$

(c) Final length of tube & spring $L_f = L + \delta_{c1} + \delta_{c2}$

$L_f = L - Qf_t + \alpha_t(\Delta T)L \quad L_f = 11.99$ in. $\longleftarrow$

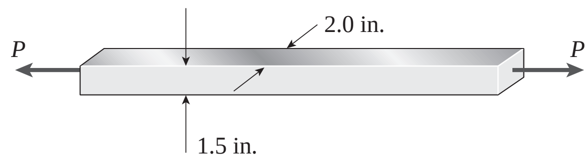
(d) Set $Q = 0$ to find ΔT required to reduce spring force to zero

$$\Delta T_{reqd} = \frac{-\delta}{(-\alpha_k L_1 + \alpha_t L)}$$

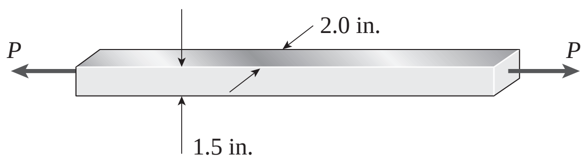
$\Delta T_{reqd} = -141.6°F$ since $\alpha_t > \alpha_k$, a temp. drop is req'd to shrink tube so that spring force goes to zero

Stresses on Inclined Sections

Problem 2.6-1 A steel bar of rectangular cross section (1.5 in. × 2.0 in.) carries a tensile load P (see figure). The allowable stresses in tension and shear are 14,500 psi and 7,100 psi, respectively. Determine the maximum permissible load P_{max}.



Solution 2.6-1



NUMERICAL DATA

$A = 3\ \text{in}^2 \qquad \sigma_a = 14500\ \text{psi}$

$\tau_a = 7100\ \text{psi}$

MAXIMUM LOAD - tension

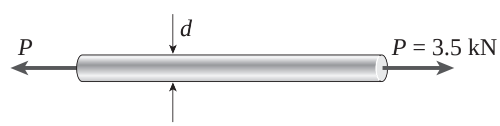
$P_{max1} = \sigma_a A \qquad P_{max1} = 43500\ \text{lbs}$

MAXIMUM LOAD - shear

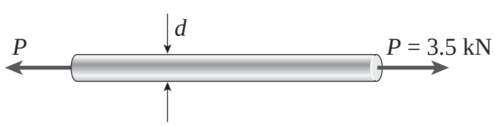
$P_{max2} = 2\tau_a A \qquad P_{max2} = 42{,}600\ \text{lbs}$

Because τ_{allow} is less than one-half of σ_{allow}, the shear stress governs.

Problem 2.6-2 A circular steel rod of diameter d is subjected to a tensile force $P = 3.5$ kN (see figure). The allowable stresses in tension and shear are 118 MPa and 48 MPa, respectively. What is the minimum permissible diameter d_{min} of the rod?



Solution 2.6-2



NUMERICAL DATA $\quad P = 3.5\ \text{kN} \qquad \sigma_a = 118\ \text{MPa}$

$\tau_a = 48\ \text{MPa}$

Find P_{max} then rod diameter

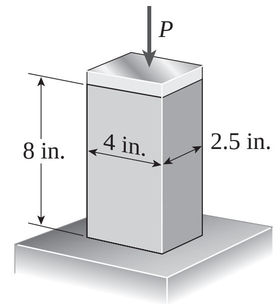
since τ_a is less than 1/2 of σ_a, shear governs

$$P_{max} = 2\tau_a\left(\frac{\pi}{4}d_{min}^{\ 2}\right)$$

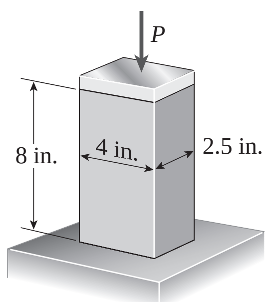
$$d_{min} = \sqrt{\frac{2}{\pi\tau_a}P}$$

$$d_{min} = 6.81\ \text{mm} \qquad \leftarrow$$

Problem 2.6-3 A standard brick (dimensions 8 in. × 4 in. × 2.5 in.) is compressed lengthwise by a force P, as shown in the figure. If the ultimate shear stress for brick is 1200 psi and the ultimate compressive stress is 3600 psi, what force P_{max} is required to break the brick?



Solution 2.6-3 Standard brick in compression



$A = 2.5 \text{ in.} \times 4.0 \text{ in.} = 10.0 \text{ in.}^2$

Maximum normal stress:

$$\sigma_x = \frac{P}{A}$$

Maximum shear stress:

$$\tau_{max} = \frac{\sigma_x}{2} = \frac{P}{2A}$$

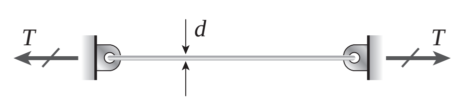
$\sigma_{ult} = 3600$ psi $\qquad \tau_{ult} = 1200$ psi

Because τ_{ult} is less than one-half of σ_{ult}, the shear stress governs.

$$\tau_{max} = \frac{P}{2A} \quad \text{or} \quad P_{max} = 2A\tau_{ult}$$

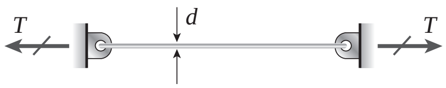
$$P_{max} = 2(10.0 \text{ in.}^2)(1200 \text{ psi}) = 24{,}000 \text{ lb} \quad \leftarrow$$

Problem 2.6-4 A brass wire of diameter $d = 2.42$ mm is stretched tightly between rigid supports so that the tensile force is $T = 98$ N (see figure). The coefficient of thermal expansion for the wire is $19.5 \times + 10^{-6}$/°C and the modulus of elasticity is E = 110 GPa



(a) What is the maximum permissible temperature drop ΔT if the allowable shear stress in the wire is 60 MPa?
(b) At what temperature changes does the wire go slack?

Solution 2.6-4 Brass wire in tension



NUMERICAL DATA

$d = 2.42$ mm $\quad$ T = 98 N

$\alpha = 19.5\ (10^{-6})/°C \quad E = 110$ GPa

(*a*) ΔT_{max} (DROP IN TEMPERATURE)

$$\sigma = \frac{T}{A} - (E\,\alpha\,\Delta T) \qquad \tau_{max} = \frac{\sigma}{2}$$

$$\tau_a = \frac{T}{2A} - \frac{E\,\alpha\,\Delta T}{2}$$

$$\tau_a = 60 \text{ MPa} \qquad A = \frac{\pi}{4}d^2$$

$$\Delta T_{max} = \frac{\frac{T}{A} - 2\,\tau_a}{E\,\alpha}$$

$$\Delta T_{max} = -46°C \text{ (drop)}$$

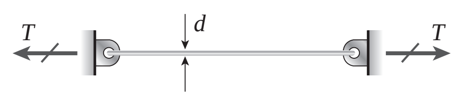
(b) ΔT AT WHICH WIRE GOES SLACK

increase ΔT until $\sigma = 0$

$$\Delta T = \frac{T}{E\,\alpha A}$$

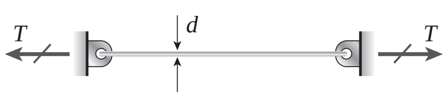
$$\Delta T = 9.93°C \text{ (increase)}$$

Problem 2.6-5 A brass wire of diameter $d = 1/16$ in. is stretched between rigid supports with an initial tension T of 37 lb (see figure). Assume that the coefficient of thermal expansion is 10.6×10^{-6}/°F and the modulus of elasticity is 15×10^6 psi.)



(a) If the temperature is lowered by 60°F, what is the maximum shear stress τ_{max} in the wire?
(b) If the allowable shear stress is 10,000 psi, what is the maximum permissible temperature drop?
(c) At what temperature change ΔT does the wire go slack?

Solution 2.6-5



NUMERICAL DATA

$$d = \frac{1}{16}\text{ in} \qquad T = 37\text{ lb} \qquad \alpha = 10.6 \times (10^{-6})/°F$$

$$E = 15 \times (10^6)\text{ psi} \qquad \Delta T = -60°F$$

$$A = \frac{\pi}{4}d^2$$

(a) τ_{max} (DUE TO DROP IN TEMPERATURE)

$$\tau_{max} = \frac{\sigma_x}{2} \qquad \tau_{max} = \frac{\frac{T}{A} - (E\,\alpha\,\Delta T)}{2}$$

$$\tau_{max} = 10800 \text{ psi} \quad \leftarrow$$

(b) ΔT_{max} FOR ALLOW. SHEAR STRESS $\qquad \tau_a = 10000$ psi

$$\Delta T_{max} = \frac{\frac{T}{A} - 2\tau_a}{E\,\alpha}$$

$$\Delta T_{max} = -49.9°F \quad \leftarrow$$

(c) ΔT AT WHICH WIRE GOES SLACK

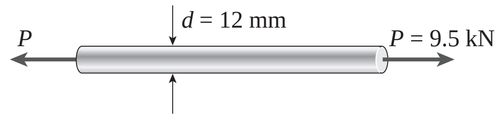
increase ΔT until $\sigma = 0$

$$\Delta T = \frac{T}{E\,\alpha A}$$

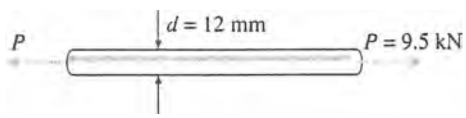
$$\Delta T = 75.9°F \text{ (increase)} \quad \leftarrow$$

Problem 2.6-6 A steel bar with diameter $d = 12$ mm is subjected to a tensile load $P = 9.5$ kN (see figure).

(a) What is the maximum normal stress σ_{max} in the bar?
(b) What is the maximum shear stress τ_{max}?
(c) Draw a stress element oriented at 45° to the axis of the bar and show all stresses acting on the faces of this element.



Solution 2.6-6 Steel bar in tension



$P = 9.5$ kN

(a) Maximum normal stress

$$\sigma_x = \frac{P}{A} = \frac{9.5 \text{ kN}}{\frac{\pi}{4}(12 \text{ mm})^2} = 84.0 \text{ MPa}$$

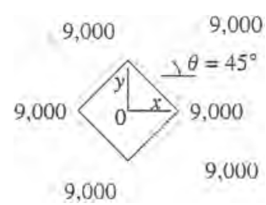
$$\sigma_{max} = 84.0 \text{ MPa} \quad \longleftarrow$$

(b) Maximum shear stress

The maximum shear stress is on a 45° plane and equals $\sigma_x/2$.

$$\tau_{max} = \frac{\sigma_x}{2} = 42.0 \text{ MPa} \quad \longleftarrow$$

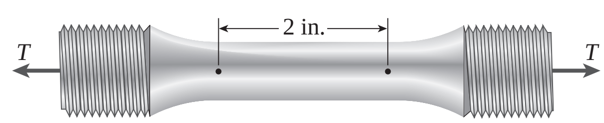
(c) Stress element at $\theta = 45°$



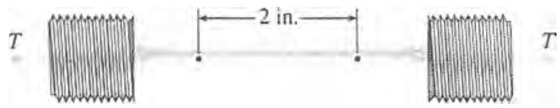
NOTE: All stresses have units of MPa.

Problem 2.6-7 During a tension test of a mild-steel specimen (see figure), the extensometer shows an elongation of 0.00120 in. with a gage length of 2 in. Assume that the steel is stressed below the proportional limit and that the modulus of elasticity $E = 30 \times 10^6$ psi.

(a) What is the maximum normal stress σ_{max} in the specimen?
(b) What is the maximum shear stress τ_{max}?
(c) Draw a stress element oriented at an angle of 45° to the axis of the bar and show all stresses acting on the faces of this element.



Solution 2.6-7 Tension test



Elongation: $\delta = 0.00120$ in.

(2 in. gage length)

$$\text{Strain: } \varepsilon = \frac{\delta}{L} = \frac{0.00120 \text{ in.}}{2 \text{ in.}} = 0.00060$$

$$\text{Hooke's law: } \sigma_x = E\varepsilon = (30 \times 10^6 \text{ psi})(0.00060)$$
$$= 18{,}000 \text{ psi}$$

(a) Maximum normal stress

σ_x is the maximum normal stress.

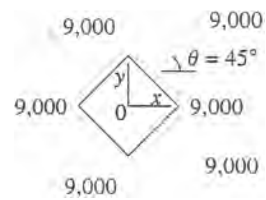
$\sigma_{max} = 18{,}000$ psi $\longleftarrow$

(b) Maximum shear stress

The maximum shear stress is on a 45° plane and equals $\sigma_x/2$.

$$\tau_{max} = \frac{\sigma_x}{2} = 9{,}000 \text{ psi} \quad \longleftarrow$$

(c) Stress element at $\theta = 45°$

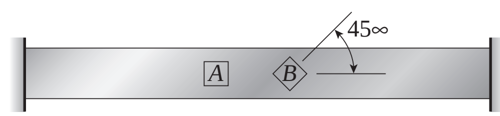


NOTE: All stresses have units of psi.

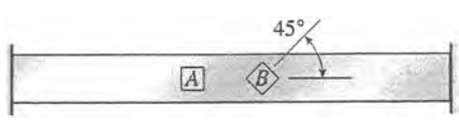
Problem 2.6-8 A copper bar with a rectangular cross section is held without stress between rigid supports (see figure). Subsequently, the temperature of the bar is raised 50°C.

Determine the stresses on all faces of the elements A and B, and show these stresses on sketches of the elements.

(Assume $\alpha = 17.5 \times 10^{-6}$/°C and $E = 120$ GPa.)



Solution 2.6-8 Copper bar with rigid supports



$\Delta T = 50°\text{C}$ (Increase)

$\alpha = 17.5 \times 10^{-6}$/°C

$E = 120$ GPa

Stress due to temperature increase

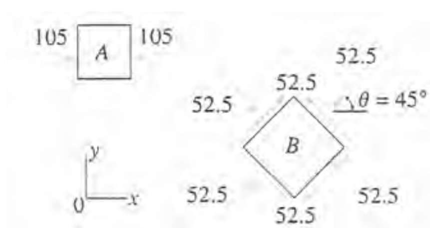
$\sigma_x = E\alpha(\Delta T)$ (See Eq. 2-18 of Section 2.5)

$= 105$ MPa (Compression)

Maximum shear stress

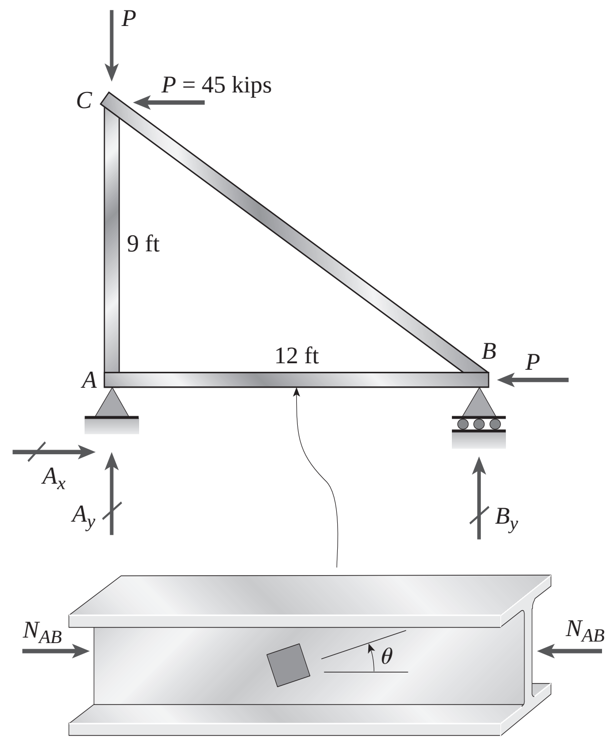
$$\tau_{max} = \frac{\sigma_x}{2} = 52.5 \text{ MPa}$$

Stresses on elements A and B



NOTE: All stresses have units of MPa.

Problem 2.6-9 The bottom chord *AB* in a small truss *ABC* (see figure) is fabricated from a W8 $\times$ 28 wide-flange steel section. The cross-sectional area $A = 8.25$ in.^{2} (Appendix E, Table E-1 (a)) and each of the three applied loads $P = 45$ k. First, find member force N_{AB}; then, determine the normal and shear stresses acting on all faces of stress elements located in the web of member *AB* and oriented at (a) an angle $\theta = 0°$, (b) an angle $\theta = 30°$, and (c) an angle $\theta = 45°$. In each case, show the stresses on a sketch of a properly oriented element.



Solution 2.6-9

Statics

$P = 45$ kips $\qquad \sum M_A = 0 \qquad B_y = \dfrac{-9}{12}P$

$B_y = -33.75$ k

$BC_V = -B_y \qquad BC_H = \dfrac{12}{9}BC_V \qquad BC_H = 45$ k

$\sum F_H = 0$ at B $\qquad N_{AB} = BC_H + P$

$N_{AB} = 90$ kips (compression) $\longleftarrow$

Normal and shear stresses on elements at 0, 30 & 45

degrees in web of AB $\quad \sigma_x = \dfrac{-N_{AB}}{A} \qquad A = 8.25\ \text{in}^2$

$\sigma_x = -10.9$ ksi $\longleftarrow$

(a) $\theta = 0 \qquad \sigma_x = -10.91$ ksi $\longleftarrow$

(b) $\theta = 30°$

on $+$x face

$\sigma_\theta = \sigma_x \cos(\theta)^2 \qquad \sigma_\theta = -8.18$ ksi $\longleftarrow$

$\tau_\theta = -\sigma_x \sin(\theta)\cos(\theta) \qquad \tau_\theta = 4.72$ ksi $\longleftarrow$

on $+$y face $\qquad \theta = \theta + \dfrac{\pi}{2}$

$\sigma_\theta = \sigma_x \cos(\theta)^2 \qquad \sigma_\theta = -2.73$ ksi

$\tau_\theta = -\sigma_x \sin(\theta)\cos(\theta) \qquad \tau_\theta = -4.72$ ksi

(c) $\theta = 45$ degrees

on +x face

$\sigma_\theta = \sigma_x \cos(\theta)^2$ $\quad \sigma_\theta = -5.45$ ksi $\longleftarrow$

$\tau_\theta = -\sigma_x \sin(\theta)\cos(\theta)$ $\quad \tau_\theta = 5.45$ ksi $\longleftarrow$

on +y face $\quad \theta = \theta + \dfrac{\pi}{2}$

$\sigma_\theta = \sigma_x \cos(\theta)^2$ $\quad \sigma_\theta = -5.45$ ksi

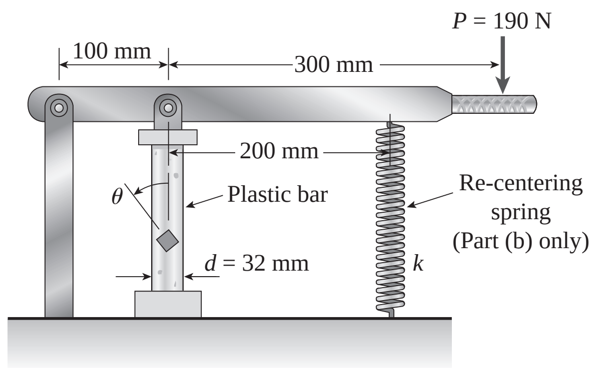
$\tau_\theta = -\sigma_x \sin(\theta)\cos(\theta)$ $\quad \tau_\theta = -5.45$ ksi

---

Problem 2.6-10 A plastic bar of diameter $d = 32$ mm is compressed in a testing device by a force $P = 190$ N applied as shown in the figure.

(a) Determine the normal and shear stresses acting on all faces of stress elements oriented at (1) an angle $\theta = 0°$, (2) an angle $\theta = 22.5°$, and (3) an angle $\theta = 45°$. In each case, show the stresses on a sketch of a properly oriented element. What are σ_{max} and τ_{max}?

(b) Find σ_{max} and τ_{max} in the plastic bar if a re-centering spring of stiffness k is inserted into the testing device, as shown in the figure. The spring stiffness is 1/6 of the axial stiffness of the plastic bar.



Solution

NUMERICAL DATA

$d = 32$ mm $\qquad A = \dfrac{\pi}{4} d^2$

$P = 190$ N $\qquad A = 804.25 \text{ mm}^2$

$a = 100$ mm

$b = 300$ mm

(a) Statics - FIND COMPRESSIVE FORCE F & STRESSES IN PLASTIC BAR

$F = \dfrac{P(a + b)}{a}$ $\qquad F = 760$ N

$\sigma_x = \dfrac{F}{A}$ $\quad \sigma_x = 0.945$ MPa $\quad$ or $\quad \sigma_x = 945$ kPa

from (1), (2) & (3) below

$\sigma_{max} = \sigma_x$ $\quad \sigma_{max} = -945$ kPa

$\tau_{max} = 472$ kPa $\quad \dfrac{\sigma_x}{2} = -472$ kPa

(1) $\theta = 0$ degrees $\quad \sigma_x = -945$ kPa $\longleftarrow$

(2) $\theta = 22.50$ degrees

on +x face

$\sigma_\theta = \sigma_x \cos(\theta)^2$
$\sigma_\theta = -807$ kPa $\longleftarrow$

$\tau_\theta = -\sigma_x \sin(\theta)\cos(\theta)$
$\tau_\theta = 334$ kPa $\longleftarrow$

on +y face $\quad \theta = \theta + \dfrac{\pi}{2}$

$\sigma_\theta = \sigma_x \cos(\theta)^2$
$\sigma_\theta = -138.39$ kPa

$\tau_\theta = -\sigma_x \sin(\theta)\cos(\theta)$
$\tau_\theta = -334.1$ kPa

(3) $\theta = 45$ degrees

on +x face

$\sigma_\theta = \sigma_x \cos(\theta)^2$
$\sigma_\theta = -472$ kPa $\longleftarrow$

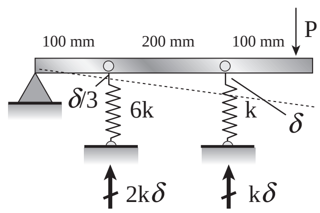
$\tau_\theta = -\sigma_x \sin(\theta)\cos(\theta)$
$\tau_\theta = 472$ kPa $\longleftarrow$

on +y face $\quad \theta = \theta + \dfrac{\pi}{2}$

$\sigma_\theta = \sigma_x \cos(\theta)^2 \qquad \sigma_\theta = -472.49\ \text{kPa}$

$\tau_\theta = -\sigma_x \sin(\theta)\cos(\theta) \qquad \tau_\theta = -472.49\ \text{kPa}$

(b) ADD SPRING - FIND MAX. NORMAL & SHEAR STRESSES IN PLASTIC BAR



$\sum M_{pin} = 0$

$P(400) = [2k\delta(100) + k\delta(300)]$

$\delta = \dfrac{4}{5}\dfrac{P}{k}$

force in plastic bar $\quad F = (2k)\left(\dfrac{4}{5}\dfrac{P}{k}\right)$

$F = \dfrac{8}{5}P \qquad F = 304\ \text{N}$

normal and shear stresses in plastic bar

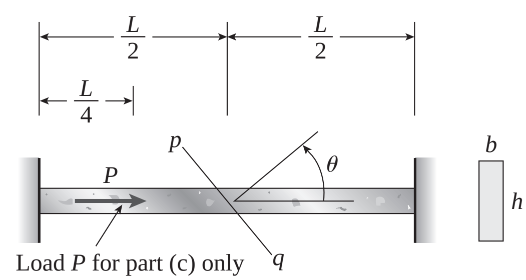
$\sigma_x = \dfrac{F}{A} \qquad \sigma_x = 0.38$

$\sigma_{max} = -378\ \text{kPa} \quad \leftarrow$

$\tau_{max} = \dfrac{\sigma_x}{2} \qquad \tau_{max} = -189\ \text{kPa} \quad \leftarrow$

Problem 2.6-11 A plastic bar of rectangular cross section ($b = 1.5$ in. and $h = 3$ in.) fits snugly between rigid supports at room temperature (68°F) but with no initial stress (see figure). When the temperature of the bar is raised to 160°F, the compressive stress on an inclined plane *pq* at midspan becomes 1700 psi.

(a) What is the shear stress on plane *pq*? (Assume $\alpha = 60 \times 10^{-6}$/°F and $E = 450 \times 10^3$ psi.)
(b) Draw a stress element oriented to plane *pq* and show the stresses acting on all faces of this element.
(c) If the allowable normal stress is 3400 psi and the allowable shear stress is 1650 psi, what is the maximum load *P* (*in* +x *direction*) which can be added at the quarter point (in addition to thermal effects above) without exceeding allowable stress values in the bar?



Solution 2.6-11

NUMERICAL DATA

$b = 1.5\ \text{in} \quad h = 3\ \text{in} \quad A = bh \quad \Delta T = (160 - 68)°\text{F}$

$\Delta T = 92°\text{F}$

$A = 4.5\ \text{in}^2 \qquad \sigma_{pq} = -1700\ \text{psi}$

$\alpha = 60 \times (10^{-6})/°\text{F}$

$E = 450 \times (10^3)\ \text{psi}$

(a) SHEAR STRESS ON PLANE PQ
STAT-INDET ANALYSIS GIVES, FOR REACTION AT RIGHT SUPPORT:

$R = -EA\alpha\Delta T \qquad R = -11178\ \text{lb}$

$\sigma_x = \dfrac{R}{A} \qquad \sigma_x = -2484\ \text{psi}$

using $\sigma_\theta = \sigma_x \cos(\theta)^2$ $\qquad \cos(\theta)^2 = \frac{\sigma_{pq}}{\sigma_x}$

$\theta = \operatorname{acos}\left(\sqrt{\frac{\sigma_{pq}}{\sigma_x}}\right)$ $\qquad \theta = 34.2°$

now with θ, can find shear stress on plane pq

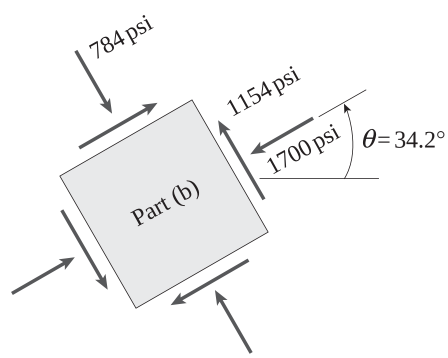
$\tau_{pq} = -\sigma_x \sin(\theta)\cos(\theta)$ $\qquad \tau_{pq} = 1154$ psi $\quad \leftarrow$

$\sigma_{pq} = \sigma_x \cos(\theta)^2$ $\qquad \sigma_{pq} = -1700$ psi

stresses at $\theta + \pi/2$ (y face)

$\sigma_y = \sigma_x \cos\left(\theta + \frac{\pi}{2}\right)^2$ $\qquad \sigma_y = -784$ psi

(b) STRESS ELEMENT FOR PLANE PQ



(c) MAX. LOAD AT QUARTER POINT $\qquad \sigma_a = 3400$ psi

$\tau_a = 1650$ psi $\qquad 2\tau_a = 3300$ $\qquad <$ less than σ_a so shear controls

stat-indet analysis for P at L/4 gives, for reactions:

$R_{R2} = \frac{-P}{4}$ $\qquad R_{L2} = \frac{-3}{4}P$

(tension for 0 to L/4 & compression for rest of bar)

from (a) (for temperature increase ΔT):

$R_{R1} = -EA\alpha\Delta T$ $\qquad R_{L1} = -EA\alpha\Delta T$

Stresses in bar (0 to L/4)

$\sigma_x = -E\alpha\Delta T + \frac{3P}{4A}$ $\qquad \tau_{max} = \frac{\sigma_x}{2}$

set $\tau_{max} = \tau_a$ & solve for P_{max1}

$\tau_a = \frac{-E\alpha\Delta T}{2} + \frac{3P}{8A}$

$P_{max1} = \frac{4A}{3}(2\tau_a + E\alpha\Delta T)$

$P_{max1} = 34704$ lb

$\tau_{max} = \frac{-E\alpha\Delta T}{2} + \frac{3P_{max1}}{8A}$

$\tau_{max} = 1650$ psi $\qquad <$ check

$\sigma_x = -E\alpha\Delta T + \frac{3P_{max1}}{4A}$

$\sigma_x = 3300$ psi $\qquad <$ less than σ_a

Stresses in bar (L/4 to L)

$\sigma_x = -E\alpha\Delta T - \frac{P}{4A}$ $\qquad \tau_{max} = \frac{\sigma_x}{2}$

set $\tau_{max} = \tau_a$ & solve for P_{max2}

$P_{max2} = -4A(-2\tau_a + E\alpha\Delta T)$

$P_{max2} = 14688$ lb $\qquad \leftarrow$ $\qquad$ shear in segment (L/4 to L) controls

$\tau_{max} = \frac{-E\alpha\Delta T}{2} - \frac{P_{max2}}{8A}$ $\qquad \tau_{max} = -1650$ psi

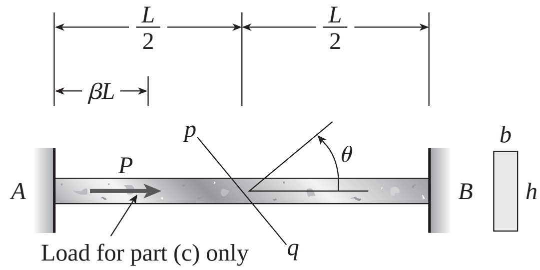
$\sigma_x = -E\alpha\Delta T - \frac{P_{max2}}{4A}$ $\qquad \sigma_x = -3300$ psi

Problem 2.6-12 A copper bar of rectangular cross section ($b = 18$ mm and $h = 40$ mm) is held snugly (but without any initial stress) between rigid supports (see figure). The allowable stresses on the inclined plane pq at midspan, for which $\theta = 55°$, are specified as 60 MPa in compression and 30 MPa in shear.

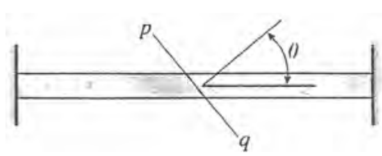
(a) What is the maximum permissible temperature rise ΔT if the allowable stresses on plane pq are not to be exceeded? (Assume $\alpha = 17 \times 10^{-6}$/°C and $E = 120$ GPa.)

(b) If the temperature increases by the maximum permissible amount, what are the stresses on plane pq?

(c) If the temperature rise $\Delta T = 28$°C, how far to the right of end A (distance βL, expressed as a fraction of length L) can load $P = 15$ kN be applied without exceeding allowable stress values in the bar? Assume that $\sigma_a = 75$ MPa and $\tau_a = 35$ MPa.



Solution 2.6-12



NUMERICAL DATA

$$\theta = 55\left(\frac{\pi}{180}\right) \quad \text{radians}$$

$b = 18$ mm $\quad h = 40$ mm

$A = bh \quad A = 720$ mm^2

$\sigma_{pqa} = 60$ MPa $\quad \tau_{pqa} = 30$ Mpa

$\alpha = 17 \times (10^{-6})$/°C $\quad E = 120$ GPa

$\Delta T = 20$°C $\quad P = 15$ kN

(a) FIND ΔT_{max} BASED ON ALLOWABLE NORMAL & SHEAR STRESS VALUES ON PLANE pq

$$\sigma_x = -E\alpha\Delta T_{max} \qquad \Delta T_{max} = \frac{-\sigma_x}{E\alpha}$$

$$\sigma_{pq} = \sigma_x \cos(\theta)^2 \qquad \tau_{pq} = -\sigma_x \sin(\theta)\cos(\theta)$$

^ set each equal to corresponding allowable & solve for σ_x

$$\sigma_{x1} = \frac{\sigma_{pqa}}{\cos(\theta)^2} \qquad \sigma_{x1} = 182.38 \text{ MPa}$$

$$\sigma_{x2} = \frac{\tau_{pqa}}{-\sin(\theta)\cos(\theta)} \qquad \sigma_{x2} = -63.85 \text{ MPa}$$

lesser value controls so allowable shear stress governs

$$\Delta T_{max} = \frac{-\sigma_{x2}}{E\alpha} \qquad \Delta T_{max} = 31.3°\text{C} \quad \leftarrow$$

(b) STRESSES ON PLANE PQ FOR max. TEMP.

$$\sigma_x = -E\alpha\Delta T_{max} \qquad \sigma_x = -63.85 \text{ MPa}$$

$$\sigma_{pq} = \sigma_x \cos(\theta)^2 \qquad \sigma_{pq} = -21.0 \text{ MPa} \quad \leftarrow$$

$$\tau_{pq} = -\sigma_x \sin(\theta)\cos(\theta) \qquad \tau_{pq} = 30 \text{ MPa} \quad \leftarrow$$

(c) ADD LOAD P IN +X-DIRECTION TO TEMPERATURE CHANGE & FIND LOCATION OF LOAD

$\Delta T = 28$ DEGREES C

$P = 15$ kN from one-degree stat-indet analysis, reactions R_A & R_B due to load P are:

$$R_A = -(1 - \beta)P \qquad R_B = \beta P$$

now add normal stresses due to P to thermal stresses due to ΔT (tension in segment 0 to βL, compression in segment βL to L)

Stresses in bar (0 to βL)

$$\sigma_x = -E\alpha\Delta T + \frac{R_A}{A} \qquad \tau_{max} = \frac{\sigma_x}{2}$$

shear controls so set $\tau_{max} = \tau_a$ & solve for β

$$2\tau_a = -E\alpha\Delta T + \frac{(1-\beta)P}{A}$$

$$\beta = 1 - \frac{A}{P}[2\tau_a + E\alpha\Delta T]$$

$\beta = -5.1$

^ impossible so evaluate segment (βL to L)

Stresses in bar (βL to L)

$$\sigma_x = -E\alpha\Delta T - \frac{R_B}{A} \qquad \tau_{max} = \frac{\sigma_x}{2}$$

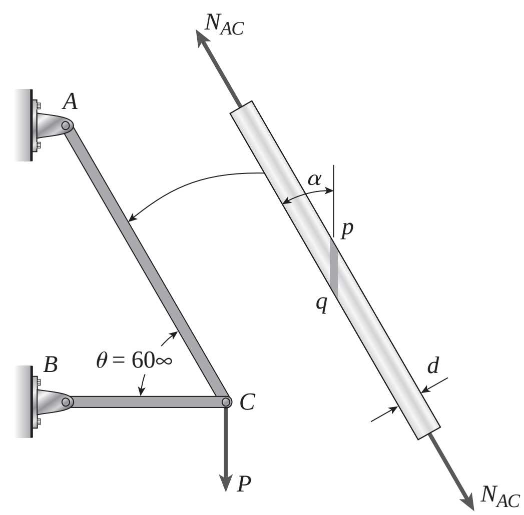
set $\tau_{max} = \tau_a$ & solve for P_{max2}

$$2\tau_a = -E\alpha\Delta T - \frac{\beta P}{A}$$

$$\beta = \frac{-A}{P}[-2\tau_a + E\alpha\Delta T]$$

$$\beta = 0.62 \quad \leftarrow$$

Problem 2.6-13 A circular brass bar of diameter d is member AC in truss ABC which has load $P = 5000$ lb applied at joint C. Bar AC is composed of two segments brazed together on a plane pq making an angle $\alpha = 36°$ with the axis of the bar (see figure). The allowable stresses in the brass are 13,500 psi in tension and 6500 psi in shear. On the brazed joint, the allowable stresses are 6000 psi in tension and 3000 psi in shear. What is the tensile force N_{AC} in bar AC? What is the minimum required diameter d_{min} of bar AC?



Solution 2.6-13

NUMERICAL DATA

$P = 5$ kips $\qquad \alpha = 36° \qquad \sigma_a = 13.5$ ksi

$\tau_a = 6.5$ ksi

$$\theta = \frac{\pi}{2} - \alpha \quad \theta = 54°$$

$\sigma_{ja} = 6.0$ ksi

$\tau_{ja} = 3.0$ ksi

tensile force N_{AC} $\qquad$ Method of Joints at C

$$N_{AC} = \frac{P}{\sin(60°)} \qquad \text{(tension)}$$

$$N_{AC} = 5.77 \text{ kips} \quad \leftarrow$$

min. required diameter of bar AC

(1) check tension and shear in bars; $\tau_a < \sigma_a/2$ so shear controls $\tau_{max} = \dfrac{\sigma_x}{2}$

$$2\tau_a = \frac{N_{AC}}{A} \qquad \sigma_x = 2\tau_a = 13 \text{ ksi}$$

$$A_{reqd} = \frac{N_{AC}}{2\tau_a} \qquad A_{reqd} = 0.44\ in^2$$

$$d_{min} = \sqrt{\frac{4}{\pi} A_{reqd}} \qquad d_{min} = 0.75\ in$$

(2) check tension and shear on brazed joint

$$\sigma_x = \frac{N_{AC}}{A} \qquad \sigma_X = \frac{N_{AC}}{\frac{\pi}{4} d^2} \qquad d_{reqd} = \sqrt{\frac{4}{\pi} \frac{N_{AC}}{\sigma_X}}$$

tension on brazed joint

$\sigma_\theta = \sigma_x \cos(\theta)^2$ set equal to σ_{ja} & solve for σ_x, then d_{reqd}

$$\sigma_x = \frac{\sigma_{ja}}{\cos(\theta)^2} \qquad \sigma_x = 17.37\ ksi$$

$$d_{reqd} = \sqrt{\frac{4}{\pi} \frac{N_{AC}}{\sigma_x}} \qquad d_{reqd} = 0.65\ in$$

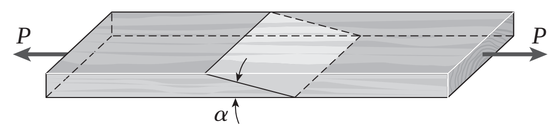
shear on brazed joint

$$\tau_\theta = -\sigma_x \sin(\theta)\cos(\theta)$$

$$\sigma_x = \left| \frac{\tau_{ja}}{-(\sin(\theta)\cos(\theta))} \right| \qquad \sigma_x = -6.31\ ksi$$

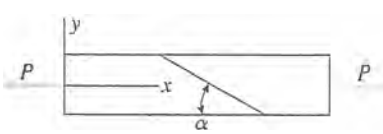
$$d_{reqd} = \sqrt{\frac{4}{\pi} \frac{N_{AC}}{\sigma_X}} \qquad d_{reqd} = 1.08\ in \quad \longleftarrow$$

Problem 2.6-14 Two boards are joined by gluing along a scarf joint, as shown in the figure. For purposes of cutting and gluing, the angle α between the plane of the joint and the faces of the boards must be between 10° and 40°. Under a tensile load P, the normal stress in the boards is 4.9 MPa.



(a) What are the normal and shear stresses acting on the glued joint if $\alpha = 20°$?
(b) If the allowable shear stress on the joint is 2.25 MPa, what is the largest permissible value of the angle α?
(c) For what angle α will the shear stress on the glued joint be numerically equal to twice the normal stress on the joint?

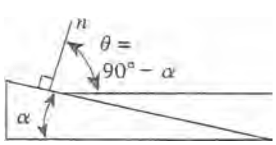
Solution 2.6-14 Two boards joined by a scarf joint



$10° \leq \alpha \leq 40°$

Due to load P: $\sigma_x = 4.9$ MPa

(a) Stresses on joint when $\alpha = 20°$



$$\theta = 90° - \alpha = 70°$$

$$\sigma_\theta = \sigma_x \cos^2\theta = (4.9\ MPa)(\cos 70°)^2$$
$$= 0.57\ MPa \quad \longleftarrow$$

$$\tau_\theta = -\sigma_x \sin\theta \cos\theta$$
$$= (-4.9\ MPa)(\sin 70°)(\cos 70°)$$
$$= -1.58\ MPa \quad \longleftarrow$$

(b) Largest angle α if $\tau_{allow} = 2.25$ MPa

$$\tau_{allow} = -\sigma_x \sin\theta \cos\theta$$

The shear stress on the joint has a negative sign. Its numerical value cannot exceed $\tau_{allow} = 2.25$ MPa. Therefore,

$-2.25 \text{ MPa} = -(4.9 \text{ MPa})(\sin\theta)(\cos\theta)$ or $\sin\theta\cos\theta = 0.4592$

From trigonometry: $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

Therefore: $\sin 2\theta = 2(0.4592) = 0.9184$

Solving: $2\theta = 66.69°$ or $113.31°$

$\theta = 33.34°$ or $56.66°$

$\alpha = 90° - \theta \qquad \therefore \alpha = 56.66°$ or $33.34°$

Since α must be between 10° and 40°, we select

$\alpha = 33.3° \quad \longleftarrow$

NOTE: If α is between 10° and 33.3°,

$|\tau_\theta| < 2.25$ MPa.

If α is between 33.3° and 40°,

$|\tau_\theta| > 2.25$ MPa.

(c) WHAT IS α if $\tau_\theta = 2\sigma_\theta$?

Numerical values only:

$|\tau_\theta| = \sigma_x \sin\theta\cos\theta \qquad |\sigma_\theta| = \sigma_x\cos^2\theta$

$$\left|\frac{\tau_0}{\sigma_0}\right| = 2$$

$\sigma_x \sin\theta\cos\theta = 2\sigma_x\cos^2\theta$

$\sin\theta = 2\cos\theta$ or $\tan\theta = 2$

$\theta = 63.43° \qquad \alpha = 90° - \theta$

$\alpha = 26.6° \quad \longleftarrow$

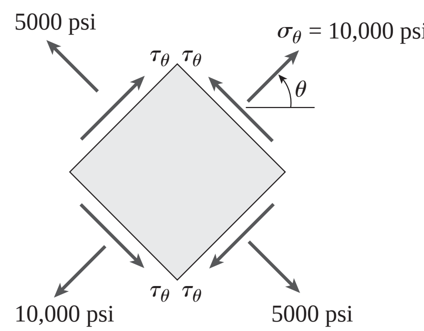
NOTE: For $\alpha = 26.6°$ and $\theta = 63.4°$, we find $\sigma_\theta = 0.98$ MPa and $\tau_\theta = -1.96$ MPa.

Thus, $\left|\frac{\tau_0}{\sigma_0}\right| = 2$ as required.

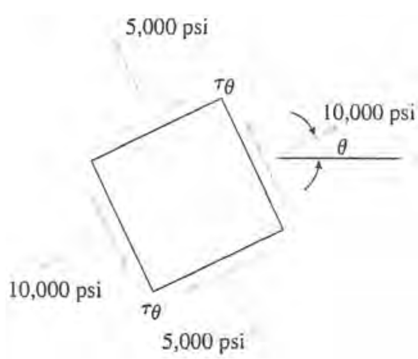
Problem 2.6-15 Acting on the sides of a stress element cut from a bar in uniaxial stress are tensile stresses of 10,000 psi and 5,000 psi, as shown in the figure.

(a) Determine the angle θ and the shear stress τ_θ and show all stresses on a sketch of the element.

(b) Determine the maximum normal stress σ_{max} and the maximum shear stress τ_{max} in the material.



Solution 2.6-15 Bar in uniaxial stress



(a) Angle θ and shear stress τ_θ

$$\sigma_\theta = \sigma_x \cos^2\theta$$

$$\sigma_\theta = 10{,}000 \text{ psi}$$

$$\sigma_x = \frac{\sigma_0}{\cos^2\theta} = \frac{10{,}000 \text{ psi}}{\cos^2\theta} \qquad (1)$$

Plane at angle $\theta + 90°$

$$\sigma_\theta + 90° = \sigma_x[\cos(\theta + 90°)]^2 = \sigma_x[-\sin\,\theta]^2$$
$$= \sigma_x \sin^2\theta$$

$$\sigma_\theta + 90° = 5{,}000 \text{ psi}$$

$$\sigma_x = \frac{\sigma_{0+90°}}{\sin^2\theta} = \frac{5{,}000 \text{ psi}}{\sin^2\theta} \qquad (2)$$

Equate (1) and (2):

$$\frac{10{,}000 \text{ psi}}{\cos^2\theta} = \frac{5{,}000 \text{ psi}}{\sin^2\theta}$$

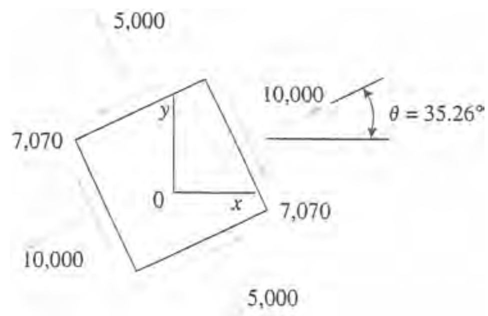
$$\tan^2\theta = \frac{1}{2} \quad \tan\theta = \frac{1}{\sqrt{2}} \quad \theta = 35.26° \quad \longleftarrow$$

From Eq. (1) or (2):

$$\sigma_x = 15{,}000 \text{ psi}$$

$$\tau_\theta = -\sigma_x \sin\,\theta \cos\,\theta$$
$$= (-15{,}000 \text{ psi})(\sin 35.26°)(\cos 35.26°)$$
$$= -7{,}070 \text{ psi} \quad \longleftarrow$$

Minus sign means that τ_θ acts clockwise on the plane for which $\theta = 35.26°$.



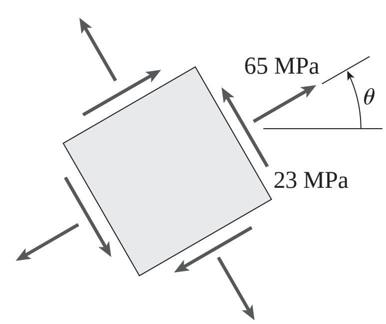
NOTE: All stresses have units of psi.

(b) Maximum normal and shear stresses

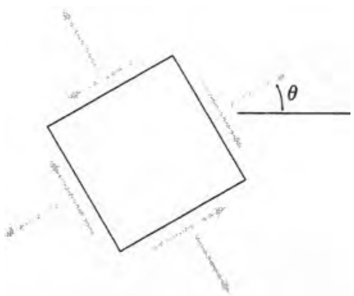
$$\sigma_{\max} = \sigma_x = 15{,}000 \text{ psi} \quad \longleftarrow$$

$$\tau_{\max} = \frac{\sigma_x}{2} = 7{,}500 \text{ psi} \quad \longleftarrow$$

Problem 2.6-16 A prismatic bar is subjected to an axial force that produces a tensile stress $\sigma_\theta = 65$ MPa and a shear stress $\tau_\theta = 23$ MPa on a certain inclined plane (see figure). Determine the stresses acting on all faces of a stress element oriented at $\theta = 30°$ and show the stresses on a sketch of the element.



Solution 2.6-16



find θ & σ_x for stress state shown above

$$\sigma_\theta = \sigma_x \cos(\theta)^2 \qquad \cos(\theta) = \sqrt{\frac{\sigma_\theta}{\sigma_x}}$$

$$\text{so} \qquad \sin(\theta) = \sqrt{1 - \frac{\sigma_\theta}{\sigma_x}}$$

$$\tau_\theta = -\sigma_x \sin(\theta)\cos(\theta)$$

$$\frac{\tau_\theta}{\sigma_x} = -\sqrt{1 - \frac{\sigma_\theta}{\sigma_x}}\sqrt{\frac{\sigma_\theta}{\sigma_x}}$$

$$\left(\frac{\tau_\theta}{\sigma_x}\right)^2 = \frac{\sigma_\theta}{\sigma_x} - \left(\frac{\sigma_\theta}{\sigma_x}\right)$$

$$\left(\frac{23}{\sigma_x}\right)^2 = \frac{65}{\sigma_x} - \left(\frac{65}{\sigma_x}\right)^2$$

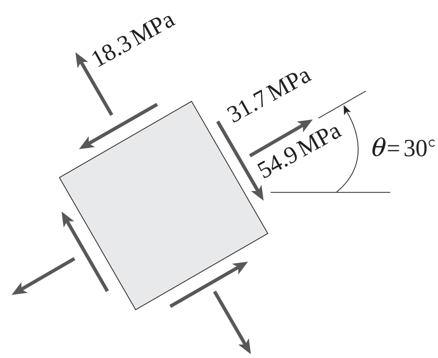
$$\left(\frac{65}{\sigma_x}\right)^2 - \left(\frac{65}{\sigma_x}\right) + \left(\frac{23}{\sigma_x}\right)^2 = 0$$

$$\frac{-(-4754 + 65\sigma_x)}{\sigma_x^2} = 0$$

$$\sigma_x = \frac{4754}{65}$$

$_x = 73.1$ MPa $\qquad \sigma_\theta = 65$ MPa

$$\theta = \operatorname{acos}\left(\sqrt{\frac{\sigma_\theta}{\sigma_x}}\right) \qquad \theta = 19.5°$$



now find σ_θ & τ_θ for $\theta = 30°$

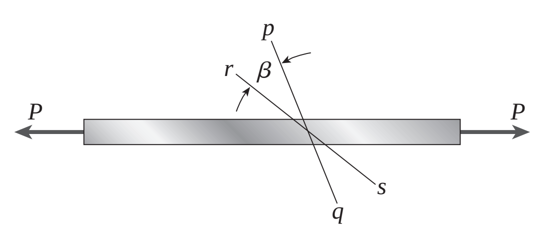
$\sigma_{\theta 1} = \sigma_x \cos(\theta)^2 \qquad \sigma_{\theta 1} = 54.9$ MPa $\leftarrow$

$\tau_\theta = -\sigma_x \sin(\theta) \cdot \cos(\theta) \qquad \tau_\theta = -31.7$ MPa $\leftarrow$

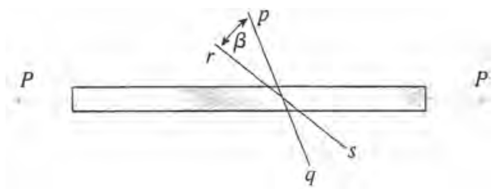
$\sigma_{\theta 2} = \sigma_x \cos\left(\theta + \frac{\pi}{2}\right)^2 \qquad \sigma_{\theta 2} = 18.3$ MPa $\leftarrow$

Problem 2.6-17 The normal stress on plane *pq* of a prismatic bar in tension (see figure) is found to be 7500 psi. On plane *rs*, which makes an angle $\beta = 30°$ with plane *pq*, the stress is found to be 2500 psi.

Determine the maximum normal stress σ_{max} and maximum shear stress τ_{max} in the bar.



Solution 2.6-17 Bar in tension



Eq. (2-29a):

$\sigma_\theta = \sigma_x \cos^2\theta$

$\beta = 30°$

PLANE *pq*: $\sigma_1 = \sigma_x \cos^2\theta_1$ $\quad\quad \sigma_1 = 7500 \text{ psi}$

PLANE *rs*: $\sigma_2 = \sigma_x \cos^2(\theta_1 + \beta)$ $\quad\quad \sigma_2 = 2500 \text{ psi}$

Equate σ_x from σ_1 and σ_2:

$$\sigma_x = \frac{\sigma_1}{\cos^2\theta_1} = \frac{\sigma_2}{\cos^2(\theta_1 + \beta)} \qquad \text{(Eq. 1)}$$

or

$$\frac{\cos^2\theta_1}{\cos^2(\theta_1 + \beta)} = \frac{\sigma_1}{\sigma_2} \quad \frac{\cos\theta_1}{\cos(\theta_1 + \beta)} = \sqrt{\frac{\sigma_1}{\sigma_2}} \qquad \text{(Eq. 2)}$$

SUBSTITUTE NUMERICAL VALUES INTO EQ. (2):

$$\frac{\cos\theta_1}{\cos(\theta_1 + 30°)} = \sqrt{\frac{7500 \text{ psi}}{2500 \text{ psi}}} = \sqrt{3} = 1.7321$$

Solve by iteration or a computer program:

$\theta_1 = 30°$

MAXIMUM NORMAL STRESS (FROM EQ. 1)

$$\sigma_{max} = \sigma_x = \frac{\sigma_1}{\cos^2\theta_1} = \frac{7500 \text{ psi}}{\cos^2 30°}$$

$$= 10{,}000 \text{ psi} \quad \longleftarrow$$

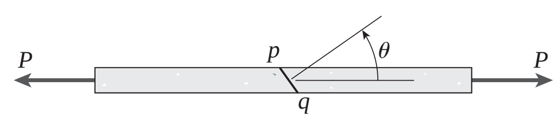
MAXIMUM SHEAR STRESS

$$\tau_{max} = \frac{\sigma_x}{2} = 5{,}000 \text{ psi} \quad \longleftarrow$$

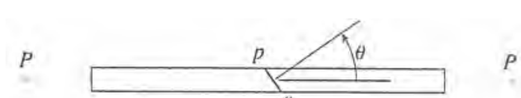
Problem 2.6-18 A tension member is to be constructed of two pieces of plastic glued along plane *pq* (see figure). For purposes of cutting and gluing, the angle θ must be between 25° and 45°. The allowable stresses on the glued joint in tension and shear are 5.0 MPa and 3.0 MPa, respectively.

(a) Determine the angle θ so that the bar will carry the largest load *P*. (Assume that the strength of the glued joint controls the design.)

(b) Determine the maximum allowable load P_{max} if the cross-sectional area of the bar is 225 mm^2.



Solution 2.6-18 Bar in tension with glued joint



$25° < \theta < 45°$

$A = 225 \text{ mm}^2$

On glued joint: $\sigma_{allow} = 5.0 \text{ MPa}$

$\tau_{allow} = 3.0 \text{ MPa}$

ALLOWABLE STRESS σ_x IN TENSION

$$\sigma_\theta = \sigma_x \cos^2\theta \qquad \sigma_x = \frac{\sigma_\theta}{\cos^2\theta} = \frac{5.0 \text{ MPa}}{\cos^2\theta} \qquad (1)$$

$\tau_\theta = -\sigma_x \sin\theta \cos\theta$

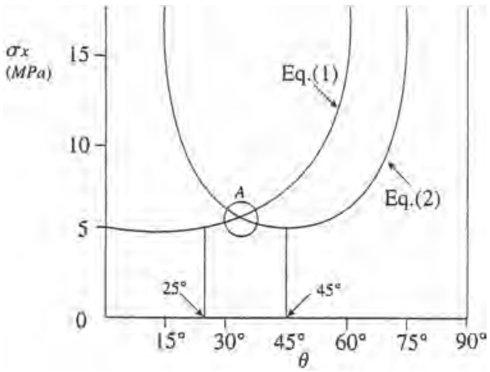
Since the direction of τ_θ is immaterial, we can write:

$|\tau_\theta| = \sigma_x \sin\theta \cos\theta$

or

$$\sigma_x = \frac{|\tau_\theta|}{\sin\theta\cos\theta} = \frac{3.0\text{ MPa}}{\sin\theta\cos\theta} \qquad (2)$$

GRAPH OF EQS. (1) AND (2)



(a) DETERMINE ANGLE Θ FOR LARGEST LOAD

Point A gives the largest value of σ_x and hence the largest load. To determine the angle θ corresponding to point A, we equate Eqs. (1) and (2).

$$\frac{5.0\text{ MPa}}{\cos^2\theta} = \frac{3.0\text{ MPa}}{\sin\theta\cos\theta}$$

$$\tan\theta = \frac{3.0}{5.0} \qquad \theta = 30.96° \quad \longleftarrow$$

(b) DETERMINE THE MAXIMUM LOAD

From Eq. (1) or Eq. (2):

$$\sigma_x = \frac{5.0\text{ MPa}}{\cos^2\theta} = \frac{3.0\text{ MPa}}{\sin\theta\cos\theta} = 6.80\text{ MPa}$$

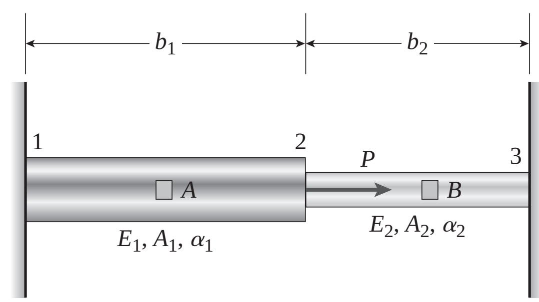
$$P_{max} = \sigma_x A = (6.80\text{ MPa})(225\text{ mm}^2)$$

$$= 1.53\text{ kN} \quad \longleftarrow$$

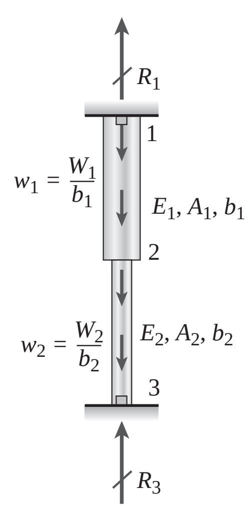
Problem 2.6-19 A nonprismatic bar 1–2–3 of rectangular cross section (cross sectional area A) and two materials is held snugly (but without any initial stress) between rigid supports (see figure). The allowable stresses in compression and in shear are specified as σ_a and τ_a, respectively. Use the following numerical data: (*Data*: $b_1 = 4b_2/3 = b$; $A_1 = 2A_2 = A$; $E_1 = 3E_2/4 = E$; $\alpha_1 = 5\alpha_2/4 = \alpha$; $\sigma_{a1} = 4\sigma_{a2}/3 = \sigma_a$, $\tau_{a1} = 2\sigma_{a1}/5$, $\tau_{a2} = 3\sigma_{a2}/5$; let $\sigma_a = 11$ ksi, $P = 12$ kips, $A = 6$ in.^{2}, $b = 8$ in. $E = 30{,}000$ ksi, $\alpha = 6.5 \times 10^{-6}$/°F; $\gamma_1 = 5\gamma_2/3 = \gamma = 490$ lb/ft^3)

(a) If load P is applied at joint 2 as shown, find an expression for the maximum permissible temperature rise ΔT_{max} so that the allowable stresses are not to be exceeded at either location A or B.

(b) If load P is removed and the bar is now rotated to a vertical position where it hangs under its own weight (load intensity $= w_1$ in segment 1–2 and w_2 in segment 2–3), find an expression for the maximum permissible temperature rise ΔT_{max} so that the allowable stresses are not exceeded at either location 1 or 3. Locations 1 and 3 are each a short distance from the supports at 1 and 3 respectively.



(a)



(b)

Solution 2.6-19

(a) STAT-INDET NONPRISMATIC BAR WITH LOAD P AT jt 2

apply load P and temp. change ΔT - use R_3 as redundant & do superposition analysis

$$\delta_{3a} = Pf_{12} + (\alpha_1 b_1 + \alpha_2 b_2)\Delta T$$

$$\delta_{3b} = R_3(f_{12} + f_{23}) \quad f_{12} = \frac{b_1}{E_1 A_1} \quad f_{23} = \frac{b_2}{E_2 A_2}$$

compatibility: $\delta_{3a} + \delta_{3b} = 0$

$$R_3 = \frac{-Pf_{12} - (\alpha_1 b_1 + \alpha_2 b_2)\Delta T}{f_{12} + f_{23}}$$

< compression at Location B due to both P and temp. increase

statics: $R_1 = -P - R_3$

$$R_1 = \frac{-Pf_{23} + (\alpha_1 b_1 + \alpha_2 b_2)\Delta T}{f_{12} + f_{23}}$$

< compression due to temp. increase, tension due to P, at Location A

numerical data & allowable stresses (normal & shear)

$$\sigma_{a1} = \sigma_a \qquad \sigma_{a2} = \frac{3}{4}\sigma_a$$

$$\tau_{a1} = \frac{2}{5}\sigma_a \qquad \tau_{a2} = \frac{9}{20}\sigma_a$$

Numerical data

$\sigma_a = 11$ ksi $\quad A = 6\ \text{in}^2 \quad P = 12$ kips
$E = 30000$ ksi $\quad \alpha = 6.5 \times (10^{-6})/°\text{F}$ steel

(1) check normal and shear stresses **at element A location** & solve for ΔT_{max} using σ_{a1} & τ_{a1}

$$\sigma_{xA} = \frac{R_1}{A_1}$$

$$\Delta T_{max} = \frac{[\sigma_{a1} A_1(f_{12} + f_{23})] + Pf_{23}}{(\alpha_1 b_1 + \alpha_2 b_2)}$$

$$f_{12} = \frac{b_1}{E_1 A_1} \qquad f_{23} = \frac{b_2}{E_2 A_2}$$

$$\Delta T_{max} = \frac{\left[\sigma_a A\left(\frac{b}{EA} + \frac{\frac{3}{4}b}{\frac{4}{3}E\frac{A}{2}}\right)\right] + \left(P\frac{\frac{3}{4}b}{\frac{4}{3}E\frac{A}{2}}\right)}{\left(\alpha b + \frac{4}{5}\alpha\frac{3}{4}b\right)}$$

$$\Delta T_{max} = \frac{85\sigma_a A + 45P}{64EA\alpha} \qquad \Delta T_{max} = 82.1°\text{F}$$

< compression due to temp. rise but tension due to P

$$\tau_{max\,A} = \frac{\sigma_{xA}}{2}$$

$$\Delta T_{max} = \frac{2\left(\frac{2}{5}\sigma_a\right)A\left(\frac{b}{EA} + \frac{\frac{3}{4}b}{\frac{4}{3}E\frac{A}{2}}\right) + P\frac{\frac{3}{4}b}{\frac{4}{3}E\frac{A}{2}}}{\left(\alpha b + \frac{4}{5}\alpha\frac{3}{4}b\right)}$$

$$\Delta T_{max} = \frac{68\sigma_a A + 45P}{64EA\alpha} \qquad \Delta T_{max} = 67.1°\text{F}$$

< shear controls for Location A where temp. rise causes compressive stress but load P causes tensile stress ←

(2) check normal and shear stresses **at element B location** & solve for ΔT_{max} using σ_{a2} & τ_{a2}

$$\sigma_{xB} = \frac{R_3}{A_2}$$

$$\Delta T_{max} = \frac{[\sigma_{a2} A_2(f_{12} + f_{23})] + Pf_{12}}{(\alpha_1 b_1 + \alpha_2 b_2)}$$

< compression due to both temp. rise & load P

$$\Delta T_{max} = \frac{\left[\frac{3}{4}\sigma_a\frac{A}{2}\left(\frac{b}{EA} + \frac{\frac{3}{4}b}{\frac{4}{3}E\frac{A}{2}}\right)\right] - P\frac{b}{EA}}{\left(\alpha b + \frac{4}{5}\alpha\frac{3}{4}b\right)}$$

$$\Delta T_{max} = \frac{255\sigma_a A - 320P}{512EA\alpha}$$

$$\Delta T_{max} = 21.7°\text{F} \quad \leftarrow$$

normal stress controls for Location B where temp. rise & load P both cause compressive stress; as a result, permissible temp. rise is reduced at B compared to Location A where temp. rise effect is offset by load P effect

$$\tau_{max\ B} = \frac{\sigma_{xB}}{2}$$

$$\Delta\ T_{max} = \frac{[2\tau_a A_2(f_{12} + f_{23})] - Pf_{12}}{(\alpha_1 b_1 + \alpha_2 b_2)}$$ Location A where temp. riseeffect is offset by load P effect

$$\Delta T_{max} = \frac{\left[2\left(\frac{9}{20}\sigma_a\right)\frac{A}{2}\left(\frac{b}{EA} + \frac{\frac{3}{4}b}{\frac{4}{3}E\frac{A}{2}}\right)\right] - P\frac{b}{EA}}{\left(\alpha b + \frac{4}{5}\alpha\frac{3}{4}b\right)}$$

$$\Delta T_{max} = \frac{153\sigma_a A - 160P}{256EA\alpha} \qquad \Delta T_{max} = 27.3°F$$

(b) STAT-INDET NONPRISMATIC BAR HANGING UNDER ITS OWN WEIGHT (GRAVITY)

apply gravity and temp. change ΔT - use R_3 as redundant & do superposition analysis

$$\delta_{3a} = \frac{-W_1}{2}f_{12} - W_2 f_{12} - \frac{W_2}{2}f_{23} - (\alpha_1 b_1 + \alpha_2 b_2)\Delta T$$

$$\delta_{3b} = R_3(f_{12} + f_{23})$$

$$f_{12} = \frac{b_1}{E_1 A_1} \qquad f_{23} = \frac{b_2}{E_2 A_2}$$

compatibility: $\delta_{3a} + \delta_{3b} = 0$ $\qquad R_3 = \dfrac{\left(\frac{W_1}{2}f_{12} + W_2 f_{12} + \frac{W_2}{2}f_{23}\right) + (\alpha_1 b_1 + \alpha_2 b_2)\Delta T}{f_{12} + f_{23}}$

^ compression at Location 3 due to both P and temp. increase

statics: $R_1 = W_1 + W_2 - R_3$ $\qquad R_1 = W_1 + W_2 - \dfrac{\left(\frac{W_1}{2}f_{12}W_2 f_{12} + \frac{W_2}{2}f_{23}\right) + (\alpha_1 b_1 + \alpha_2 b_2)\Delta T}{f_{12} + f_{23}}$

^ compression at Location 1 due to temp. increase, tension due to W_1 & W_2

$$\sigma_{x1} = \frac{R_1}{A_1} \qquad \tau_{max1} = \frac{\sigma_{x1}}{2} \qquad \sigma_{x3} = \frac{R_3}{A_2} \qquad \tau_{max3} = \frac{\sigma_{x3}}{A_2}$$

numerical data & allowables stresses (normal & shear)

$$\sigma_{a1} = \sigma_a \qquad \sigma_{a2} = \frac{3}{4}\sigma_a \qquad \tau_{a1} = \frac{2}{5}\sigma_a \qquad \tau_{a2} = \frac{9}{20}\sigma_a$$

$$\sigma_a = 11\text{ ksi} \qquad A = 6\text{ in}^2 \qquad E = 30000\text{ ksi} \qquad \alpha = 6.5 \times (10^{-6})/°F$$

$$b = 8\text{ in.} \qquad \gamma = \frac{0.490}{12^3}\text{ k/in}^3$$

(1) check normal and shear stresses **at element 1 location** & solve for ΔT_{max} using σ_{a1} & τ_{a1} normal stress

$$\sigma_{a1}A_1 = W_1 + W_2 - \frac{\left(\frac{W_1}{2}f_{12} + W_2 f_{12} + \frac{W_2}{2}f_{23}\right) + (\alpha_1 b_1 + \alpha_2 b_2)\Delta T}{f_{12} + f_{23}}$$

$$\Delta T_{max} = \frac{\gamma_1 A_1 b_1 f_{12} + 2(\gamma_1 A_1 b_1)f_{23} + \gamma_2 A_2 b_2 f_{23} - 2\sigma_{a1}A_1(f_{12} + f_{23})}{2(\alpha_1 b_1 + \alpha_2 b_2)}$$

$$\Delta T_{max} = \frac{\left[-\gamma Ab\frac{b}{EA} + 2(\gamma Ab)\frac{\frac{3}{4}b}{\frac{4}{3}E\frac{A}{2}} + \frac{3}{5}\gamma\frac{A}{2}\left(\frac{3}{4}b\right)\frac{\frac{3}{4}b}{\frac{4}{3}E\frac{A}{2}}\right] + 2\sigma_{al}A\left(\frac{b}{EA} + \frac{\frac{3}{4}b}{\frac{4}{3}E\frac{A}{2}}\right)}{2\left(\alpha b + \frac{4}{5}\alpha\frac{3}{4}b\right)}$$

^ sign difference because gravity offsets effect of temp. rise

$$\Delta T_{max} = \frac{-1121\gamma b + 1360\sigma_a}{1024E\alpha} \qquad \Delta T_{max} = 74.9°F$$

Next, shear stress

$$\Delta T_{max} = \frac{-1121\gamma b + 1360\left(2\frac{2}{5}\sigma_a\right)}{1024E\alpha} \qquad \Delta T_{max} = 59.9°F$$

(2) check normal and shear stresses **at element 3 location** & solve for ΔT_{max} using σ_{a2} & τ_{a2} normal stress

$$\Delta T_{max} = \frac{\sigma_{a2}A_2(f_{12} + f_{23}) + \left(\frac{W_1}{2}f_{12} + W_2 f_{12} + \frac{W_2}{2}f_{23}\right)}{\alpha_1 b_1 + \alpha_2 b_2}$$

<same sign because temp. rise & gravity both produce compressive stress at element 3

$$\Delta T_{max} = \frac{\left(\frac{3}{4}\sigma_a\right)\frac{A}{2}\left[\frac{b}{EA} + \frac{\frac{3}{4}b}{\frac{4}{3}E\frac{A}{2}}\right] + \left[\frac{\gamma Ab}{2}\frac{b}{EA} + \frac{3}{5}\gamma\frac{A}{2}\left(\frac{3}{4}b\right)\frac{b}{EA} + \frac{\frac{3}{5}\gamma\frac{A}{2}\left(\frac{3}{4}b\right)}{2} + \frac{\frac{3}{4}b}{\frac{4}{3}E\frac{A}{2}}\right]}{\alpha b + \frac{4}{5}\alpha\frac{3}{4}b}$$

$$\Delta T_{max} = \frac{510\sigma_a + 545\gamma b}{1024E\alpha} \qquad \Delta T_{max} = 28.1°F$$

shear stress

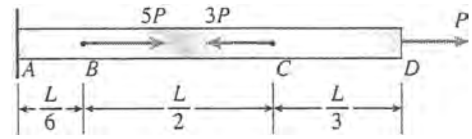
$$\Delta T_{max} = \frac{510\left(2\frac{9}{20}\sigma_a\right) + 545\gamma b}{1024E\alpha} \qquad \Delta T_{max} = 25.3°F \quad \leftarrow$$

shear at element 3 location controls

Strain Energy

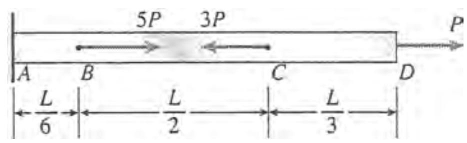
When solving the problems for Section 2.7, assume that the material behaves linearly elastically.

Problem 2.7-1 A prismatic bar AD of length L, cross-sectional area A, and modulus of elasticity E is subjected to loads $5P$, $3P$, and P acting at points B, C, and D, respectively (see figure). Segments AB, BC, and CD have lengths $L/6$, $L/2$, and $L/3$, respectively.



(a) Obtain a formula for the strain energy U of the bar.
(b) Calculate the strain energy if $P = 6$ k, $L = 52$ in., $A = 2.76$ in.^{2}, and the material is aluminum with $E = 10.4 \times 10^6$ psi.

Solution 2.7-1 Bar with three loads



$P = 6$ k

$L = 52$ in.

$E = 10.4 \times 10^6$ psi

$A = 2.76$ in.^{2}

Internal axial forces

$$N_{AB} = 3P \qquad N_{BC} = -2P \qquad N_{CD} = P$$

Lengths

$$L_{AB} = \frac{L}{6} \qquad L_{BC} = \frac{L}{2} \qquad L_{CD} = \frac{L}{3}$$

(a) Strain energy of the bar (Eq. 2-40)

$$U = \Sigma \frac{N_i^2 L_i}{2E_i A_i}$$

$$= \frac{1}{2EA}\left[(3P)^2\left(\frac{L}{6}\right) + (-2P)^2\left(\frac{L}{2}\right) + (P)^2\left(\frac{L}{3}\right)\right]$$

$$= \frac{P^2 L}{2EA}\left(\frac{23}{6}\right) = \frac{23P^2 L}{12EA} \quad \leftarrow$$

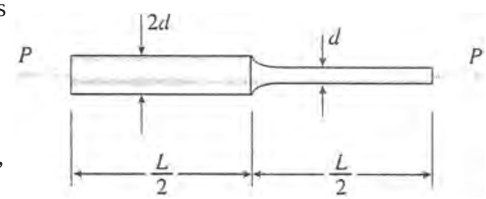
(b) Substitute numerical values:

$$U = \frac{23(6\text{ k})^2(52\text{ in.})}{12(10.4 \times 10^6\text{ psi})(2.76\text{ in.}^2)}$$

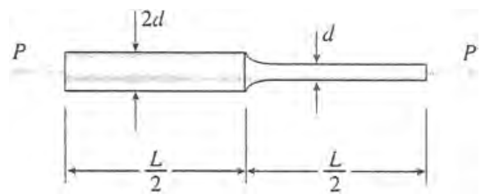
$$= 125\text{ in.-lb} \quad \leftarrow$$

Problem 2.7-2 A bar of circular cross section having two different diameters d and $2d$ is shown in the figure. The length of each segment of the bar is $L/2$ and the modulus of elasticity of the material is E.

(a) Obtain a formula for the strain energy U of the bar due to the load P.
(b) Calculate the strain energy if the load $P = 27$ kN, the length $L = 600$ mm, the diameter $d = 40$ mm, and the material is brass with $E = 105$ GPa.



Solution 2.7-2 **Bar with two segments**



(a) Strain energy of the bar

Add the strain energies of the two segments of the bar (see Eq. 2-40).

$$U = \sum_{i=1}^{2} \frac{N_i^2 L_i}{2 E_i A_i} = \frac{P^2(L/2)}{2E}\left[\frac{1}{\frac{\pi}{4}(2d)^2} + \frac{1}{\frac{\pi}{4}(d^2)}\right]$$

$$= \frac{P^2 L}{\pi E}\left(\frac{1}{4d^2} + \frac{1}{d^2}\right) = \frac{5P^2 L}{4\pi E d^2} \quad \leftarrow$$

(b) Substitute numerical values:

$P = 27$ kN $\quad L = 600$ mm

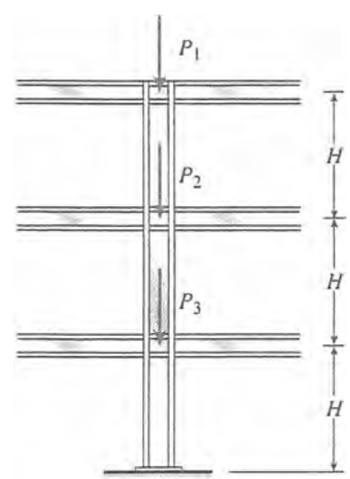
$d = 40$ mm $\quad E = 105$ GPa

$$U = \frac{5(27\text{ kN}^2)(600\text{ mm})}{4\pi(105\text{ GPa})(40\text{ mm})^2}$$

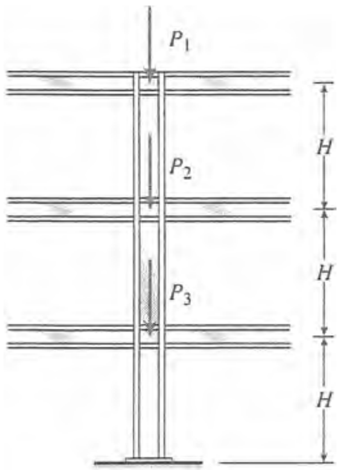
$$= 1.036\text{ N}\cdot\text{m} = 1.036\text{ J} \quad \leftarrow$$

Problem 2.7-3 A three-story steel column in a building supports roof and floor loads as shown in the figure. The story height H is 10.5 ft, the cross-sectional area A of the column is 15.5 in.^{2}, and the modulus of elasticity E of the steel is 30×10^6 psi.

Calculate the strain energy U of the column assuming $P_1 = 40$ k and $P_2 = P_3 = 60$ k.



Solution 2.7-3 Three-story column



$H = 10.5$ ft $\quad E = 30 \times 10^6$ psi

$A = 15.5$ in.^{2} $\quad P_1 = 40$ k

$P_2 = P_3 = 60$ k

To find the strain energy of the column, add the strain energies of the three segments (see Eq. 2-40).

Upper segment: $N_1 = -P_1$

Middle segment: $N_2 = -(P_1 + P_2)$

Lower segment: $N_3 = -(P_1 + P_2 + P_3)$

STRAIN ENERGY

$$U = \Sigma \frac{N_i^2 L_i}{2E_i A_i}$$

$$= \frac{H}{2EA}[P_1^2 + (P_1 + P_2)^2 + (P_1 + P_2 + P_3)^2]$$

$$= \frac{H}{2EA}[Q]$$

$$[Q] = (40\text{ k})^2 + (100\text{ k})^2 + (160\text{ k})^2 = 37{,}200\text{ k}^2$$

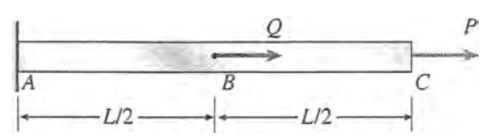
$$2EA = 2(30 \times 10^6\text{ psi})(15.5\text{ in.}^2) = 930 \times 10^6\text{ lb}$$

$$U = \frac{(10.5\text{ ft})(12\text{ in./ft})}{930 \times 10^6\text{ lb}}[37{,}200\text{ k}^2]$$

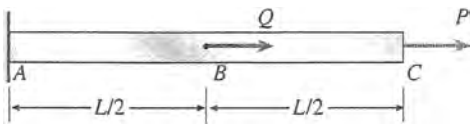
$$= 5040\text{ in.-lb} \quad \leftarrow$$

Problem 2.7-4 The bar ABC shown in the figure is loaded by a force P acting at end C and by a force Q acting at the midpoint B. The bar has constant axial rigidity EA.

(a) Determine the strain energy U_1 of the bar when the force P acts alone ($Q = 0$).
(b) Determine the strain energy U_2 when the force Q acts alone ($P = 0$).
(c) Determine the strain energy U_3 when the forces P and Q act simultaneously upon the bar.



Solution 2.7-4 Bar with two loads



(a) Force P acts alone ($Q = 0$)

$$U_1 = \frac{P^2L}{2EA} \quad \leftarrow$$

(b) Force Q acts alone ($P = 0$)

$$U_2 = \frac{Q^2(L/2)}{2EA} = \frac{Q^2L}{4EA} \quad \leftarrow$$

(c) Forces P and Q act simultaneously

$$\text{Segment } BC\text{: } U_{BC} = \frac{P^2(L/2)}{2EA} = \frac{P^2L}{4EA}$$

$$\text{Segment } AB\text{: } U_{AB} = \frac{(P+Q)^2(L/2)}{2EA}$$

$$= \frac{P^2L}{4EA} + \frac{PQL}{2EA} + \frac{Q^2L}{4EA}$$

$$U_3 = U_{BC} + U_{AB} = \frac{P^2L}{2EA} + \frac{PQL}{2EA} + \frac{Q^2L}{4EA} \quad \leftarrow$$

(Note that U_3 is *not* equal to $U_1 + U_2$. In this case, $U_3 > U_1 + U_2$. However, if Q is reversed in direction, $U_3 < U_1 + U_2$. Thus, U_3 may be larger or smaller than $U_1 + U_2$.)

Problem 2.7-5 Determine the strain energy per unit volume (units of psi) and the strain energy per unit weight (units of in.) that can be stored in each of the materials listed in the accompanying table, assuming that the material is stressed to the proportional limit.

DATA FOR PROBLEM 2.7-5

Material	Weight density (lb/in.^{3})	Modulus of elasticity (ksi)	Proportional limit (psi)
Mild steel	0.284	30,000	36,000
Tool steel	0.284	30,000	75,000
Aluminum	0.0984	10,500	60,000
Rubber (soft)	0.0405	0.300	300

Solution 2.7-5 Strain-energy density

DATA:

Material	Weight density (lb/in.^{3})	Modulus of elasticity (ksi)	Proportional limit (psi)
Mild steel	0.284	30,000	36,000
Tool steel	0.284	30,000	75,000
Aluminum	0.0984	10,500	60,000
Rubber (soft)	0.0405	0.300	300

Strain energy per unit volume

$$U = \frac{P^2L}{2EA} \qquad \text{Volume } V = AL$$

$$\text{Stress } \sigma = \frac{P}{A}$$

$$u = \frac{U}{V} = \frac{\sigma_{PL}^2}{2E}$$

At the proportional limit:

$u = u_R$ = modulus of resistance

$$u_R = \frac{\sigma_{PL}^2}{2E} \qquad \text{(Eq. 1)}$$

STRAIN ENERGY PER UNIT WEIGHT

$U = \dfrac{P^2L}{2EA}$ Weight $W = \gamma AL$

γ = weight density

$$u_W = \frac{U}{W} = \frac{\sigma^2}{2\gamma E}$$

At the proportional limit:

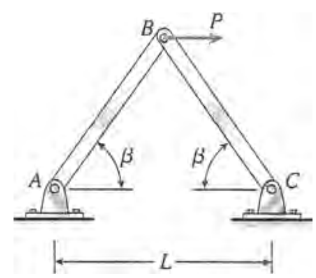
$$u_W = \frac{\sigma_{PL}^2}{2\gamma E} \qquad \text{(Eq. 2)}$$

RESULTS

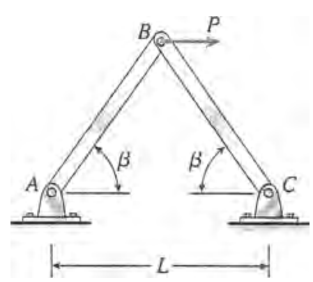
	u_R (psi)	u_w (in.)
Mild steel	22	76
Tool steel	94	330
Aluminum	171	1740
Rubber (soft)	150	3700

Problem 2.7-6 The truss *ABC* shown in the figure is subjected to a horizontal load *P* at joint *B*. The two bars are identical with cross-sectional area *A* and modulus of elasticity *E*.

(a) Determine the strain energy *U* of the truss if the angle $\beta = 60°$.
(b) Determine the horizontal displacement δ_B of joint *B* by equating the strain energy of the truss to the work done by the load.



Solution 2.7-6 Truss subjected to a load *P*



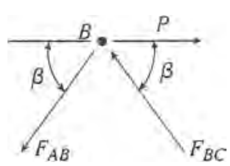
$\beta = 60°$

$L_{AB} = L_{BC} = L$

$\sin\beta = \sqrt{3}/2$

$\cos\beta = 1/2$

FREE-BODY DIAGRAM OF JOINT *B*



$\Sigma F_{vert} = 0 \quad \uparrow_+ \quad \downarrow^-$

$-F_{AB}\sin\beta + F_{BC}\sin\beta = 0$

$$F_{AB} = F_{BC} \qquad \text{(Eq. 1)}$$

$\Sigma F_{horiz} = 0 \rightarrow \leftarrow$

$-F_{AB}\cos\beta - F_{BC}\cos\beta + P = 0$

$$F_{AB} = F_{BC} = \frac{P}{2\cos\beta} = \frac{P}{2(1/2)} = P \qquad \text{(Eq. 2)}$$

Axial forces: $N_{AB} = P$ (tension)

$N_{BC} = -P$ (compression)

(a) STRAIN ENERGY OF TRUSS (EQ. 2-40)

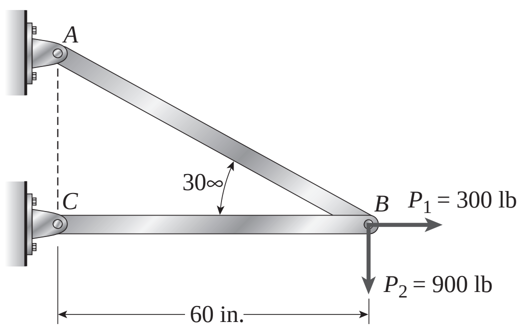
$$U = \Sigma \frac{N_i^2 L_i}{2E_i A_i} = \frac{(N_{AB})^2 L}{2EA} + \frac{(N_{BC})^2 L}{2EA} = \frac{P^2 L}{EA} \quad \longleftarrow$$

(b) HORIZONTAL DISPLACEMENT OF JOINT B (EQ. 2-42)

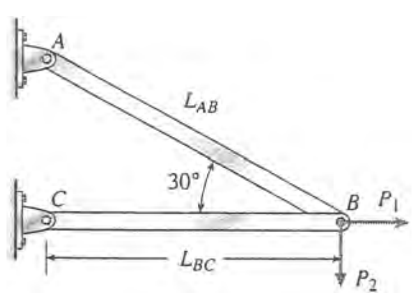
$$\delta_B = \frac{2U}{P} = \frac{2}{P}\left(\frac{P^2 L}{EA}\right) = \frac{2PL}{EA} \quad \longleftarrow$$

Problem 2.7-7 The truss ABC shown in the figure supports a horizontal load $P_1 = 300$ lb and a vertical load $P_2 = 900$ lb. Both bars have cross-sectional area $A = 2.4$ in.^{2} and are made of steel with $E = 30 \times 10^6$ psi.

(a) Determine the strain energy U_1 of the truss when the load P_1 acts alone ($P_2 = 0$).
(b) Determine the strain energy U_2 when the load P_2 acts alone ($P_1 = 0$).
(c) Determine the strain energy U_3 when both loads act simultaneously.



Solution 2.7-7 Truss with two loads



$P_1 = 300$ lb

$P_2 = 900$ lb

$A = 2.4$ in.^{2}

$E = 30 \times 10^6$ psi

$L_{BC} = 60$ in.

$\beta = 30°$

$$\sin\beta = \sin 30° = \frac{1}{2}$$

$$\cos\beta = \cos 30° = \frac{\sqrt{3}}{2}$$

$$L_{AB} = \frac{L_{BC}}{\cos 30°} = \frac{120}{\sqrt{3}} \text{ in.} = 69.282 \text{ in.}$$

$$2EA = 2(30 \times 10^6 \text{ psi})(2.4 \text{ in.}^2) = 144 \times 10^6 \text{ lb}$$

FORCES F_{AB} AND F_{BC} IN THE BARS

From equilibrium of joint B:

$$F_{AB} = 2P_2 = 1800 \text{ lb}$$

$$F_{BC} = P_1 - P_2\sqrt{3} = 300 \text{ lb} - 1558.8 \text{ lb}$$

Force	P_1 alone	P_2 alone	P_1 and P_2
F_{AB}	0	1800 lb	1800 lb
F_{BC}	300 lb	−1558.8 lb	−1258.8 lb

(a) LOAD P_1 ACTS ALONE

$$U_1 = \frac{(F_{BC})^2 L_{BC}}{2EA} = \frac{(300 \text{ lb})^2 (60 \text{ in.})}{144 \times 10^6 \text{ lb}}$$

$$= 0.0375 \text{ in.-lb} \quad \longleftarrow$$

(b) LOAD P_2 ACTS ALONE

$$U_2 = \frac{1}{2EA}\left[(F_{AB})^2 L_{AB} + (F_{BC})^2 L_{BC}\right]$$

$$= \frac{1}{2EA}\Big[(1800\ \text{lb})^2(69.282\ \text{in.})$$

$$+ (-1558.8\ \text{lb})^2(60\ \text{in.})\Big]$$

$$= \frac{370.265 \times 10^6\ \text{lb}^2\text{-in.}}{144 \times 10^6\ \text{lb}} = 2.57\ \text{in.-lb} \quad \leftarrow$$

(c) LOADS P_1 AND P_2 ACT SIMULTANEOUSLY

$$U_3 = \frac{1}{2EA}\Big[(F_{AB})^2 L_{AB} + (F_{BC})^2 L_{BC}\Big]$$

$$= \frac{1}{2EA}\Big[(1800\ \text{lb})^2(69.282\ \text{in.})$$

$$+ (-1258.8\ \text{lb})^2(60\ \text{in.})\Big]$$

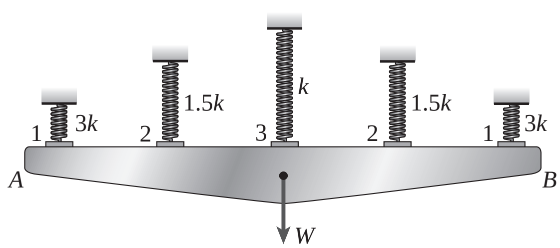
$$= \frac{319.548 \times 10^6\ \text{lb}^2\text{-in.}}{144 \times 10^6\ \text{lb}}$$

$$= 2.22\ \text{in.-lb} \quad \leftarrow$$

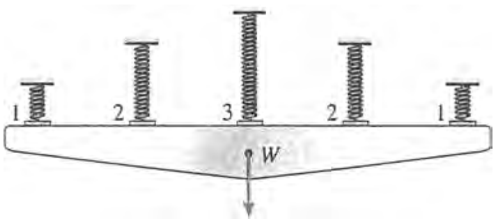
NOTE: The strain energy U_3 is *not* equal to $U_1 + U_2$.

Problem 2.7-8 The statically indeterminate structure shown in the figure consists of a horizontal rigid bar AB supported by five equally spaced springs. Springs 1, 2, and 3 have stiffnesses $3k$, $1.5k$, and k, respectively. When unstressed, the lower ends of all five springs lie along a horizontal line. Bar AB, which has weight W, causes the springs to elongate by an amount δ.

(a) Obtain a formula for the total strain energy U of the springs in terms of the downward displacement δ of the bar.
(b) Obtain a formula for the displacement δ by equating the strain energy of the springs to the work done by the weight W.
(c) Determine the forces F_1, F_2, and F_3 in the springs.
(d) Evaluate the strain energy U, the displacement δ, and the forces in the springs if $W = 600$ N and $k = 7.5$ N/mm.



Solution 2.7-8 Rigid bar supported by springs



$k_1 = 3k$

$k_2 = 1.5k$

$k_3 = k$

δ = downward displacement of rigid bar

For a spring: $U = \dfrac{k\delta^2}{2}$ Eq. (2-38b)

(a) Strain energy U of all springs

$$U = 2\left(\frac{3k\delta^2}{2}\right) + 2\left(\frac{1.5k\delta^2}{2}\right) + \frac{k\delta^2}{2} = 5k\delta^2 \quad \leftarrow$$

(b) Displacement δ

Work done by the weight W equals $\dfrac{W\delta}{2}$

Strain energy of the springs equals $5k\delta^2$

$$\therefore \frac{W\delta}{2} = 5k\delta^2 \quad \text{and} \quad \delta = \frac{W}{10k} \quad \leftarrow$$

(c) Forces in the springs

$$F_1 = 3k\delta = \frac{3W}{10} \quad F_2 = 1.5k\delta = \frac{3W}{20} \quad \leftarrow$$

$$F_3 = k\delta = \frac{W}{10} \quad \leftarrow$$

(d) Numerical values

$W = 600$ N $\quad k = 7.5$ N/mm $= 7500$ N/mm

$$U = 5k\delta^2 = 5k\left(\frac{W}{10k}\right)^2 = \frac{W^2}{20k}$$

$$= 2.4 \text{ N} \cdot \text{m} = 2.4 \text{ J} \quad \leftarrow$$

$$\delta = \frac{W}{10k} = 8.0 \text{ mm} \quad \leftarrow$$

$$F_1 = \frac{3W}{10} = 180 \text{ N} \quad \leftarrow$$

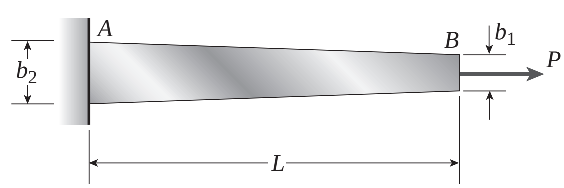
$$F_2 = \frac{3W}{20} = 90 \text{ N} \quad \leftarrow$$

$$F_3 = \frac{W}{10} = 60 \text{ N} \quad \leftarrow$$

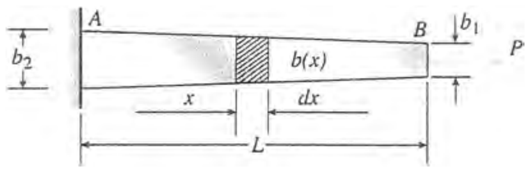
NOTE: $W = 2F_1 + 2F_2 + F_3 = 600$ N (Check)

Problem 2.7-9 A slightly tapered bar AB of rectangular cross section and length L is acted upon by a force P (see figure). The width of the bar varies uniformly from b_2 at end A to b_1 at end B. The thickness t is constant.

(a) Determine the strain energy U of the bar.
(b) Determine the elongation δ of the bar by equating the strain energy to the work done by the force P.



Solution 2.7-9 Tapered bar of rectangular cross section



$$b(x) = b_2 - \frac{(b_2 - b_1)x}{L}$$

$$A(x) = tb(x)$$

$$= t\left[b_2 - \frac{(b_2 - b_1)x}{L}\right]$$

(a) Strain energy of the bar

$$U = \int \frac{[N(x)]^2 dx}{2EA(x)} \quad \text{(Eq. 2-41)}$$

$$= \int_0^L \frac{P^2 dx}{2Etb(x)} = \frac{P^2}{2Et}\int_0^L \frac{dx}{b_2 - (b_2 - b_1)\frac{x}{L}} \qquad (1)$$

From Appendix C: $\displaystyle\int \frac{dx}{a + bx} = \frac{1}{b}\ln(a + bx)$

Apply this integration formula to Eq. (1):

$$U = \frac{P^2}{2Et}\left[\frac{1}{-(b_2 - b_1)(\frac{1}{L})}\ln\left[b_2 - \frac{(b_2 - b_1)x}{L}\right]\right]_0^L$$

$$= \frac{P^2}{2Et}\left[\frac{-L}{(b_2 - b_1)}\ln b_1 - \frac{-L}{(b_2 - b_1)}\ln b_2\right]$$

$$U = \frac{P^2 L}{2Et(b_2 - b_1)}\ln\frac{b_2}{b_1} \quad \leftarrow$$

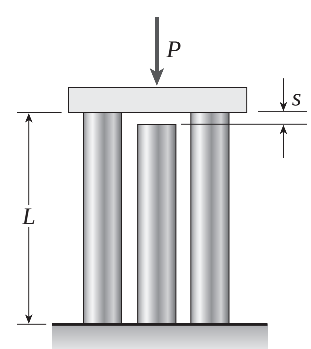
(b) Elongation of the bar (Eq. 2-42)

$$\delta = \frac{2U}{P} = \frac{PL}{Et(b_2 - b_1)}\ln\frac{b_2}{b_1} \quad \leftarrow$$

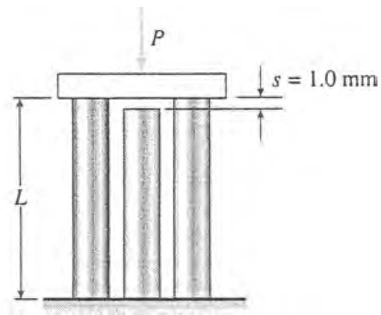
NOTE: This result agrees with the formula derived in Prob. 2.3-13.

Problem 2.7-10 A compressive load P is transmitted through a rigid plate to three magnesium-alloy bars that are identical except that initially the middle bar is slightly shorter than the other bars (see figure). The dimensions and properties of the assembly are as follows: length $L = 1.0$ m, cross-sectional area of each bar $A = 3000\ \text{mm}^2$, modulus of elasticity $E = 45$ GPa, and the gap $s = 1.0$ mm.

(a) Calculate the load P_1 required to close the gap.
(b) Calculate the downward displacement δ of the rigid plate when $P = 400$ kN.
(c) Calculate the total strain energy U of the three bars when $P = 400$ kN.
(d) Explain why the strain energy U is *not* equal to $P\delta/2$.
(*Hint*: Draw a load-displacement diagram.)



Solution 2.7-10 Three bars in compression



$s = 1.0$ mm

$L = 1.0$ m

For each bar:

$A = 3000\ \text{mm}^2$

$E = 45$ GPa

$$\frac{EA}{L} = 135 \times 10^6\ \text{N/m}$$

(a) Load P_1 required to close the gap

In general, $\delta = \dfrac{PL}{EA}$ and $P = \dfrac{EA\delta}{L}$

For two bars, we obtain:

$$P_1 = 2\left(\frac{EAs}{L}\right) = 2(135 \times 10^6\ \text{N/m})(1.0\ \text{mm})$$

$$P_1 = 270\ \text{kN} \quad \longleftarrow$$

(b) Displacement δ for $P = 400$ kN

Since $P > P_1$, all three bars are compressed. The force P equals P_1 plus the additional force required to compress all three bars by the amount $\delta - s$.

$$P = P_1 + 3\left(\frac{EA}{L}\right)(\delta - s)$$

or 400 kN = 270 kN + 3(135 × 10^6 N/m) (δ − 0.001 m)

Solving, we get $\delta = 1.321$ mm $\longleftarrow$

(c) Strain energy U for $P = 400$ kN

$$U = \sum \frac{EA\delta^2}{2L}$$

Outer bars: $\delta = 1.321$ mm

Middle bar: $\delta = 1.321\ \text{mm} - s$

$= 0.321$ mm

$$U = \frac{EA}{2L}[2(1.321\ \text{mm})^2 + (0.321\ \text{mm})^2]$$

$$= \frac{1}{2}(135 \times 10^6\ \text{N/m})(3.593\ \text{mm}^2)$$

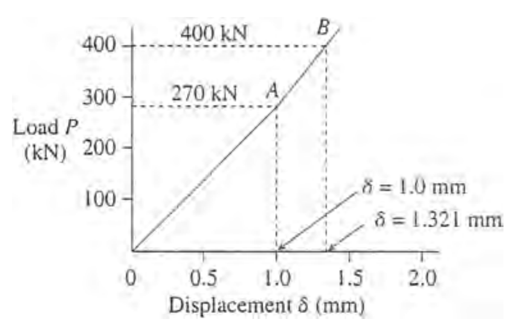
$$= 243\ \text{N} \cdot \text{m} = 243\ \text{J} \quad \longleftarrow$$

(d) Load-displacement diagram

$U = 243\ \text{J} = 243\ \text{N} \cdot \text{m}$

$$\frac{P\delta}{2} = \frac{1}{2}(400\ \text{kN})(1.321\ \text{mm}) = 264\ \text{N} \cdot \text{m}$$

The strain energy U is *not* equal to $\dfrac{P\delta}{2}$ = because the load-displacement relation is not linear.

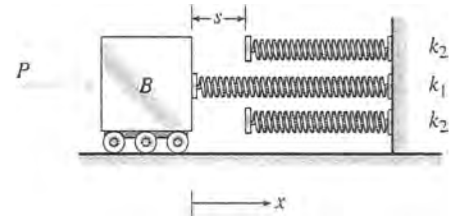


U = area under line OAB.

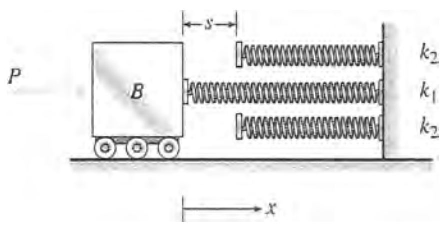
$\dfrac{P\delta}{2}$ = area under a straight line from O to B, which is larger than U.

Problem 2.7-11 A block B is pushed against three springs by a force P (see figure). The middle spring has stiffness k_1 and the outer springs each have stiffness k_2. Initially, the springs are unstressed and the middle spring is longer than the outer springs (the difference in length is denoted s).

(a) Draw a force-displacement diagram with the force P as ordinate and the displacement x of the block as abscissa.
(b) From the diagram, determine the strain energy U_1 of the springs when $x = 2s$.
(c) Explain why the strain energy U_1 is not equal to $P\delta/2$, where $\delta = 2s$.



Solution 2.7-11 Block pushed against three springs



Force P_0 required to close the gap:

$$P_0 = k_1 s \tag{1}$$

FORCE-DISPLACEMENT RELATION BEFORE GAP IS CLOSED

$$P = k_1 x \qquad (0 \le x \le s)(0 \le P \le P_0) \tag{2}$$

FORCE-DISPLACEMENT RELATION AFTER GAP IS CLOSED

All three springs are compressed. Total stiffness equals $k_1 + 2k_2$. Additional displacement equals $x - s$. Force P equals P_0 plus the force required to compress all three springs by the amount $x - s$.

$$P = P_0 + (k_1 + 2k_2)(x - s)$$

$$= k_1 s + (k_1 + 2k_2)x - k_1 s - 2k_2 s$$

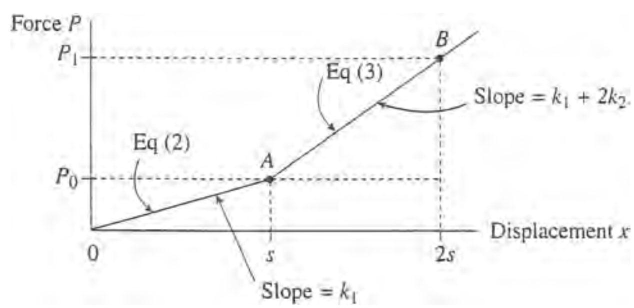
$$P = (k_1 + 2k_2)x - 2k_2 s \qquad (x \ge s);\ (P \ge P_0) \tag{3}$$

P_1 = force P when $x = 2s$

Substitute $x = 2s$ into Eq. (3):

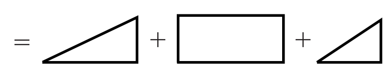
$$P_1 = 2(k_1 + k_2)s \tag{4}$$

(a) FORCE-DISPLACEMENT DIAGRAM



(b) STRAIN ENERGY U_1 WHEN $x = 2s$

U_1 = Area below force - displacement curve



$$= \frac{1}{2}P_0 s + P_0 s + \frac{1}{2}(P_1 - P_0)s = P_0 s + \frac{1}{2}P_1 s$$

$$= k_1 s^2 + (k_1 + k_2)s^2$$

$$U_1 = (2k_1 + k_2)s^2 \quad \leftarrow \tag{5}$$

(c) STRAIN ENERGY U_1 IS NOT EQUAL TO $\dfrac{P\delta}{2}$

For $\delta = 2s$: $\dfrac{P\delta}{2} = \dfrac{1}{2} P_1(2\,s) = P_1 s = 2(k_1 + k_2)s^2$

(This quantity is greater than U_1.)

U_1 = area under line OAB.

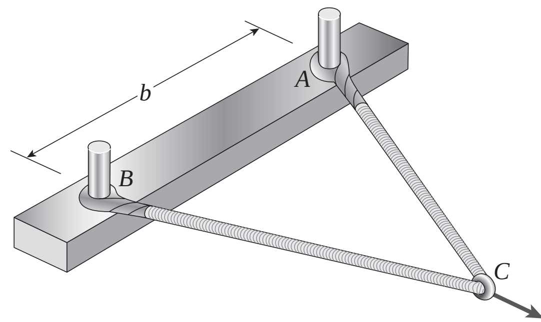
$\dfrac{P\delta}{2}$ = area under a straight line from O to B, which is larger than U_1.

Thus, $\dfrac{P\delta}{2}$ is *not* equal to the strain energy because the force-displacement relation is not linear.

Problem 2.7-12 A bungee cord that behaves linearly elastically has an unstressed length $L_0 = 760$ mm and a stiffness $k = 140$ N/m.The cord is attached to two pegs, distance $b = 380$ mm apart, and pulled at its midpoint by a force $P = 80$ N (see figure).

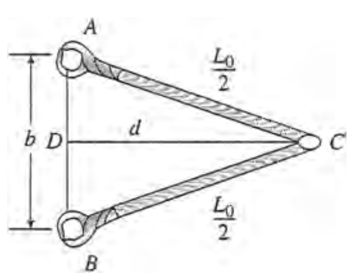
(a) How much strain energy U is stored in the cord?
(b) What is the displacement δ_C of the point where the load is applied?
(c) Compare the strain energy U with the quantity $P\delta_C/2$.

(*Note*: The elongation of the cord is *not* small compared to its original length.)



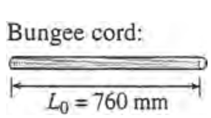
Solution 2.7-12 Bungee cord subjected to a load *P*.

DIMENSIONS BEFORE THE LOAD P IS APPLIED



$L_0 = 760$ mm $\qquad \dfrac{L_0}{2} = 380$ mm

$b = 380$ mm

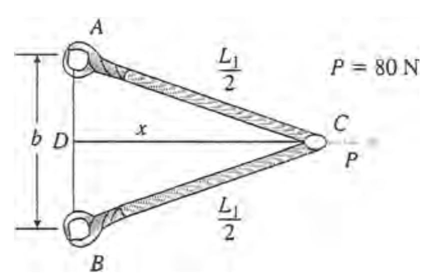


$k = 140$ N/m

From triangle ACD:

$$d = \frac{1}{2}\sqrt{L_0^2 - b^2} = 329.09 \text{ mm} \qquad (1)$$

DIMENSIONS AFTER THE LOAD P IS APPLIED



Let x = distance CD

Let L_1 = stretched length of bungee cord

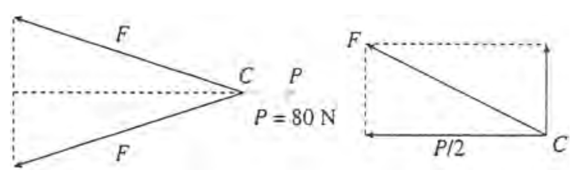
From triangle *ACD*:

$$\frac{L_1}{2} = \sqrt{\left(\frac{b}{2}\right)^2 + x^2} \qquad (2)$$

$$L_1 = \sqrt{b^2 + 4x^2} \qquad (3)$$

EQUILIBRIUM AT POINT C

Let F = tensile force in bungee cord



$$\frac{F}{P/2} = \frac{L_1/2}{x} \qquad F = \left(\frac{P}{2}\right)\left(\frac{L_1}{2}\right)\left(\frac{1}{x}\right)$$

$$= \frac{P}{2}\sqrt{1 + \left(\frac{b}{2x}\right)^2} \qquad (4)$$

ELONGATION OF BUNGEE CORD

Let δ = elongation of the entire bungee cord

$$\delta = \frac{F}{k} = \frac{P}{2k}\sqrt{1 + \frac{b^2}{4x^2}} \qquad (5)$$

Final length of bungee cord = original length + δ

$$L_1 = L_0 + \delta = L_0 + \frac{P}{2k}\sqrt{1 + \frac{b^2}{4x^2}} \qquad (6)$$

SOLUTION OF EQUATIONS

Combine Eqs. (6) and (3):

$$L_1 = L_0 + \frac{P}{2k}\sqrt{1 + \frac{b^2}{4x^2}} = \sqrt{b^2 + 4x^2}$$

or $\quad L_1 = L_0 + \dfrac{P}{4kx}\sqrt{b^2 + 4x^2} = \sqrt{b^2 + 4x^2}$

$$L_0 = \left(1 - \frac{P}{4kx}\right)\sqrt{b^2 + 4x^2} \qquad (7)$$

This equation can be solved for x.

SUBSTITUTE NUMERICAL VALUES INTO EQ. (7):

$$760 \text{ mm} = \left[1 - \frac{(80 \text{ N})(1000 \text{ mm/m})}{4(140 \text{ N/m})x}\right]$$

$$\times \sqrt{(380 \text{ mm})^2 + 4x^2} \qquad (8)$$

$$760 = \left(1 - \frac{142.857}{x}\right)\sqrt{144{,}400 + 4x^2} \qquad (9)$$

Units: x is in millimeters
Solve for x (Use trial & error or a computer program):

$x = 497.88$ mm

(a) STRAIN ENERGY U OF THE BUNGEE CORD

$$U = \frac{k\delta^2}{2} \qquad k = 140 \text{ N/m} \qquad P = 80 \text{ N}$$

From Eq. (5):

$$\delta = \frac{P}{2k}\sqrt{1 + \frac{b^2}{4x^2}} = 305.81 \text{ mm}$$

$$U = \frac{1}{2}(140 \text{ N/m})(305.81 \text{ mm})^2 = 6.55 \text{ N.m}$$

$U = 6.55$ J $\leftarrow$

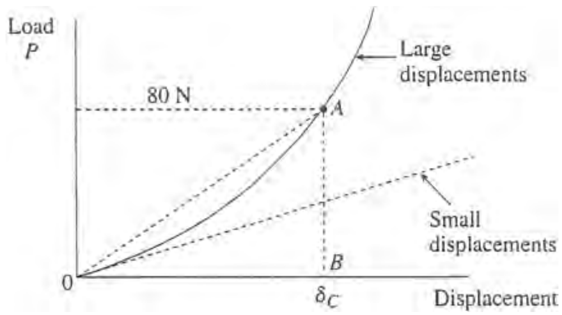
(b) DISPLACEMENT δ_C OF POINT C

$$\delta_C = x - d = 497.88 \text{ mm} - 329.09 \text{ mm}$$
$$= 168.8 \text{ mm} \quad \leftarrow$$

(c) COMPARISON OF STRAIN ENERGY U WITH THE QUANTITY $P\delta_C/2$

$U = 6.55$ J

$$\frac{P\delta_C}{2} = \frac{1}{2}(80\text{ N})(168.8\text{ mm}) = 6.75\text{ J}$$



The two quantities are not the same. The work done by the load P is *not* equal to $P\delta_C/2$ because the load-displacement relation (see below) is non-linear when the displacements are large. (The *work* done by the load P is equal to the strain energy because the bungee cord behaves elastically and there are no energy losses.)

U = area OAB under the curve OA.

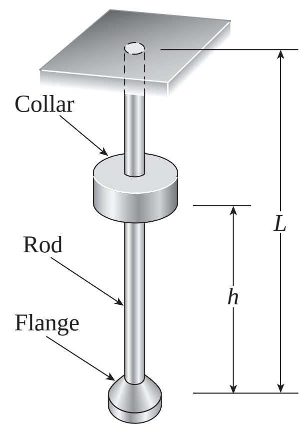
$\frac{P\delta_C}{2}$ = area of triangle OAB, which is greater than U.

Impact Loading

The problems for Section 2.8 are to be solved on the basis of the assumptions and idealizations described in the text. In particular, assume that the material behaves linearly elastically and no energy is lost during the impact.

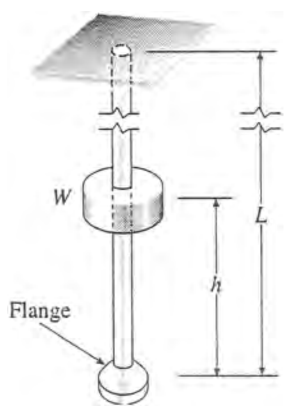
Problem 2.8-1 A sliding collar of weight $W = 150$ lb falls from a height $h = 2.0$ in. onto a flange at the bottom of a slender vertical rod (see figure). The rod has length $L = 4.0$ ft, cross-sectional area $A = 0.75$ in.^{2}, and modulus of elasticity $E = 30 \times 10^6$ psi.

Calculate the following quantities: (a) the maximum downward displacement of the flange, (b) the maximum tensile stress in the rod, and (c) the impact factor.



Probs. 2.8-1, 2.8-2, 2.8-3

Solution 2.8-1 Collar falling onto a flange



$W = 150$ lb

$h = 2.0$ in. $\quad L = 4.0$ ft $= 48$ in.

$E = 30 \times 10^6$ psi $\quad A = 0.75$ in.^{2}

(a) Downward displacement of flange

$$\delta_{st} = \frac{WL}{EA} = 0.00032 \text{ in.}$$

Eq. of (2-53):

$$\delta_{max} = \delta_{st}\left[1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}\right]$$

$$= 0.0361 \text{ in.} \quad \leftarrow$$

(b) Maximum tensile stress (Eq. 2-55)

$$\sigma_{max} = \frac{E\delta_{max}}{L} = 22{,}600 \text{ psi} \quad \leftarrow$$

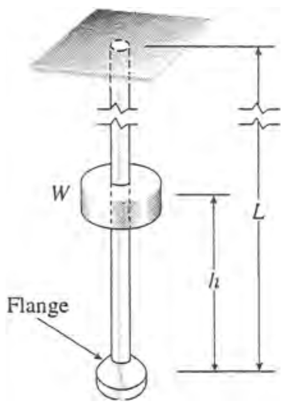
(c) Impact factor (Eq. 2-61)

$$\text{Impact factor} = \frac{\delta_{max}}{\delta_{st}} = \frac{0.0361 \text{ in.}}{0.00032 \text{ in.}}$$

$$= 113 \quad \leftarrow$$

Problem 2.8-2 Solve the preceding problem if the collar has mass $M = 80$ kg, the height $h = 0.5$ m, the length $L = 3.0$ m, the cross-sectional area $A = 350\ \text{mm}^2$, and the modulus of elasticity $E = 170$ GPa.

Solution 2.8-2 Collar falling onto a flange



$M = 80$ kg

$W = Mg = (80\ \text{kg})(9.81\ \text{m/s}^2)$

$= 784.8$ N

$h = 0.5$ m $\qquad L = 3.0$ m

$E = 170$ GPa $\qquad A = 350\ \text{mm}^2$

(a) Downward displacement of flange

$$\delta_{st} = \frac{WL}{EA} = 0.03957\ \text{mm}$$

$$\text{Eq. (2-53):}\quad \delta_{max} = \delta_{st}\left[1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}\right]$$

$$= 6.33\ \text{mm} \quad \leftarrow$$

(b) Maximum tensile stress (Eq. 2-55)

$$\sigma_{max} = \frac{E\delta_{max}}{L} = 359\ \text{MPa} \quad \leftarrow$$

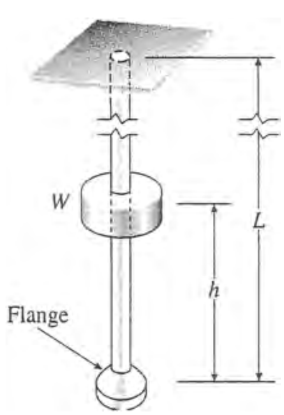
(c) Impact factor (Eq. 2–61)

$$\text{Impact factor} = \frac{\delta_{max}}{\delta_{st}} = \frac{6.33\ \text{mm}}{0.03957\ \text{mm}}$$

$$= 160 \quad \leftarrow$$

Problem 2.8-3 Solve Problem 2.8-1 if the collar has weight $W = 50$ lb, the height $h = 2.0$ in., the length $L = 3.0$ ft, the cross-sectional area $A = 0.25\ \text{in.}^2$, and the modulus of elasticity $E = 30{,}000$ ksi.

Solution 2.8-3 Collar falling onto a flange



$W = 50$ lb $\qquad h = 2.0$ in.

$L = 3.0$ ft $= 36$ in.

$E = 30{,}000$ psi $\qquad A = 0.25\ \text{in.}^2$

(a) Downward displacement of flange

$$\delta_{st} = \frac{WL}{EA} = 0.00024\ \text{in.}$$

$$\text{Eq. (2 - 53):}\quad \delta_{max} = \delta_{st}\left[1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}\right]$$

$$= 0.0312\ \text{in.} \quad \leftarrow$$

(b) Maximum tensile stress (Eq. 2–55)

$$\sigma_{max} = \frac{E\delta_{max}}{L} = 26{,}000 \text{ psi} \quad \leftarrow$$

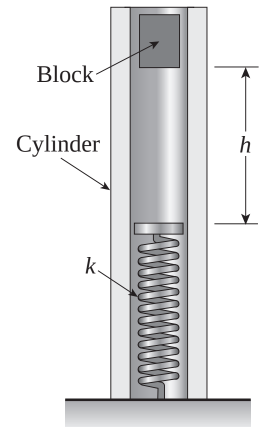
(c) Impact factor (Eq. 2-61)

$$\text{Impact factor} = \frac{\delta_{max}}{\delta_{st}} = \frac{0.0312 \text{ in.}}{0.00024 \text{ in.}}$$

$$= 130 \quad \leftarrow$$

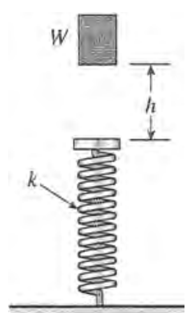
Problem 2.8-4 A block weighing $W = 5.0$ N drops inside a cylinder from a height $h = 200$ mm onto a spring having stiffness $k = 90$ N/m (see figure).

(a) Determine the maximum shortening of the spring due to the impact, and (b) determine the impact factor.



Prob. 2.8-4 and 2.8-5

Solution 2.8-4 Block dropping onto a spring



$W = 5.0$ N $\qquad h = 200$ mm $\qquad k = 90$ N/m

(a) Maximum shortening of the spring

$$\delta_{st} = \frac{W}{k} = \frac{5.0 \text{ N}}{90 \text{ N/m}} = 55.56 \text{ mm}$$

$$\text{Eq. (2-53):} \quad \delta_{max} = \delta_{st}\left[1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}\right]$$

$$= 215 \text{ mm} \quad \leftarrow$$

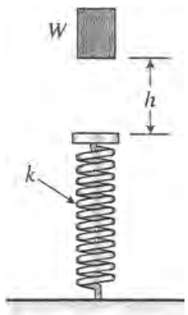
(b) Impact factor (Eq. 2-61)

$$\text{Impact factor} = \frac{\delta_{max}}{\delta_{st}} = \frac{215 \text{ mm}}{55.56 \text{ mm}}$$

$$= 3.9 \quad \leftarrow$$

Problem 2.8-5 Solve the preceding problem if the block weighs $W = 1.0$ lb, $h = 12$ in., and $k = 0.5$ lb/in.

Solution 2.8-5 Block dropping onto a spring



$W = 1.0$ lb $\quad h = 12$ in. $\quad k = 0.5$ lb/in.

(a) MAXIMUM SHORTENING OF THE SPRING

$$\delta_{st} = \frac{W}{k} = \frac{1.0 \text{ lb}}{0.5 \text{ lb/in.}} = 2.0 \text{ in.}$$

$$\text{Eq. (2-53):} \quad \delta_{max} = \delta_{st}\left[1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}\right]$$

$$= 9.21 \text{ in.} \quad \leftarrow$$

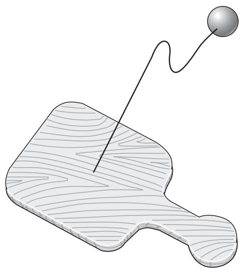
(b) IMPACT FACTOR (EQ. 2-61)

$$\text{Impact factor} = \frac{\delta_{max}}{\delta_{st}} = \frac{9.21 \text{ in.}}{2.0 \text{ in.}}$$

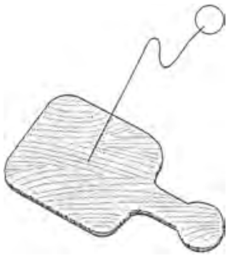
$$= 4.6 \quad \leftarrow$$

Problem 2.8-6 A small rubber ball (weight $W = 450$ mN) is attached by a rubber cord to a wood paddle (see figure). The natural length of the cord is $L_0 = 200$ mm, its cross-sectional area is $A = 1.6 \text{ mm}^2$, and its modulus of elasticity is $E = 2.0$ MPa. After being struck by the paddle, the ball stretches the cord to a total length $L_1 = 900$ mm.

What was the velocity v of the ball when it left the paddle? (Assume linearly elastic behavior of the rubber cord, and disregard the potential energy due to any change in elevation of the ball.)



Solution 2.8-6 Rubber ball attached to a paddle



$g = 9.81 \text{ m/s}^2 \qquad E = 2.0$ MPa

$A = 1.6 \text{ mm}^2 \qquad L_0 = 200$ mm

$L_1 = 900$ mm $\qquad W = 450$ mN

WHEN THE BALL LEAVES THE PADDLE

$$KE = \frac{Wv^2}{2g}$$

WHEN THE RUBBER CORD IS FULLY STRETCHED:

$$U = \frac{EA\delta^2}{2L_0} = \frac{EA}{2L_0}(L_1 - L_0)^2$$

CONSERVATION OF ENERGY

$$KE = U \quad \frac{Wv^2}{2g} = \frac{EA}{2L_0}(L_1 - L_0)^2$$

$$v^2 = \frac{gEA}{WL_0}(L_1 - L_0)^2$$

$$v = (L_1 - L_0)\sqrt{\frac{gEA}{WL_0}} \quad \leftarrow$$

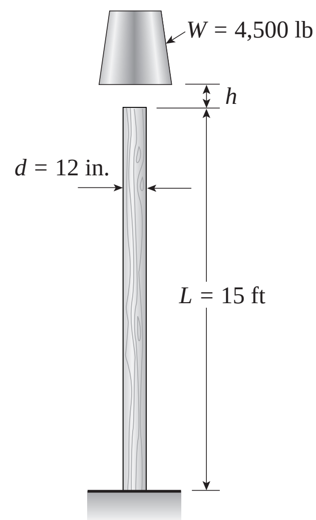
SUBSTITUTE NUMERICAL VALUES:

$$v = (700 \text{ mm})\sqrt{\frac{(9.81 \text{ m/s}^2)(2.0 \text{ MPa})(1.6 \text{ mm}^2)}{(450 \text{ mN})(200 \text{ mm})}}$$

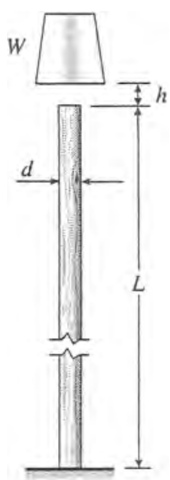
$$= 13.1 \text{ m/s} \quad \leftarrow$$

Problem 2.8-7 A weight $W = 4500$ lb falls from a height h onto a vertical wood pole having length $L = 15$ ft, diameter $d = 12$ in., and modulus of elasticity $E = 1.6 \times 10^6$ psi (see figure).

If the allowable stress in the wood under an impact load is 2500 psi, what is the maximum permissible height h?



Solution 2.8-7 Weight falling on a wood pole



$W = 4500$ lb $\quad d = 12$ in.

$L = 15$ ft $= 180$ in.

$$A = \frac{\pi d^2}{4} = 113.10 \text{ in.}^2$$

$E = 1.6 \times 10^6$ psi

$\sigma_{allow} = 2500$ psi $(= \sigma_{max})$

Find h_{max}

STATIC STRESS

$$\sigma_{st} = \frac{W}{A} = \frac{4500 \text{ lb}}{113.10 \text{ in.}^2} = 39.79 \text{ psi}$$

MAXIMUM HEIGHT h_{max}

$$\text{Eq. (2-59):} \quad \sigma_{max} = \sigma_{st}\left[1 + \left(1 + \frac{2hE}{L\sigma_{st}}\right)^{1/2}\right]$$

or

$$\frac{\sigma_{max}}{\sigma_{st}} - 1 = \left(1 + \frac{2hE}{L\sigma_{st}}\right)^{1/2}$$

Square both sides and solve for h:

$$h = h_{max} = \frac{L\sigma_{max}}{2E}\left(\frac{\sigma_{max}}{\sigma_{st}} - 2\right) \quad \leftarrow$$

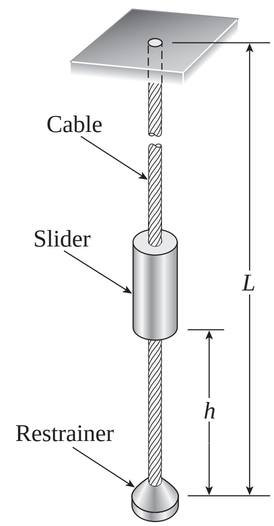
SUBSTITUTE NUMERICAL VALUES:

$$h_{max} = \frac{(180 \text{ in.})(2500 \text{ psi})}{2(1.6 \times 10^6 \text{ psi})}\left(\frac{2500 \text{ psi}}{39.79 \text{ psi}} - 2\right)$$

$$= 8.55 \text{ in.} \quad \leftarrow$$

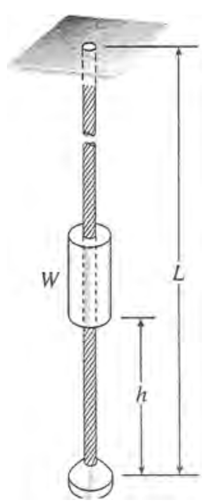
Problem 2.8-8 A cable with a restrainer at the bottom hangs vertically from its upper end (see figure). The cable has an effective cross-sectional area $A = 40\ \text{mm}^2$ and an effective modulus of elasticity $E = 130$ GPa. A slider of mass $M = 35$ kg drops from a height $h = 1.0$ m onto the restrainer.

If the allowable stress in the cable under an impact load is 500 MPa, what is the minimum permissible length L of the cable?



Probs. 2.8-8, 2.8-2, 2.8-9

Solution 2.8-8 Slider on a cable



$W = Mg = (35\ \text{kg})(9.81\ \text{m/s}^2) = 343.4\ \text{N}$

$A = 40\ \text{mm}^2 \qquad E = 130\ \text{GPa}$

$h = 1.0\ \text{m} \qquad \sigma_{allow} = \sigma_{max} = 500\ \text{MPa}$

Find minimum length L_{min}

STATIC STRESS

$$\sigma_{st} = \frac{W}{A} = \frac{343.4\ \text{N}}{40\ \text{mm}^2} = 8.585\ \text{MPa}$$

MINIMUM LENGTH L_{min}

$$\text{Eq. (2-59):} \quad \sigma_{max} = \sigma_{st}\left[1 + \left(1 + \frac{2hE}{L\sigma_{st}}\right)^{1/2}\right]$$

or

$$\frac{\sigma_{max}}{\sigma_{st}} - 1 = \left(1 + \frac{2hE}{L\sigma_{st}}\right)^{1/2}$$

Square both sides and solve for L:

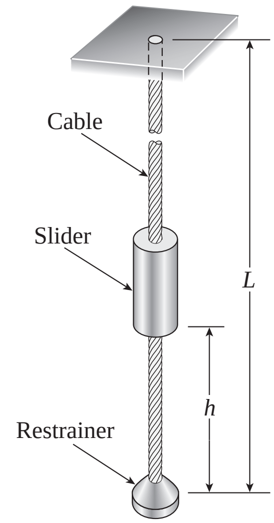
$$L = L_{min} = \frac{2Eh\sigma_{st}}{\sigma_{max}(\sigma_{max} - 2\sigma_{st})} \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

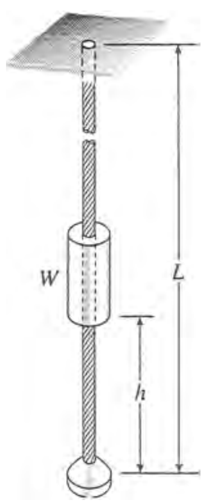
$$L_{min} = \frac{2(130\ \text{GPa})\ (1.0\ \text{m})\ (8.585\ \text{MPa})}{(500\ \text{MPa})\ [500\ \text{MPa} - 2(8.585\ \text{MPa})]}$$

$$= 9.25\ \text{mm} \quad \leftarrow$$

Problem 2.8-9 Solve the preceding problem if the slider has weight $W = 100$ lb, $h = 45$ in., $A = 0.080$ in.^{2}, $E = 21 \times 10^6$ psi, and the allowable stress is 70 ksi.



Solution 2.8-9 Slider on a cable



$W = 100$ lb

$A = 0.080$ in.^{2} $E = 21 \times 10^6$ psi

$h = 45$ in $\sigma_{allow} = \sigma_{max} = 70$ ksi

Find minimum length L_{min}

STATIC STRESS

$$\sigma_{st} = \frac{W}{A} = \frac{100 \text{ lb}}{0.080 \text{ in.}^2} = 1250 \text{ psi}$$

MINIMUM LENGTH L_{min}

Eq. (2 − 59): $\sigma_{max} = \sigma_{st}\left[1 + \left(1 + \frac{2hE}{L\sigma_{st}}\right)^{1/2}\right]$

or

$$\frac{\sigma_{max}}{\sigma_{st}} - 1 = \left(1 + \frac{2hE}{L\sigma_{st}}\right)^{1/2}$$

Square both sides and solve for L:

$$L = L_{min} = \frac{2Eh\sigma_{st}}{\sigma_{max}(\sigma_{max} - 2\sigma_{st})} \quad \leftarrow$$

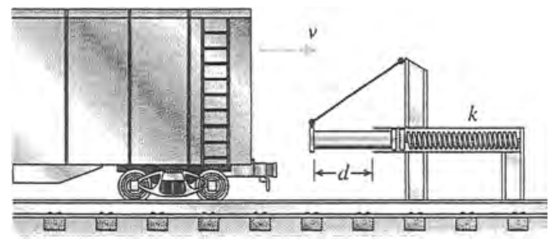
SUBSTITUTE NUMERICAL VALUES:

$$L_{min} = \frac{2(21 \times 10^6 \text{ psi})(45 \text{ in.})(1250 \text{ psi})}{(70{,}000 \text{ psi})[70{,}000 \text{ psi} - 2(1250 \text{ psi})]}$$

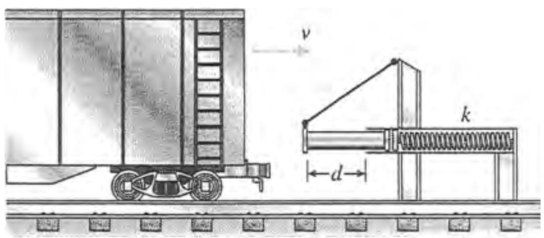
$$= 500 \text{ in.} \quad \leftarrow$$

Problem 2.8-10 A bumping post at the end of a track in a railway yard has a spring constant $k = 8.0$ MN/m (see figure). The maximum possible displacement d of the end of the striking plate is 450 mm.

What is the maximum velocity v_{max} that a railway car of weight $W = 545$ kN can have without damaging the bumping post when it strikes it?



Solution 2.8-10 Bumping post for a railway car



$k = 8.0$ MN/m $\qquad W = 545$ kN

d = maximum displacement of spring

$d = \delta_{max} = 450$ mm

Find v_{max}

Kinetic energy before impact

$$KE = \frac{Mv^2}{2} = \frac{Wv^2}{2g}$$

Strain energy when spring is compressed to the maximum allowable amount

$$U = \frac{k\delta_{max}^2}{2} = \frac{kd^2}{2}$$

Conservation of energy

$$KE = U \quad \frac{Wv^2}{2g} = \frac{kd^2}{2} \quad v^2 = \frac{kd^2}{W/g}$$

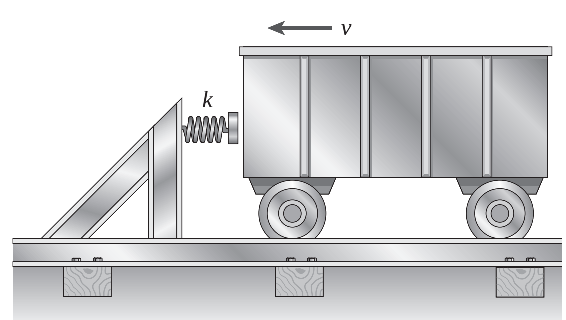
$$v = v_{max} = d\sqrt{\frac{k}{W/g}} \qquad \leftarrow$$

Substitute numerical values:

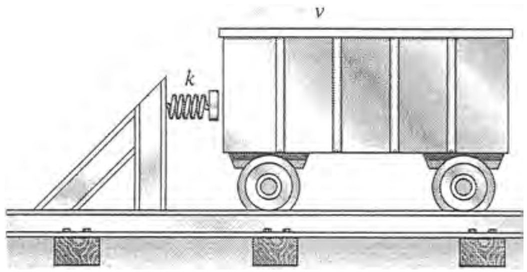
$$v_{max} = (450 \text{ mm})\sqrt{\frac{8.0 \text{ MN/m}}{(545 \text{ kN})/(9.81 \text{ m/s}^2)}}$$

$$= 5400 \text{ mm/s} = 5.4 \text{ m/s} \qquad \leftarrow$$

Problem 2.8-11 A bumper for a mine car is constructed with a spring of stiffness $k = 1120$ lb/in. (see figure). If a car weighing 3450 lb is traveling at velocity $v = 7$ mph when it strikes the spring, what is the maximum shortening of the spring?



Solution 2.8-11 Bumper for a mine car



$k = 1120$ lb/in. $W = 3450$ lb

$v = 7$ mph $= 123.2$ in./sec

$g = 32.2$ ft/sec^2 $= 386.4$ in./sec^2

Find the shortening δ_{max} of the spring.

KINETIC ENERGY JUST BEFORE IMPACT

$$KE = \frac{Mv^2}{2} = \frac{Wv^2}{2g}$$

STRAIN ENERGY WHEN SPRING IS FULLY COMPRESSED

$$U = \frac{k\delta_{max}^2}{2}$$

Conservation of energy

$$KE = U \quad \frac{Wv^2}{2g} = \frac{k\delta_{max}^2}{2}$$

Solve for δ_{max}: $\delta_{max} = \sqrt{\frac{Wv^2}{gk}}$ ←

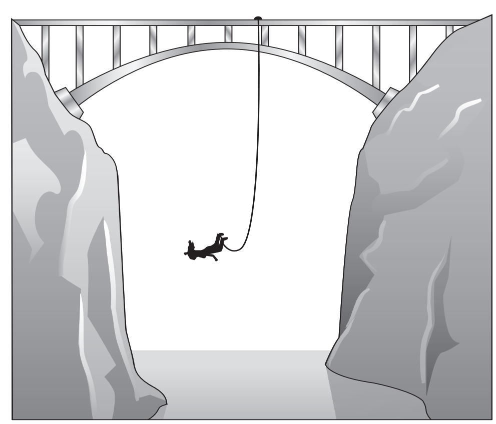
SUBSTITUTE NUMERICAL VALUES:

$$\delta_{max} = \sqrt{\frac{(3450 \text{ lb})(123.2 \text{ in./sec})^2}{(386.4 \text{ in./sec}^2)(1120 \text{ lb/in.})}}$$

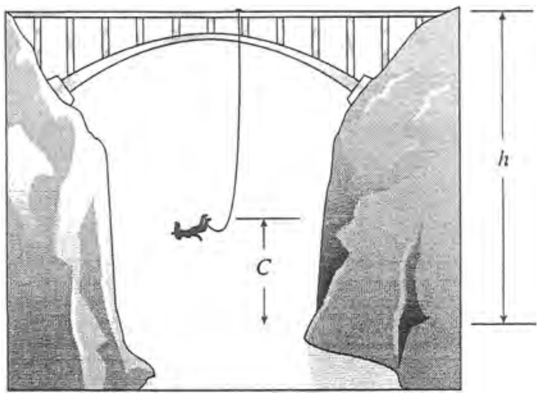
$$= 11.0 \text{ in.} \quad \leftarrow$$

Problem 2.8-12 A bungee jumper having a mass of 55 kg leaps from a bridge, braking her fall with a long elastic shock cord having axial rigidity $EA = 2.3$ kN (see figure).

If the jumpoff point is 60 m above the water, and if it is desired to maintain a clearance of 10 m between the jumper and the water, what length L of cord should be used?



Solution 2.8-12 Bungee jumper



$W = Mg = (55 \text{ kg})(9.81 \text{ m/s}^2)$

$= 539.55 \text{ N}$

$EA = 2.3 \text{ kN}$

Height: $h = 60$ m

Clearance: $C = 10$ m

Find length L of the bungee cord.

$P.E.$ = Potential energy of the jumper at the top of bridge (with respect to lowest position)

$= W(L + \delta_{max})$

U = strain energy of cord at lowest position

$$= \frac{EA\delta_{max}^2}{2L}$$

CONSERVATION OF ENERGY

$$P.E. = U \qquad W(L + \delta_{max}) = \frac{EA\delta_{max}^2}{2L}$$

or $$\delta_{max}^2 - \frac{2WL}{EA}\delta_{max} - \frac{2WL^2}{EA} = 0$$

SOLVE QUADRATIC EQUATION FOR δ_{max}:

$$\delta_{max} = \frac{WL}{EA} + \left[\left(\frac{WL}{EA}\right)^2 + 2L\left(\frac{WL}{EA}\right)\right]^{1/2}$$

$$= \frac{WL}{EA}\left[1 + \left(1 + \frac{2EA}{W}\right)^{1/2}\right]$$

VERTICAL HEIGHT

$$h = C + L + \delta_{max}$$

$$h - C = L + \frac{WL}{EA}\left[1 + \left(1 + \frac{2EA}{W}\right)^{1/2}\right]$$

SOLVE FOR L:

$$L = \frac{h - C}{1 + \frac{W}{EA}\left[1 + \left(1 + \frac{2EA}{W}\right)^{1/2}\right]} \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$$\frac{W}{EA} = \frac{539.55 \text{ N}}{2.3 \text{ kN}} = 0.234587$$

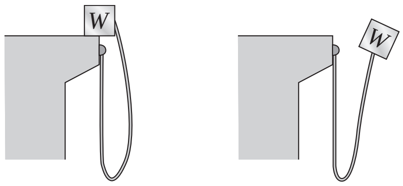
Numerator $= h - C = 60 \text{ m} - 10 \text{ m} = 50 \text{ m}$

$$\text{Denominator} = 1 + (0.234587) \times \left[1 + \left(1 + \frac{2}{0.234587}\right)^{1/2}\right]$$

$$= 1.9586$$

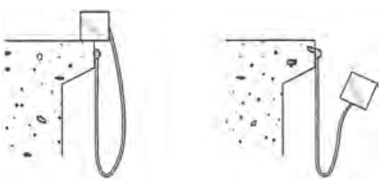
$$L = \frac{50 \text{ m}}{1.9586} = 25.5 \text{ m} \quad \leftarrow$$

Problem 2.8-13 A weight W rests on top of a wall and is attached to one end of a very flexible cord having cross-sectional area A and modulus of elasticity E (see figure). The other end of the cord is attached securely to the wall. The weight is then pushed off the wall and falls freely the full length of the cord.



(a) Derive a formula for the impact factor.
(b) Evaluate the impact factor if the weight, when hanging statically, elongates the band by 2.5% of its original length.

Solution 2.8-13 Weight falling off a wall



W = Weight

Properties of elastic cord:

E = modulus of elasticity

A = cross-sectional area

L = original length

δ_{max} = elongation of elastic cord

$P.E.$ = potential energy of weight before fall (with respect to lowest position)

$P.E. = W(L + \delta_{max})$

Let U = strain energy of cord at lowest position

$$U = \frac{EA\delta_{max}^2}{2L}$$

CONSERVATION OF ENERGY

$$P.E. = U \qquad W(L + \delta_{max}) = \frac{EA\delta_{max}^2}{2\,L}$$

$$\text{or} \qquad \delta_{max}^2 - \frac{2WL}{EA}\delta_{max} - \frac{2WL^2}{EA} = 0$$

SOLVE QUADRATIC EQUATION FOR δ_{max}:

$$\delta_{max} = \frac{WL}{EA} + \left[\left(\frac{WL}{EA}\right)^2 + 2L\left(\frac{WL}{EA}\right)\right]^{1/2}$$

STATIC ELONGATION

$$\delta_{st} = \frac{WL}{EA}$$

IMPACT FACTOR

$$\frac{\delta_{max}}{\delta_{st}} = 1 + \left[1 + \frac{2EA}{W}\right]^{1/2} \quad \leftarrow$$

NUMERICAL VALUES

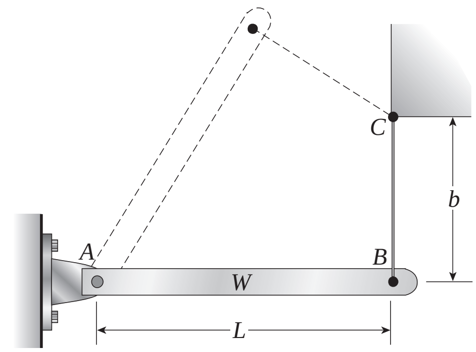
$$\delta_{st} = (2.5\%)(L) = 0.025L$$

$$\delta_{st} = \frac{WL}{EA} \qquad \frac{W}{EA} = 0.025 \qquad \frac{EA}{W} = 40$$

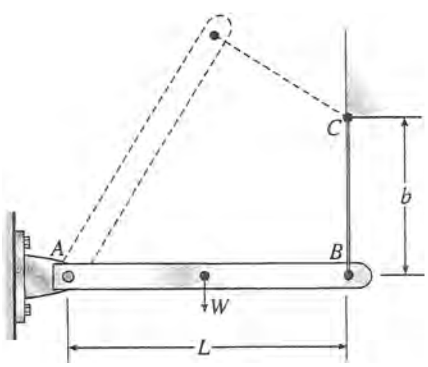
$$\text{Impact factor} = 1 + [1 + 2(40)]^{1/2} = 10 \quad \leftarrow$$

Problem 2.8-14 A rigid bar AB having mass $M = 1.0$ kg and length $L = 0.5$ m is hinged at end A and supported at end B by a nylon cord BC (see figure). The cord has cross-sectional area $A = 30\ \text{mm}^2$, length $b = 0.25$ m, and modulus of elasticity $E = 2.1$ GPa.

If the bar is raised to its maximum height and then released, what is the maximum stress in the cord?



Solution 2.8-14 Falling bar *AB*



RIGID BAR:

$W = Mg = (1.0 \text{ kg})(9.81 \text{ m/s}^2)$

$\quad = 9.81$ N

$L = 0.5$ m

NYLON CORD:

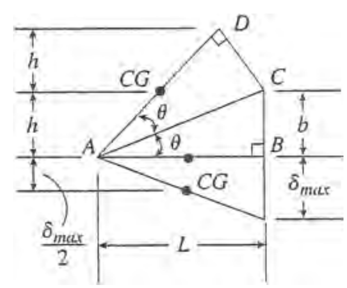
$A = 30\ \text{mm}^2$

$b = 0.25$ m

$E = 2.1$ GPa

Find maximum stress σ_{max} in cord BC.

GEOMETRY OF BAR AB AND CORD BC



$\overline{CD} = \overline{CB} = b$

$\overline{AD} = \overline{AB} = L$

h = height of center of gravity of raised bar AD

δ_{max} = elongation of cord

From triangle ABC: $\sin\theta = \dfrac{b}{\sqrt{b^2 + L^2}}$

$$\cos\theta = \frac{L}{\sqrt{b^2 + L^2}}$$

From line AD: $\sin 2\theta = \dfrac{2h}{AD} = \dfrac{2h}{L}$

From Appendix C: $\sin 2\theta = 2 \sin\theta \cos\theta$

$$\therefore \frac{2h}{L} = 2\left(\frac{b}{\sqrt{b^2 + L^2}}\right)\left(\frac{L}{\sqrt{b^2 + L^2}}\right) = \frac{2bL}{b^2 + L^2}$$

and $\quad h = \dfrac{bL^2}{b^2 + L^2} \qquad$ (Eq. 1)

CONSERVATION OF ENERGY

$P.E.$ = potential energy of raised bar AD

$$= W\left(h + \frac{\delta_{max}}{2}\right)$$

$$U = \text{strain energy of stretched cord} = \frac{EA\delta_{max}^2}{2b}$$

$$P.E. = U \quad W\left(h + \frac{\delta_{max}}{2}\right) = \frac{EA\delta_{max}^2}{2b} \qquad \text{(Eq. 2)}$$

For the cord: $\delta_{max} = \dfrac{\sigma_{max} b}{E}$

Substitute into Eq. (2) and rearrange:

$$\sigma_{max}^2 - \frac{W}{A}\sigma_{max} - \frac{2WhE}{bA} = 0 \qquad \text{(Eq. 3)}$$

Substitute from Eq. (1) into Eq. (3):

$$\sigma_{max}^2 - \frac{W}{A}\sigma_{max} - \frac{2WL^2E}{A(b^2 + L^2)} = 0 \qquad \text{(Eq. 4)}$$

SOLVE FOR σ_{max}:

$$\sigma_{max} = \frac{W}{2A}\left[1 + \sqrt{1 + \frac{8L^2EA}{W(b^2 + L^2)}}\right] \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$\sigma_{max} = 33.3$ MPa $\leftarrow$

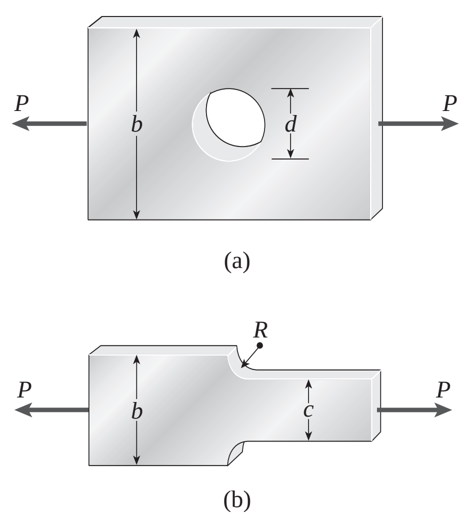
Stress Concentrations

The problems for Section 2.10 are to be solved by considering the stress-concentration factors and assuming linearly elastic behavior.

Problem 2.10-1 The flat bars shown in parts (a) and (b) of the figure are subjected to tensile forces $P = 3.0$ k. Each bar has thickness $t = 0.25$ in.

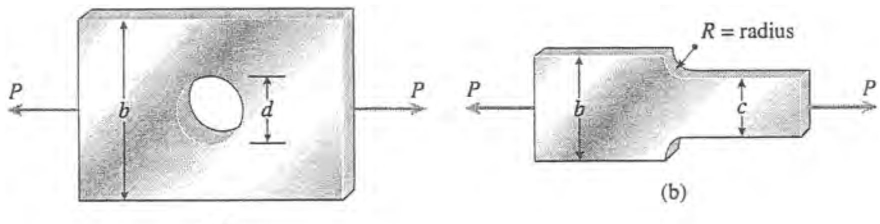
(a) For the bar with a circular hole, determine the maximum stresses for hole diameters $d = 1$ in. and $d = 2$ in. if the width $b = 6.0$ in.

(b) For the stepped bar with shoulder fillets, determine the maximum stresses for fillet radii $R = 0.25$ in. and $R = 0.5$ in. if the bar widths are $b = 4.0$ in. and $c = 2.5$ in.



Probs. 2.10-1 and 2.10-2

Solution 2.10-1 Flat bars in tension



(a)

$P = 3.0$ k $\quad t = 0.25$ in.

(a) Bar with circular hole ($b = 6$ in.)

Obtain K from Fig. 2-63

For $d = 1$ in.: $\quad c = b - d = 5$ in.

$$\sigma_{\text{nom}} = \frac{P}{ct} = \frac{3.0\text{ k}}{(5\text{ in.})(0.25\text{ in.})} = 2.40\text{ ksi}$$

$$d/b = \frac{1}{6} \quad K \approx 2.60$$

$$\sigma_{\max} = k\sigma_{\text{nom}} \approx 6.2\text{ ksi} \quad \leftarrow$$

For $d = 2$ in.: $c = b - d = 4$ in.

$$\sigma_{\text{nom}} = \frac{P}{ct} = \frac{3.0\text{ k}}{(4\text{ in.})(0.25\text{ in.})} = 3.00\text{ ksi}$$

$$d/b = \frac{1}{3} \quad K \approx 2.31$$

$$\sigma_{\max} = K\sigma_{\text{nom}} \approx 6.9\text{ ksi} \quad \leftarrow$$

(b) Stepped bar with shoulder fillets

$b = 4.0$ in. $\quad c = 2.5$ in.; Obtain k from Fig. 2-64

$$\sigma_{\text{nom}} = \frac{P}{ct} = \frac{3.0\text{ k}}{(2.5\text{ in.})(0.25\text{ in.})} = 4.80\text{ ksi}$$

For $R = 0.25$ in.: $R/c = 0.1 \quad b/c = 1.60$

$k \approx 2.30 \quad \sigma_{\max} = K\sigma_{\text{nom}} \approx 11.0\text{ ksi} \quad \leftarrow$

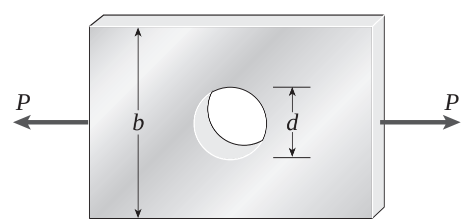
For $R = 0.5$ in.: $R/c = 0.2 \quad b/c = 1.60$

$K \approx 1.87 \quad \sigma_{\max} = K\sigma_{\text{nom}} \approx 9.0\text{ ksi} \quad \leftarrow$

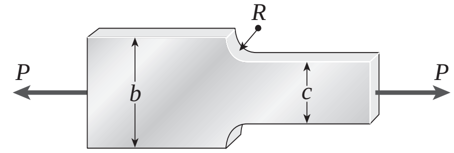
Problem 2.10-2 The flat bars shown in parts (a) and (b) of the figure are subjected to tensile forces $P = 2.5$ kN. Each bar has thickness $t = 5.0$ mm.

(a) For the bar with a circular hole, determine the maximum stresses for hole diameters $d = 12$ mm and $d = 20$ mm if the width $b = 60$ mm.

(b) For the stepped bar with shoulder fillets, determine the maximum stresses for fillet radii $R = 6$ mm and $R = 10$ mm if the bar widths are $b = 60$ mm and $c = 40$ mm.

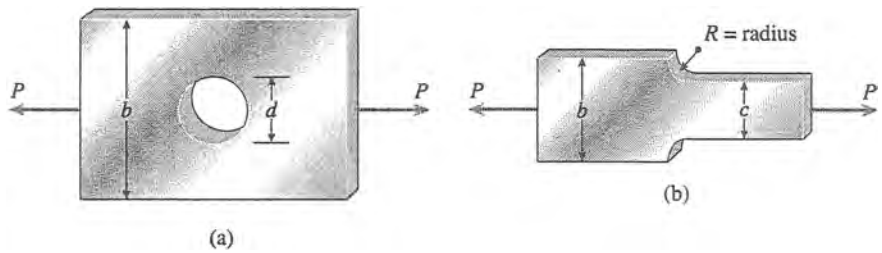


(a)



(b)

Solution 2.10-2 Flat bars in tension



$P = 2.5$ kN $\quad t = 5.0$ mm

(a) Bar with circular hole ($b = 60$ mm)

Obtain K from Fig. 2-63

For $d = 12$ mm: $c = b - d = 48$ mm

$$\sigma_{\text{nom}} = \frac{P}{ct} = \frac{2.5 \text{ kN}}{(48 \text{ mm})(5 \text{ mm})} = 10.42 \text{ MPa}$$

$$d/b = \frac{1}{5} \quad K \approx 2.51$$

$$\sigma_{\max} = K\sigma_{\text{nom}} \approx 26 \text{ MPa} \quad \leftarrow$$

For $d = 20$ mm: $c = b - d = 40$ mm

$$\sigma_{\text{nom}} = \frac{P}{ct} = \frac{2.5 \text{ kN}}{(40 \text{ mm})(5 \text{ mm})} = 12.50 \text{ MPa}$$

$$d/b = \frac{1}{3} \quad K \approx 2.31$$

$$\sigma_{\max} = K\sigma_{\text{nom}} \approx 29 \text{ MPa} \quad \leftarrow$$

(b) Stepped bar with shoulder fillets

$b = 60$ mm $\quad c = 40$ mm;

Obtain K from Fig. 2-64

$$\sigma_{\text{nom}} = \frac{P}{ct} = \frac{2.5 \text{ kN}}{(40 \text{ mm})(5 \text{ mm})} = 12.50 \text{ MPa}$$

For $R = 6$ mm: $R/c = 0.15 \quad b/c = 1.5$

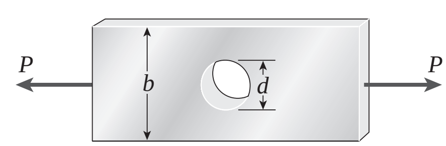
$K \approx 2.00 \quad \sigma_{\max} = K\sigma_{\text{nom}} \approx 25$ MPa $\leftarrow$

For $R = 10$ mm: $R/c = 0.25 \quad b/c = 1.5$

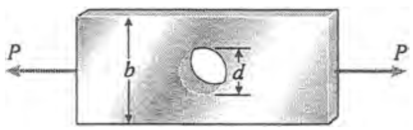
$K \approx 1.75 \quad \sigma_{\max} = K\sigma_{\text{nom}} \approx 22$ MPa $\leftarrow$

Problem 2.10-3 A flat bar of width b and thickness t has a hole of diameter d drilled through it (see figure). The hole may have any diameter that will fit within the bar.

What is the maximum permissible tensile load $P_{\max}$ if the allowable tensile stress in the material is σ_t?



Solution 2.10-3 Flat bar in tension



t = thickness

σ_t = allowable tensile stress

Find P_{max}

Find K from Fig. 2-64

$$P_{max} = \sigma_{nom}\, ct = \frac{\sigma_{max}}{K} ct = \frac{\sigma_t}{K}(b - d)t$$

$$= \frac{\sigma_t}{K} bt \left(1 - \frac{d}{b}\right)$$

Because σ_t, b, and t are constants, we write:

$$P^* = \frac{P_{max}}{\sigma_t bt} = \frac{1}{K}\left(1 - \frac{d}{b}\right)$$

$\frac{d}{b}$	K	P^*
0	3.00	0.333
0.1	2.73	0.330
0.2	2.50	0.320
0.3	2.35	0.298
0.4	2.24	0.268

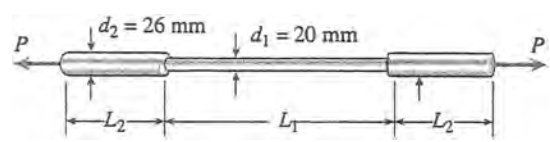
We observe that P_{max} decreases as d/b increases. Therefore, the maximum load occurs when the hole becomes very small.

$$\left(\frac{d}{b} \to 0 \quad \text{and} \quad K \to 3\right)$$

$$P_{max} = \frac{\sigma_t bt}{3} \quad \leftarrow$$

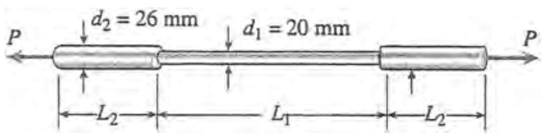
Problem 2.10-4 A round brass bar of diameter $d_1 = 20$ mm has upset ends of diameter $d_2 = 26$ mm (see figure). The lengths of the segments of the bar are $L_1 = 0.3$ m and $L_2 = 0.1$ m. Quarter-circular fillets are used at the shoulders of the bar, and the modulus of elasticity of the brass is $E = 100$ GPa.

If the bar lengthens by 0.12 mm under a tensile load P, what is the maximum stress σ_{max} in the bar?



Probs. 2.10-4 and 2.10-5

Solution 2.10-4 Round brass bar with upset ends



$E = 100$ GPa

$\delta = 0.12$ mm

$L_2 = 0.1$ m

$L_1 = 0.3$ m

$$R = \text{radius of fillets} = \frac{26\text{ mm} - 20\text{ mm}}{2} = 3\text{ mm}$$

$$\delta = 2\left(\frac{PL_2}{EA_2}\right) + \frac{PL_1}{EA_1}$$

Solve for P: $\quad P = \dfrac{\delta E A_1 A_2}{2L_2A_1 + L_1A_2}$

Use Fig. 2-65 for the stress-concentration factor:

$$\sigma_{\text{nom}} = \frac{P}{A_1} = \frac{\delta E A_2}{2L_2A_1 + L_1A_2} = \frac{\delta E}{2L_2\left(\frac{A_1}{A_2}\right) + L_1}$$

$$= \frac{\delta E}{2L_2\left(\frac{d_1}{d_2}\right)^2 + L_1}$$

Substitute numerical values:

$$\sigma_{\text{nom}} = \frac{(0.12\text{ mm})\,(100\text{ GPa})}{2(0.1\text{ m})\left(\frac{20}{26}\right)^2 + 0.3\text{ m}} = 28.68\text{ MPa}$$

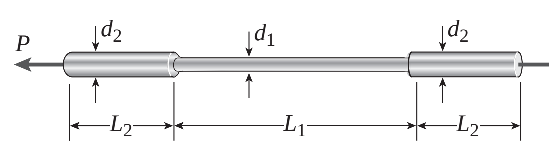
$$\frac{R}{D_1} = \frac{3\text{ mm}}{20\text{ mm}} = 0.15$$

Use the dashed curve in Fig. 2-65. $K \approx 1.6$

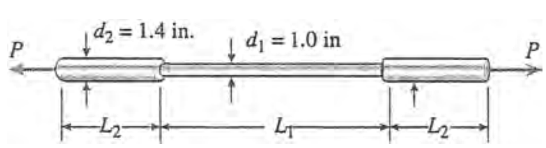
$$\sigma_{\max} = K\sigma_{\text{nom}} \approx (1.6)\,(28.68\text{ MPa})$$

$$\approx 46\text{ MPa} \quad \leftarrow$$

Problem 2.10-5 Solve the preceding problem for a bar of monel metal having the following properties: $d_1 = 1.0$ in., $d_2 = 1.4$ in., $L_1 = 20.0$ in., $L_2 = 5.0$ in., and $E = 25 \times 10^6$ psi. Also, the bar lengthens by 0.0040 in. when the tensile load is applied.



Solution 2.10-5 Round bar with upset ends



$E = 25 \times 10^6$ psi

$\delta = 0.0040$ in.

$L_1 = 20$ in.

$L_2 = 5$ in.

$$R = \text{radius of fillets} \quad R = \frac{1.4\text{ in.} - 1.0\text{ in.}}{2}$$

$$= 0.2\text{ in.}$$

$$\delta = 2\left(\frac{PL_2}{EA_2}\right) + \frac{PL_1}{EA_1}$$

Solve for P: $P = \dfrac{\delta E A_1 A_2}{2L_2A_1 + L_1A_2}$

Use Fig. 2-65 for the stress-concentration factor.

$$\sigma_{\text{nom}} = \frac{P}{A_1} = \frac{\delta E A_2}{2L_2A_1 + L_1A_2} = \frac{\delta E}{2L_2\left(\frac{A_1}{A_2}\right) + L_1}$$

$$= \frac{\delta E}{2L_2\left(\frac{d_1}{d_2}\right)^2 + L_1}$$

SUBSTITUTE NUMERICAL VALUES:

$$\sigma_{nom} = \frac{(0.0040 \text{ in.})(25 \times 10^6 \text{ psi})}{2(5 \text{ in.})\left(\frac{1.0}{1.4}\right)^2 + 20 \text{ in.}} = 3{,}984 \text{ psi}$$

$$\frac{R}{D_1} = \frac{0.2 \text{ in.}}{1.0 \text{ in.}} = 0.2$$

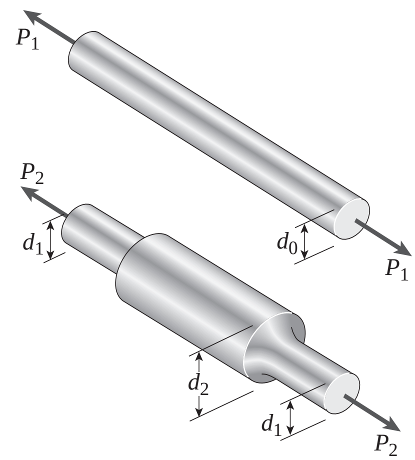
Use the dashed curve in Fig. 2-65. $K \approx 1.53$

$$\sigma_{max} = K\sigma_{nom} \approx (1.53)(3984 \text{ psi})$$

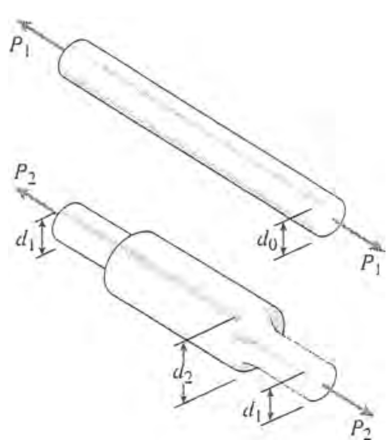
$$\approx 6100 \text{ psi} \quad \leftarrow$$

Problem 2.10-6 A prismatic bar of diameter $d_0 = 20$ mm is being compared with a stepped bar of the same diameter ($d_1 = 20$ mm) that is enlarged in the middle region to a diameter $d_2 = 25$ mm (see figure). The radius of the fillets in the stepped bar is 2.0 mm.

(a) Does enlarging the bar in the middle region make it stronger than the prismatic bar? Demonstrate your answer by determining the maximum permissible load P_1 for the prismatic bar and the maximum permissible load P_2 for the enlarged bar, assuming that the allowable stress for the material is 80 MPa.
(b) What should be the diameter d_0 of the prismatic bar if it is to have the same maximum permissible load as does the stepped bar?



Solution 2.10-6 Prismatic bar and stepped bar



$d_0 = 20$ mm

$d_1 = 20$ mm

$d_2 = 25$ mm

Fillet radius: $R = 2$ mm

Allowable stress: $\sigma_t = 80$ MPa

(a) COMPARISON OF BARS

$$\text{Prismatic bar: } P_1 = \sigma_t A_0 = \sigma_t\left(\frac{\pi d_0^2}{4}\right)$$

$$= (80 \text{ MPa})\left(\frac{\pi}{4}\right)(20\text{mm})^2 = 25.1 \text{ kN} \quad \leftarrow$$

Stepped bar: See Fig. 2-65 for the stress-concentration factor.

$R = 2.0$ mm $\quad D_1 = 20$ mm $\quad D_2 = 25$ mm

$R/D_1 = 0.10 \quad D_2/D_1 = 1.25 \quad K \approx 1.75$

$$\sigma_{nom} = \frac{P_2}{\frac{\pi}{4}d_1^2} = \frac{P_2}{A_1} \qquad \sigma_{nom} = \frac{\sigma_{max}}{K}$$

$$P_2 = \sigma_{\text{nom}} A_1 = \frac{\sigma_{\max}}{K} A_1 = \frac{\sigma_t}{K} A_1$$

$$= \left(\frac{80 \text{ MPa}}{1.75}\right)\left(\frac{\pi}{4}\right)(20 \text{ mm})^2$$

$$\approx 14.4 \text{ kN} \quad \leftarrow$$

Enlarging the bar makes it *weaker*, not stronger. The ratio of loads is $P_1/P_2 = K = 1.75$

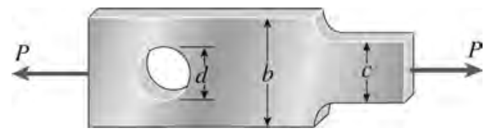
(b) Diameter of prismatic bar for the same allowable load

$$P_1 = P_2 \quad \sigma_t\left(\frac{\pi d_0^2}{4}\right) = \frac{\sigma_t}{K}\left(\frac{\pi d_1^2}{4}\right) \quad d_0^2 = \frac{d_1^2}{K}$$

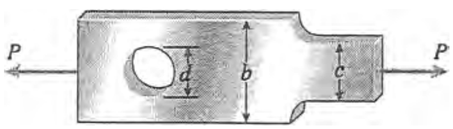
$$d_0 = \frac{d_1}{\sqrt{K}} \approx \frac{20 \text{ mm}}{\sqrt{1.75}} \approx 15.1 \text{ mm} \quad \leftarrow$$

Problem 2.10-7 A stepped bar with a hole (see figure) has widths $b = 2.4$ in. and $c = 1.6$ in. The fillets have radii equal to 0.2 in.

What is the diameter $d_{\max}$ of the largest hole that can be drilled through the bar without reducing the load-carrying capacity?



Solution 2.10-7 Stepped bar with a hole



$b = 2.4$ in.

$c = 1.6$ in.

Fillet radius: $R = 0.2$ in.

Find $d_{\max}$

Based upon fillets (Use Fig. 2-64)

$b = 2.4$ in. $\quad c = 1.6$ in. $\quad R = 0.2$ in.

$R/c = 0.125 \quad b/c = 1.5 \quad K \approx 2.10$

$$P_{\max} = \sigma_{\text{nom}} ct = \frac{\sigma_{\max}}{K} ct = \frac{\sigma_{\max}}{K}\left(\frac{c}{b}\right)(bt)$$

$$\approx 0.317\, bt\, \sigma_{\max}$$

Based upon hole (Use Fig. 2-63)

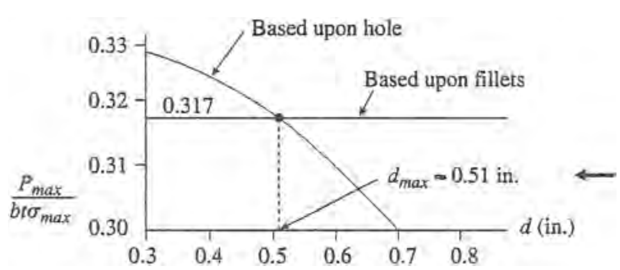
$b = 2.4$ in. $\quad d$ = diameter of the hole (in.)

$c_1 = b - d$

$$P_{\max} = \sigma_{\text{nom}}\, c_1 t = \frac{\sigma_{\max}}{K}(b - d)t$$

$$= \frac{1}{K}\left(1 - \frac{d}{b}\right) bt\sigma_{\max}$$

d(in.)	d/b	K	$P_{max}/bt\sigma_{max}$
0.3	0.125	2.66	0.329
0.4	0.167	2.57	0.324
0.5	0.208	2.49	0.318
0.6	0.250	2.41	0.311
0.7	0.292	2.37	0.299



Nonlinear Behavior (Changes in Lengths of Bars)

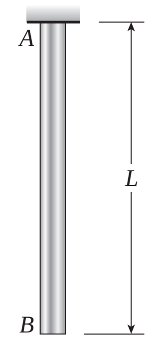
Problem 2.11-1 A bar AB of length L and weight density γ hangs vertically under its own weight (see figure). The stress-strain relation for the material is given by the Ramberg-Osgood equation (Eq. 2-71):

$$\epsilon = \frac{\sigma}{E} + \frac{\sigma_0 \alpha}{E}\left(\frac{\sigma}{\sigma_0}\right)^m$$

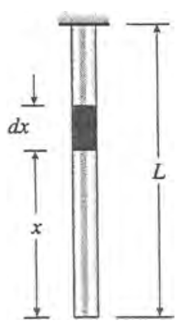
Derive the following formula

$$\delta = \frac{\gamma L^2}{2E} + \frac{\sigma_0 \alpha L}{(m + 1)E}\left(\frac{\gamma L}{\sigma_0}\right)^m$$

for the elongation of the bar.



Solution 2.11-1 Bar hanging under its own weight



Let A = cross-sectional area

Let N = axial force at distance x

$N = \gamma A x$

$$\sigma = \frac{N}{A} = \gamma x$$

STRAIN AT DISTANCE x

$$\varepsilon = \frac{\sigma}{E} + \frac{\sigma_0 \alpha}{E}\left(\frac{\sigma}{\sigma_0}\right)^m = \frac{\gamma x}{E} + \frac{\sigma_0}{\alpha E}\left(\frac{\gamma x}{\sigma_0}\right)^m$$

ELONGATION OF BAR

$$\delta = \int_0^L \varepsilon dx = \int_0^L \frac{\gamma x}{E} dx + \frac{\sigma_0 \alpha}{E}\int_0^L \left(\frac{\gamma x}{\sigma_0}\right)^m dx$$

$$= \frac{\gamma L^2}{2E} + \frac{\sigma_0 \alpha L}{(m + 1)E}\left(\frac{\gamma L}{\sigma_0}\right)^m \qquad \text{Q.E.D.} \quad \leftarrow$$

Problem 2.11-2 A prismatic bar of length $L = 1.8$ m and cross-sectional area $A = 480\ \text{mm}^2$ is loaded by forces $P_1 = 30$ kN and $P_2 = 60$ kN (see figure). The bar is constructed of magnesium alloy having a stress-strain curve described by the following Ramberg-Osgood equation:

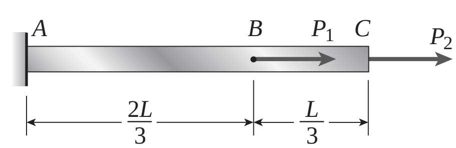
$$\epsilon = \frac{\sigma}{45{,}000} + \frac{1}{618}\left(\frac{\sigma}{170}\right)^{10} \qquad (\sigma = \text{MPa})$$

in which σ has units of megapascals.

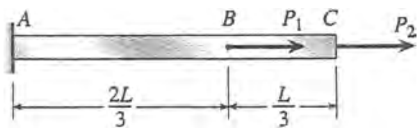
(a) Calculate the displacement δ_C of the end of the bar when the load P_1 acts alone.

(b) Calculate the displacement when the load P_2 acts alone.

(c) Calculate the displacement when both loads act simultaneously.



Solution 2.11-2 Axially loaded bar



$L = 1.8$ m $\quad A = 480$ mm^2

$P_1 = 30$ kN $\quad P_2 = 60$ kN

Ramberg–Osgood Equation:

$$\varepsilon = \frac{\sigma}{45{,}000} + \frac{1}{618}\left(\frac{\sigma}{170}\right)^{10} \quad (\sigma = \text{MPa})$$

Find displacement at end of bar.

(a) P_1 ACTS ALONE

$$AB\text{: } \sigma = \frac{P_1}{A} = \frac{30 \text{ kN}}{480 \text{ mm}^2} = 62.5 \text{ MPa}$$

$$\varepsilon = 0.001389$$

$$\delta_c = \varepsilon\left(\frac{2L}{3}\right) = 1.67 \text{ mm} \quad \leftarrow$$

(b) P_2 ACTS ALONE

$$ABC\text{:}\sigma = \frac{P_2}{A} = \frac{60 \text{ kN}}{480 \text{ mm}^2} = 125 \text{ MPa}$$

$$\varepsilon = 0.002853$$

$$\delta_c = \varepsilon L = 5.13 \text{ mm} \quad \leftarrow$$

(c) BOTH P_1 AND P_2 ARE ACTING

$$AB\text{:}\sigma = \frac{P_1 + P_2}{A} = \frac{90 \text{ kN}}{480 \text{ mm}^2} = 187.5 \text{ MPa}$$

$$\varepsilon = 0.008477$$

$$\delta_{AB} = \varepsilon\left(\frac{2L}{3}\right) = 10.17 \text{ mm}$$

$$BC\text{:}\sigma = \frac{P_2}{A} = \frac{60 \text{ kN}}{480 \text{ mm}^2} = 125 \text{ MPa}$$

$$\varepsilon = 0.002853$$

$$\delta_{BC} = \varepsilon\left(\frac{L}{3}\right) = 1.71 \text{ mm}$$

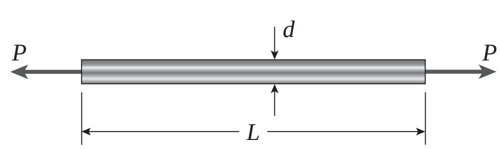
$$\delta_C = \delta_{AB} + \delta_{BC} = 11.88 \text{ mm} \quad \leftarrow$$

(Note that the displacement when both loads act simultaneously is *not* equal to the sum of the displacements when the loads act separately.)

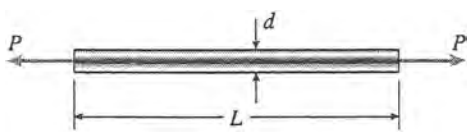
Problem 2.11-3 A circular bar of length $L = 32$ in. and diameter $d = 0.75$ in. is subjected to tension by forces P (see figure). The wire is made of a copper alloy having the following *hyperbolic stress-strain relationship*:

$$\sigma = \frac{18{,}000\epsilon}{1 + 300\epsilon} \quad 0 \leq \epsilon \leq 0.03 \quad (\sigma = \text{ksi})$$

(a) Draw a stress-strain diagram for the material.
(b) If the elongation of the wire is limited to 0.25 in. and the maximum stress is limited to 40 ksi, what is the allowable load P?



Solution 2.11-3 Copper bar in tension

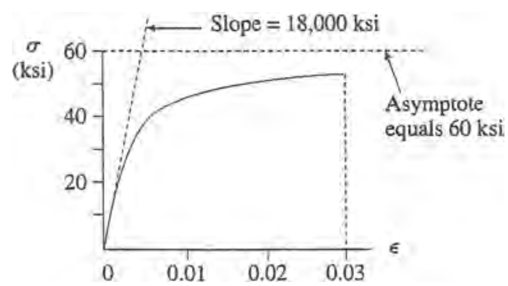


$L = 32$ in. $\quad d = 0.75$ in.

$$A = \frac{\pi d^2}{4} = 0.4418 \text{ in.}^2$$

(a) STRESS-STRAIN DIAGRAM

$$\sigma = \frac{18{,}000\varepsilon}{1 + 300\varepsilon} \quad 0 \leq \varepsilon \leq 0.03 \quad (\sigma = \text{ksi})$$



(b) ALLOWABLE LOAD P

Max. elongation $\delta_{max} = 0.25$ in.

Max. stress $\sigma_{max} = 40$ ksi

Based upon elongation:

$$\varepsilon_{max} = \frac{\delta_{max}}{L} = \frac{0.25 \text{ in.}}{32 \text{ in.}} = 0.007813$$

$$\sigma_{max} = \frac{18{,}000\varepsilon_{max}}{1 + 300\varepsilon_{max}} = 42.06 \text{ ksi}$$

BASED UPON STRESS:

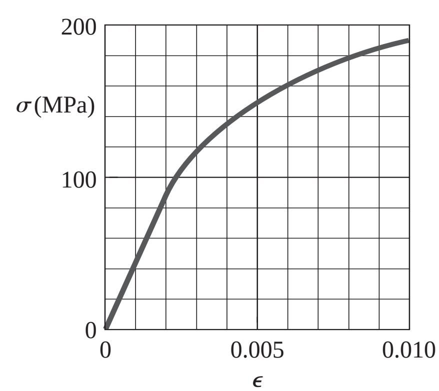
$\sigma_{max} = 40$ ksi

Stress governs. $P = \sigma_{max} A = (40 \text{ ksi})(0.4418 \text{ in.}^2)$

$= 17.7 \text{ k} \quad \leftarrow$

Problem 2.11-4 A prismatic bar in tension has length $L = 2.0$ m and cross-sectional area $A = 249$ mm^2. The material of the bar has the stress-strain curve shown in the figure.

Determine the elongation δ of the bar for each of the following axial loads: $P = 10$ kN, 20 kN, 30 kN, 40 kN, and 45 kN. From these results, plot a diagram of load P versus elongation δ (load-displacement diagram).



Solution 2.11-4 Bar in tension

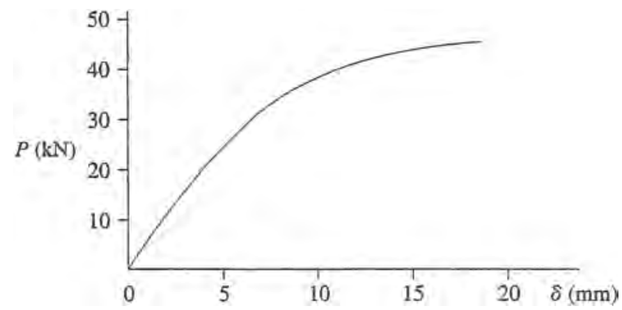
$L = 2.0$ m

$A = 249$ mm^2

STRESS-STRAIN DIAGRAM
(See the problem statement for the diagram)

LOAD-DISPLACEMENT DIAGRAM

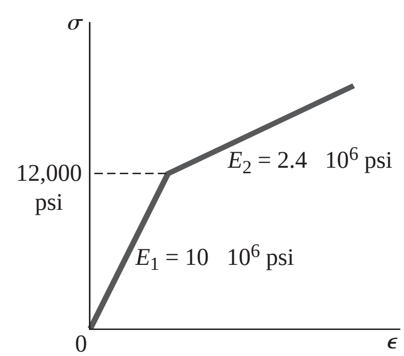
P (kN)	$\sigma = P/A$ (MPa)	ε (from diagram)	$\delta = \varepsilon L$ (mm)
10	40	0.0009	1.8
20	80	0.0018	3.6
30	120	0.0031	6.2
40	161	0.0060	12.0
45	181	0.0081	16.2



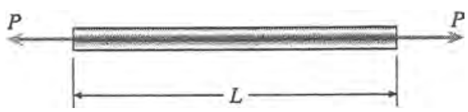
NOTE: The load-displacement curve has the same shape as the stress-strain curve.

Problem 2.11-5 An aluminum bar subjected to tensile forces P has length $L = 150$ in. and cross-sectional area $A = 2.0$ in.^{2} The stress-strain behavior of the aluminum may be represented approximately by the bilinear stress-strain diagram shown in the figure.

Calculate the elongation δ of the bar for each of the following axial loads: $P = 8$ k, 16 k, 24 k, 32 k, and 40 k. From these results, plot a diagram of load P versus elongation δ (load-displacement diagram).



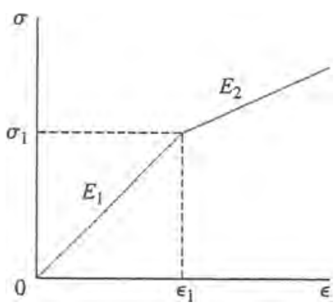
Solution 2.11-5 Aluminum bar in tension



$L = 150$ in.

$A = 2.0$ in.^{2}

STRESS-STRAIN DIAGRAM



$$E_1 = 10 \times 10^6 \text{ psi}$$

$$E_2 = 2.4 \times 10^6 \text{ psi}$$

$$\sigma_1 = 12{,}000 \text{ psi}$$

$$\varepsilon_1 = \frac{\sigma_1}{E_1} = \frac{12{,}000 \text{ psi}}{10 \times 10^6 \text{ psi}}$$

$$= 0.0012$$

For $0 \leq \sigma \leq \sigma_1$:

$$\varepsilon = \frac{\sigma}{E_2} = \frac{\sigma}{10 \times 10^6 \text{psi}} (\sigma = \text{psi}) \qquad \text{Eq. (1)}$$

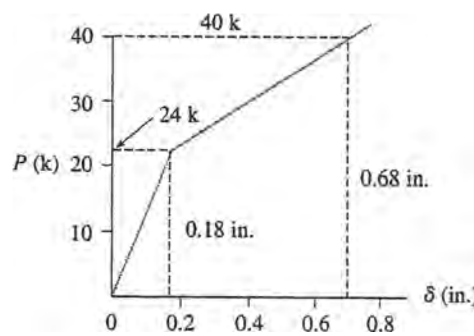
For $\sigma \geq \sigma_1$:

$$\varepsilon = \varepsilon_1 + \frac{\sigma - \sigma_1}{E_2} = 0.0012 + \frac{\sigma - 12{,}000}{2.4 \times 10^6}$$

$$= \frac{\sigma}{2.4 \times 10^6} - 0.0038 \quad (\sigma = \text{psi}) \quad \text{Eq. (2)}$$

LOAD-DISPLACEMENT DIAGRAM

P (k)	$\sigma = P/A$ (psi)	ε (from Eq. 1 or Eq. 2)	$\delta = \varepsilon L$ (in.)
8	4,000	0.00040	0.060
16	8,000	0.00080	0.120
24	12,000	0.00120	0.180
32	16,000	0.00287	0.430
40	20,000	0.00453	0.680



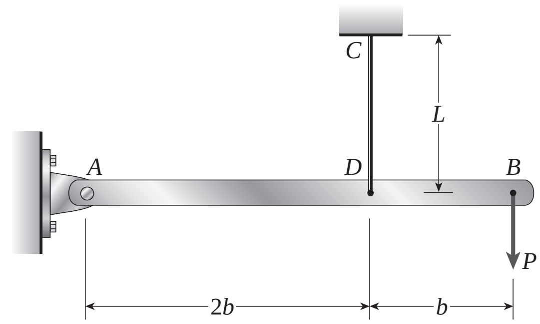
Problem 2.11-6 A rigid bar AB, pinned at end A, is supported by a wire CD and loaded by a force P at end B (see figure). The wire is made of high-strength steel having modulus of elasticity $E = 210$ GPa and yield stress $\sigma_Y = 820$ MPa. The length of the wire is $L = 1.0$ m and its diameter is $d = 3$ mm. The stress-strain diagram for the steel is defined by the *modified power law*, as follows:

$$\sigma = E\epsilon \quad 0 \le \sigma \le \sigma_Y$$

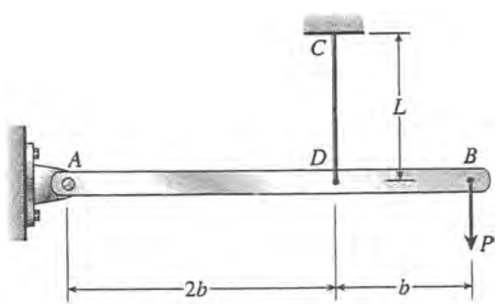
$$\sigma = \sigma_Y\left(\frac{E\epsilon}{\sigma_Y}\right)^n \quad \sigma \ge \sigma_Y$$

(a) Assuming $n = 0.2$, calculate the displacement δ_B at the end of the bar due to the load P. Take values of P from 2.4 kN to 5.6 kN in increments of 0.8 kN.

(b) Plot a load-displacement diagram showing P versus δ_B.



Solution 2.11-6 Rigid bar supported by a wire



Wire: $E = 210$ GPa

$\sigma_Y = 820$ MPa

$L = 1.0$ m

$d = 3$ mm

$$A = \frac{\pi d^2}{4} = 7.0686 \text{ mm}^2$$

STRESS-STRAIN DIAGRAM

$$\sigma = E\varepsilon \qquad (0 \le \sigma \le \sigma_Y) \tag{1}$$

$$\sigma = \sigma_Y\left(\frac{E\varepsilon}{\sigma_Y}\right)^n \qquad (\sigma \ge \sigma_Y) \qquad (n = 0.2) \tag{2}$$

(a) DISPLACEMENT δ_B AT END OF BAR

$$\delta = \text{elongation of wire } \delta_B = \frac{3}{2}\delta = \frac{3}{2}\varepsilon L \tag{3}$$

Obtain ε from stress-strain equations:

$$\text{From Eq. (1): } \varepsilon = \frac{\sigma E}{(0 \le \sigma \le \sigma_Y)} \tag{4}$$

$$\text{From Eq. (2): } \varepsilon = \frac{\sigma_Y}{E}\left(\frac{\sigma}{\sigma_Y}\right)^{1/n} \tag{5}$$

Axial force in wire: $F = \dfrac{3P}{2}$

$$\text{Stress in wire: } \sigma = \frac{F}{A} = \frac{3P}{2A} \tag{6}$$

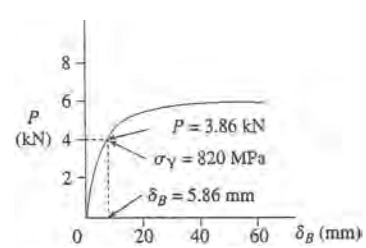
PROCEDURE: Assume a value of P
Calculate σ from Eq. (6)
Calculate ε from Eq. (4) or (5)
Calculate δ_B from Eq. (3)

P (kN)	σ (MPa) Eq. (6)	ε Eq. (4) or (5)	δ_B (mm) Eq. (3)
2.4	509.3	0.002425	3.64
3.2	679.1	0.003234	4.85
4.0	848.8	0.004640	6.96
4.8	1018.6	0.01155	17.3
5.6	1188.4	0.02497	37.5

For $\sigma = \sigma_Y = 820$ MPa:

$\varepsilon = 0.0039048 \qquad P = 3.864$ kN $\qquad \delta_B = 5.86$ mm

(b) LOAD-DISPLACEMENT DIAGRAM

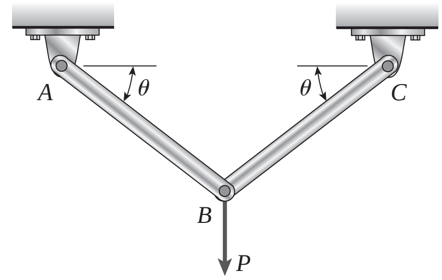


Elastoplastic Analysis

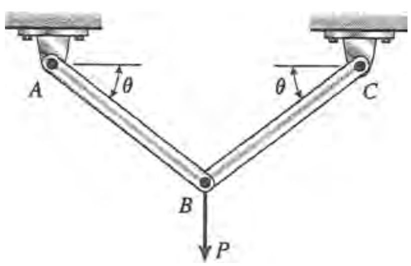
The problems for Section 2.12 are to be solved assuming that the material is elastoplastic with yield stress σ_Y, yield strain ϵ_Y, and modulus of elasticity E in the linearly elastic region (see Fig. 2-70).

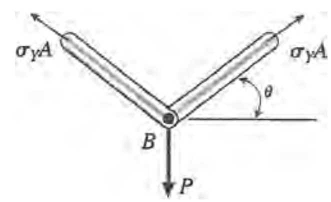
Problem 2.12-1 Two identical bars AB and BC support a vertical load P (see figure). The bars are made of steel having a stress-strain curve that may be idealized as elastoplastic with yield stress σ_Y. Each bar has cross-sectional area A.

Determine the yield load P_Y and the plastic load P_P.



Solution 2.12-1 Two bars supporting a load P





Structure is statically determinate. The yield load P_Y and the plastic lead P_P occur at the same time, namely, when both bars reach the yield stress.

JOINT B

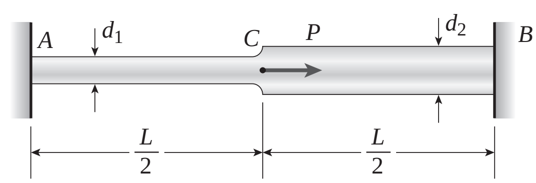
$\Sigma F_{\text{vert}} = 0$

$(2\sigma_Y A) \sin \theta = P$

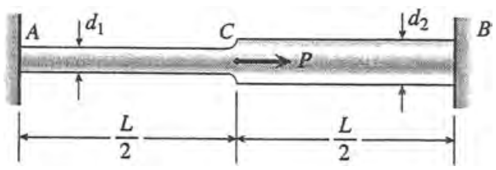
$P_Y = P_P = 2\sigma_Y A \sin \theta \quad \leftarrow$

Problem 2.12-2 A stepped bar ACB with circular cross sections is held between rigid supports and loaded by an axial force P at midlength (see figure). The diameters for the two parts of the bar are $d_1 = 20$ mm and $d_2 = 25$ mm, and the material is elastoplastic with yield stress $\sigma_Y = 250$ MPa.

Determine the plastic load P_P.



Solution 2.12-2 Bar between rigid supports

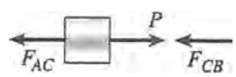


$d_1 = 20$ mm $\quad d_2 = 25$ mm $\quad \sigma_Y = 250$ MPa

DETERMINE THE PLASTIC LOAD P_P:

At the plastic load, all parts of the bar are stressed to the yield stress.

Point C:



$$F_{AC} = \sigma_Y A_1 \qquad F_{CB} = \sigma_Y A_2$$

$$P = F_{AC} + F_{CB}$$

$$P_P = \sigma_Y A_1 + \sigma_Y A_2 = \sigma_Y (A_1 + A_2) \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

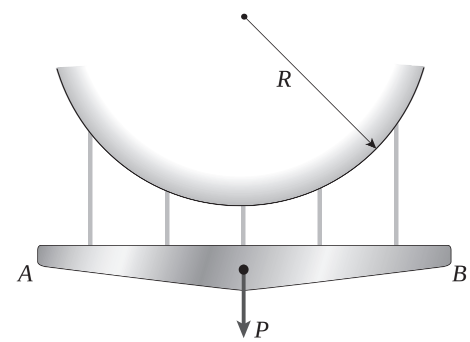
$$P_P = (250 \text{ MPa})\left(\frac{\pi}{4}\right)(d_1^2 + d_2^2)$$

$$= (250 \text{ MPa})\left(\frac{\pi}{4}\right)[(20 \text{ mm})^2 + (25 \text{ mm})^2]$$

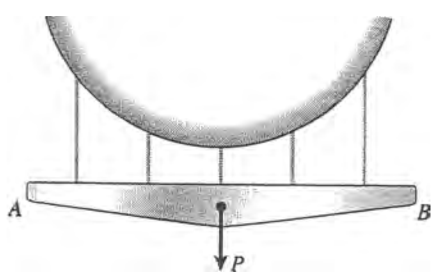
$$= 201 \text{ kN} \quad \leftarrow$$

Problem 2.12-3 A horizontal rigid bar AB supporting a load P is hung from five symmetrically placed wires, each of cross-sectional area A (see figure). The wires are fastened to a curved surface of radius R.

(a) Determine the plastic load P_P if the material of the wires is elastoplastic with yield stress σ_Y.
(b) How is P_P changed if bar AB is flexible instead of rigid?
(c) How is P_P changed if the radius R is increased?



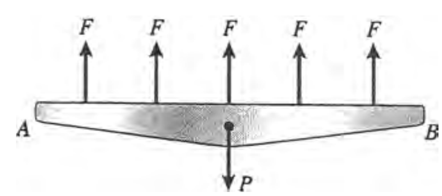
Solution 2.12-3 Rigid bar supported by five wires



(a) PLASTIC LOAD P_P

At the plastic load, each wire is stressed to the yield stress. $\therefore P_P = 5\sigma_Y A \quad \leftarrow$

$F = \sigma_Y A$



(b) BAR AB IS FLEXIBLE

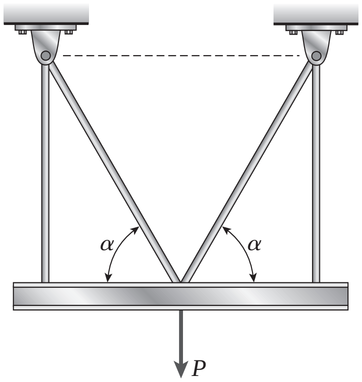
At the plastic load, each wire is stressed to the yield stress, so the plastic load is not changed. $\leftarrow$

(c) RADIUS R IS INCREASED

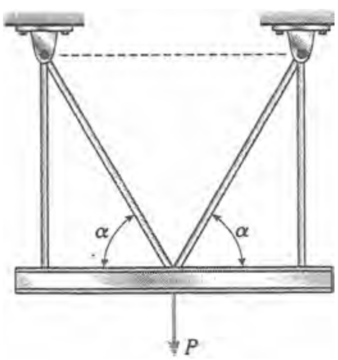
Again, the forces in the wires are not changed, so the plastic load is not changed. $\leftarrow$

Problem 2.12-4 A load P acts on a horizontal beam that is supported by four rods arranged in the symmetrical pattern shown in the figure. Each rod has cross-sectional area A and the material is elastoplastic with yield stress σ_Y.

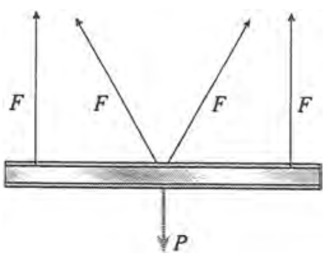
Determine the plastic load P_P.



Solution 2.12-4 Beam supported by four rods



At the plastic load, all four rods are stressed to the yield stress.



$F = \sigma_Y A$

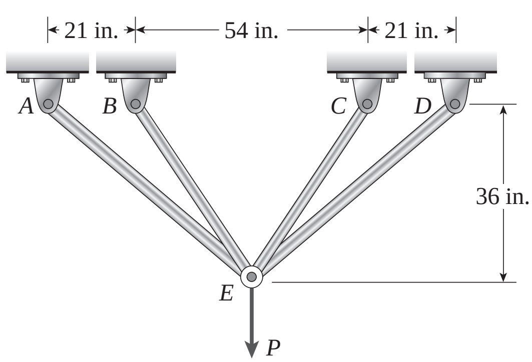
Sum forces in the vertical direction and solve for the load:

$P_P = 2F + 2F \sin \alpha$

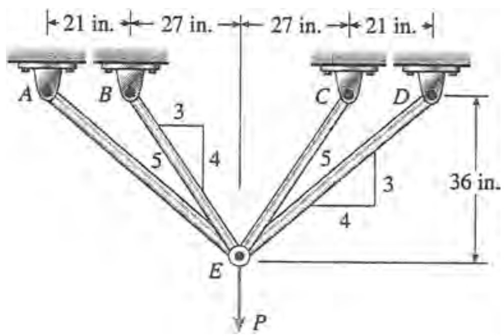
$P_P = 2\sigma_Y A\,(1 + \sin \alpha) \quad \leftarrow$

Problem 2.12-5 The symmetric truss $ABCDE$ shown in the figure is constructed of four bars and supports a load P at joint E. Each of the two outer bars has a cross-sectional area of 0.307 in.^{2}, and each of the two inner bars has an area of 0.601 in.^{2} The material is elastoplastic with yield stress $\sigma_Y = 36$ ksi.

Determine the plastic load P_P.

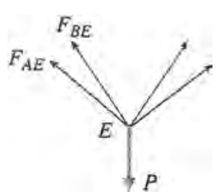


Solution 2.12-5 Truss with four bars



$L_{AE} = 60$ in. $\quad L_{BE} = 45$ in.

JOINT E



Equilibrium:

$$2F_{AE}\left(\frac{3}{5}\right) + 2F_{BE}\left(\frac{4}{5}\right) = P$$

or

$$P = \frac{6}{5}F_{AE} + \frac{8}{5}F_{BE}$$

PLASTIC LOAD P_P

At the plastic load, all bars are stressed to the yield stress.

$$F_{AE} = \sigma_Y A_{AE} \qquad F_{BE} = \sigma_Y A_{BE}$$

$$P_P = \frac{6}{5}\sigma_Y A_{AE} + \frac{8}{5}\sigma_Y A_{BE} \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

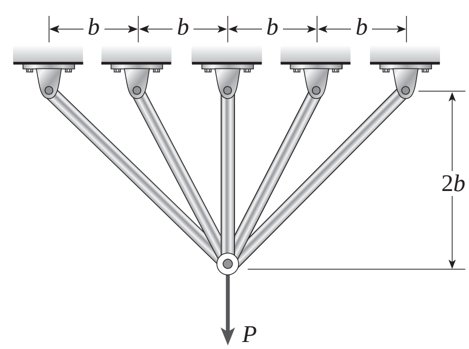
$$A_{AE} = 0.307 \text{ in.}^2 \quad A_{BE} = 0.601 \text{ in.}^2$$

$$\sigma_Y = 36 \text{ ksi}$$

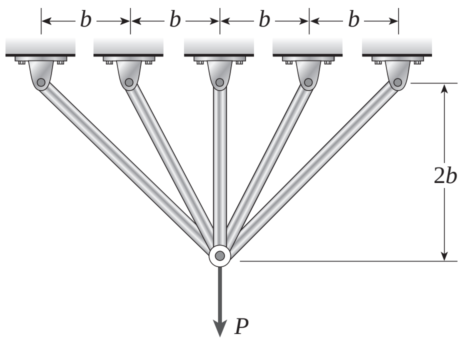
$$P_P = \frac{6}{5}(36 \text{ ksi})(0.307 \text{ in.}^2) + \frac{8}{5}(36 \text{ ksi})(0.601 \text{ in.}^2)$$

$$= 13.26 \text{ k} + 34.62 \text{ k} = 47.9 \text{ k} \quad \leftarrow$$

Problem 2.12-6 Five bars, each having a diameter of 10 mm, support a load P as shown in the figure. Determine the plastic load P_P if the material is elastoplastic with yield stress $\sigma_Y = 250$ MPa.



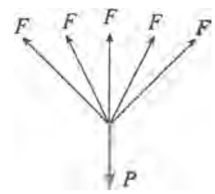
Solution 2.12-6 Truss consisting of five bars



$d = 10\text{ mm}$

$$A = \frac{\pi d^2}{4} = 78.54\text{ mm}^2$$

$\sigma_Y = 250\text{ MPa}$



At the plastic load, all five bars are stressed to the yield stress

$F = \sigma_Y A$

Sum forces in the vertical direction and solve for the load:

$$P_P = 2F\left(\frac{1}{\sqrt{2}}\right) + 2F\left(\frac{2}{\sqrt{5}}\right) + F$$

$$= \frac{\sigma_Y A}{5}(5\sqrt{2} + 4\sqrt{5} + 5)$$

$$= 4.2031\sigma_Y A \quad \leftarrow$$

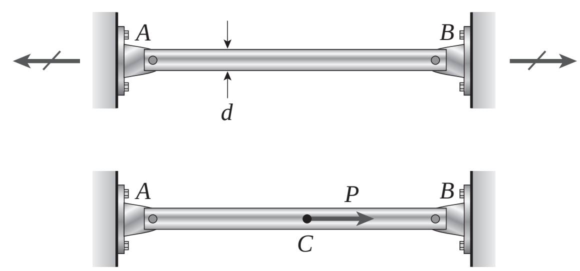
Substitute numerical values:

$$P_P = (4.2031)(250\text{ MPa})(78.54\text{ mm}^2)$$

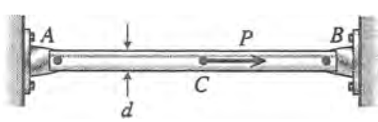
$$= 82.5\text{ kN} \quad \leftarrow$$

Problem 2.12-7 A circular steel rod AB of diameter $d = 0.60$ in. is stretched tightly between two supports so that initially the tensile stress in the rod is 10 ksi (see figure). An axial force P is then applied to the rod at an intermediate location C.

(a) Determine the plastic load P_P if the material is elastoplastic with yield stress $\sigma_Y = 36$ ksi.
(b) How is P_P changed if the initial tensile stress is doubled to 20 ksi?



Solution 2.12-7 Bar held between rigid supports



$d = 0.6$ in.

$\sigma_Y = 36$ ksi

Initial tensile stress = 10 ksi

(a) Plastic load P_P

The presence of the initial tensile stress does not affect the plastic load. Both parts of the bar must yield in order to reach the plastic load.

Point C:

$$\xleftarrow{\sigma_Y A} C \xrightarrow{P} \xleftarrow{\sigma_Y A}$$

$$P_P = 2\sigma_Y A = (2)\,(36\text{ ksi})\left(\frac{\pi}{4}\right)(0.60\text{ in.})^2$$

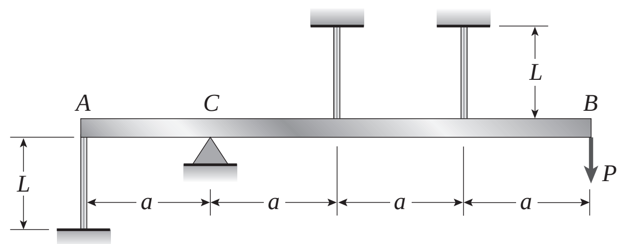
$$= 20.4\text{ k} \quad \leftarrow$$

(b) Initial tensile stress is doubled

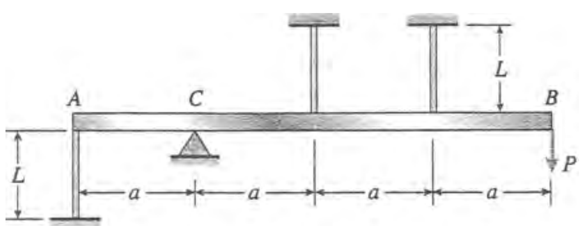
P_P is not changed. $\leftarrow$

Problem 2.12-8 A rigid bar *ACB* is supported on a fulcrum at *C* and loaded by a force *P* at end *B* (see figure). Three identical wires made of an elastoplastic material (yield stress σ_Y and modulus of elasticity *E*) resist the load *P*. Each wire has cross-sectional area *A* and length *L*.

(a) Determine the yield load P_Y and the corresponding yield displacement δ_Y at point *B*.
(b) Determine the plastic load P_P and the corresponding displacement δ_P at point *B* when the load just reaches the value P_P.
(c) Draw a load-displacement diagram with the load *P* as ordinate and the displacement δ_B of point *B* as abscissa.

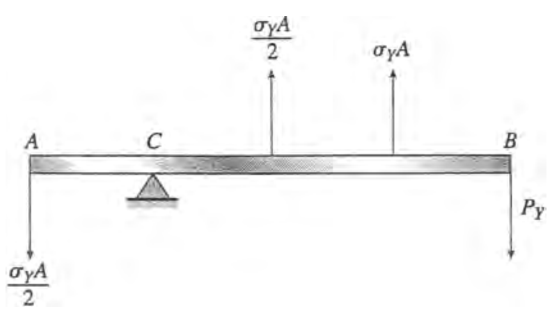


Solution 2.12-8 Rigid bar supported by wires



(a) YIELD LOAD P_Y

Yielding occurs when the most highly stressed wire reaches the yield stress σ_Y



$\Sigma M_C = 0$

$P_Y = \sigma_Y A \quad \leftarrow$

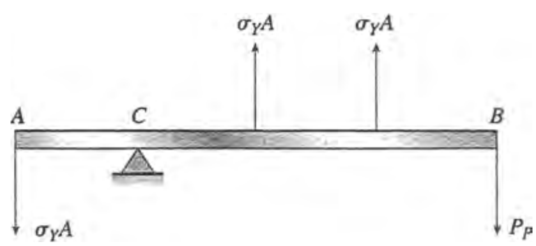
At point *A*:

$$\delta_A = \left(\frac{\sigma_Y A}{2}\right)\left(\frac{L}{EA}\right) = \frac{\sigma_Y L}{2E}$$

At point *B*:

$$\delta_B = 3\delta_A = \delta_Y = \frac{3\sigma_Y L}{2E} \quad \leftarrow$$

(b) PLASTIC LOAD P_P



At the plastic load, all wires reach the yield stress.

$\Sigma M_C = 0$

$$P_P = \frac{4\sigma_Y A}{3} \quad \leftarrow$$

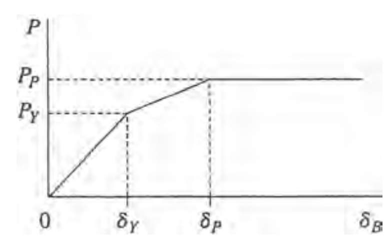
At point *A*:

$$\delta_A = (\sigma_Y A)\left(\frac{L}{EA}\right) = \frac{\sigma_Y L}{E}$$

At point *B*:

$$\delta_B = 3\delta_A = \delta_P = \frac{3\sigma_Y L}{E} \quad \leftarrow$$

(c) LOAD-DISPLACEMENT DIAGRAM

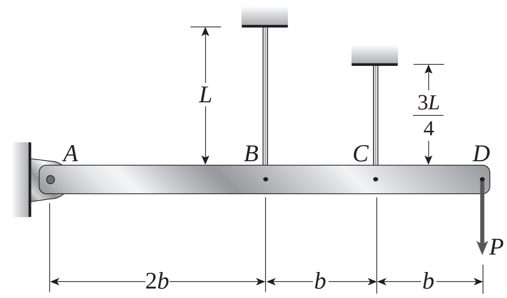


$$P_P = \frac{4}{3}P_Y$$

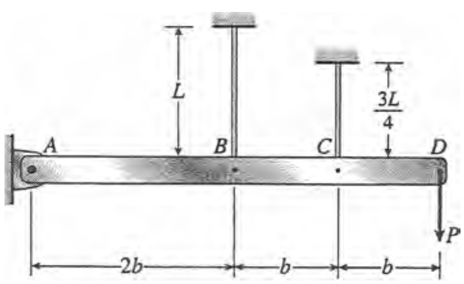
$$\delta_P = 2\delta_Y$$

Problem 2.12-9 The structure shown in the figure consists of a horizontal rigid bar $ABCD$ supported by two steel wires, one of length L and the other of length $3L/4$. Both wires have cross-sectional area A and are made of elastoplastic material with yield stress σ_Y and modulus of elasticity E. A vertical load P acts at end D of the bar.

(a) Determine the yield load P_Y and the corresponding yield displacement δ_Y at point D.
(b) Determine the plastic load P_P and the corresponding displacement δ_P at point D when the load just reaches the value P_P.
(c) Draw a load-displacement diagram with the load P as ordinate and the displacement δ_D of point D as abscissa.



Solution 2.12-9 Rigid bar supported by two wires

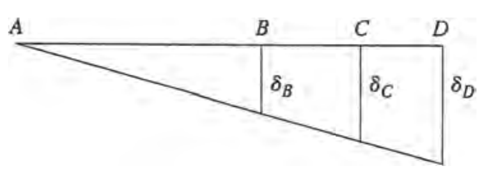


A = cross-sectional area

σ_Y = yield stress

E = modulus of elasticity

Displacement diagram

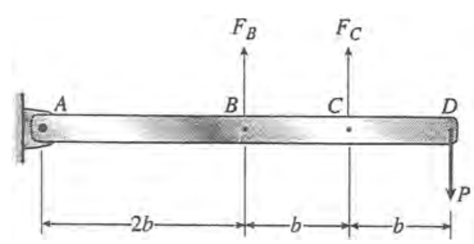


Compatibility:

$$\delta_C = \frac{3}{2}\delta_B \qquad (1)$$

$$\delta_D = 2\delta_B \qquad (2)$$

Free-body diagram



Equilibrium:

$\Sigma M_A = 0$ $\qquad F_B(2b) + F_C(3b) = P(4b)$

$$2F_B + 3F_C = 4P$$

(3)

Force-displacement relations

$$\delta_B = \frac{F_BL}{EA} \qquad \delta_C = \frac{F_C\left(\frac{3}{4}L\right)}{EA} \qquad (4, 5)$$

Substitute into Eq. (1):

$$\frac{3F_CL}{4EA} = \frac{3F_BL}{2EA}$$

$$F_C = 2F_B \qquad (6)$$

STRESSES

$$\sigma_B = \frac{F_B}{A} \quad \sigma_C = \frac{F_C}{A} \quad \sigma_C = 2\sigma_B \qquad (7)$$

Wire C has the larger stress. Therefore, it will yield first.

(a) YIELD LOAD

$$\sigma_C = \sigma_Y \qquad \sigma_B = \frac{\sigma_C}{2} = \frac{\sigma_Y}{2} \qquad \text{(From Eq. 7)}$$

$$F_C = \sigma_Y A \qquad F_B = \frac{1}{2}\sigma_Y A$$

From Eq. (3):

$$2\left(\frac{1}{2}\sigma_Y A\right) + 3(\sigma_Y A) = 4P$$

$$P = P_Y = \sigma_Y A \quad \leftarrow$$

From Eq. (4):

$$\delta_B = \frac{F_B L}{EA} = \frac{\sigma_Y L}{2E}$$

From Eq. (2):

$$\delta_D = \delta_Y = 2\delta_B = \frac{\sigma_Y L}{E} \quad \leftarrow$$

(b) PLASTIC LOAD

At the plastic load, both wires yield.

$$\sigma_B = \sigma_Y = \sigma_C \qquad F_B = F_C = \sigma_Y A$$

From Eq. (3):

$$2(\sigma_Y A) + 3(\sigma_Y A) = 4P$$

$$P = P_P = \frac{5}{4}\sigma_Y A \quad \leftarrow$$

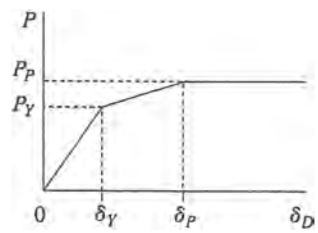
From Eq. (4):

$$\delta_B = \frac{F_B L}{EA} = \frac{\sigma_Y L}{E}$$

From Eq. (2):

$$\delta_D = \delta_P = 2\delta_B = \frac{2\sigma_Y L}{E} \quad \leftarrow$$

(c) LOAD-DISPLACEMENT DIAGRAM



$$P_P = \frac{5}{4}P_Y$$

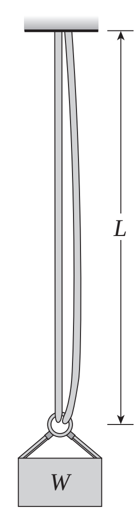
$$\delta_P = 2\delta_Y$$

Problem 2.12-10 Two cables, each having a length L of approximately 40 m, support a l oaded container of weight W (see figure). The cables, which have effective cross-sectional area $A = 48.0\ \text{mm}^2$ and effective modulus of elasticity $E = 160$ GPa, are identical except that one cable is longer than the other when they are hanging separately and unloaded. The difference in lengths is $d = 100$ mm. The cables are made of steel having an elastoplastic stress-strain diagram with $\sigma_Y = 500$ MPa. Assume that the weight W is initially zero and is slowly increased by the addition of material to the container.

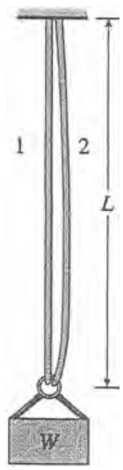
(a) Determine the weight W_Y that first produces yielding of the shorter cable. Also, determine the corresponding elongation δ_Y of the shorter cable.

(b) Determine the weight W_P that produces yielding of both cables. Also, determine the elongation δ_P of the shorter cable when the weight W just reaches the value W_P.

(c) Construct a load-displacement diagram showing the weight W as ordinate and the elongation δ of the shorter cable as abscissa. (*Hint*: The load displacement diagram is not a single straight line in the region $0 \leq W \leq W_Y$.)



Solution 2.12-10 Two cables supporting a load



$L = 40$ m $\quad A = 48.0 \text{ mm}^2$

$E = 160$ GPa

$d =$ difference in length $= 100$ mm

$\sigma_Y = 500$ MPa

INITIAL STRETCHING OF CABLE 1

Initially, cable 1 supports all of the load. Let $W_1 =$ load required to stretch cable 1 to the same length as cable 2

$$W_1 = \frac{EA}{L}d = 19.2 \text{ kN}$$

$\delta_1 = 100$ mm (elongation of cable 1)

$$\sigma_1 = \frac{W_1}{A} = \frac{Ed}{L} = 400 \text{ MPa } (\sigma_1 < \sigma_Y \therefore > \text{OK})$$

(a) YIELD LOAD W_Y

Cable 1 yields first. $F_1 = \sigma_Y A = 24$ kN

$\delta_{1Y} =$ total elongation of cable 1

$\delta_{1Y} =$ total elongation of cable 1

$$\delta_{1Y} = \frac{F_1 L}{EA} = \frac{\sigma_Y L}{E} = 0.125 \text{ m} = 125 \text{ mm}$$

$$\delta_Y = \delta_{1Y} = 125 \text{ mm} \quad \leftarrow$$

$\delta_{2Y} =$ elongation of cable 2

$$= \delta_{1Y} - d = 25 \text{ mm}$$

$$F_2 = \frac{EA}{L}\delta_{2Y} = 4.8 \text{ kN}$$

$$W_Y = F_1 + F_2 = 24 \text{ kN} + 4.8 \text{ kN}$$

$$= 28.8 \text{ kN} \quad \leftarrow$$

(b) PLASTIC LOAD W_P

$F_1 = \sigma_Y A \qquad F_2 = \sigma_Y A$

$$W_P = 2\sigma_Y A = 48 \text{ kN} \quad \leftarrow$$

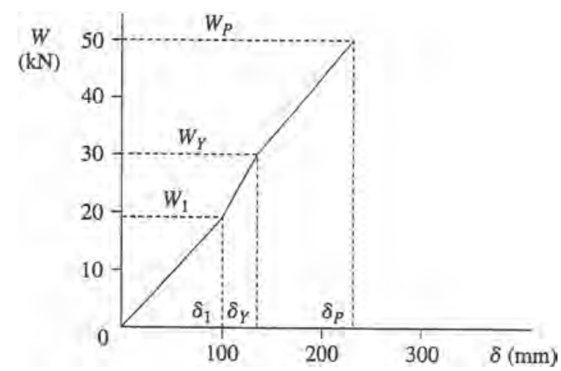
$\delta_{2P} =$ elongation of cable 2

$$= F_2\left(\frac{L}{EA}\right) = \frac{\sigma_Y L}{E} = 0.125 \text{ mm} = 125 \text{ mm}$$

$$\delta_{1P} = \delta_{2P} + d = 225 \text{ mm}$$

$$\delta_P = \delta_{1P} = 225 \text{ mm} \quad \leftarrow$$

(c) LOAD-DISPLACEMENT DIAGRAM



$$\frac{W_Y}{W_1} = 1.5 \qquad \frac{\delta_Y}{\delta_1} = 1.25$$

$$\frac{W_P}{W_Y} = 1.667 \qquad \frac{\delta_P}{\delta_Y} = 1.8$$

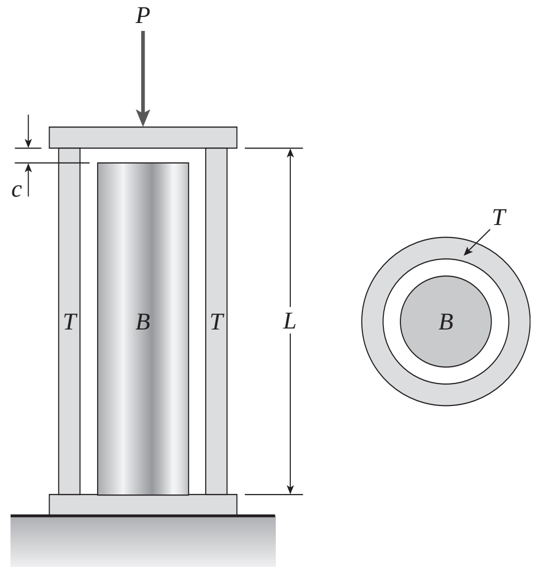
$0 < W < W_1$: slope $= 192{,}000$ N/m

$W_1 < W < W_Y$: slope $= 384{,}000$ N/m

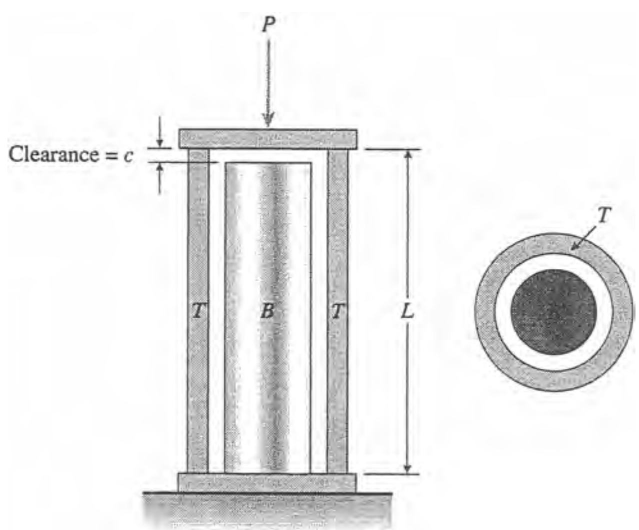
$W_Y < W < W_P$: slope $= 192{,}000$ N/m

Problem 2.12-11 A hollow circular tube T of length $L = 15$ in. is uniformly compressed by a force P acting through a rigid plate (see figure). The outside and inside diameters of the tube are 3.0 and 2.75 in., repectively. A concentric solid circular bar B of 1.5 in. diameter is mounted inside the tube. When no load is present, there is a clearance $c = 0.010$ in. between the bar B and the rigid plate. Both bar and tube are made of steel having an elastoplastic stress-strain diagram with $E = 29 \times 10^3$ ksi and $\sigma_Y = 36$ ksi.

(a) Determine the yield load P_Y and the corresponding shortening δ_Y of the tube.
(b) Determine the plastic load P_P and the corresponding shortening δ_P of the tube.
(c) Construct a load-displacement diagram showing the load P as ordinate and the shortening δ of the tube as abscissa. (*Hint*: The load-displacement diagram is not a single straight line in the region $0 \le P \le P_Y$.)



Solution 2.12-11 Tube and bar supporting a load



$L = 15$ in.

$c = 0.010$ in.

$E = 29 \times 10^3$ ksi

$\sigma_Y = 36$ ksi

TUBE:

$d_2 = 3.0$ in.

$d_1 = 2.75$ in.

$$A_T = \frac{\pi}{4}(d_2^2 - d_1^2) = 1.1290 \text{ in.}^2$$

BAR

$d = 1.5$ in.

$$A_B = \frac{\pi d^2}{4} = 1.7671 \text{ in.}^2$$

INITIAL SHORTENING OF TUBE T

Initially, the tube supports all of the load.
Let P_1 = load required to close the clearance

$$P_1 = \frac{EA_T}{L}c = 21{,}827 \text{ lb}$$

Let δ_1 = shortening of tube $\qquad \delta_1 = c = 0.010$ in.

$$\sigma_1 = \frac{P_1}{A_T} = 19{,}330 \text{ psi} \qquad (\sigma_1 < \sigma_Y \therefore \text{OK})$$

(a) YIELD LOAD P_Y

Because the tube and bar are made of the same material, and because the strain in the tube is larger than the strain in the bar, the tube will yield first.

$F_T = \sigma_Y A_T = 40{,}644$ lb

δ_{TY} = shortening of tube at the yield stress

$$\sigma_{TY} = \frac{F_T L}{EA_T} = \frac{\sigma_Y L}{E} = 0.018621 \text{ in.}$$

$$\delta_Y = \delta_{TY} = 0.018621 \text{ in.} \quad \leftarrow$$

δ_{BY} = shortening of bar

$= \delta_{TY} - c = 0.008621$ in.

$$F_B = \frac{EA_B}{L}\delta_{BY} = 29{,}453 \text{ lb}$$

$P_Y = F_T + F_B = 40{,}644 \text{ lb} + 29{,}453 \text{ lb}$

$= 70{,}097$ lb

$$P_Y = 70{,}100 \text{ lb} \quad \leftarrow$$

(b) PLASTIC LOAD P_P

$F_T = \sigma_Y A_T \qquad F_B = \sigma_Y A_B$

$P_P = F_T + F_B = \sigma_Y(A_T + A_B)$

$= 104{,}300 \text{ lb} \quad \leftarrow$

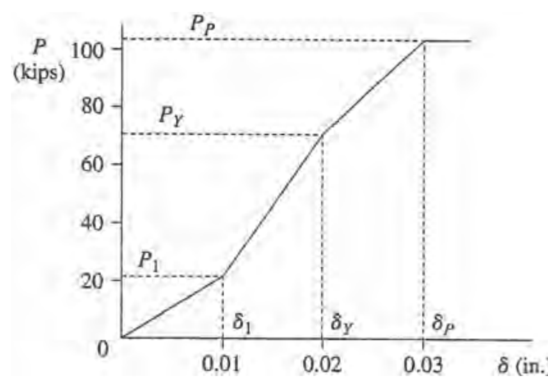
δ_{BP} = shortening of bar

$$= F_B\left(\frac{L}{EA_B}\right) = \frac{\sigma_Y L}{E} = 0.018621 \text{ in.}$$

$\delta_{TP} = \delta_{BP} + c = 0.028621$ in.

$$\delta_P = \delta_{TP} = 0.02862 \text{ in.} \quad \leftarrow$$

(c) LOAD-DISPLACEMENT DIAGRAM



$$\frac{P_Y}{P_1} = 3.21 \qquad \frac{\delta_Y}{\delta_1} = 1.86$$

$$\frac{P_P}{P_Y} = 1.49 \qquad \frac{\delta_P}{\delta_Y} = 1.54$$

$0 < P < P_1$: slope = 2180 k/in.

$P_1 < P < P_Y$: slope = 5600 k/in.

$P_Y < P < P_P$: slope = 3420 k/in.

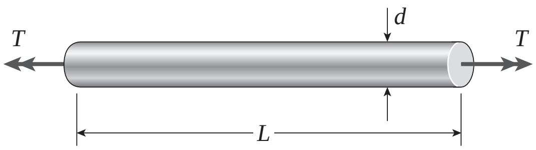
3

Torsion

Torsional Deformations

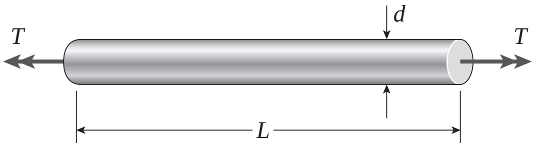
Problem 3.2-1 A copper rod of length $L = 18.0$ in. is to be twisted by torques T (see figure) until the angle of rotation between the ends of the rod is 3.0°.

If the allowable shear strain in the copper is 0.0006 rad, what is the maximum permissible diameter of the rod?



Probs. 3.2-1 and 3.2-2

Solution 3.2-1 Copper rod in torsion



$L = 18.0$ in.

$$\phi = 3.0° = (3.0)\left(\frac{\pi}{180}\right) \text{ rad}$$

$$= 0.05236 \text{ rad}$$

$\gamma_{allow} = 0.0006$ rad

Find d_{max}

From Eq. (3-3):

$$\gamma_{max} = \frac{r\phi}{L} = \frac{d\phi}{2L}$$

$$d_{max} = \frac{2L\gamma_{allow}}{\phi} = \frac{(2)(18.0 \text{ in.})(0.0006 \text{ rad})}{0.05236 \text{ rad}}$$

$$d_{max} = 0.413 \text{ in.} \longleftarrow$$

Problem 3.2-2 A plastic bar of diameter $d = 56$ mm is to be twisted by torques T (see figure) until the angle of rotation between the ends of the bar is 4.0°.

If the allowable shear strain in the plastic is 0.012 rad, what is the minimum permissible length of the bar?

Solution 3.2-2

NUMERICAL DATA

$$d = 56 \text{ mm}$$

$$\gamma_a = 0.012 \text{ radians}$$

$$\phi = 4\left(\frac{\pi}{180}\right) \text{ radians}$$

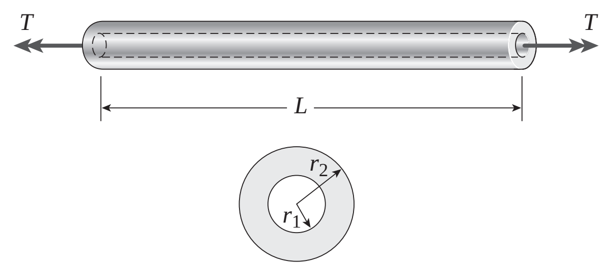
solution based on Equ. (3-3): $L_{min} = \dfrac{d\phi}{2\gamma_a}$

$L_{min} = 162.9 \text{ mm}$ $\longleftarrow$

Problem 3.2-3 A circular aluminum tube subjected to pure torsion by torques T (see figure) has an outer radius r_2 equal to 1.5 times the inner radius r_1.

(a) If the maximum shear strain in the tube is measured as 400×10^{-6} rad, what is the shear strain γ_1 at the inner surface?

(b) If the maximum allowable rate of twist is 0.125 degrees per foot and the maximum shear strain is to be kept at 400×10^{-6} rad by adjusting the torque T, what is the minimum required outer radius $(r_2)_{min}$?



Probs. 3.2-3, 3.2-4, and 3.2-5

Solution 3.2-3

NUMERICAL DATA

$$r_2 = 1.5 r_1 \quad \gamma_{max} = 400 \times (10^{-6}) \text{ radians}$$

$$\theta = 0.125\left(\frac{\pi}{180}\right)\left(\frac{1}{12}\right)$$

$$\theta = 1.818 \times 10^{-4} \text{ rad/m.}$$

(a) SHEAR STRAIN AT INNER SURFACE AT RADIUS r_1

$$\gamma_1 = \frac{r_1}{r_2}\gamma_{max} \quad \gamma_1 = \frac{1}{1.5}\gamma_{max}$$

$\gamma_1 = 267 \times 10^{-6}$ radians $\longleftarrow$

(b) MIN. REQUIRED OUTER RADIUS

$$r_{2min} = \frac{\gamma_{max}}{\theta} \quad r_{2min} = \frac{\gamma_{max}}{\theta}$$

$r_{2min} = 2.2$ inches $\longleftarrow$

Problem 3.2-4 A circular steel tube of length $L = 1.0$ m is loaded in torsion by torques T (see figure).

(a) If the inner radius of the tube is $r_1 = 45$ mm and the measured angle of twist between the ends is 0.5°, what is the shear strain γ_1 (in radians) at the inner surface?

(b) If the maximum allowable shear strain is 0.0004 rad and the angle of twist is to be kept at 0.45° by adjusting the torque T, what is the maximum permissible outer radius $(r_2)_{max}$?

Solution 3.2-4

NUMERICAL DATA

$L = 1000$ mm

$r_1 = 45$ mm

$\phi = 0.5\left(\frac{\pi}{180}\right)$ radians

(a) SHEAR STRAIN AT INNER SURFACE

$\gamma_1 = r_1\frac{\phi}{L}$ $\quad \gamma_1 = 393 \times 10^{-6}$ radians $\longleftarrow$

(b) MAX. PERMISSIBLE OUTER RADIUS

$\phi = 0.45\left(\frac{\pi}{180}\right)$ radians $\quad \gamma_{max} = r_2\frac{\phi}{L}$

$\gamma_{max} = 0.0004$ radians $\quad r_{2max} = \gamma_{max}\frac{L}{\phi}$

$r_{2max} = 50.9$ mm $\longleftarrow$

Problem 3.2-5 Solve the preceding problem if the length $L = 56$ in., the inner radius $r_1 = 1.25$ in., the angle of twist is 0.5°, and the allowable shear strain is 0.0004 rad.

Solution 3.2-5

NUMERICAL DATA

$L = 56$ inches $\quad r_1 = 1.25$ inches

$\phi = 0.5\left(\frac{\pi}{180}\right)$ radians

$\gamma_a = 0.0004$ radians

(a) SHEAR STRAIN g1 (IN RADIANS) AT THE INNER SURFACE

$\gamma_1 = r_1\frac{\phi}{L}$ $\quad \gamma_1 = 195 \times 10^{-6}$ radians $\longleftarrow$

(b) MAXIMUM PERMISSIBLE OUTER RADIUS $(r_2)_{max}$

$\phi = 0.5\left(\frac{\pi}{180}\right)$ radians

$\gamma_{max} = r_2\frac{\phi}{L}$

$\gamma_a = 0.0004$ radians

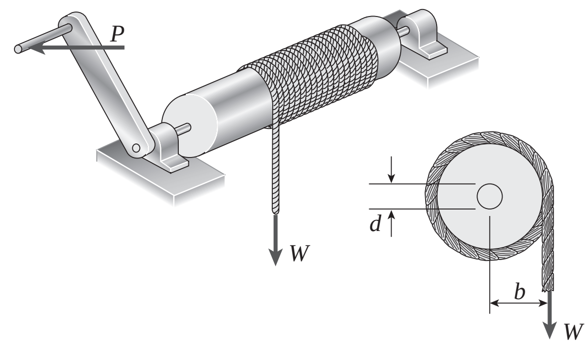
$r_{2max} = \gamma_a\frac{L}{\phi}$

$r_{2max} = 2.57$ inches $\longleftarrow$

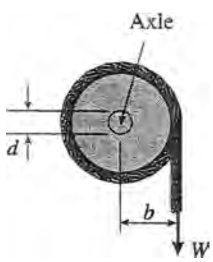
Circular Bars and Tubes

Problem 3.3-1 A prospector uses a hand-powered winch (see figure) to raise a bucket of ore in his mine shaft. The axle of the winch is a steel rod of diameter $d = 0.625$ in. Also, the distance from the center of the axle to the center of the lifting rope is $b = 4.0$ in.

If the weight of the loaded bucket is $W = 100$ lb, what is the maximum shear stress in the axle due to torsion?



Solution 3.3-1 Hand-powered winch



$d = 0.625$ in.

$b = 4.0$ in.

$W = 100$ lb

Torque T applied to the axle:

$T = Wb = 400$ lb-in.

Maximum shear stress in the axle

From Eq. (3-12):

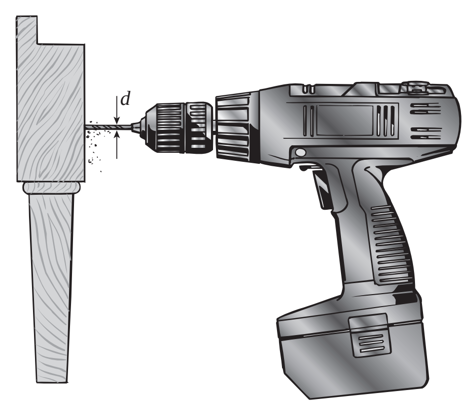
$$\tau_{max} = \frac{16T}{\pi d^3}$$

$$\tau_{max} = \frac{(16)(400 \text{ lb-in.})}{\pi(0.625 \text{ in.})^3}$$

$$\tau_{max} = 8{,}340 \text{ psi} \quad \leftarrow$$

Problem 3.3-2 When drilling a hole in a table leg, a furniture maker uses a hand-operated drill (see figure) with a bit of diameter $d = 4.0$ mm.

(a) If the resisting torque supplied by the table leg is equal to 0.3 N·m, what is the maximum shear stress in the drill bit?
(b) If the shear modulus of elasticity of the steel is $G = 75$ GPa, what is the rate of twist of the drill bit (degrees per meter)?



Solution 3.3-2 Torsion of a drill bit

$d = 4.0$ mm $\quad T = 0.3$ N·m $\quad G = 75$ GPa

(a) Maximum shear stress

From Eq. (3-12):

$$\tau_{\max} = \frac{16T}{\pi d^3}$$

$$\tau_{\max} = \frac{16(0.3\ \text{N·m})}{\pi(4.0\ \text{mm})^3}$$

$$\tau_{\max} = 23.8\ \text{MPa} \quad \leftarrow$$

(b) Rate of twist

From Eq. (3-14):

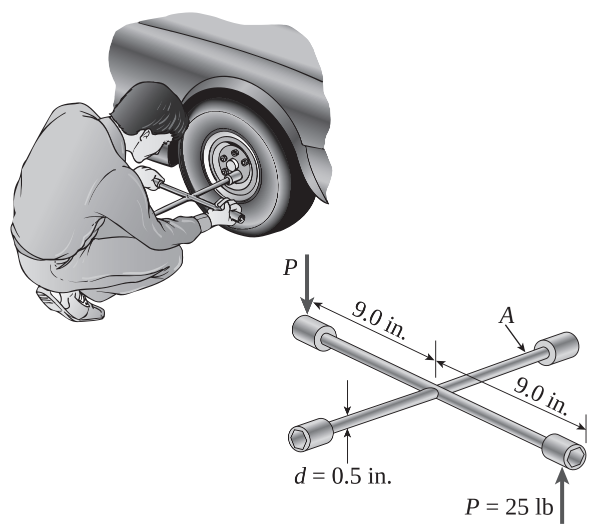
$$\theta = \frac{T}{GI_P}$$

$$\theta = \frac{0.3\ \text{N·m}}{(75\ \text{GPa})\left(\dfrac{\pi}{32}\right)(4.0\ \text{mm})^4}$$

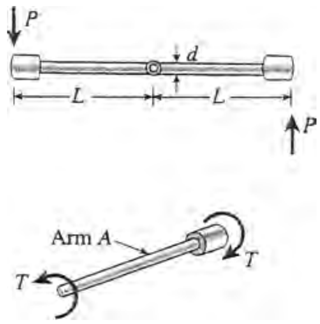
$$\theta = 0.1592\ \text{rad/m} = 9.12^{\circ}/\text{m} \quad \leftarrow$$

Problem 3.3-3 While removing a wheel to change a tire, a driver applies forces $P = 25$ lb at the ends of two of the arms of a lug wrench (see figure). The wrench is made of steel with shear modulus of elasticity $G = 11.4 \times 10^6$ psi. Each arm of the wrench is 9.0 in. long and has a solid circular cross section of diameter $d = 0.5$ in.

(a) Determine the maximum shear stress in the arm that is turning the lug nut (arm A).
(b) Determine the angle of twist (in degrees) of this same arm.



Solution 3.3-3 Lug wrench



$P = 25$ lb

$L = 9.0$ in.

$d = 0.5$ in.

$G = 11.4 \times 10^6$ psi

T = torque acting on arm A

$T = P(2L) = 2(25\text{ lb})(9.0\text{ in.})$

$= 450$ lb-in.

(a) MAXIMUM SHEAR STRESS

From Eq. (3-12):

$$\tau_{max} = \frac{16T}{\pi d^3} = \frac{(16)(450\text{ lb-in.})}{\pi(0.5\text{ in.})^3}$$

$$\tau_{max} = 18{,}300\text{ psi} \quad \leftarrow$$

(b) ANGLE OF TWIST

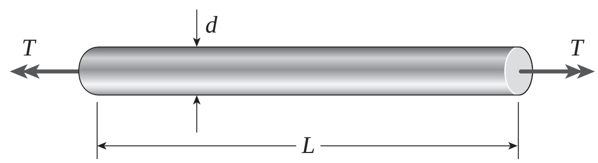
From Eq. (3-15):

$$\phi = \frac{TL}{GI_P} = \frac{(450\text{ lb-in.})(9.0\text{ in.})}{(11.4 \times 10^6\text{ psi})\left(\frac{\pi}{32}\right)(0.5\text{ in.})^4}$$

$$\phi = 0.05790\text{ rad} = 3.32^\circ \quad \leftarrow$$

Problem 3.3-4 An aluminum bar of solid circular cross section is twisted by torques T acting at the ends (see figure). The dimensions and shear modulus of elasticity are as follows: $L = 1.4$ m, $d = 32$ mm, and $G = 28$ GPa.

(a) Determine the torsional stiffness of the bar.
(b) If the angle of twist of the bar is 5°, what is the maximum shear stress? What is the maximum shear strain (in radians)?



Solution 3.3-4

(a) TORSIONAL STIFFNESS OF BAR

$d = 32$ mm $\qquad G = 28$ GPa

$$k_T = \frac{GI_p}{L} \qquad I_p = \frac{\pi}{32}d^4$$

$$I_p = 1.029 \times 10^5\text{ mm}^4$$

$$k_T = \frac{28(10^9)\left(\frac{\pi}{32}0.032^4\right)}{1.4}$$

$$k_T = 2059\text{ N}\cdot\text{m} \quad \leftarrow$$

(b) MAX SHEAR STRESS AND STRAIN

$$\phi = 5\left(\frac{\pi}{180}\right)\text{ radians}$$

$$\text{T} = k_T\,\phi \qquad \tau_{max} = \frac{T\left(\frac{d}{2}\right)}{I_p}$$

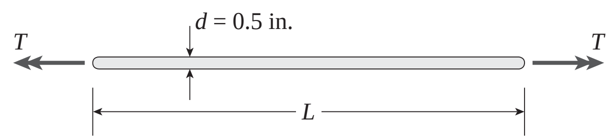
$$\tau_{max} = 27.9\text{ MPa} \quad \leftarrow$$

$$\gamma_{max} = \frac{\tau_{max}}{\text{G}}$$

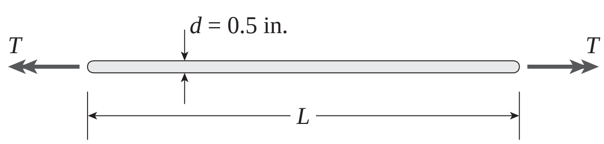
$$\gamma_{max} = 997 \times 10^{-6}\text{ radians} \quad \leftarrow$$

Problem 3.3-5 A high-strength steel drill rod used for boring a hole in the earth has a diameter of 0.5 in. (see figure). The allowable shear stress in the steel is 40 ksi and the shear modulus of elasticity is 11,600 ksi.

What is the minimum required length of the rod so that one end of the rod can be twisted 30° with respect to the other end without exceeding the allowable stress?



Solution 3.3-5 Steel drill rod



$G = 11{,}600$ psi

$d = 0.5$ in.

$$\phi = 30° = (30°)\left(\frac{\pi}{180}\right)\text{rad} = 0.52360 \text{ rad}$$

$\tau_{allow} = 40$ ksi

MINIMUM LENGTH

From Eq. (3-12): $$\tau_{max} = \frac{16T}{\pi d^3} \qquad (1)$$

From Eq. (3-15): $$\phi = \frac{TL}{GI_P} = \frac{32TL}{G\pi d^4}$$

$$T = \frac{G\pi d^4 \phi}{32L}$$, substitute T into Eq. (1):

$$\tau_{max} = \left(\frac{16}{\pi d^3}\right)\left(\frac{G\pi d^4\phi}{32L}\right) = \frac{Gd\phi}{2L}$$

$$L_{min} = \frac{Gd\phi}{2\tau_{allow}}$$

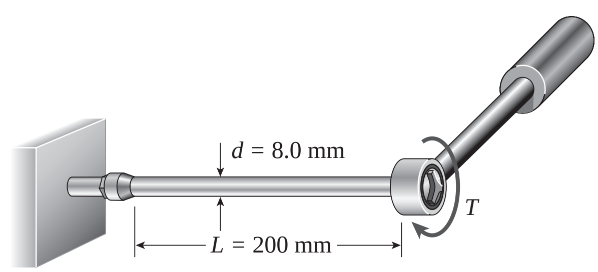
$$= \frac{(11{,}600 \text{ ksi})(0.5 \text{ in.})(0.52360 \text{ rad})}{2(40 \text{ ksi})}$$

$$L_{min} = 38.0 \text{ in.} \quad \leftarrow$$

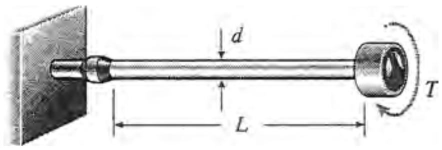
Problem 3.3-6 The steel shaft of a socket wrench has a diameter of 8.0 mm. and a length of 200 mm (see figure).

If the allowable stress in shear is 60 MPa, what is the maximum permissible torque T_{max} that may be exerted with the wrench?

Through what angle ϕ (in degrees) will the shaft twist under the action of the maximum torque? (Assume $G = 78$ GPa and disregard any bending of the shaft.)



Solution 3.3-6 Socket wrench



$d = 8.0$ mm $\quad L = 200$ mm

$\tau_{allow} = 60$ MPa $\quad G = 78$ GPa

MAXIMUM PERMISSIBLE TORQUE

From Eq. (3-12): $\tau_{max} = \dfrac{16T}{\pi d^3}$

$$T_{max} = \frac{\pi d^3 \tau_{max}}{16}$$

$$T_{max} = \frac{\pi (8.0 \text{ mm})^3 (60 \text{ MPa})}{16}$$

$$T_{max} = 6.03 \text{ N} \cdot \text{m} \quad \leftarrow$$

ANGLE OF TWIST

From Eq. (3-15): $\phi = \dfrac{T_{max} L}{G I_P}$

From Eq. (3-12): $T_{max} = \dfrac{\pi d^3 \tau_{max}}{16}$

$$\phi = \left(\frac{\pi d^3 \tau_{max}}{16}\right)\left(\frac{L}{G I_P}\right) \qquad I_P = \frac{\pi d^4}{32}$$

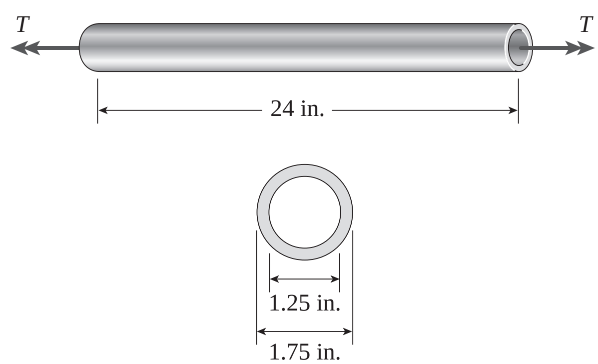
$$\phi = \frac{\pi d^3 \tau_{max} L (32)}{16 G (\pi d^4)} = \frac{2 \tau_{max} L}{G d}$$

$$\phi = \frac{2(60 \text{ MPa})(200 \text{ mm})}{(78 \text{ GPa})(8.0 \text{ mm})} = 0.03846 \text{ rad}$$

$$\phi = (0.03846 \text{ rad})\left(\frac{180}{\pi} \text{ deg/rad}\right) = 2.20^\circ \quad \leftarrow$$

Problem 3.3-7 A circular tube of aluminum is subjected to torsion by torques T applied at the ends (see figure). The bar is 24 in. long, and the inside and outside diameters are 1.25 in. and 1.75 in., respectively. It is determined by measurement that the angle of twist is 4° when the torque is 6200 lb-in.

Calculate the maximum shear stress τ_{max} in the tube, the shear modulus of elasticity G, and the maximum shear strain γ_{max} (in radians).



Solution 3.3-7

NUMERICAL DATA

$L = 24 \text{ in.} \qquad r_2 = \frac{1.75}{2} \text{ in.} \qquad r_1 = \frac{1.25}{2} \text{ in.}$

$\phi = 4\left(\frac{\pi}{180}\right) \text{ radians} \qquad T = 6200 \text{ lb-in.}$

MAX. SHEAR STRESS $\quad \tau_{max} = \frac{Tr_2}{I_p}$

$I_p = \frac{\pi}{2}\left(r_2^4 - r_1^4\right) \qquad I_p = 0.681 \text{ in.}^4$

$\tau_{max} = \frac{Tr_2}{I_p} \qquad \tau_{max} = 7965 \text{ psi} \quad \leftarrow$

MAX. SHEAR STRAIN $\quad \gamma_{max} = \frac{r_2}{L}\phi$

$\gamma_{max} = 0.00255 \text{ radians} \quad \leftarrow$

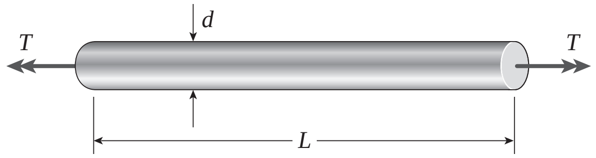
SHEAR MODULUS OF ELASTICITY G $\quad G = \frac{\tau_{max}}{\gamma_{max}}$

$G = 3.129 \times 10^6 \text{ psi}$

or $\quad G = \frac{TL}{\phi I_p} \qquad G = 3.13 \times 10^6 \text{ psi} \quad \leftarrow$

Problem 3.3-8 A propeller shaft for a small yacht is made of a solid steel bar 104 mm in diameter. The allowable stress in shear is 48 MPa, and the allowable rate of twist is 2.0° in 3.5 meters.

Assuming that the shear modulus of elasticity is $G = 80$ GPa, determine the maximum torque T_{max} that can be applied to the shaft.



Solution 3.3-8

NUMERICAL DATA

$d = 104 \text{ mm} \qquad \tau_a = 48 \text{ MPa} \qquad \theta = \frac{\phi}{L}$

$\theta = \frac{2\left(\frac{\pi}{180}\right)}{3.5} \frac{\text{rad}}{\text{m}} \qquad G = 80 \text{ GPa}$

$I_p = \frac{\pi}{32}d^4 \qquad I_p = 1.149 \times 10^7 \text{ mm}^4$

FIND MAX. TORQUE BASED ON ALLOWABLE RATE OF TWIST

$T_{max} = \frac{GI_p\phi}{L} \qquad T_{max} = GI_p\theta$

$T_{max} = 9164 \text{ N} \cdot \text{m} \quad \leftarrow$

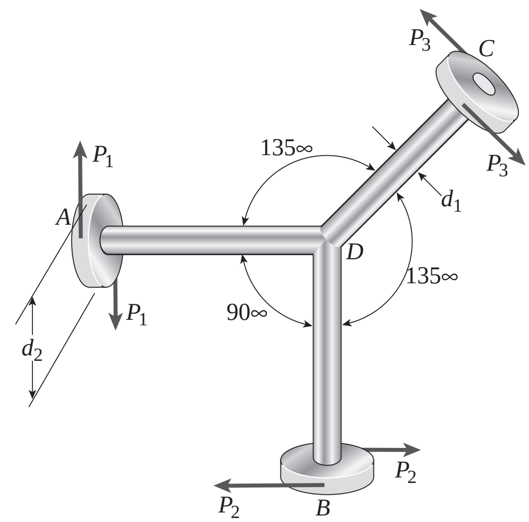
^ governs

FIND MAX. TORQUE BASED ON ALLOWABLE SHEAR STRESS

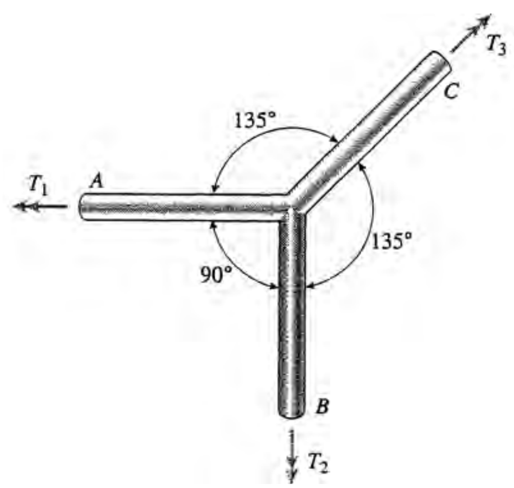
$T_{max} = \frac{\tau_a I_p}{\frac{d}{2}} \qquad T_{max} = 10{,}602 \text{ N} \cdot \text{m}$

Problem 3.3-9 Three identical circular disks A, B, and C are welded to the ends of three identical solid circular bars (see figure). The bars lie in a common plane and the disks lie in planes perpendicular to the axes of the bars. The bars are welded at their intersection D to form a rigid connection. Each bar has diameter $d_1 = 0.5$ in. and each disk has diameter $d_2 = 3.0$ in.

Forces P_1, P_2, and P_3 act on disks A, B, and C, respectively, thus subjecting the bars to torsion. If $P_1 = 28$ lb, what is the maximum shear stress τ_{max} in any of the three bars?



Solution 3.3-9 Three circular bars



d_1 = diameter of bars

$= 0.5$ in.

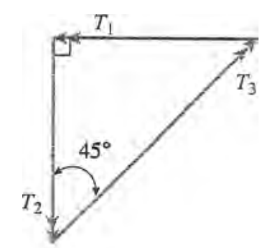
d_2 = diameter of disks

$= 3.0$ in.

$P_1 = 28$ lb

$T_1 = P_1 d_2 \quad T_2 = P_2 d_2 \quad T_3 = P_3 d_2$

THE THREE TORQUES MUST BE IN EQUILIBRIUM



T_3 is the largest torque

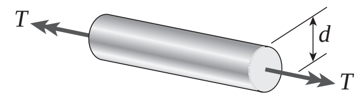
$T_3 = T_1\sqrt{2} = P_1 d_2 \sqrt{2}$

MAXIMUM SHEAR STRESS (Eq. 3-12)

$$\tau_{max} = \frac{16T}{\pi d^3} = \frac{16T_3}{\pi d_1^3} = \frac{16P_1 d_2 \sqrt{2}}{\pi d_1^3}$$

$$\tau_{max} = \frac{16(28 \text{ lb})(3.0 \text{ in.})\sqrt{2}}{\pi (0.5 \text{ in.})^3} = 4840 \text{ psi} \quad \leftarrow$$

Problem 3.3-10 The steel axle of a large winch on an ocean liner is subjected to a torque of 1.65 kN · m (see figure). What is the minimum required diameter d_{min} if the allowable shear stress is 48 MPa and the allowable rate of twist is 0.75°/m? (Assume that the shear modulus of elasticity is 80 GPa.)



Solution 3.3-10

NUMERICAL DATA

$T = 1.65\ \text{kN} \cdot \text{m} \qquad \tau_a = 48\ \text{MPa} \qquad G = 80\ \text{GPa}$

$$\theta_a = 0.75\left(\frac{\pi}{180}\right)\frac{\text{rad}}{\text{m}}$$

MIN. REQUIRED DIAMETER OF SHAFT BASED ON ALLOWABLE RATE OF TWIST

$$\theta = \frac{T}{GI_p} \qquad I_p = \frac{T}{G\theta} \qquad \frac{\pi}{32}d^4 = \frac{T}{G\theta}$$

$$d^4 = \frac{32T}{\pi G\theta_a} \qquad d_{min} = \left(\frac{32T}{\pi G\theta_a}\right)^{\frac{1}{4}}$$

$d_{min} = 0.063\ \text{m} \qquad d_{min} = 63.3\ \text{mm} \quad \leftarrow$
^ governs

MIN. REQUIRED DIAMETER OF SHAFT BASED ON ALLOWABLE SHEAR STRESS

$$\tau = \frac{Td}{2I_p} \qquad \tau = \frac{Td}{2\left(\frac{\pi}{32}d^4\right)}$$

$$d_{min} = \left[\frac{16T}{\pi\tau_a}\right]^{\frac{1}{3}} \qquad d_{min} = 0.056\ \text{m}$$

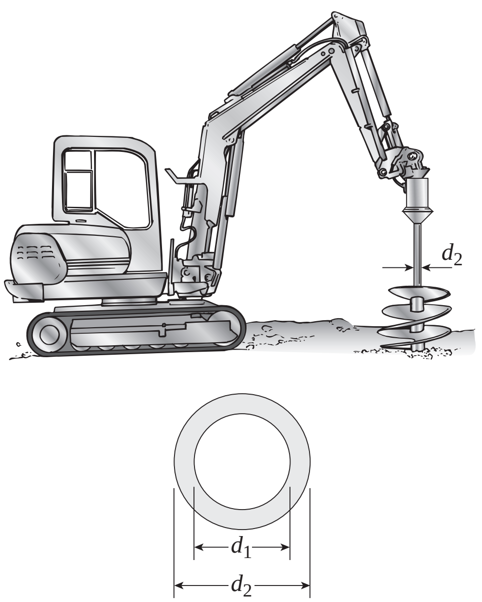
$d_{min} = 55.9\ \text{mm}$

Problem 3.3-11 A hollow steel shaft used in a construction auger has outer diameter $d_2 = 6.0$ in. and inner diameter $d_1 = 4.5$ in. (see figure). The steel has shear modulus of elasticity $G = 11.0 \times 10^6$ psi.

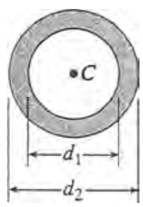
For an applied torque of 150 k-in., determine the following quantities:

(a) shear stress τ_2 at the outer surface of the shaft,
(b) shear stress τ_1 at the inner surface, and
(c) rate of twist θ (degrees per unit of length).

Also, draw a diagram showing how the shear stresses vary in magnitude along a radial line in the cross section.



Solution 3.3-11 Construction auger



$d_2 = 6.0$ in. $\quad r_2 = 3.0$ in.

$d_1 = 4.5$ in. $\quad r_1 = 2.25$ in.

$G = 11 \times 10^6$ psi

$T = 150$ k-in.

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4) = 86.98 \text{ in.}^4$$

(a) Shear stress at outer surface

$$\tau_2 = \frac{Tr_2}{I_P} = \frac{(150 \text{ k-in.})(3.0 \text{ in.})}{86.98 \text{ in.}^4}$$

$$= 5170 \text{ psi} \quad \leftarrow$$

(b) Shear stress at inner surface

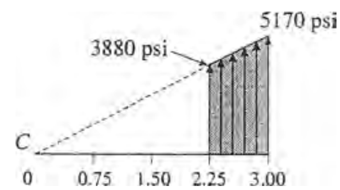
$$\tau_1 = \frac{Tr_1}{I_P} = \frac{r_1}{r_2}\tau_2 = 3880 \text{ psi} \quad \leftarrow$$

(c) Rate of twist

$$\theta = \frac{T}{GI_P} = \frac{(150 \text{ k-in.})}{(11 \times 10^6 \text{ psi})(86.98 \text{ in.})^4}$$

$$\theta = 157 \times 10^{-6} \text{ rad/in.} = 0.00898^\circ/\text{in.} \quad \leftarrow$$

(d) Shear stress diagram



Problem 3.3-12 Solve the preceding problem if the shaft has outer diameter $d_2 = 150$ mm and inner diameter $d_1 = 100$ mm. Also, the steel has shear modulus of elasticity $G = 75$ GPa and the applied torque is $16 \text{ kN} \cdot \text{m}$.

Solution 3.3-12 Construction auger

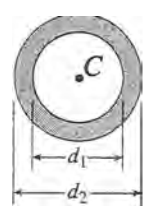
$d_2 = 150$ mm $\quad r_2 = 75$ mm

$d_1 = 100$ mm $\quad r_1 = 50$ mm

$G = 75$ GPa

$T = 16 \text{ kN} \cdot \text{m}$

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4) = 39.88 \times 10^6 \text{ mm}^4$$



(a) Shear stress at outer surface

$$\tau_2 = \frac{Tr_2}{I_P} = \frac{(16 \text{ kN} \cdot \text{m})(75 \text{ mm})}{39.88 \times 10^6 \text{ mm}^4}$$

$$= 30.1 \text{ MPa} \quad \leftarrow$$

(b) Shear stress at inner surface

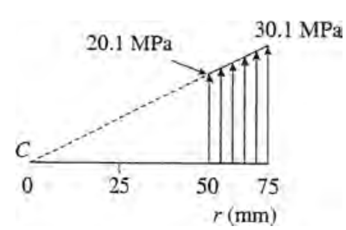
$$\tau_1 = \frac{Tr_1}{I_P} = \frac{r_1}{r_2}\tau_2 = 20.1 \text{ MPa} \quad \leftarrow$$

(c) Rate of twist

$$\theta = \frac{T}{GI_P} = \frac{16 \text{ kN} \cdot \text{m}}{(75 \text{ GPa})(39.88 \times 10^6 \text{ mm}^4)}$$

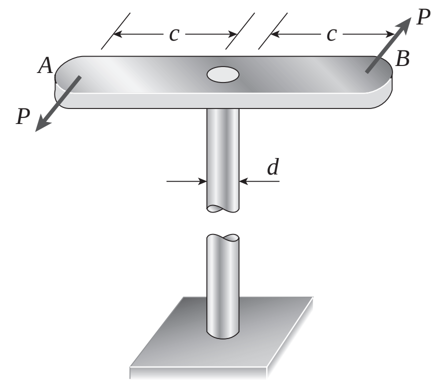
$$\theta = 0.005349 \text{ rad/m} = 0.306^\circ/\text{m} \quad \leftarrow$$

(d) Shear stress diagram

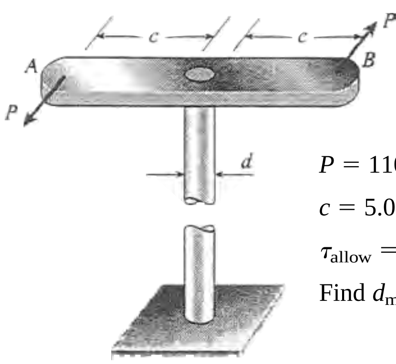


Problem 3.3-13 A vertical pole of solid circular cross section is twisted by horizontal forces $P = 1100$ lb acting at the ends of a horizontal arm AB (see figure). The distance from the outside of the pole to the line of action of each force is $c = 5.0$ in.

If the allowable shear stress in the pole is 4500 psi, what is the minimum required diameter d_{min} of the pole?



Solution 3.3-13 Vertical pole



$P = 1100$ lb

$c = 5.0$ in.

$\tau_{allow} = 4500$ psi

Find d_{min}

TORSION FORMULA

$$\tau_{max} = \frac{Tr}{I_P} = \frac{Td}{2I_P}$$

$$T = P(2c + d) \qquad I_P = \frac{\pi d^4}{32}$$

$$\tau_{max} = \frac{P(2c + d)d}{\pi d^4/16} = \frac{16P(2c + d)}{\pi d^3}$$

$$(\pi \tau_{max})d^3 - (16P)d - 32Pc = 0$$

SUBSTITUTE NUMERICAL VALUES:

UNITS: Pounds, Inches

$$(\pi)(4500)d^3 - (16)(1100)d - 32(1100)(5.0) = 0$$

or

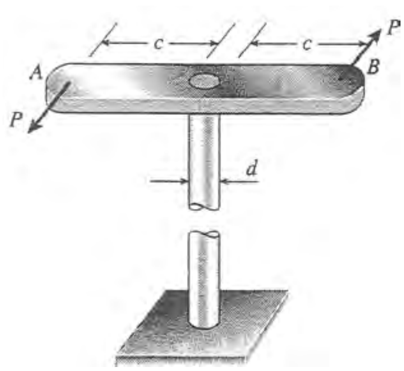
$$d^3 - 1.24495d - 12.4495 = 0$$

Solve numerically: $d = 2.496$ in.

$d_{min} = 2.50$ in. $\leftarrow$

Problem 3.3-14 Solve the preceding problem if the horizontal forces have magnitude $P = 5.0$ kN, the distance $c = 125$ mm, and the allowable shear stress is 30 MPa.

Solution 3.3-14 Vertical pole



$P = 5.0$ kN

$c = 125$ mm

$\tau_{\text{allow}} = 30$ MPa

Find $d_{\min}$

Torsion formula

$$\tau_{\max} = \frac{Tr}{I_P} = \frac{Td}{2I_P}$$

$$T = P(2c + d) \qquad I_P = \frac{\pi d^4}{32}$$

$$\tau_{\max} = \frac{P(2c + d)d}{\pi d^4/16} = \frac{16P(2c + d)}{\pi d^3}$$

$$(\pi\tau_{\max})d^3 - (16P)d - 32Pc = 0$$

Substitute numerical values:

Units: Newtons, Meters

$$(\pi)(30 \times 10^6)d^3 - (16)(5000)d - 32(5000)(0.125) = 0$$

or

$$d^3 - 848.826 \times 10^{-6}d - 212.207 \times 10^{-6} = 0$$

Solve numerically: $\quad d = 0.06438$ m

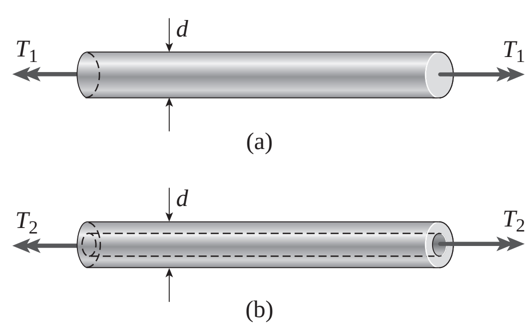
$$d_{\min} = 64.4 \text{ mm} \quad \leftarrow$$

Problem 3.3-15 A solid brass bar of diameter $d = 1.25$ in. is subjected to torques T_1, as shown in part (a) of the figure. The allowable shear stress in the brass is 12 ksi.

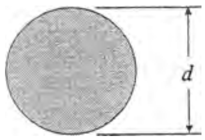
(a) What is the maximum permissible value of the torques T_1?

(b) If a hole of diameter 0.625 in. is drilled longitudinally through the bar, as shown in part (b) of the figure, what is the maximum permissible value of the torques T_2?

(c) What is the percent decrease in torque and the percent decrease in weight due to the hole?



Solution 3.3-15



(a) MAX. PERMISSIBILE VALUE OF TORQUE T_1 – SOLID BAR

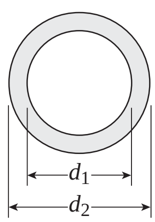
$$T_{1max} = \frac{\tau_a I_p}{\frac{d}{2}} \qquad T_{1max} = \frac{\tau_a \frac{\pi}{32} d^4}{\frac{d}{2}}$$

$$T_{1max} = \frac{1}{16}\tau_a \pi d^3$$

$$T_{1max} = \frac{1}{16}(12)\pi(1.25)^3$$

$$T_{1max} = 4.60 \text{ in.-k} \quad \longleftarrow$$

(b) MAX. PERMISSIBILE VALUE OF TORQUE T_2 – HOLLOW BAR



$d_2 = 1.25$ in. $\quad d_1 = 0.625$ in. $\qquad \tau_a = 12$ ksi

$$T_{2max} = \frac{\tau_a \frac{\pi}{32}\left(d_2^4 - d_1^4\right)}{\frac{d_2}{2}}$$

$$T_{2max} = \frac{1}{16}\tau_a \pi \frac{d_2^4 - d_1^2}{d_2}$$

$$T_{2max} = 4.31 \text{ in.-k} \quad \longleftarrow$$

(c) PERCENT DECREASE IN TORQUE & PERCENT DECREASE IN WEIGHT DUE TO HOLE IN (b)

percent decrease in torque

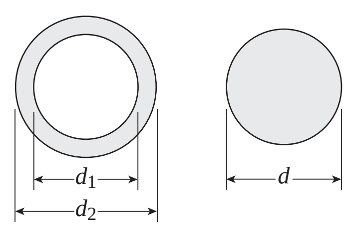
$$\frac{T_{1max} - T_{2max}}{T_{1max}}(100) = 6.25\% \quad \longleftarrow$$

percent decrease in weight (weight is proportional to x-sec area)

$$A_1 = \frac{\pi}{4}d_2^2 \qquad A_2 = \frac{\pi}{4}\left(d_2^2 - d_1^2\right)$$

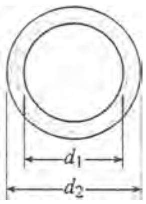
$$\frac{A_1 - A_2}{A_1}(100) = 25\ \% \quad \longleftarrow$$

Problem 3.3-16 A hollow aluminum tube used in a roof structure has an outside diameter $d_2 = 104$ mm and an inside diameter $d_1 = 82$ mm (see figure). The tube is 2.75 m long, and the aluminum has shear modulus $G = 28$ GPa.



(a) If the tube is twisted in pure torsion by torques acting at the ends, what is the angle of twist (in degrees) when the maximum shear stress is 48 MPa?
(b) What diameter d is required for a solid shaft (see figure) to resist the same torque with the same maximum stress?
(c) What is the ratio of the weight of the hollow tube to the weight of the solid shaft?

Solution 3.3-16



NUMERICAL DATA

$d_2 = 104$ mm

$d_1 = 82$ mm

$L = 2.75 \times 10^3$ mm

$G = 28$ GPa

$I_p = (\pi/32)(d_2{}^4 - d_1{}^4)$

$I_p = 7.046 \times 10^6 \text{ mm}^4$

(a) FIND ANGLE OF TWIST $\quad \tau_{max} = 48$ MPa

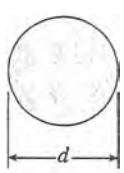
$$\phi = \frac{TL}{GI_p} \qquad \phi = \left(\frac{Td_2}{2I_p}\right)\frac{2L}{Gd_2}$$

$$\phi = (\tau_{max})\frac{2L}{Gd_2}$$

$\phi = 0.091$ radians

$\phi = 5.19°$ $\leftarrow$

(b) REPLACE HOLLOW SHAFT WITH SOLID SHAFT - FIND DIAMETER



$$\tau_{max} = \frac{T\dfrac{d}{2}}{\dfrac{\pi}{32}d^4} \qquad \tau_{max} = \frac{16T}{d^3\pi}$$

set τ_{max} expression equal to $\dfrac{Td_2^2}{\dfrac{\pi}{32}\left(d_2^4 - d_1^4\right)}$

$$= \frac{32Td_2}{\pi\left(d_2{}^4 - d_1{}^4\right)} \text{ then solve for d}$$

$$d^3 = \frac{d_2{}^4 - d_1{}^4}{d_2}$$

$$d_{reqd} = \left(\frac{d_2{}^4 - d_1{}^4}{d_2}\right)^{\frac{1}{3}} \quad d_{reqd} = 88.4 \text{ mm} \quad \leftarrow$$

(c) RATIO OF WEIGHTS OF HOLLOW & SOLID SHAFTS
WEIGHT IS PROPORTIONAL TO CROSS SECTIONAL AREA

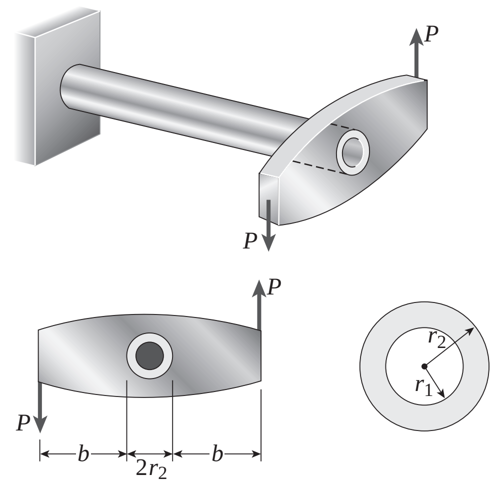
$$A_h = \frac{\pi}{4}\left(d_2{}^2 - d_1{}^2\right)$$

$$A_s = \frac{\pi}{4}d_{reqd}{}^2 \qquad \frac{A_h}{A_s} = 0.524 \quad \leftarrow$$

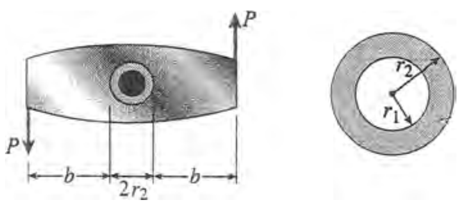
So the weight of the tube is 52% of the solid shaft, but they resist the same torque.

Problem 3.3-17 A circular tube of inner radius r_1 and outer radius r_2 is subjected to a torque produced by forces $P = 900$ lb (see figure). The forces have their lines of action at a distance $b = 5.5$ in. from the outside of the tube.

If the allowable shear stress in the tube is 6300 psi and the inner radius $r_1 = 1.2$ in., what is the minimum permissible outer radius r_2?



Solution 3.3-17 Circular tube in torsion



$P = 900$ lb

$b = 5.5$ in.

$\tau_{allow} = 6300$ psi

$r_1 = 1.2$ in.

Find minimum permissible radius r_2

TORSION FORMULA

$$T = 2P(b + r_2)$$

$$I_P = \frac{\pi}{2}(r_2^4 - r_1^4)$$

$$\tau_{max} = \frac{Tr_2}{I_P} = \frac{2P(b + r_2)r_2}{\frac{\pi}{2}(r_2^4 - r_1^4)} = \frac{4P(b + r_2)r_2}{\pi(r_2^4 - r_1^4)}$$

All terms in this equation are known except r_2.

SOLUTION OF EQUATION

UNITS: Pounds, Inches

Substitute numerical values:

$$6300 \text{ psi} = \frac{4(900 \text{ lb})(5.5 \text{ in.} + r_2)(r_2)}{\pi[(r_2^4) - (1.2 \text{ in.})^4]}$$

or

$$\frac{r_2^4 - 2.07360}{r_2(r_2 + 5.5)} - 0.181891 = 0$$

or

$$r_2^4 - 0.181891\, r_2^2 - 1.000402\, r_2 - 2.07360 = 0$$

Solve numerically:

$r_2 = 1.3988$ in.

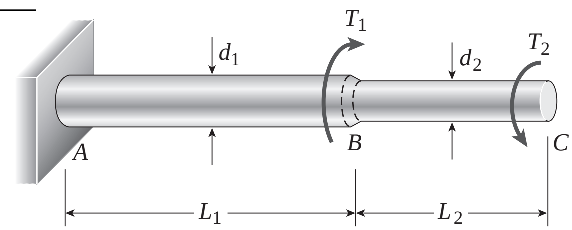
MINIMUM PERMISSIBLE RADIUS

$r_2 = 1.40$ in. $\leftarrow$

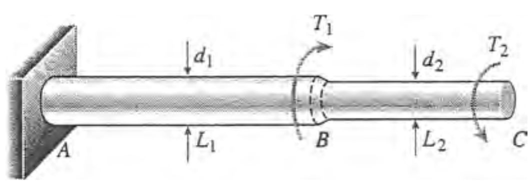
Nonuniform Torsion

Problem 3.4-1 A stepped shaft ABC consisting of two solid circular segments is subjected to torques T_1 and T_2 acting in opposite directions, as shown in the figure. The larger segment of the shaft has diameter $d_1 = 2.25$ in. and length $L_1 = 30$ in.; the smaller segment has diameter $d_2 = 1.75$ in. and length $L_2 = 20$ in. The material is steel with shear modulus $G = 11 \times 10^6$ psi, and the torques are $T_1 = 20{,}000$ lb-in. and $T_2 = 8{,}000$ lb-in.

Calculate the following quantities: (a) the maximum shear stress τ_{max} in the shaft, and (b) the angle of twist ϕ_C (in degrees) at end C.



Solution 3.4-1 Stepped shaft



$d_1 = 2.25$ in. $\quad L_1 = 30$ in.

$d_2 = 1.75$ in. $\quad L_2 = 20$ in.

$G = 11 \times 10^6$ psi

$T_1 = 20{,}000$ lb-in.

$T_2 = 8{,}000$ lb-in.

Segment AB

$$T_{AB} = T_2 - T_1 = -12{,}000 \text{ lb-in.}$$

$$\tau_{AB} = \left| \frac{16\,T_{AB}}{\pi d_1^3} \right| = \frac{16(12{,}000 \text{ lb-in.})}{\pi(2.25 \text{ in.})^3} = 5365 \text{ psi}$$

$$\phi_{AB} = \frac{T_{AB}L_1}{G(I_p)_{AB}} = \frac{(-12{,}000 \text{ lb-in.})(30 \text{ in.})}{(11 \times 10^6 \text{ psi})\left(\dfrac{\pi}{32}\right)(2.25 \text{ in.})^4}$$

$$= -0.013007 \text{ rad}$$

Segment BC

$$T_{BC} = +T_2 = 8{,}000 \text{ lb-in.}$$

$$\tau_{BC} = \frac{16\,T_{BC}}{\pi d_2^3} = \frac{16(8{,}000 \text{ lb-in.})}{\pi(1.75 \text{ in.})^3} = 7602 \text{ psi}$$

$$\phi_{BC} = \frac{T_{BC}L_2}{G(I_p)_{BC}} = \frac{(8{,}000 \text{ lb-in.})(20 \text{ in.})}{(11 \times 10^6 \text{ psi})\left(\dfrac{\pi}{32}\right)(1.75 \text{ in.})^4}$$

$$= +0.015797 \text{ rad}$$

(a) Maximum shear stress

Segment BC has the maximum stress

$$\tau_{max} = 7600 \text{ psi} \quad \leftarrow$$

(b) Angle of twist at end C

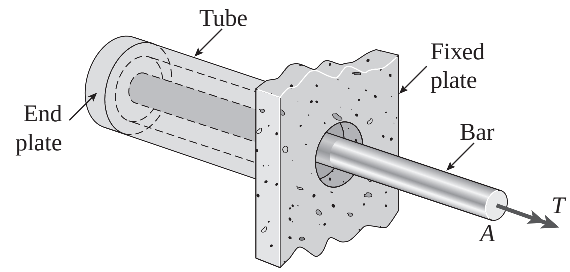
$$\phi_C = \phi_{AB} + \phi_{BC} = (-0.013007 + 0.015797) \text{ rad}$$

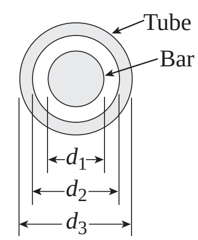
$$\phi_C = 0.002790 \text{ rad} = 0.16° \quad \leftarrow$$

Problem 3.4-2 A circular tube of outer diameter $d_3 = 70$ mm and inner diameter $d_2 = 60$ mm is welded at the right-hand end to a fixed plate and at the left-hand end to a rigid end plate (see figure). A solid circular bar of diameter $d_1 = 40$ mm is inside of, and concentric with, the tube. The bar passes through a hole in the fixed plate and is welded to the rigid end plate.

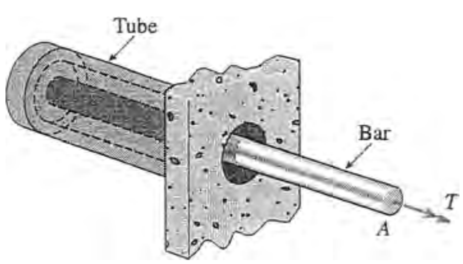
The bar is 1.0 m long and the tube is half as long as the bar. A torque $T = 1000$ N · m acts at end A of the bar. Also, both the bar and tube are made of an aluminum alloy with shear modulus of elasticity $G = 27$ GPa.

(a) Determine the maximum shear stresses in both the bar and tube.
(b) Determine the angle of twist (in degrees) at end A of the bar.





Solution 3.4-2 Bar and tube



TUBE

$d_3 = 70$ mm $\qquad d_2 = 60$ mm

$L_{\text{tube}} = 0.5$ m $\qquad G = 27$ GPa

$$(I_p)_{\text{tube}} = \frac{\pi}{32}(d_3^4 - d_2^4)$$

$$= 1.0848 \times 10^6 \text{ mm}^4$$

BAR

$d_1 = 40$ mm $\qquad L_{\text{bar}} = 1.0$ m $\qquad G = 27$ GPa

$$(I_p)_{\text{bar}} = \frac{\pi d_1^4}{32} = 251.3 \times 10^3 \text{ mm}^4$$

TORQUE

$T = 1000$ N · m

(a) MAXIMUM SHEAR STRESSES

Bar: $\tau_{\text{bar}} = \dfrac{16T}{\pi d_1^3} = 79.6 \text{ MPa} \quad \leftarrow$

Tube: $\tau_{\text{tube}} = \dfrac{T(d_3/2)}{(I_p)_{\text{tube}}} = 32.3 \text{ MPa} \quad \leftarrow$

(b) ANGLE OF TWIST AT END A

Bar: $\phi_{\text{bar}} = \dfrac{TL_{\text{bar}}}{G(I_p)_{\text{bar}}} = 0.1474 \text{ rad}$

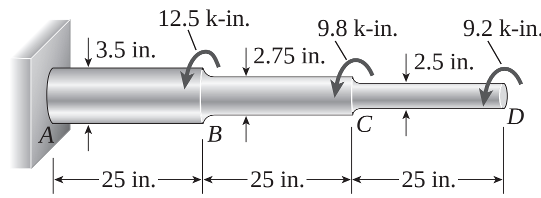
Tube: $\phi_{\text{tube}} = \dfrac{TL_{\text{tube}}}{G(I_p)_{\text{tube}}} = 0.0171 \text{ rad}$

$\phi_A = \phi_{\text{bar}} + \phi_{\text{tube}} = 0.1474 + 0.0171 = 0.1645 \text{ rad}$

$\phi_A = 9.43° \quad \leftarrow$

Problem 3.4-3 A stepped shaft *ABCD* consisting of solid circular segments is subjected to three torques, as shown in the figure. The torques have magnitudes 12.5 k-in., 9.8 k-in., and 9.2 k-in. The length of each segment is 25 in. and the diameters of the segments are 3.5 in., 2.75 in., and 2.5 in. The material is steel with shear modulus of elasticity $G = 11.6 \times 10^3$ ksi.

(a) Calculate the maximum shear stress τ_{max} in the shaft.
(b) Calculate the angle of twist ϕ_D (in degrees) at end *D*.



Solution 3.4-3

NUMERICAL DATA (INCHES, KIPS)

$T_B = 12.5$ k-in. $\quad T_C = 9.8$k-in.

$T_D = 9.2$ k-in. $\quad L = 25$ in.

$d_{AB} = 3.5$ in. $\quad d_{BC} = 2.75$ in.

$d_{CD} = 2.5$ in. $\quad G = 11.6 \times (10^3)$ ksi

(a) MAX. SHEAR STRESS IN SHAFT

torque reaction at A: $R_A = -(T_B + T_C + T_D)$
$R_A = -31.5$ in.-kip

$$\tau_{\mathrm{AB}} = \frac{|R_A|\dfrac{d_{AB}}{2}}{\dfrac{\pi}{32}\,d_{AB}{}^4} \qquad \tau_{\max} = 3.742 \text{ ksi}$$

Check CD: $$\tau_{CD} = \frac{T_D\dfrac{d_{CD}}{2}}{\dfrac{\pi}{32}d_{CD}{}^4} \qquad \tau_{CD} = 2.999 \text{ ksi}$$

Check BC: $$\tau_{BC} = \frac{(T_C + T_D)\dfrac{d_{BC}}{2}}{\dfrac{\pi}{32}d_{BC}{}^4}$$

$\tau_{BC} = 4.65$ ksi $\quad \leftarrow$ controls

(b) ANGLE OF TWIST AT END D

$T_1 = |R_A| \qquad T_2 = T_C + T_D \qquad T_3 = T_D$

$$I_{P1} = \frac{\pi}{32}d_{AB}{}^4 \qquad I_{P2} = \frac{\pi}{32}d_{BC}{}^4$$

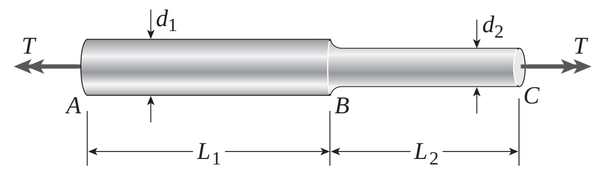
$$I_P3 = \frac{\pi}{32}d\,{}_{CD}{}^4$$

$$\phi_D = \sum\frac{T_iL_i}{GI_{pi}} \qquad \phi_D = \frac{L}{G}\left(\frac{T_1}{I_{P1}} + \frac{T_2}{I_{P2}} + \frac{T_3}{I_{P3}}\right)$$

$\phi_D = 0.017$ radians $\quad \phi_D = 0.978$ degrees $\quad \leftarrow$

Problem 3.4-4 A solid circular bar *ABC* consists of two segments, as shown in the figure. One segment has diameter $d_1 = 56$ mm and length $L_1 = 1.45$ m; the other segment has diameter $d_2 = 48$ mm and length $L_2 = 1.2$ m.

What is the allowable torque T_{allow} if the shear stress is not to exceed 30 MPa and the angle of twist between the ends of the bar is not to exceed 1.25°? (Assume $G = 80$ GPa.)



Solution 3.4-4

NUMERICAL DATA

$d_1 = 56 \text{ mm} \qquad d_2 = 48 \text{ mm}$

$L_1 = 1450 \text{ mm} \qquad L_2 = 1200 \text{ mm} \qquad G = 80 \text{ GPa}$

$\tau_a = 30 \text{ MPa} \qquad \phi_a = 1.25\left(\dfrac{\pi}{180}\right) \text{ radians}$

Allowable torque

T_{allow} based on shear stress

$$\tau_{max} = \frac{16T}{d_2^3\pi} \qquad T_{allow} = \frac{\tau_a \pi d_2^3}{16}$$

$T_{allow} = 651.441 \text{ N} \cdot \text{m}$

T_{allow} based on angle of twist

$$\phi_{max} = \frac{T}{G}\left[\frac{L_1}{\left(\frac{\pi}{32}d_1^4\right)} + \frac{L_2}{\left(\frac{\pi}{32}d_2^4\right)}\right]$$

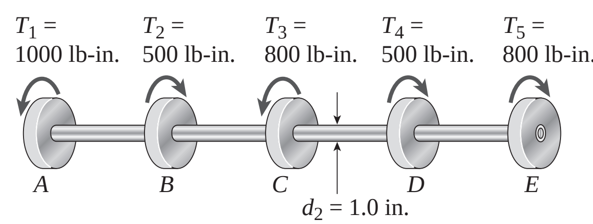
$$T_{allow} = \frac{G\phi_a}{\dfrac{L_1}{\left(\frac{\pi}{32}d_1^4\right)} + \dfrac{L_2}{\left(\frac{\pi}{32}d_2^4\right)}}$$

$T_{allow} = 459 \text{ N} \cdot \text{m} \qquad \leftarrow \qquad$ governs

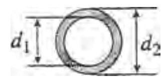
---

Problem 3.4-5 A hollow tube *ABCDE* constructed of monel metal is subjected to five torques acting in the directions shown in the figure. The magnitudes of the torques are $T_1 = 1000$ lb-in., $T_2 = T_4 = 500$ lb-in., and $T_3 = T_5 = 800$ lb-in. The tube has an outside diameter $d_2 = 1.0$ in. The allowable shear stress is 12,000 psi and the allowable rate of twist is 2.0°/ft.

Determine the maximum permissible inside diameter d_1 of the tube.



Solution 3.4-5 Hollow tube of monel metal



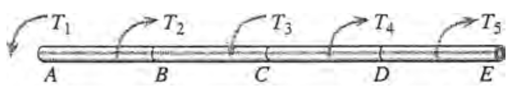
$d_2 = 1.0$ in. $\quad \tau_{allow} = 12{,}000$ psi

$\theta_{allow} = 2°/\text{ft} = 0.16667°/\text{in.}$

$= 0.002909$ rad/in.

From Table H-2, Appendix H: $G = 9500$ ksi

TORQUES



$T_1 = 1000$ lb-in. $\quad T_2 = 500$ lb-in. $\quad T_3 = 800$ lb-in.

$T_4 = 500$ lb-in. $\quad T_5 = 800$ lb-in.

INTERNAL TORQUES

$T_{AB} = -T_1 = -1000$ lb-in.

$T_{BC} = -T_1 + T_2 = -500$ lb-in.

$T_{CD} = -T_1 + T_2 - T_3 = -1300$ lb-in.

$T_{DE} = -T_1 + T_2 - T_3 + T_4 = -800$ lb-in.

Largest torque (absolute value only):

$T_{max} = 1300$ lb-in.

REQUIRED POLAR MOMENT OF INERTIA BASED UPON ALLOWABLE SHEAR STRESS

$$\tau_{max} = \frac{T_{max} r}{I_p} \qquad I_P = \frac{T_{max}(d_2/2)}{\tau_{allow}} = 0.05417 \text{ in.}^4$$

REQUIRED POLAR MOMENT OF INERTIA BASED UPON ALLOWABLE ANGLE OF TWIST

$$\theta = \frac{T_{max}}{GI_P} \quad I_P = \frac{T_{max}}{G\theta_{allow}} = 0.04704 \text{ in.}^4$$

SHEAR STRESS GOVERNS

Required $I_P = 0.05417$ in.^{4}

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4)$$

$$d_1^4 = d_3^4 - \frac{32 I_P}{\pi} = (1.0 \text{ in.})^4 - \frac{32(0.05417 \text{ in.}^4)}{\pi}$$

$$= 0.4482 \text{ in.}^4$$

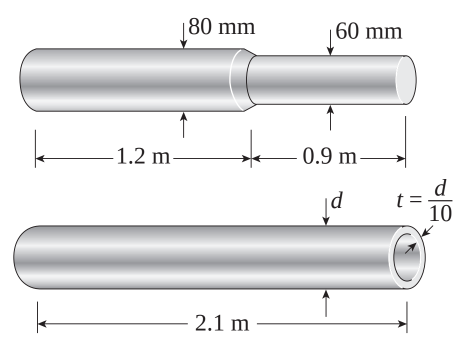
$d_1 = 0.818$ in. $\leftarrow$

(Maximum permissible inside diameter)

Problem 3.4-6 A shaft of solid circular cross section consisting of two segments is shown in the first part of the figure. The left-hand segment has diameter 80 mm and length 1.2 m; the right-hand segment has diameter 60 mm and length 0.9 m.

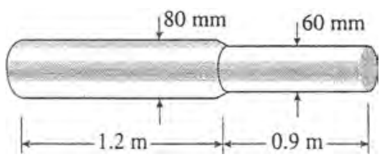
Shown in the second part of the figure is a hollow shaft made of the same material and having the same length. The thickness t of the hollow shaft is $d/10$, where d is the outer diameter. Both shafts are subjected to the same torque.

If the hollow shaft is to have the same torsional stiffness as the solid shaft, what should be its outer diameter d?



Solution 3.4-6 Solid and hollow shafts

SOLID SHAFT CONSISTING OF TWO SEGMENTS

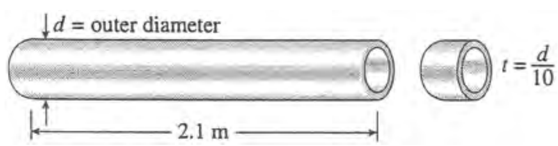


$$\phi_1 = \Sigma \frac{TL_i}{GI_{Pi}} = \frac{T(1.2\text{ m})}{G\left(\frac{\pi}{32}\right)(80\text{ mm})^4} + \frac{T(0.9\text{ m})}{G\left(\frac{\pi}{32}\right)(60\text{ mm})^4}$$

$$= \frac{32T}{\pi G}(29{,}297\text{ m}^{-3} + 69{,}444\text{ m}^{-3})$$

$$= \frac{32T}{\pi G}(98{,}741\text{ m}^{-3})$$

HOLLOW SHAFT



d_0 = inner diameter = $0.8d$

$$\phi_2 = \frac{TL}{GI_p} = \frac{T(2.1\text{ m})}{G\left(\frac{\pi}{32}\right)[d^4 - (0.8d)^4]}$$

$$= \frac{32T}{\pi G}\left(\frac{2.1\text{ m}}{0.5904\,d^4}\right) = \frac{32T}{\pi G}\left(\frac{3.5569\text{ m}}{d^4}\right)$$

UNITS: d = meters

TORSIONAL STIFFNESS

$k_T = \dfrac{T}{\phi}$ Torque T is the same for both shafts.

$\therefore$ For equal stiffnesses, $\phi_1 = \phi_2$

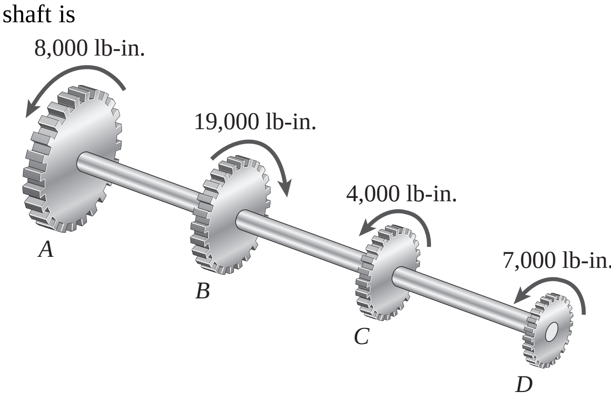
$$98{,}741\text{ m}^{-3} = \frac{3.5569\text{ m}}{d^4}$$

$$d^4 = \frac{3.5569}{98{,}741} = 36.023 \times 10^{-6}\text{ m}^4$$

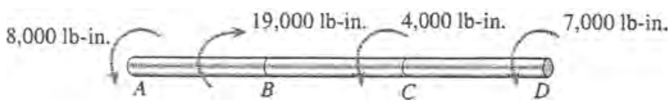
$$d = 0.0775\text{ m} = 77.5\text{ mm} \quad \leftarrow$$

Problem 3.4-7 Four gears are attached to a circular shaft and transmit the torques shown in the figure. The allowable shear stress in the shaft is 10,000 psi.

(a) What is the required diameter d of the shaft if it has a solid cross section?
(b) What is the required outside diameter d if the shaft is hollow with an inside diameter of 1.0 in.?



Solution 3.4-7 Shaft with four gears



$\tau_{\text{allow}} = 10{,}000$ psi $\quad T_{BC} = +11{,}000$ lb-in.

$T_{AB} = -8000$ lb-in. $\quad T_{CD} = +7000$ lb-in.

(a) SOLID SHAFT

$$\tau_{\max} = \frac{16T}{\pi d^3}$$

$$d^3 = \frac{16T_{\max}}{\pi\tau_{\text{allow}}} = \frac{16(11{,}000 \text{ lb-in.})}{\pi(10{,}000 \text{ psi})} = 5.602 \text{ in.}^3$$

Required $d = 1.78$ in. $\leftarrow$

(b) HOLLOW SHAFT

Inside diameter $d_0 = 1.0$ in.

$$\tau_{\max} = \frac{Tr}{I_p} \qquad \tau_{\text{allow}} = \frac{T_{\max}\left(\frac{d}{2}\right)}{I_p}$$

$$10{,}000 \text{ psi} = \frac{(11{,}000 \text{ lb-in.})\left(\frac{d}{2}\right)}{\left(\frac{\pi}{32}\right)[d^4 - (1.0 \text{ in.})^4]}$$

UNITS: d = inches

$$10{,}000 = \frac{56{,}023\, d}{d^4 - 1}$$

or

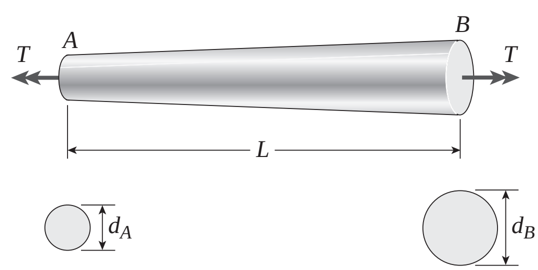
$$d^4 - 5.6023\, d - 1 = 0$$

Solving, $d = 1.832$

Required $d = 1.83$ in. $\leftarrow$

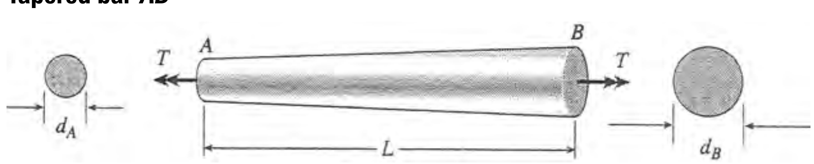
Problem 3.4-8 A tapered bar *AB* of solid circular cross section is twisted by torques *T* (see figure). The diameter of the bar varies linearly from d_A at the left-hand end to d_B at the right-hand end.

For what ratio d_B/d_A will the angle of twist of the tapered bar be one-half the angle of twist of a prismatic bar of diameter d_A? (The prismatic bar is made of the same material, has the same length, and is subjected to the same torque as the tapered bar.) *Hint*: Use the results of Example 3-5.



Problems 3.4-8, 3.4-9 and 3.4-10

Solution 3.4-8 Tapered bar *AB*



TAPERED BAR (From Eq. 3-27)

$$\phi_1 = \frac{TL}{G(I_P)_A}\left(\frac{\beta^2 + \beta + 1}{3\beta^3}\right) \qquad \beta = \frac{d_B}{d_A}$$

PRISMATIC BAR

$$\phi_2 = \frac{TL}{G(I_P)_A}$$

ANGLE OF TWIST

$$\phi_1 = \frac{1}{2}\phi_2 \qquad \frac{\beta^2 + \beta + 1}{3\beta^3} = \frac{1}{2}$$

or $\quad 3\beta^3 - 2\beta^2 - 2\beta - 2 = 0$

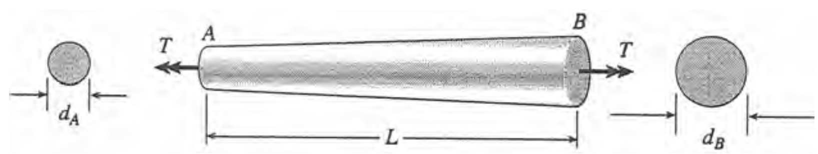
SOLVE NUMERICALLY:

$$\beta = \frac{d_B}{d_A} = 1.45 \quad \leftarrow$$

Problem 3.4-9 A tapered bar AB of solid circular cross section is twisted by torques $T = 36{,}000$ lb-in. (see figure). The diameter of the bar varies linearly from d_A at the left-hand end to d_B at the right-hand end. The bar has length $L = 4.0$ ft and is made of an aluminum alloy having shear modulus of elasticity $G = 3.9 \times 10^6$ psi. The allowable shear stress in the bar is 15,000 psi and the allowable angle of twist is 3.0°.

If the diameter at end B is 1.5 times the diameter at end A, what is the minimum required diameter d_A at end A? (*Hint*: Use the results of Example 3–5).

Solution 3.4-9 Tapered bar



$$d_B = 1.5\, d_A$$
$$T = 36{,}000 \text{ lb-in.}$$
$$L = 4.0 \text{ ft} = 48 \text{ in.}$$
$$G = 3.9 \times 10^6 \text{ psi}$$
$$\tau_{\text{allow}} = 15{,}000 \text{ psi}$$
$$\phi_{\text{allow}} = 3.0° = 0.0523599 \text{ rad}$$

MINIMUM DIAMETER BASED UPON ALLOWABLE SHEAR STRESS

$$\tau_{\max} = \frac{16T}{\pi d_A^3} \qquad d_A^3 = \frac{16\,T}{\pi \tau_{\text{allow}}} = \frac{16(36{,}000 \text{ lb-in.})}{\pi(15{,}000 \text{ psi})}$$
$$= 12.2231 \text{ in.}^3$$
$$d_A = 2.30 \text{ in.}$$

MINIMUM DIAMETER BASED UPON ALLOWABLE ANGLE OF TWIST (From Eq. 3-27)

$$\beta = d_B/d_A = 1.5$$
$$\phi = \frac{TL}{G(I_P)_A}\left(\frac{\beta^2 + \beta + 1}{3\beta^3}\right) = \frac{TL}{G(I_P)_A}(0.469136)$$
$$= \frac{(36{,}000 \text{ lb-in.})(48 \text{ in.})}{(3.9 \times 10^6 \text{ psi})\left(\frac{\pi}{32}\right)d_A^4}(0.469136)$$
$$= \frac{2.11728 \text{ in.}^4}{d_A^4}$$
$$d_A^4 = \frac{2.11728 \text{ in.}^4}{\phi_{\text{allow}}} = \frac{2.11728 \text{ in.}^4}{0.0523599 \text{ rad}}$$
$$= 40.4370 \text{ in.}^4$$
$$d_A = 2.52 \text{ in.}$$

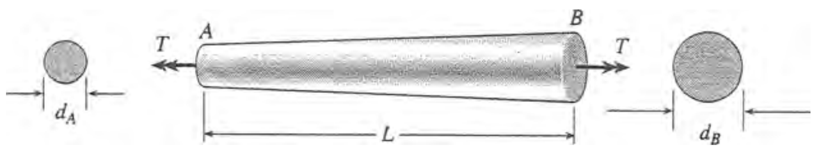
ANGLE OF TWIST GOVERNS

Min. $d_A = 2.52$ in. $\leftarrow$

Problem 3.4-10 The bar shown in the figure is tapered linearly from end A to end B and has a solid circular cross section. The diameter at the smaller end of the bar is $d_A = 25$ mm and the length is $L = 300$ mm. The bar is made of steel with shear modulus of elasticity $G = 82$ GPa.

If the torque $T = 180\ \text{N} \cdot \text{m}$ and the allowable angle of twist is 0.3°, what is the minimum allowable diameter d_B at the larger end of the bar? (*Hint*: Use the results of Example 3-5.)

Solution 3.4-10 Tapered bar



$d_A = 25$ mm

$L = 300$ mm

$G = 82$ GPa

$T = 180\ \text{N} \cdot \text{m}$

$\phi_{\text{allow}} = 0.3°$

Find d_B

Diameter based upon allowable angle of twist

(From Eq. 3-27)

$$\beta = \frac{d_B}{d_A}$$

$$\phi = \frac{TL}{G(I_P)_A}\left(\frac{\beta^2 + \beta + 1}{3\beta^3}\right) \quad (I_P)_A = \frac{\pi}{32} d_A^4$$

$$(0.3^{\circ})\left(\frac{\pi}{180}\frac{\text{rad}}{\text{degrees}}\right) = \frac{(180\ \text{N}\cdot\text{m})(0.3\ \text{m})}{(82\ \text{GPa})\left(\frac{\pi}{32}\right)(25\ \text{mm})^4}\left(\frac{\beta^2 + \beta + 1}{3\beta^3}\right)$$

$$0.304915 = \frac{\beta^2 + \beta + 1}{3\beta^3}$$

$$0.914745\beta^3 - \beta^2 - 1 = 0$$

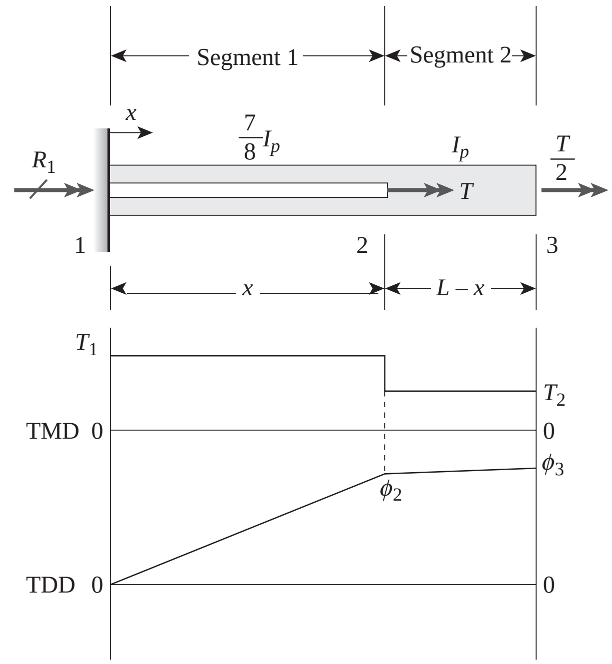
Solve numerically:

$\beta = 1.94452$

Min. $d_B = \beta d_A = 48.6$ mm $\longleftarrow$

Problem 3.4-11 The nonprismatic cantilever circular bar shown has an internal cylindrical hole from 0 to x, so the net polar moment of inertia of the cross section for segment 1 is $(7/8)I_p$. Torque T is applied at x and torque $T/2$ is applied at $x = L$. Assume that G is constant.

(a) Find reaction moment R_1.
(b) Find internal torsional moments T_i in segments 1 & 2.
(c) Find x required to obtain twist at joint 3 of $\varphi_3 = TL/GI_p$
(d) What is the rotation at joint 2, φ_2?
(e) Draw the torsional moment (TMD: $T(x)$, $0 \le x \le L$) and displacement (TDD: $\varphi(x)$, $0 \le x \le L$) diagrams.



Solution 3.4-11

(a) Reaction torque R_1

$$\sum M_x = 0 \quad R_1 = -\left(T + \frac{T}{2}\right) \quad R_1 = \frac{-3}{2}T \quad \leftarrow$$

(b) Internal moments in segments 1 & 2

$$T_1 = -R_1 \qquad T_1 = 1.5\,T \qquad T_2 = \frac{T}{2}$$

(c) Find x required to obtain trwist at jt 3

$$\phi_3 = \sum \frac{T_i L_i}{G I_{Pi}}$$

$$\frac{TL}{GI_P} = \frac{T_1 x}{G\left(\frac{7}{8}I_P\right)} + \frac{T_2(L - x)}{GI_P}$$

$$\frac{TL}{GI_P} = \frac{\left(\frac{3}{2}T\right)x}{G\left(\frac{7}{8}I_P\right)} + \frac{\left(\frac{T}{2}\right)(L - x)}{GI_P}$$

$$L = \frac{\left(\frac{3}{2}\right)x}{\left(\frac{7}{8}\right)} + \frac{1}{2}(L - x)$$

$$L = \frac{17}{14}x + \frac{1}{2}L$$

$$x = \frac{14}{17}\left(\frac{L}{2}\right) \qquad x = \frac{7}{17}L \qquad \leftarrow$$

(d) Rotation at joint 2 for x value in (c)

$$\phi_2 = \frac{T_1 x}{G\left(\frac{7}{8}I_p\right)} \qquad \phi_2 = \frac{\left(\frac{3}{2}T\right)\left(\frac{7}{17}L\right)}{G\left(\frac{7}{8}I_p\right)}$$

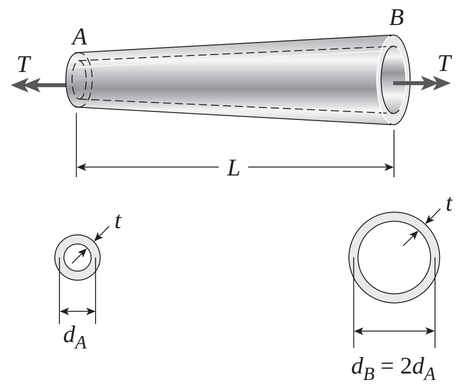
$$\phi_2 = \frac{12TL}{17GI_P} \qquad \leftarrow$$

(e) TMD & TDD – see plots above

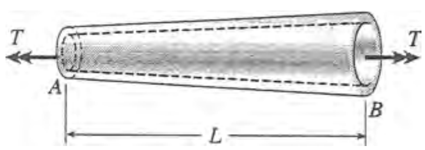
TMD is constant - T_1 for 0 to x & T_2 for x to L; hence TDD is linear - zero at jt 1, ϕ_2 at jt 2 & ϕ_3 at jt 3

Problem 3.4-12 A uniformly tapered tube AB of hollow circular cross section is shown in the figure. The tube has constant wall thickness t and length L. The average diameters at the ends are d_A and $d_B = 2d_A$. The polar moment of inertia may be represented by the approximate formula $I_P \approx \pi d^3 t/4$ (see Eq. 3-18).

Derive a formula for the angle of twist ϕ of the tube when it is subjected to torques T acting at the ends.



Solution 3.4-12 Tapered tube

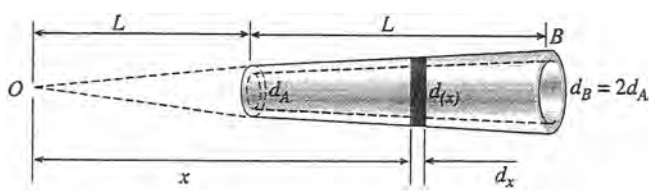


t = thickness (constant)

d_A, d_B = average diameters at the ends

$d_B = 2d_A \qquad I_p = \dfrac{\pi d^3 t}{4}$ (approximate formula)

ANGLE OF TWIST



Take the origin of coordinates at point O.

$$d(x) = \frac{x}{2L}(d_B) = \frac{x}{L}d_A$$

$$L_p(x) = \frac{\pi[d(x)]^3 t}{4} = \frac{\pi t d_A^3}{4L^3}x^3$$

For element of length dx:

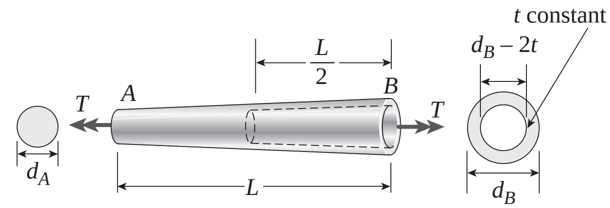
$$d\phi = \frac{Tdx}{GI_P(x)} = \frac{Tdx}{G\left(\dfrac{\pi t d_A^3}{4L^3}\right)x^3} = \frac{4TL^3 dx}{\pi G t d_A^3 x^3}$$

$$\phi = \int_L^{2L} d\phi = \frac{4TL^3}{\pi G t d_A^3}\int_L^{2L}\frac{dx}{x^3} = \frac{3TL}{2\pi G t d_A^3} \qquad \leftarrow$$

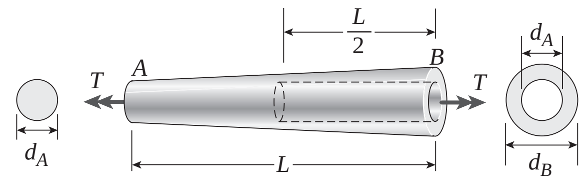
Problem 3.4-13 A uniformly tapered aluminum-alloy tube AB of circular cross section and length L is shown in the figure. The outside diameters at the ends are d_A and $d_B = 2d_A$. A hollow section of length $L/2$ and constant thickness $t = d_A/10$ is cast into the tube and extends from B halfway toward A.

(a) Find the angle of twist φ of the tube when it is subjected to torques T acting at the ends. Use numerical values as follows: $d_A = 2.5$ in., $L = 48$ in., $G = 3.9 \times 10^6$ psi, and $T = 40{,}000$ in.-lb.

(b) Repeat (a) if the hollow section has constant diameter d_A. (See figure part b.)



(a)



(b)

Solution 3.4-13

PART (A) - CONSTANT THICKNESS

use x as integration variable measured from B toward A

<u>from B to centerline</u>

outer and inner diameters as function of x

$$0 \le x \le \frac{L}{2} \qquad d_0(x) = d_B - \left(\frac{d_B - d_A}{L}\right)x$$

$$d_0(x) = 2d_A - \frac{xd_A}{L}$$

$$d_i(x) = (d_B - 2t)\frac{[(2d_A - 2t)-(d_A - 2t)]}{L}x$$

$$d_i(x) = \frac{-1}{5}d_A\frac{-9L + 5x}{L}$$

<u>solid from centerline to A</u>

$$\frac{L}{2} \le x \le L \qquad d_0(x) = 2d_A - \frac{xd_A}{L}$$

$$\phi = \frac{T}{G}\left(\frac{32}{\pi}\right)\left(\int_0^{\frac{L}{2}}\frac{1}{d_0^{\,4} - d_i^{\,4}}\,dx + \int_{\frac{L}{2}}^{L}\frac{1}{d_0^{\,4}}\,dx\right)$$

$$\phi = \frac{T}{G}\left(\frac{32}{\pi}\right)\left[\int_0^{\frac{L}{2}} \frac{1}{\left(2d_A - \frac{xd_A}{L}\right)^4 - \left(\frac{-1}{5}d_A\frac{-9L+5x}{L}\right)^4}dx + \int_{\frac{L}{2}}^{L}\frac{1}{\left(2d_A - \frac{xd_A}{L}\right)^4}dx\right]$$

$$\phi = 32\frac{T}{G\pi}\left(\frac{-125}{2}L\frac{3\ln(2)+2\ln(7)-\ln(197)}{d_A{}^4} - \frac{125}{2}L\frac{-2\ln(19)+\ln(181)}{d_A{}^4} + \frac{19}{81d_A{}^4}L\right)$$

Simplifying: $\phi = \dfrac{16TL}{81G\pi d_A^4}\left(38 + 10125\ln\left(\dfrac{71117}{70952}\right)\right)$ or $\phi_a = 3.868\dfrac{TL}{Gd_A^4}$

Use numerical properties as follows $L = 48$ in. $G = 3.9 \times 10^6$ psi $d_A = 2.5$ in. $t = \dfrac{d_A}{10}$ T = 40000 in.-lb

$\phi_a = 0.049$ radians $\phi_a = 2.79°$ $\leftarrow$

PART (B) - CONSTANT HOLE DIAMETER

$0 \le x \le \dfrac{L}{2}$ $d_0(x) = d_B - \left(\dfrac{d_B - d_A}{L}\right)x$ $d_0(x) = 2d_A - \dfrac{xd_A}{L}$ $d_i(x) = d_A$

$\dfrac{L}{2} \le x \le L$ $d_0(x) = 2d_A - \dfrac{xd_A}{L}$

$$\phi = \frac{T}{G}\left(\frac{32}{\pi}\right)\left(\int_0^{\frac{L}{2}}\frac{1}{d_0{}^4 - d_i{}^4}dx + \int_{\frac{L}{2}}^{L}\frac{1}{d_0{}^4}dx\right)$$

$$\phi = \frac{T}{G}\left(\frac{32}{\pi}\right)\left[\int_0^{\frac{L}{2}}\frac{1}{\left(2d_A - \frac{xd_A}{L}\right)^4 - d_A{}^4}dx + \int_{\frac{1}{2}}^{L}\frac{1}{\left(2d_A - \frac{xd_A}{L}\right)^4}dx\right]$$

$$\phi_b = 32\frac{T}{G\pi}\left(\frac{1}{4}L\frac{\ln(5)+2\operatorname{atan}\left(\frac{3}{2}\right)}{d_A{}^4} - \frac{1}{4}L\frac{\ln(3)+2\operatorname{atan}(2)}{d_A{}^4} + \frac{19}{81d_A{}^4}L\right)$$

Simplifying, $\phi_b = 3.057\dfrac{TL}{Gd_A^4}$

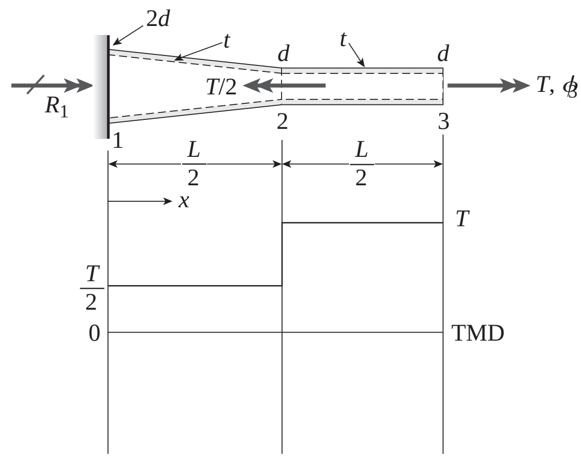
Use numerical properties given above

$\phi_b = 0.039$ radians $\phi_b = 2.21°$ $\leftarrow$

$\dfrac{\phi_a}{\phi_b} = 1.265$ so tube (a) is more flexible than tube (b)

Problem 3.4-14 For the *thin* nonprismatic steel pipe of constant thickness t and variable diameter d shown with applied torques at joints 2 and 3, determine the following.

(a) Find reaction moment R_1.
(b) Find an expression for twist rotation φ_3 at joint 3. Assume that G is constant.
(c) Draw the torsional moment diagram (TMD: $T(x)$, $0 \le x \le L$).



Solution 3.4-14

(a) REACTION TORQUE R_1

statics: $\Sigma T = 0$

$$R_1 - \frac{T}{2} + T = 0 \quad R_1 = \frac{-T}{2} \quad \leftarrow$$

(b) ROTATION AT JOINT 3

$$d_{12}(x) = 2d\left(1 - \frac{x}{L}\right) \quad 0 \le x \le \frac{L}{2}$$

$$d_{23}(x) = d \quad \frac{L}{2} \le x \le L$$

$$\phi_3 = \int_0^{\frac{L}{2}} \frac{\frac{T}{2}}{G\left(\frac{\pi}{4} d_{12}(x)^3 t\right)} dx + \int_{\frac{L}{2}}^{L} \frac{T}{G\left(\frac{\pi}{4} d_{23}(x)^3 t\right)} dx$$

use I_P expression for thin walled tubes

$$\phi_3 = \frac{2T}{G\pi t} \int_0^{\frac{L}{2}} \frac{1}{\left[2d\left(1 - \frac{x}{L}\right)\right]^3} dx + \frac{4T}{G\pi d^3 t} \int_{\frac{L}{2}}^{L} dx$$

$$\phi_3 = \frac{2T}{G\pi t} \int_0^{\frac{L}{2}} \frac{1}{\left[2d\left(1 - \frac{x}{L}\right)\right]^3} dx + \frac{2TL}{G\pi d^3 t}$$

$$\phi_3 = \frac{3TL}{8G\pi d^3 t} + \frac{2TL}{G\pi d^3 t}$$

$$\phi_3 = \frac{19TL}{8G\pi d^3 t} \quad \leftarrow$$

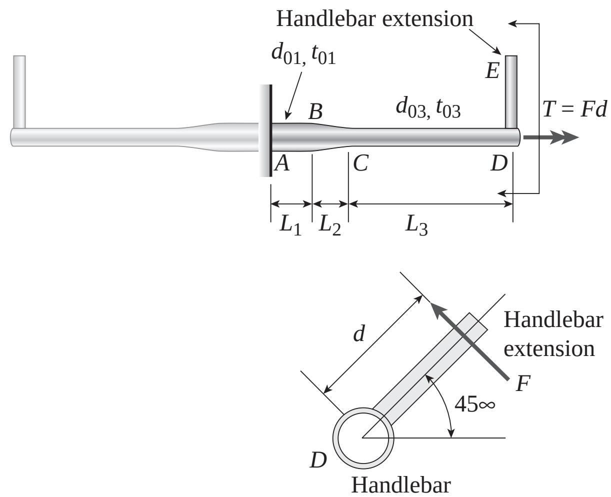
(c) TMD

TMD is piecewise constant: $T(x) = +T/2$ for segment 1-2 & $T(x) = +T$ for segment 2-3 (see plot above)

Problem 3.4-15 A mountain-bike rider going uphill applies torque $T = Fd$ ($F = 15$ lb, $d = 4$ in.) to the end of the handlebars *ABCD* (by pulling on the handlebar extenders *DE*). Consider the right half of the handlebar assembly only (assume the bars are fixed at the fork at *A*). Segments *AB* and *CD* are prismatic with lengths $L_1 = 2$ in. and $L_3 = 8.5$ in., and with outer diameters and thicknesses $d_{01} = 1.25$ in., $t_{01} = 0.125$ in., and $d_{03} = 0.87$ in., $t_{03} = 0.115$ in., respectively as shown. Segment *BC* of length $L_2 = 1.2$ in., however, is tapered, and outer diameter and thickness vary linearly between dimensions at *B* and *C*.

Consider torsion effects only. Assume $G = 4000$ ksi is constant.

Derive an integral expression for the angle of twist φD of half of the handlebar tube when it is subjected to torque $T = Fd$ acting at the end. Evaluate φD for the given numerical values.



Solution 3.4-15

ASSUME THIN WALLED TUBES

Segments AB & CD

$$I_{P1} = \frac{\pi}{4} d_{01}{}^3 t_{01} \qquad I_{P3} = \frac{\pi}{4} d_{03}{}^3 t_{03}$$

Segment BC $\quad 0 \le x \le L_2$

$$d_{02}(x) = d_{01}\left(1 - \frac{x}{L_2}\right) + d_{03}\left(\frac{x}{L_2}\right)$$

$$d_{02}(x) = \frac{d_{01}L_2 - d_{01}x + d_{03}x}{L_2}$$

$$t_{02}(x) = t_{01}\left(1 - \frac{x}{L_2}\right) + t_{03}\left(\frac{x}{L_2}\right)$$

$$t_{02}(x) = \frac{t_{01}L_2 - t_{01}x + t_{03}x}{L_2}$$

$$\phi_D = \frac{Fd}{G}\left(\frac{L_1}{I_{P1}} + \int_0^{L_2} \frac{1}{\frac{\pi}{4} d_{02}(x)^3 t_{02}(x)}\,dx + \frac{L_3}{I_{P3}}\right)$$

$$\phi_D = \frac{4Fd}{G\pi}\left[\frac{L_1}{d_{01}{}^3 t_{01}} + \int_0^{L_2} \frac{L_2{}^4}{(d_{01}L_2 - d_{01}x + d_{03}x)^3 \times (t_{01}L_2 - t_{01}x + t_{03}x)}\,dx + \frac{L_3}{d_{03}{}^3 t_{03}}\right] \longleftarrow$$

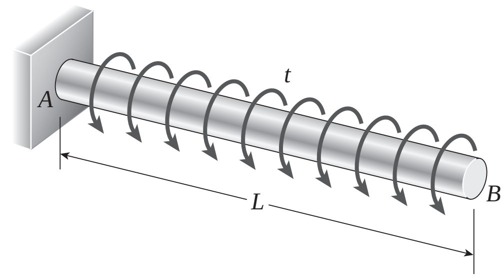
NUMERICAL DATA

$L_1 = 2$ in. $\quad L_2 = 1.2$ in. $\quad L_3 = 8.5$ in.
$t_{01} = 0.125$ in. $\quad t_{03} = 0.115$ in. $\quad d_{01} = 1.25$ in.
$d_{03} = 0.87$ in. $\quad F = 15$ lb $\quad d = 4$ in.
$G = 4 \times (10^6)$ psi

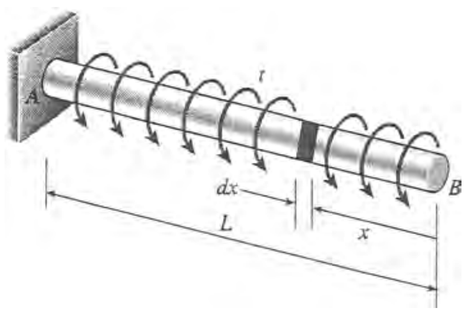
$\phi_D = 0.142°$ $\longleftarrow$

Problem 3.4-16 A prismatic bar *AB* of length *L* and solid circular cross section (diameter *d*) is loaded by a distributed torque of constant intensity *t* per unit distance (see figure).

(a) Determine the maximum shear stress τ_{max} in the bar.
(b) Determine the angle of twist ϕ between the ends of the bar.



Solution 3.4-16 Bar with distributed torque



t = intensity of distributed torque

d = diameter

G = shear modulus of elasticity

(a) Maximum shear stress

$$T_{max} = tL \quad \tau_{max} = \frac{16T_{max}}{\pi d^3} = \frac{16tL}{\pi d^3} \quad \leftarrow$$

(b) Angle of twist

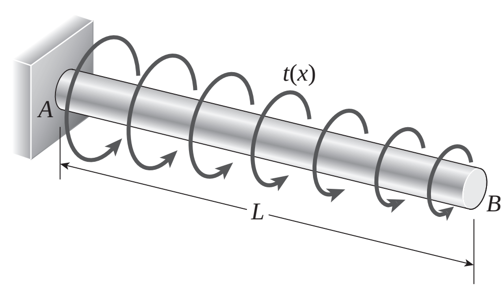
$$T(x) = tx \quad I_P = \frac{\pi d^4}{32}$$

$$d\phi = \frac{T(x)dx}{GI_P} = \frac{32\ tx\ dx}{\pi G d^4}$$

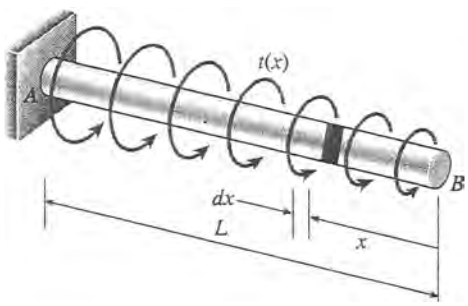
$$\phi = \int_0^L d\phi = \frac{32t}{\pi G d^4}\int_0^L x\,dx = \frac{16tL^2}{\pi G d^4} \quad \leftarrow$$

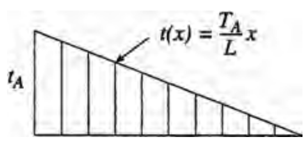
Problem 3.4-17 A prismatic bar *AB* of solid circular cross section (diameter *d*) is loaded by a distributed torque (see figure). The intensity of the torque, that is, the torque per unit distance, is denoted $t(x)$ and varies linearly from a maximum value t_A at end *A* to zero at end *B*. Also, the length of the bar is *L* and the shear modulus of elasticity of the material is *G*.

(a) Determine the maximum shear stress τ_{max} in the bar.
(b) Determine the angle of twist ϕ between the ends of the bar.



Solution 3.4-17 Bar with linearly varying torque





$t(x)$ = intensity of distributed torque
t_A = maximum intensity of torque
d = diameter
G = shear modulus
T_A = maximum torque
$= \frac{1}{2}\, t_A L$

(a) Maximum shear stress

$$\tau_{\max} = \frac{16T_{\max}}{\pi d^3} = \frac{16T_A}{\pi d^3} = \frac{8t_A L}{\pi d^3} \longleftarrow$$

(b) Angle of twist

$T(x)$ = torque at distance x from end B

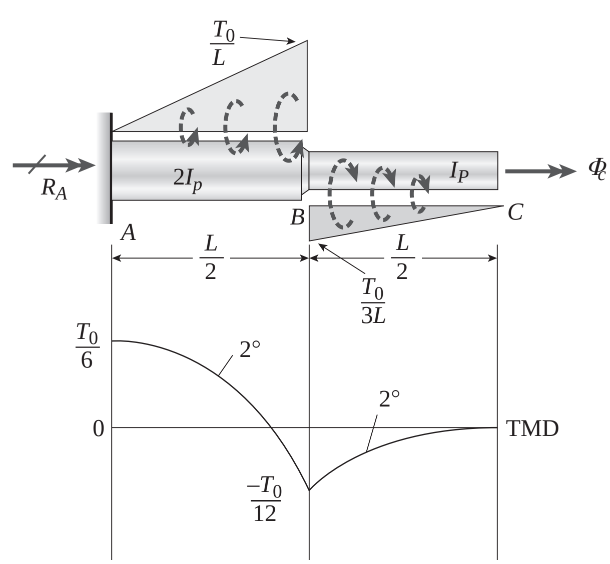
$$T(x) = \frac{t(x)x}{2} = \frac{t_A x^2}{2L} \qquad I_P = \frac{\pi d^4}{32}$$

$$d\phi = \frac{T(x)\,dx}{GI_P} = \frac{16 t_A x^2\,dx}{\pi G L d^4}$$

$$\phi = \int_0^L d\phi = \frac{16t_A}{\pi G L d^4}\int_0^L x^2\,dx = \frac{16t_A L^2}{3\pi G d^4} \longleftarrow$$

Problem 3.4-18 A nonprismatic bar ABC of solid circular cross section is loaded by distributed torques (see figure). The intensity of the torques, that is, the torque per unit distance, is denoted $t(x)$ and varies linearly from zero at A to a maximum value T_0/L at B. Segment BC has linearly distributed torque of intensity $t(x) = T_0/3L$ of opposite sign to that applied along AB. Also, the polar moment of inertia of AB is twice that of BC, and the shear modulus of elasticity of the material is G.

(a) Find reaction torque R_A.
(b) Find internal torsional moments $T(x)$ in segments AB and BC.
(c) Find rotation ϕ_C.
(d) Find the maximum shear stress $\tau_{\max}$ and its location along the bar.
(e) Draw the torsional moment diagram (TMD: $T(x)$, $0 \le x \le L$).



Solution 3.4-18

(a) TORQUE REACTION R_A

STATICS: $\Sigma T = 0$

$$R_A + \frac{1}{2}\left(\frac{T_0}{L}\right)\left(\frac{L}{2}\right) - \frac{1}{2}\left(\frac{T_0}{3L}\right)\left(\frac{L}{2}\right) = 0$$

$$R_A + \frac{1}{6}T_0 = 0 \qquad R_A = \frac{-T_0}{6} \quad \leftarrow$$

(b) INTERNAL TORSIONAL MOMENTS IN AB & BC

$$T_{AB}(x) = \frac{T_0}{6} - \left(\frac{x}{\frac{L}{2}}\right)\left(\frac{T_0}{L}\right)\frac{x}{2}$$

$$T_{AB}(x) = \left(\frac{T_0}{6} - \frac{x^2}{L^2}T_0\right) \quad 0 \le x \le \frac{L}{2} \quad \leftarrow$$

$$T_{BC}(x) = \frac{-(L-x)}{\frac{L}{2}}\left(\frac{T_0}{3L}\right)\frac{(L-x)}{2}$$

$$T_{BC}(x) = -\left[\left(\frac{x-L}{L}\right)^2\frac{T_0}{3}\right]$$

$$\frac{L}{2} \le x \le L \quad \leftarrow$$

(c) ROTATION AT C

$$\phi_C = \int_0^{\frac{L}{2}} \frac{T_{AB}(x)}{G(2I_P)}\,dx + \int_{\frac{L}{2}}^{L} \frac{T_{BC}(x)}{GI_P}\,dx$$

$$\phi_C = \int_0^{\frac{L}{2}} \frac{\frac{T_0}{6} - \frac{x^2\,T_0}{3L^2}}{G(2I_P)}\,dx + \int_{\frac{L}{2}}^{L} \frac{-\left[\left(\frac{x-L}{L}\right)^2\frac{T_0}{3}\right]}{GI_P}\,dx$$

$$\phi_C = \frac{T_0L}{48GI_P} - \frac{T_0L}{72GI_P}$$

$$\phi_C = \frac{T_0L}{144GI_P} \quad \leftarrow$$

(d) MAXIMUM SHEAR STRESS ALONG BAR

For AB $\;2I_P = \frac{\pi}{32}d_{AB}{}^4$

For BC $\;I_P = \frac{\pi}{32}d_{BC}{}^4$

$$d_{BC} = \left(\frac{1}{2}\right)^{\frac{1}{4}} d_{AB}$$

At A, $T = T_0/6$ $\qquad \tau_{max} = \frac{\frac{T_0}{6}\,\frac{d_AB}{2}}{\frac{\pi}{32}d_{AB}{}^4}$

$$\tau_{max} = \frac{8T_0}{3\pi d_{AB}{}^3} \quad \leftarrow \text{controls}$$

Just to right of B, $T = -T_0/12$

$$\tau_{max} = \frac{\frac{T_0}{12}\left(\frac{d_{BC}}{2}\right)}{\frac{\pi}{32}d_{BC}{}^4}$$

$$\tau_{max} = \frac{\frac{T_0}{12}\left(\frac{0.841d_{AB}}{2}\right)}{\frac{\pi}{32}(0.841d_{AB})^4}$$

$$\tau_{max} = \frac{2.243T_0}{\pi d_{AB}{}^3}$$

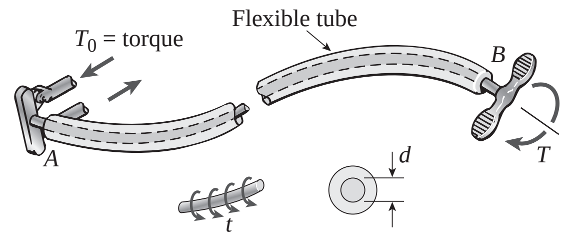
(e) TMD = two 2nd degree curves: from $T_0/6$ at A, to $-T_0/12$ at B, to zero at C (with zero slopes at A & C since slope on TMD is proportional to ordinate on torsional loading) – see plot of $T(x)$ above

Problem 3.4-19 A magnesium-alloy wire of diameter $d = 4$ mm and length L rotates inside a flexible tube in order to open or close a switch from a remote location (see figure). A torque T is applied manually (either clockwise or counterclockwise) at end B, thus twisting the wire inside the tube. At the other end A, the rotation of the wire operates a handle that opens or closes the switch.

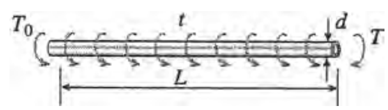
A torque $T_0 = 0.2$ N · m is required to operate the switch. The torsional stiffness of the tube, combined with friction between the tube and the wire, induces a distributed torque of constant intensity $t = 0.04$ N · m/m (torque per unit distance) acting along the entire length of the wire.

(a) If the allowable shear stress in the wire is $\tau_{allow} = 30$ MPa, what is the longest permissible length L_{max} of the wire?

(b) If the wire has length $L = 4.0$ m and the shear modulus of elasticity for the wire is $G = 15$ GPa, what is the angle of twist ϕ (in degrees) between the ends of the wire?



Solution 3.4-19 Wire inside a flexible tube



$d = 4$ mm
$T_0 = 0.2$ N · m
$t = 0.04$ N · m/m

(a) Maximum length L_{max}

$\tau_{allow} = 30$ MPa

Equilibrium: $T = tL + T_0$

From Eq. (3-12): $\tau_{max} = \dfrac{16T}{\pi d^3} \quad T = \dfrac{\pi d^3 \tau_{max}}{16}$

$$tL + T_0 = \frac{\pi d^3 \tau_{max}}{16}$$

$$L = \frac{1}{16t}(\pi d^3 \tau_{max} - 16T_0)$$

$$L_{max} = \frac{1}{16t}(\pi d^3 \tau_{allow} - 16T_0) \quad \leftarrow$$

Substitute numerical values: $L_{max} = 4.42$ m $\leftarrow$

(b) Angle of twist ϕ

$L = 4$ m $\quad G = 15$ GPa

ϕ_1 = angle of twist due to distributed torque t

$$= \frac{16tL^2}{\pi G d^4} \text{ (from problem 3.4-16)}$$

ϕ_2 = angle of twist due to torque T_0

$$= \frac{T_0 L}{G I_P} = \frac{32\, T_0 L}{\pi G d^4} \text{ (from Eq. 3-15)}$$

ϕ = total angle of twist
$= \phi_1 + \phi_2$

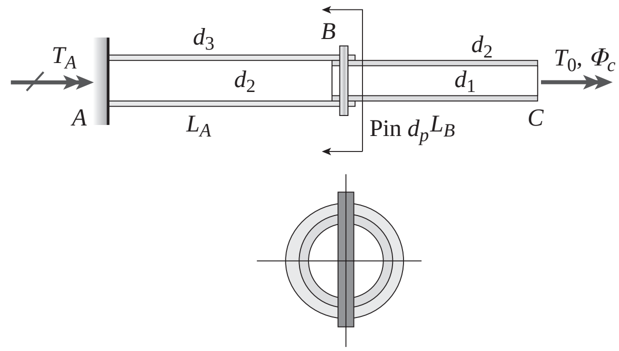
$$\phi = \frac{16L}{\pi G d^4}(tL + 2T_0) \quad \leftarrow$$

Substitute numerical values:
$\phi = 2.971$ rad $= 170°$ $\leftarrow$

Problem 3.4-20 Two hollow tubes are connected by a pin at B which is inserted into a hole drilled through both tubes at B (see cross-section view at B). Tube BC fits snugly into tube AB but neglect any friction on the interface. Tube inner and outer diameters d_i ($i = 1, 2, 3$) and pin diameter d_p are labeled in the figure. Torque T_0 is applied at joint C. The shear modulus of elasticity of the material is G.

Find expressions for the maximum torque $T_{0,max}$ which can be applied at C for each of the following conditions.

(a) The shear in the connecting pin is less than some allowable value ($\tau_{pin} < \tau_{p,allow}$).
(b) The shear in tube AB or BC is less than some allowable value ($\tau_{tube} < \tau_{t,allow}$).
(c) What is the maximum rotation ϕ_C for each of cases (a) and (b) above?



Cross-section at B

Solution 3.4-20

(a) $T_{0,max}$ BASED ON ALLOWABLE SHEAR IN PIN AT B

Pin at B is in shear at interface between the two tubes; force couple $V \cdot d_2 = T_0$

$$V = \frac{T_0}{d_2} \qquad \tau_{pin} = \frac{V}{A_S}$$

$$\tau_{pin} = \frac{\dfrac{T_0}{d_2}}{\left(\dfrac{\pi d_p^2}{4}\right)} \qquad \tau_{pin} = \frac{4T_0}{\pi d_2 d_p^2}$$

$$T_{0,max} = \tau_{p,allow}\left(\frac{\pi d_2 d_p^2}{4}\right) \quad \leftarrow$$

(b) $T_{0,max}$ BASED ON ALLOWABLE SHEAR IN TUBES AB & BC

$$I_{PAB} = \frac{\pi}{32}\left(d_3^4 - d_2^4\right)$$

$$I_{PAC} = \frac{\pi}{32}\left(d_2^4 - d_1^4\right)$$

$$\tau_{tubeAB} = \frac{T_0\left(\dfrac{d_3}{2}\right)}{I_{PAB}}$$

$$\tau_{tubeAB} = \frac{T_0\left(\dfrac{d_3}{2}\right)}{\dfrac{\pi}{32}(d_3^4 - d_2^4)}$$

$$\tau_{tubeAB} = \frac{16T_0 d_3}{\pi(d_3^4 - d_2^4)}$$

so based on tube AB:

$$T_{0,max} = \tau_{t,allow}\left[\frac{\pi(d_3^4 - d_2^4)}{16d_3}\right] \quad \leftarrow$$

and based on tube BC:

$$\tau_{tubeBC} = \frac{T_0\left(\dfrac{d_2}{2}\right)}{\dfrac{\pi}{32}(d_2^4 - d_1^4)}$$

$$\tau_{tubeBC} = \frac{16T_0 d_2}{\pi(d_2^4 - d_1^4)}$$

$$T_{0,max} = \tau_{t,allow}\left[\frac{\pi(d_2^4 - d_1^4)}{16d_2}\right] \quad \leftarrow$$

(c) MAX. ROTATION AT C BASED ON EITHER ALLOWABLE SHEAR IN PIN AT B OR ALLOWABLE SHEAR STRESS IN TUBES

MAX. ROTATION BASED ON ALLOWABLE *SHEAR IN PIN AT* B

$$\phi_C = \frac{T_{0,\max}}{G}\left(\frac{L_A}{I_{PAB}} + \frac{L_B}{I_{PBC}}\right)$$

$$\phi_{C\max} = \frac{\tau_{\text{p,allow}}\left(\frac{\pi d_2 d_p^2}{4}\right)}{G} \left[\frac{L_A}{\frac{\pi}{32}(d_3^{\,4} - d_2^{\,4})} + \frac{L_B}{\frac{\pi}{32}(d_2^{\,4} - d_1^{\,4})}\right]$$

$$\phi_{C\max} = \tau_{\text{p, allow}}\left(\frac{8 d_2 d_p^{\,2}}{G}\right) \left[\frac{L_A}{(d_3^{\,4} - d_2^{\,4})} + \frac{L_B}{(d_2^{\,4} - d_1^{\,4})}\right] \quad \leftarrow$$

MAX. ROTATION BASED ON ALLOWABLE *SHEAR STRESS IN TUBE* AB

$$\phi_{C\max} = \tau_{\text{t, allow}}\left[\frac{2(d_3^{\,4} - d_2^{\,4})}{G d_3}\right] \left[\frac{L_A}{(d_3^{\,4} - d_2^{\,4})} + \frac{L_B}{(d_2^{\,4} - d_1^{\,4})}\right] \quad \leftarrow$$

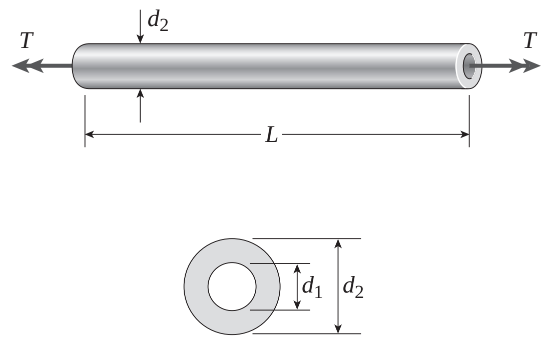
MAX. ROTATION BASED ON ALLOWABLE *SHEAR STRESS IN TUBE* BC

$$\phi_{C\max} = \tau_{\text{t, allow}}\left[\frac{2(d_2^{\,4} - d_1^{\,4})}{G d_2}\right] \left[\frac{L_A}{(d_3^{\,4} - d_2^{\,4})} + \frac{L_B}{(d_2^{\,4} - d_1^{\,4})}\right] \quad \leftarrow$$

Pure Shear

Problem 3.5-1 A hollow aluminum shaft (see figure) has outside diameter $d_2 = 4.0$ in. and inside diameter $d_1 = 2.0$ in. When twisted by torques T, the shaft has an angle of twist per unit distance equal to 0.54°/ft. The shear modulus of elasticity of the aluminum is $G = 4.0 \times 10^6$ psi.

(a) Determine the maximum tensile stress σ_{max} in the shaft.
(b) Determine the magnitude of the applied torques T.



Probs. 3.5-1, 3.5-2, and 3.5-3

Solution 3.5-1 Hollow aluminum shaft

$d_2 = 4.0$ in. $\quad d_1 = 2.0$ in. $\quad \theta = 0.54°/\text{ft}$

$G = 4.0 \times 10^6$ psi

MAXIMUM SHEAR STRESS

$\tau_{max} = Gr\theta$ (from Eq. 3-7a)

$$r = d_2/2 = 2.0 \text{ in.}$$

$$\theta = (0.54°/\text{ft})\left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)\left(\frac{\pi \text{rad}}{180 \text{ degree}}\right)$$

$$= 785.40 \times 10^{-6} \text{ rad/in.}$$

$$\tau_{max} = (4.0 \times 10^6 \text{ psi})(2.0 \text{ in.})(785.40 \times 10^{-6} \text{ rad/in.})$$

$$= 6283.2 \text{ psi}$$

(a) MAXIMUM TENSILE STRESS

σ_{max} occurs on a 45° plane and is equal to τ_{max}.

$$\sigma_{max} = \tau_{max} = 6280 \text{ psi} \quad \leftarrow$$

(b) APPLIED TORQUE

Use the torsion formula $\tau_{max} = \dfrac{Tr}{I_P}$

$$T = \frac{\tau_{max} I_P}{r} \quad I_P = \frac{\pi}{32}[(4.0 \text{ in.})^4 - (2.0 \text{ in.})^4]$$

$$= 23.562 \text{ in.}^4$$

$$T = \frac{(6283.2 \text{ psi})(23.562 \text{ in.}^4)}{2.0 \text{ in.}}$$

$$= 74{,}000 \text{ lb-in.} \quad \leftarrow$$

Problem 3.5-2 A hollow steel bar ($G = 80$ GPa) is twisted by torques T (see figure). The twisting of the bar produces a maximum shear strain $\gamma_{max} = 640 \times 10^{-6}$ rad. The bar has outside and inside diameters of 150 mm and 120 mm, respectively.

(a) Determine the maximum tensile strain in the bar.
(b) Determine the maximum tensile stress in the bar.
(c) What is the magnitude of the applied torques T?

Solution 3.5-2 Hollow steel bar

$G = 80$ GPa $\quad \gamma_{max} = 640 \times 10^{-6}$ rad

$d_2 = 150$ mm $\quad d_1 = 120$ mm

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4)$$

$$= \frac{\pi}{32}[(150\text{ mm})^4 - (120\text{ mm})^4]$$

$$= 29.343 \times 10^6 \text{ mm}^4$$

(a) Maximum tensile strain

$$\varepsilon_{max} = \frac{\gamma_{max}}{2} = 320 \times 10^{-6} \quad \leftarrow$$

(b) Maximum tensile stress

$$\tau_{max} = G\gamma_{max} = (80\text{ GPa})(640 \times 10^{-6})$$

$$= 51.2 \text{ MPa}$$

$$\sigma_{max} = \tau_{max} = 51.2 \text{ MPa} \quad \leftarrow$$

(c) Applied torques

Torsion formula: $\tau_{max} = \dfrac{Tr}{I_P} = \dfrac{Td_2}{2I_P}$

$$T = \frac{2I_P\tau_{max}}{d_2} = \frac{2(29.343 \times 10^6 \text{ mm}^4)(51.2 \text{ MPa})}{150 \text{ mm}}$$

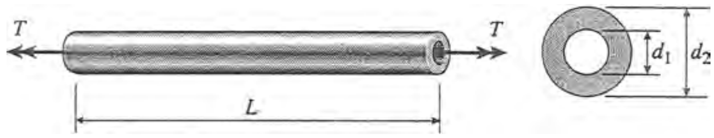
$$= 20{,}030 \text{ N}\cdot\text{m}$$

$$= 20.0 \text{ kN}\cdot\text{m} \quad \leftarrow$$

Problem 3.5-3 A tubular bar with outside diameter $d_2 = 4.0$ in. is twisted by torques $T = 70.0$ k-in. (see figure). Under the action of these torques, the maximum tensile stress in the bar is found to be 6400 psi.

(a) Determine the inside diameter d_1 of the bar.
(b) If the bar has length $L = 48.0$ in. and is made of aluminum with shear modulus $G = 4.0 \times 10^6$ psi, what is the angle of twist ϕ (in degrees) between the ends of the bar?
(c) Determine the maximum shear strain γ_{max} (in radians)?

Solution 3.5-3 Tubular bar



$d_2 = 4.0$ in. $\quad T = 70.0$ k-in. $= 70{,}000$ lb-in.

$\sigma_{max} = 6400$ psi $\quad \tau_{max} = \sigma_{max} = 6400$ psi

(a) Inside diameter d_1

Torsion formula: $\tau_{max} = \dfrac{Tr}{I_P} = \dfrac{Td_2}{2I_P}$

$$I_P = \frac{Td_2}{2\tau_{max}} = \frac{(70.0 \text{ k-in.})(4.0 \text{ in.})}{2(6400 \text{ psi})}$$
$$= 21.875 \text{ in.}^4$$

Also, $I_p = \dfrac{\pi}{32}(d_2^4 - d_1^4) = \dfrac{\pi}{32}[(4.0 \text{ in.})^4 - d_1^4]$

Equate formulas:

$$\frac{\pi}{32}[256 \text{ in.}^4 - d_1^4] = 21.875 \text{ in.}^4$$

Solve for d_1: $d_1 = 2.40$ in. $\quad \leftarrow$

(b) Angle of twist ϕ

$L = 48$ in. $\quad G = 4.0 \times 10^6$ psi

$$\phi = \frac{TL}{GI_p}$$

From torsion formula, $T = \dfrac{2I_P\tau_{max}}{d_2}$

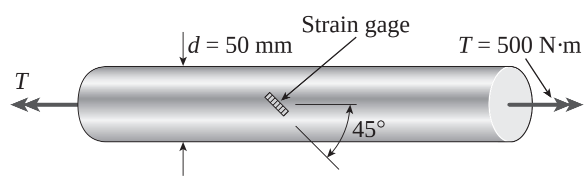
$$\therefore \phi = \frac{2I_P\tau_{max}}{d_2}\left(\frac{L}{GI_P}\right) = \frac{2L\tau_{max}}{Gd_2}$$
$$= \frac{2(48 \text{ in.})(6400 \text{ psi})}{(4.0 \times 10^6 \text{ psi})(4.0 \text{ in.})} = 0.03840 \text{ rad}$$
$$\phi = 2.20° \quad \leftarrow$$

(c) Maximum shear strain

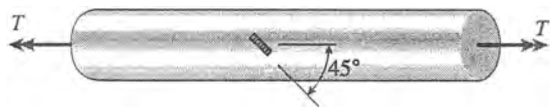
$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{6400 \text{ psi}}{4.0 \times 10^6 \text{ psi}}$$
$$= 1600 \times 10^{-6} \text{ rad} \quad \leftarrow$$

Problem 3.5-4 A solid circular bar of diameter $d = 50$ mm (see figure) is twisted in a testing machine until the applied torque reaches the value $T = 500$ N · m. At this value of torque, a strain gage oriented at 45° to the axis of the bar gives a reading $\epsilon = 339 \times 10^{-6}$.

What is the shear modulus G of the material?



Solution 3.5-4 Bar in a testing machine



Strain gage at 45°:

$\varepsilon_{max} = 339 \times 10^{-6}$

$d = 50$ mm

$T = 500$ N · m

Shear strain (from Eq. 3-33)

$\gamma_{max} = 2\varepsilon_{max} = 678 \times 10^{-6}$

Shear stress (from Eq. 3-12)

$$\tau_{max} = \frac{16T}{\pi d^3} = \frac{16(500\text{ N}\cdot\text{m})}{\pi(0.050\text{ m})^3} = 20.372\text{ MPa}$$

Shear modulus

$$G = \frac{\tau_{max}}{\gamma_{max}} = \frac{20.372\text{ MPa}}{678 \times 10^{-6}} = 30.0\text{ GPa} \quad \leftarrow$$

Problem 3.5-5 A steel tube ($G = 11.5 \times 10^6$ psi) has an outer diameter $d_2 = 2.0$ in. and an inner diameter $d_1 = 1.5$ in. When twisted by a torque T, the tube develops a maximum normal strain of 170×10^{-6}.

What is the magnitude of the applied torque T?

Solution 3.5-5 Steel tube

$G = 11.5 \times 10^6$ psi $\quad d_2 = 2.0$ in. $\quad d_1 = 1.5$ in.

$\varepsilon_{max} = 170 \times 10^{-6}$

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4) = \frac{\pi}{32}[(2.0\text{ in.})^4 - (1.5\text{ in.})^4]$$

$$= 1.07379\text{ in.}^4$$

Shear strain (from Eq. 3-33)

$\gamma_{max} = 2\varepsilon_{max} = 340 \times 10^{-6}$

Shear Stress (from torsion formula)

$$\tau_{max} = \frac{Tr}{I_P} = \frac{Td_2}{2I_P}$$

Also, $\tau_{max} = G\gamma_{max}$

Equate expressions:

$$\frac{Td_2}{2I_P} = G\gamma_{max}$$

Solve for torque

$$T = \frac{2GI_P\gamma_{max}}{d_2}$$

$$= \frac{2(11.5 \times 10^6\text{ psi})(1.07379\text{ in.}^4)(340 \times 10^{-6})}{2.0\text{ in.}}$$

$$= 4200\text{ lb-in.} \quad \leftarrow$$

Problem 3.5-6 A solid circular bar of steel ($G = 78$ GPa) transmits a torque $T = 360$ N · m. The allowable stresses in tension, compression, and shear are 90 MPa, 70 MPa, and 40 MPa, respectively. Also, the allowable tensile strain is 220×10^{-6}.

Determine the minimum required diameter d of the bar.

Solution 3.5-6 Solid circular bar of steel

$T = 360$ N · m $\quad G = 78$ GPa

ALLOWABLE STRESSES

Tension: 90 MPa Compression: 70 MPa
Shear: 40 MPa
Allowable tensile strain: $\varepsilon_{max} = 220 \times 10^{-6}$

DIAMETER BASED UPON ALLOWABLE STRESS

The maximum tensile, compressive, and shear stresses in a bar in pure torsion are numerically equal. Therefore, the lowest allowable stress (shear stress) governs.

$\tau_{allow} = 40$ MPa

$$\tau_{max} = \frac{16T}{\pi d^3} \quad d^3 = \frac{16T}{\pi \tau_{allow}} = \frac{16(360 \text{ N} \cdot \text{m})}{\pi(40 \text{ MPa})}$$

$$d^3 = 45.837 \times 10^{-6} \text{ m}^3$$

$$d = 0.0358 \text{ m} = 35.8 \text{ mm}$$

DIAMETER BASED UPON ALLOWABLE TENSILE STRAIN

$$\gamma_{max} = 2\varepsilon_{max};\ \tau_{max} = G\gamma_{max} = 2G\varepsilon_{max}$$

$$\tau_{max} = \frac{16T}{\pi d^3} \quad d^3 = \frac{16T}{\pi \tau_{max}} = \frac{16T}{2\pi G \varepsilon_{max}}$$

$$d^3 = \frac{16(360 \text{ N} \cdot \text{m})}{2\pi(78 \text{ GPa})(220 \times 10^{-6})}$$

$$= 53.423 \times 10^{-6} \text{ m}^3$$

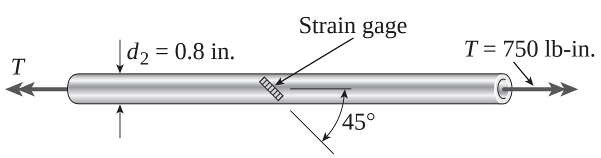
$$d = 0.0377 \text{ m} = 37.7 \text{ mm}$$

TENSILE STRAIN GOVERNS

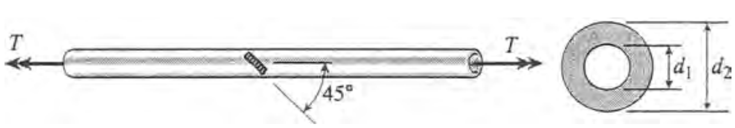
$d_{min} = 37.7$ mm $\quad \leftarrow$

Problem 3.5-7 The normal strain in the 45° direction on the surface of a circular tube (see figure) is 880×10^{-6} when the torque $T = 750$ lb-in. The tube is made of copper alloy with $G = 6.2 \times 10^6$ psi.

If the outside diameter d_2 of the tube is 0.8 in., what is the inside diameter d_1?



Solution 3.5-7 Circular tube with strain gage



$d_2 = 0.80$ in. $\quad T = 750$ lb-in. $\quad G = 6.2 \times 10^6$ psi

Strain gage at 45°: $\varepsilon_{max} = 880 \times 10^{-6}$

MAXIMUM SHEAR STRAIN

$$\gamma_{max} = 2\varepsilon_{max}$$

MAXIMUM SHEAR STRESS

$$\tau_{max} = G\gamma_{max} = 2G\varepsilon_{max}$$

$$\tau_{max} = \frac{T(d_2/2)}{I_P} \qquad I_P = \frac{Td_2}{2\tau_{max}} = \frac{Td_2}{4G\varepsilon_{max}}$$

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4) = \frac{Td_2}{4G\varepsilon_{max}}$$

$$d_2^4 - d_1^4 = \frac{8Td_2}{\pi G\varepsilon_{max}} \qquad d_1^4 = d_2^4 - \frac{8Td_2}{\pi G\varepsilon_{max}}$$

INSIDE DIAMETER

Substitute numerical values:

$$d_2^4 = (0.8 \text{ in.})^4 - \frac{8(750 \text{ lb-in.})(0.80 \text{ in.})}{\pi(6.2 \times 10^6 \text{ psi})(880 \times 10^{-6})}$$

$$= 0.4096 \text{ in.}^4 - 0.2800 \text{ in.}^4 = 0.12956 \text{ in.}^4$$

$$d_1 = 0.60 \text{ in.} \quad \leftarrow$$

Problem 3.5-8 An aluminium tube has inside diameter $d_1 = 50$ mm, shear modulus of elasticity $G = 27$ GPa, and torque $T = 4.0$ kN · m. The allowable shear stress in the aluminum is 50 MPa and the allowable normal strain is 900×10^{-6}.

Determine the required outside diameter d_2.

Solution 3.5-8 Aluminum tube

$d_1 = 50$ mm $\quad G = 27$ GPa

$T = 4.0$ kN · m $\quad \tau_{allow} = 50$ MPa $\quad \varepsilon_{allow} = 900 \times 10^{-6}$

Determine the required diameter d_2.

ALLOWABLE SHEAR STRESS

$$(\tau_{allow})_1 = 50 \text{ MPa}$$

ALLOWABLE SHEAR STRESS BASED ON NORMAL STRAIN

$$\varepsilon_{max} = \frac{\gamma}{2} = \frac{\tau}{2G} \qquad \tau = 2G\varepsilon_{max}$$

$$(\tau_{allow})_2 = 2G\varepsilon_{allow} = 2(27 \text{ GPa})(900 \times 10^{-6})$$
$$= 48.6 \text{ MPa}$$

NORMAL STRAIN GOVERNS

$$\tau_{allow} = 48.60 \text{ MPa}$$

REQUIRED DIAMETER

$$\tau = \frac{Tr}{I_P} \qquad 48.6 \text{ MPa} = \frac{(4000 \text{ N} \cdot \text{m})(d_2/2)}{\frac{\pi}{32}[d_2^4 - (0.050 \text{ m})^4]}$$

Rearrange and simplify:

$$d_2^4 - (419.174 \times 10^{-6})d_2 - 6.25 \times 10^{-6} = 0$$

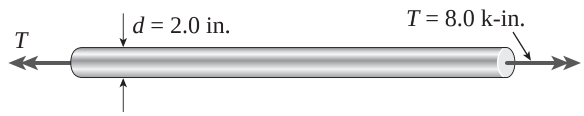
Solve numerically:

$$d_2 = 0.07927 \text{ m}$$

$$d_2 = 79.3 \text{ mm} \quad \leftarrow$$

Problem 3.5-9 A solid steel bar ($G = 11.8 \times 10^6$ psi) of diameter $d = 2.0$ in. is subjected to torques $T = 8.0$ k-in. acting in the directions shown in the figure.

(a) Determine the maximum shear, tensile, and compressive stresses in the bar and show these stresses on sketches of properly oriented stress elements.
(b) Determine the corresponding maximum strains (shear, tensile, and compressive) in the bar and show these strains on sketches of the deformed elements.



Solution 3.5-9 Solid steel bar

$T = 8.0$ k-in.

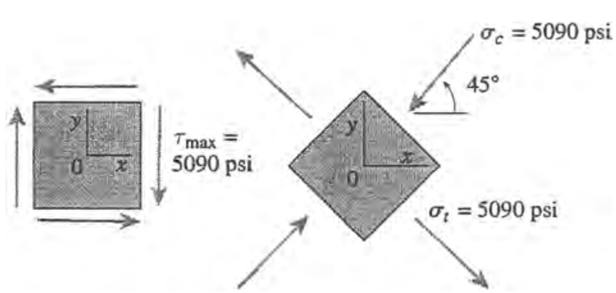
$G = 11.8 \times 10^6$ psi

(a) Maximum stresses

$$\tau_{max} = \frac{16T}{\pi d^3} = \frac{16(8000 \text{ lb-in.})}{\pi (2.0 \text{ in.})^3}$$

$$= 5093 \text{ psi} \quad \leftarrow$$

$$\sigma_t = 5090 \text{ psi} \quad \sigma_c = -5090 \text{ psi} \quad \leftarrow$$



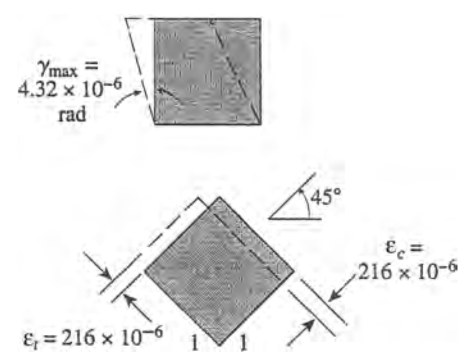
(b) Maximum strains

$$\gamma_{max} = \frac{\gamma_{max}}{G} = \frac{5093 \text{ psi}}{11.8 \times 10^6 \text{ psi}}$$

$$= 432 \times 10^{-6} \text{ rad} \quad \leftarrow$$

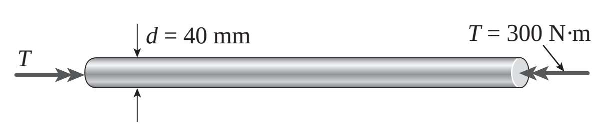
$$\varepsilon_{max} = \frac{\gamma_{max}}{2} = 216 \times 10^{-6}$$

$$\varepsilon_t = 216 \times 10^{-6} \quad \varepsilon_c = -216 \times 10^{-6} \quad \leftarrow$$



Problem 3.5-10 A solid aluminum bar ($G = 27$ GPa) of diameter $d = 40$ mm is subjected to torques $T = 300$ N · m acting in the directions shown in the figure.

(a) Determine the maximum shear, tensile, and compressive stresses in the bar and show these stresses on sketches of properly oriented stress elements.
(b) Determine the corresponding maximum strains (shear, tensile, and compressive) in the bar and show these strains on sketches of the deformed elements.



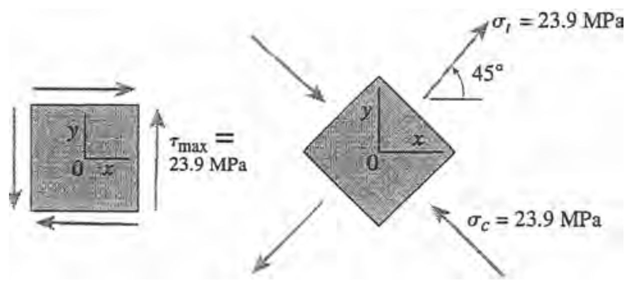
Solution 3.5-10 Solid aluminum bar

(a) MAXIMUM STRESSES

$$\tau_{max} = \frac{16T}{\pi d^3} = \frac{16(300 \text{ N} \cdot \text{ m})}{\pi(0.040 \text{ m})^3}$$

$$= 23.87 \text{ MPa} \quad \leftarrow$$

$$\sigma_t = 23.9 \text{ MPa} \quad \sigma_c = -23.9 \text{ MPa} \quad \leftarrow$$



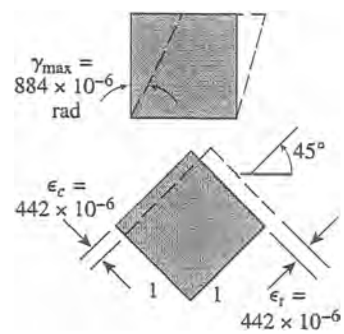
(b) MAXIMUM STRAINS

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{23.87 \text{ MPa}}{27 \text{ GPa}}$$

$$= 884 \times 10^{-6} \text{ rad} \quad \leftarrow$$

$$\varepsilon_{max} = \frac{\gamma_{max}}{2} = 442 \times 10^{-6}$$

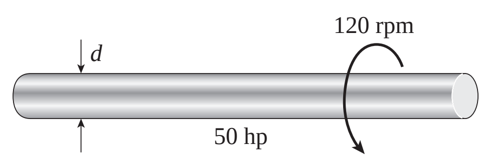
$$\varepsilon_t = 442 \times 10^{-6} \quad \varepsilon_c = -442 \times 10^{-6} \quad \leftarrow$$



Transmission of Power

Problem 3.7-1 A generator shaft in a small hydroelectric plant turns at 120 rpm and delivers 50 hp (see figure).

(a) If the diameter of the shaft is $d = 3.0$ in., what is the maximum shear stress τ_{max} in the shaft?
(b) If the shear stress is limited to 4000 psi, what is the minimum permissible diameter d_{min} of the shaft?



Solution 3.7-1 Generator shaft

$n = 120$ rpm $\quad H = 50$ hp $\quad d =$ diameter

TORQUE

$$H = \frac{2\pi n T}{33{,}000} \quad H = \text{hp} \quad n = \text{rpm} \quad T = \text{1b-ft}$$

$$T = \frac{33{,}000\, H}{2\pi n} = \frac{(33{,}000)(50 \text{ hp})}{2\pi(120 \text{ rpm})}$$

$$= 2188 \text{ 1b-ft} = 26{,}260 \text{ 1b-in.}$$

(a) MAXIMUM SHEAR STRESS τ_{max}

$d = 3.0$ in.

$$\tau_{max} = \frac{16T}{\pi d^3} = \frac{16(26{,}260 \text{ 1b-in.})}{\pi(3.0 \text{ in.})^3}$$

$$\tau_{max} = 4950 \text{ psi} \quad \leftarrow$$

(b) MINIMUM DIAMETER d_{min}

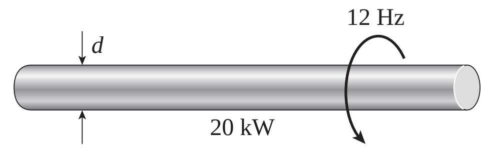
$$\tau_{allow} = 4000 \text{ psi}$$

$$d^3 = \frac{16T}{\pi \tau_{allow}} = \frac{16(26{,}260 \text{ 1b-in.})}{\pi(4000 \text{ psi})} = 33.44 \text{ in.}^3$$

$$d_{min} = 3.22 \text{ in.} \quad \leftarrow$$

Problem 3.7-2 A motor drives a shaft at 12 Hz and delivers 20 kW of power (see figure).

(a) If the shaft has a diameter of 30 mm, what is the maximum shear stress τ_{max} in the shaft?
(b) If the maximum allowable shear stress is 40 MPa, what is the minimum permissible diameter d_{min} of the shaft?



Solution 3.7-2 Motor-driven shaft

$f = 12$ Hz $\quad P = 20$ kW $= 20{,}000$ N · m/s

TORQUE

$P = 2\pi f T \quad P =$ watts $\quad f =$ Hz $=$ s^{-1}

$T =$ Newton meters

$$T = \frac{P}{2\pi f} = \frac{20{,}000 \text{ W}}{2\pi(12 \text{ Hz})} = 265.3 \text{ N} \cdot \text{m}$$

(a) MAXIMUM SHEAR STRESS τ_{max}

$d = 30$ mm

$$\tau_{max} = \frac{16T}{\pi d^3} = \frac{16(265.3 \text{ N} \cdot \text{m})}{\pi(0.030 \text{ m})^3}$$

$$= 50.0 \text{ MPa} \quad \leftarrow$$

(b) MINIMUM DIAMETER d_{min}

$\tau_{allow} = 40$ MPa

$$d^3 = \frac{16T}{\pi\tau_{allow}} = \frac{16(265.3 \text{ N} \cdot \text{m})}{\pi(40 \text{ MPa})}$$

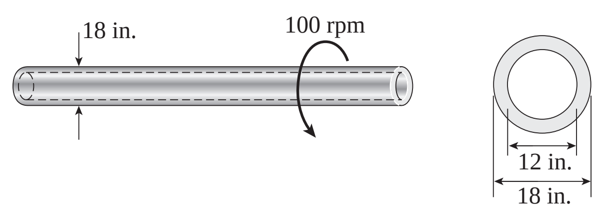
$$= 33.78 \times 10^{-6} \text{ m}^3$$

$$d_{min} = 0.0323 \text{ m} = 32.3 \text{ mm} \quad \leftarrow$$

Problem 3.7-3 The propeller shaft of a large ship has outside diameter 18 in. and inside diameter 12 in., as shown in the figure. The shaft is rated for a maximum shear stress of 4500 psi.

(a) If the shaft is turning at 100 rpm, what is the maximum horsepower that can be transmitted without exceeding the allowable stress?

(b) If the rotational speed of the shaft is doubled but the power requirements remain unchanged, what happens to the shear stress in the shaft?



Solution 3.7-3 Hollow propeller shaft

$d_2 = 18$ in. $\quad d_1 = 12$ in. $\quad \tau_{allow} = 4500$ psi

$$I_P = \frac{\pi}{32}(d_2^4 - d_2^4) = 8270.2 \text{ in.}^4$$

TORQUE

$$\tau_{max} = \frac{T(d_2/2)}{I_P} \qquad T = \frac{2\tau_{allow} I_P}{d_2}$$

$$T = \frac{2(4500 \text{ psi})(8270.2 \text{ in.}^4)}{18 \text{ in.}}$$

$$= 4.1351 \times 10^6 \text{ 1b-in.}$$

$$= 344{,}590 \text{ 1b-ft.}$$

(a) HORSEPOWER

$$n = 100 \text{ rpm} \qquad H = \frac{2\pi n T}{33{,}000}$$

$n =$ rpm $\quad T =$ lb-ft $\quad H =$ hp

$$H = \frac{2\pi(100 \text{ rpm})(344{,}590 \text{ lb-ft})}{33{,}000}$$

$$= 6560 \text{ hp} \quad \leftarrow$$

(b) ROTATIONAL SPEED IS DOUBLED

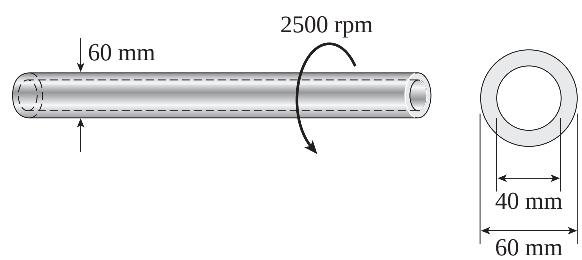
$$H = \frac{2\pi n T}{33{,}000}$$

If n is doubled but H remains the same, then T is halved.
If T is halved, so is the maximum shear stress.

$\therefore$ Shear stress is halved $\leftarrow$

Problem 3.7-4 The drive shaft for a truck (outer diameter 60 mm and inner diameter 40 mm) is running at 2500 rpm (see figure).

(a) If the shaft transmits 150 kW, what is the maximum shear stress in the shaft?
(b) If the allowable shear stress is 30 MPa, what is the maximum power that can be transmitted?



Solution 3.7-4 Drive shaft for a truck

$d_2 = 60$ mm $\quad d_1 = 40$ mm $\quad n = 2500$ rpm

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4) = 1.0210 \times 10^{-6}\ \text{m}^4$$

(a) MAXIMUM SHEAR STRESS τ_{max}

P = power (watts) $\quad P = 150$ kW $= 150{,}000$ W

T = torque (newton meters) $\quad n$ = rpm

$$P = \frac{2\pi n T}{60} \qquad T = \frac{60P}{2\pi n}$$

$$T = \frac{60(150{,}000\ \text{W})}{2\pi(2500\ \text{rpm})} = 572.96\ \text{N}\cdot\text{m}$$

$$\tau_{max} = \frac{Td_2}{2\,I_P} = \frac{(572.96\ \text{N}\cdot\text{m})(0.060\ \text{m})}{2(1.0210 \times 10^{-6}\ \text{m}^4)}$$

$$= 16.835\ \text{MPa}$$

$$\tau_{max} = 16.8\ \text{MPa} \quad \leftarrow$$

(b) MAXIMUM POWER P_{max}

$\tau_{allow} = 30$ MPa

$$P_{max} = P\,\frac{\tau_{allow}}{\tau_{max}} = (150\ \text{kW})\left(\frac{30\ \text{MPa}}{16.835\ \text{MPa}}\right)$$

$$= 267\ \text{kW} \quad \leftarrow$$

Problem 3.7-5 A hollow circular shaft for use in a pumping station is being designed with an inside diameter equal to 0.75 times the outside diameter. The shaft must transmit 400 hp at 400 rpm without exceeding the allowable shear stress of 6000 psi.

Determine the minimum required outside diameter d.

Solution 3.7-5 Hollow shaft

d = outside diameter

d_0 = inside diameter

$= 0.75\, d$

$H = 400$ hp $\quad n = 400$ rpm

$\tau_{allow} = 6000$ psi

$$I_P = \frac{\pi}{32}[d^4 - (0.75\, d)^4] = 0.067112\, d^4$$

TORQUE

$$H = \frac{2\pi n T}{33{,}000}$$

H = hp $\quad n$ = rpm $\quad T$ = lb-ft

$$T = \frac{33{,}000\ H}{2\pi n} = \frac{(33{,}000)(400\ \text{hp})}{2\pi(400\ \text{rpm})}$$

$= 5252.1$ lb-ft $= 63{,}025$ lb-in.

MINIMUM OUTSIDE DIAMETER

$$\tau_{max} = \frac{Td}{2I_P} \quad I_P = \frac{Td}{2\tau_{max}} = \frac{Td}{2\tau_{allow}}$$

$$0.067112\, d^4 = \frac{(63{,}025\ \text{lb-in.})(d)}{2(6000\ \text{psi})}$$

$d^3 = 78.259$ in.^{3} $\quad d_{min} = 4.28$ in. $\quad \leftarrow$

Problem 3.7-6 A tubular shaft being designed for use on a construction site must transmit 120 kW at 1.75 Hz. The inside diameter of the shaft is to be one-half of the outside diameter.

If the allowable shear stress in the shaft is 45 MPa, what is the minimum required outside diameter d?

Solution 3.7-6 Tubular shaft

d = outside diameter

d_0 = inside diameter

$= 0.5\, d$

$P = 120$ kW $= 120{,}000$ W $\quad f = 1.75$ Hz

$\tau_{allow} = 45$ MPa

$$I_P = \frac{\pi}{32}[d^4 - (0.5\, d)^4] = 0.092039\, d^4$$

TORQUE

$P = 2\pi f T \quad P$ = watts $\quad f$ = Hz

T = newton meters

$$T = \frac{P}{2\pi f} = \frac{120{,}000\ \text{W}}{2\pi(1.75\ \text{Hz})} = 10{,}913.5\ \text{N} \cdot \text{m}$$

MINIMUM OUTSIDE DIAMETER

$$\tau_{max} = \frac{Td}{2I_P} \quad I_P = \frac{Td}{2\tau_{max}} = \frac{Td}{2\tau_{allow}}$$

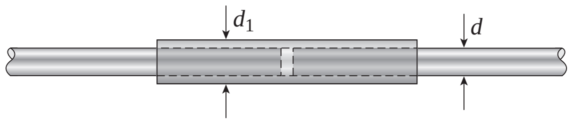
$$0.092039\, d^4 = \frac{(10{,}913.5\ \text{N} \cdot \text{m})(d)}{2(45\ \text{MPa})}$$

$d^3 = 0.0013175$ m^3 $\quad d = 0.1096$ m

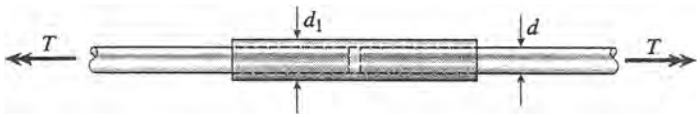
$d_{min} = 110$ mm $\quad \leftarrow$

Problem 3.7-7 A propeller shaft of solid circular cross section and diameter d is spliced by a collar of the same material (see figure). The collar is securely bonded to both parts of the shaft.

What should be the minimum outer diameter d_1 of the collar in order that the splice can transmit the same power as the solid shaft?



Solution 3.7-7 Splice in a propeller shaft



SOLID SHAFT

$$\tau_{max} = \frac{16\,T_1}{\pi d^3} \qquad T_1 = \frac{\pi d^3 \tau_{max}}{16}$$

HOLLOW COLLAR

$$I_P = \frac{\pi}{32}(d_1^4 - d^4) \qquad \tau_{max} = \frac{T_2 r}{I_P} = \frac{T_2(d_1/2)}{I_P}$$

$$T_2 = \frac{2\tau_{max} I_P}{d_1} = \frac{2\tau_{max}}{d_1}\left(\frac{\pi}{32}\right)(d_1^4 - d^4)$$

$$= \frac{\pi\tau_{max}}{16\,d_1}(d_1^4 - d^4)$$

EQUATE TORQUES

For the same power, the torques must be the same.
For the same material, both parts can be stressed to the same maximum stress.

$$\therefore\; T_1 = T_2 \quad \frac{\pi d^3 \tau_{max}}{16} = \frac{\pi\tau_{max}}{16 d_1}(d_1^4 - d^4)$$

$$\text{or}\; \left(\frac{d_1}{d}\right)^4 - \frac{d_1}{d} - 1 = 0 \qquad \text{(Eq. 1)}$$

MINIMUM OUTER DIAMETER

Solve Eq. (1) numerically:

Min. $d_1 = 1.221\,d$ ←

Problem 3.7-8 What is the maximum power that can be delivered by a hollow propeller shaft (outside diameter 50 mm, inside diameter 40 mm, and shear modulus of elasticity 80 GPa) turning at 600 rpm if the allowable shear stress is 100 MPa and the allowable rate of twist is 3.0°/m?

Solution 3.7-8 Hollow propeller shaft

$d_2 = 50$ mm $\quad d_1 = 40$ mm

$G = 80$ GPa $\quad n = 600$ rpm

$\tau_{allow} = 100$ MPa $\quad \theta_{allow} = 3.0°/\text{m}$

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4) = 362.3 \times 10^{-9}\ \text{m}^4$$

BASED UPON ALLOWABLE SHEAR STRESS

$$\tau_{max} = \frac{T_1(d_2/2)}{I_P} \qquad T_1 = \frac{2\tau_{allow} I_P}{d_2}$$

$$T_1 = \frac{2(100\ \text{MPa})(362.3 \times 10^{-9}\ \text{m}^4)}{0.050\ \text{m}}$$

$$= 1449\ \text{N} \cdot \text{m}$$

BASED UPON ALLOWABLE RATE OF TWIST

$$\theta = \frac{T_2}{GI_P} \qquad T_2 = GI_P\theta_{allow}$$

$$T_2 = (80\ \text{GPa})(362.3 \times 10^{-9}\ \text{m}^4)(3.0°/\text{m}) \times \left(\frac{\pi}{180}\ \text{rad/degree}\right)$$

$T_2 = 1517$ N · m

SHEAR STRESS GOVERNS

$T_{allow} = T_1 = 1449$ N · m

MAXIMUM POWER

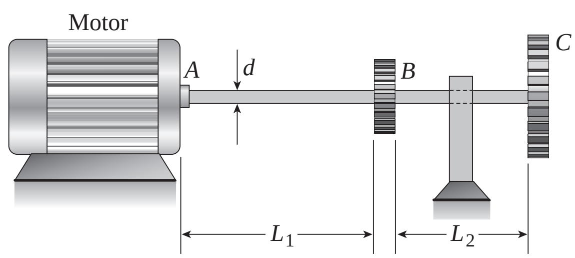
$$P = \frac{2\pi n T}{60} = \frac{2\pi(600\ \text{rpm})(1449\ \text{N} \cdot \text{m})}{60}$$

$P = 91{,}047$ W

$P_{max} = 91.0$ kW ←

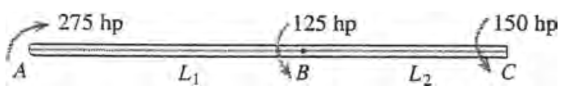
Problem 3.7-9 A motor delivers 275 hp at 1000 rpm to the end of a shaft (see figure). The gears at B and C take out 125 and 150 hp, respectively.

Determine the required diameter d of the shaft if the allowable shear stress is 7500 psi and the angle of twist between the motor and gear C is limited to 1.5°. (Assume $G = 11.5 \times 10^6$ psi, $L_1 = 6$ ft, and $L_2 = 4$ ft.)



PROBS. 3.7-9 and 3.7-10

Solution 3.7-9 Motor-driven shaft



$L_1 = 6$ ft

$L_2 = 4$ ft

$d =$ diameter

$n = 1000$ rpm

$\tau_{allow} = 7500$ psi

$(\phi_{AC})_{allow} = 1.5° = 0.02618$ rad

$G = 11.5 \times 10^6$ psi

TORQUES ACTING ON THE SHAFT

$$H = \frac{2\pi n T}{33{,}000} \quad H = \text{hp} \quad n = \text{rpm} \quad T = \text{lb-ft}$$

$$T = \frac{33{,}000\text{ H}}{2\pi n}$$

$$\text{At point } A\text{: } T_A = \frac{33{,}000(275\text{ hp})}{2\pi(1000\text{ rpm})}$$

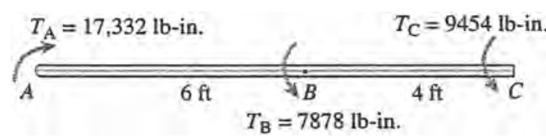
$$= 1444 \text{ lb-ft}$$

$$= 17{,}332 \text{ lb-in.}$$

$$\text{At point } B\text{: } T_B = \frac{125}{275}\quad T_A = 7878 \text{ lb-in.}$$

$$\text{At point } C\text{: } T_C = \frac{150}{275}\quad T_A = 9454 \text{ lb-in.}$$

FREE-BODY DIAGRAM



$T_A = 17{,}332$ lb-in.

$T_C = 9454$ lb-in.

$d =$ diameter

$T_B = 7878$ lb-in.

INTERNAL TORQUES

$T_{AB} = 17{,}332$ lb-in.

$T_{BC} = 9454$ lb-in.

DIAMETER BASED UPON ALLOWABLE SHEAR STRESS

The larger torque occurs in segment AB

$$\tau_{max} = \frac{16T_{AB}}{\pi d^3} \quad d^3 = \frac{16T_{AB}}{\pi\tau_{allow}}$$

$$= \frac{16(17{,}332\text{ lb-in.})}{\pi(7500\text{ psi})} = 11.77\text{ in.}^3$$

$d = 2.27$ in.

DIAMETER BASED UPON ALLOWABLE ANGLE OF TWIST

$$I_P = \frac{\pi d^4}{32} \quad \phi = \frac{TL}{GI_P} = \frac{32TL}{\pi G d^4}$$

Segment AB:

$$\phi_{AB} = \frac{32 T_{AB} L_{AB}}{\pi G d^4}$$

$$= \frac{32(17{,}330 \text{ lb} - \text{in.})(6 \text{ ft})(12 \text{ in./ft})}{\pi(11.5 \times 10^6 \text{ psi}) d^4}$$

$$\phi_{AB} = \frac{1.1052}{d^4}$$

Segment BC:

$$\phi_{BC} = \frac{32\, T_{BC} L_{BC}}{\pi G d^4}$$

$$= \frac{32(9450 \text{ lb-in.})(4 \text{ ft})(12 \text{ in./ft})}{\pi(11.5 \times 10^6 \text{ psi}) d^4}$$

$$\phi_{BC} = \frac{0.4018}{d^4}$$

From A to C: $\phi_{AC} = \phi_{AB} + \phi_{BC} = \dfrac{1.5070}{d^4}$

$(\phi_{AC})_{\text{allow}} = 0.02618$ rad

$\therefore \quad 0.02618 = \dfrac{1.5070}{d^4}$ and $d = 2.75$ in.

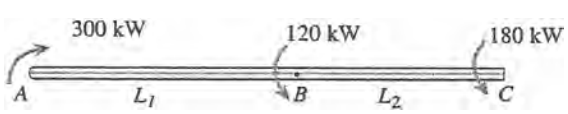
Angle of twist governs

$d = 2.75$ in. $\leftarrow$

Problem 3.7-10 The shaft ABC shown in the figure is driven by a motor that delivers 300 kW at a rotational speed of 32 Hz. The gears at B and C take out 120 and 180 kW, respectively. The lengths of the two parts of the shaft are $L_1 = 1.5$ m and $L_2 = 0.9$ m.

Determine the required diameter d of the shaft if the allowable shear stress is 50 MPa, the allowable angle of twist between points A and C is 4.0°, and $G = 75$ GPa.

Solution 3.7-10 Motor-driven shaft



$L_1 = 1.5$ m

$L_2 = 0.9$ m

d = diameter

$f = 32$ Hz

$\tau_{\text{allow}} = 50$ MPa

$G = 75$ GPa

$(\phi_{AC})_{\text{allow}} = 4° = 0.06981$ rad

TORQUES ACTING ON THE SHAFT

$P = 2\pi f T \quad P = \text{watts} \quad f = \text{Hz}$

T = newton meters

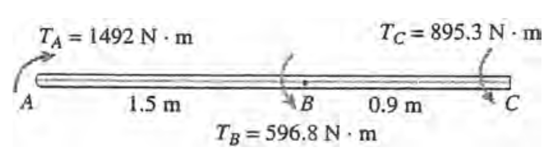
$$T = \frac{P}{2\pi f}$$

At point A: $T_A = \dfrac{300{,}000 \text{ W}}{2\pi(32 \text{ Hz})} = 1492 \text{ N} \cdot \text{m}$

At point B: $T_B = \dfrac{120}{300} T_A = 596.8 \text{ N} \cdot \text{m}$

At point C: $T_C = \dfrac{180}{300} T_A = 895.3 \text{ N} \cdot \text{m}$

FREE-BODY DIAGRAM



$T_A = 1492 \text{ N} \cdot \text{m}$

$T_B = 596.8 \text{ N} \cdot \text{m}$

$T_C = 895.3 \text{ N} \cdot \text{m}$

d = diameter

INTERNAL TORQUES

$T_{AB} = 1492 \text{ N} \cdot \text{m}$

$T_{BC} = 895.3 \text{ N} \cdot \text{m}$

DIAMETER BASED UPON ALLOWABLE SHEAR STRESS

The larger torque occurs in segment AB

$$\tau_{\max} = \frac{16\ T_{AB}}{\pi d^3} \quad d^3 = \frac{16\ T_{AB}}{\pi \tau_{\text{allow}}} = \frac{16(1492 \text{ N}\cdot\text{ m})}{\pi(50 \text{ MPa})}$$

$d^3 = 0.0001520 \text{ m}^3 \quad d = 0.0534 \text{ m} = 53.4 \text{ mm}$

DIAMETER BASED UPON ALLOWABLE ANGLE OF TWIST

$$I_P = \frac{\pi d^4}{32} \quad \phi = \frac{TL}{GI_P} = \frac{32TL}{\pi G d^4}$$

Segment AB:

$$\phi_{AB} = \frac{32\ T_{AB} L_{AB}}{\pi G d^4} = \frac{32(1492 \text{ N}\cdot\text{ m})(1.5 \text{ m})}{\pi(75 \text{ GPa})d^4}$$

$$\phi_{AB} = \frac{0.3039 \times 10^{-6}}{d^4}$$

Segment BC:

$$\phi_{BC} = \frac{32\ T_{BC} L_{BC}}{\pi G d^4} = \frac{32(895.3 \text{ N}\cdot\text{ m})(0.9 \text{ m})}{\pi(75 \text{ GPa})d^4}$$

$$\phi_{BC} = \frac{0.1094 \times 10^{-6}}{d^4}$$

From A to C: $\phi_{AC} = \phi_{AB} + \phi_{BC} = \dfrac{0.4133 \times 10^{-6}}{d^4}$

$(\phi_{AC})_{\text{allow}} = 0.06981 \text{ rad}$

$$\therefore\ 0.06981 = \frac{0.1094 \times 10^{-6}}{d^4}$$

and $d = 0.04933 \text{ m}$

$= 49.3\text{mm}$

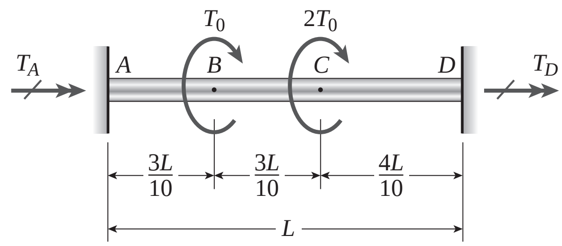
SHEAR STRESS GOVERNS

$d = 53.4\text{mm} \quad \leftarrow$

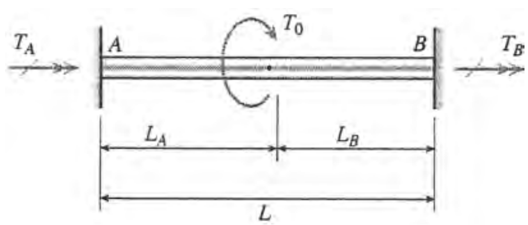
Statically Indeterminate Torsional Members

Problem 3.8-1 A solid circular bar *ABCD* with fixed supports is acted upon by torques T_0 and $2T_0$ at the locations shown in the figure.

Obtain a formula for the maximum angle of twist ϕ_{max} of the bar. (*Hint*: Use Eqs. 3-46a and b of Example 3-9 to obtain the reactive torques.)



Solution 3.8-1 Circular bar with fixed ends



From Eqs. (3-46a and b):

$$T_A = \frac{T_0 L_B}{L}$$

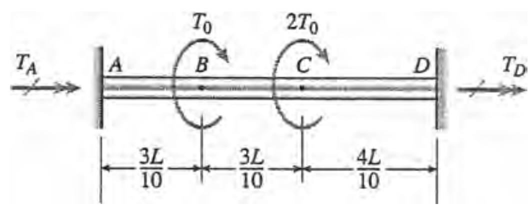
$$T_B = \frac{T_0 L_A}{L}$$

Apply the above formulas to the given bar:

$$T_A = T_0\left(\frac{7}{10}\right) + 2T_0\left(\frac{4}{10}\right) = \frac{15T_0}{10}$$

$$T_D = T_0\left(\frac{3}{10}\right) + 2T_0\left(\frac{6}{10}\right) = \frac{15T_0}{10}$$

Angle of twist at section B



$$\phi_B = \phi_{AB} = \frac{T_A(3L/10)}{GI_P} = \frac{9T_0 L}{20GI_P}$$

Angle of twist at section C

$$\phi_C = \phi_{CD} = \frac{T_D(4L/10)}{GI_P} = \frac{3T_0 L}{5GI_P}$$

Maximum angle of twist

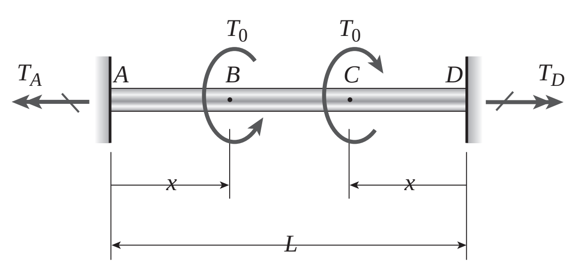
$$\phi_{max} = \phi_C = \frac{3T_0 L}{5GI_P} \quad \leftarrow$$

Problem 3.8-2 A solid circular bar *ABCD* with fixed supports at ends A and D is acted upon by two equal and oppositely directed torques T_0, as shown in the figure. The torques are applied at points B and C, each of which is located at distance x from one end of the bar. (The distance x may vary from zero to $L/2$.)

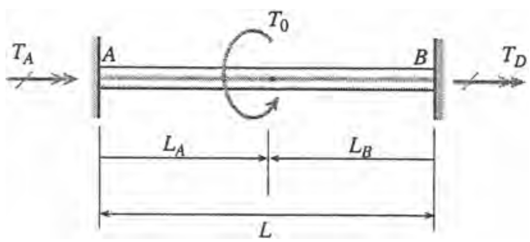
(a) For what distance x will the angle of twist at points B and C be a maximum?

(b) What is the corresponding angle of twist ϕ_{max}?

(*Hint*: Use Eqs. 3-46a and b of Example 3-9 to obtain the reactive torques.)



Solution 3.8-2 Circular bar with fixed ends

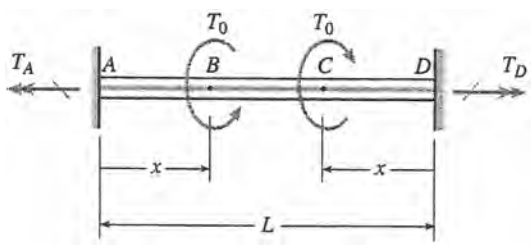


From Eqs. (3-46a and b):

$$T_A = \frac{T_0 L_B}{L}$$

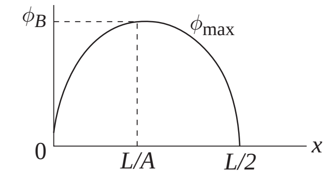
$$T_B = \frac{T_0 L_A}{L}$$

APPLY THE ABOVE FORMULAS TO THE GIVEN BAR:



$$T_A = \frac{T_0(L - x)}{L} - \frac{T_0 x}{L} = \frac{T_0}{L}(L - 2x) \quad T_D = T_A$$

(a) ANGLE OF TWIST AT SECTIONS B AND C



$$\phi_B = \phi_{AB} = \frac{T_A x}{GI_P} = \frac{T_0}{GI_P L}(L - 2x)(x)$$

$$\frac{d\phi_B}{dx} = \frac{T_0}{GI_P L}(L - 4x)$$

$$\frac{d\phi_B}{dx} = 0; L - 4x = 0$$

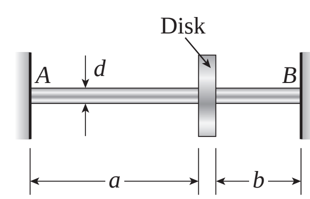
$$\text{or} \quad x = \frac{L}{4} \quad \longleftarrow$$

(b) MAXIMUM ANGLE OF TWIST

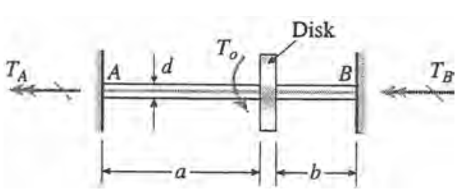
$$\phi_{max} = (\phi_B)_{max} = (\phi_B)_{x=\frac{L}{4}} = \frac{T_0 L}{8GI_P} \quad \longleftarrow$$

Problem 3.8-3 A solid circular shaft AB of diameter d is fixed against rotation at both ends (see figure). A circular disk is attached to the shaft at the location shown.

What is the largest permissible angle of rotation ϕ_{max} of the disk if the allowable shear stress in the shaft is τ_{allow}? (Assume that $a > b$. Also, use Eqs. 3-46a and b of Example 3-9 to obtain the reactive torques.)



Solution 3.8-3 Shaft fixed at both ends



$L = a + b$

$a > b$

Assume that a torque T_0 acts at the disk.

The reactive torques can be obtained from Eqs. (3-46a and b):

$$T_A = \frac{T_0 b}{L} \quad T_B = \frac{T_0 a}{L}$$

Since $a > b$, the larger torque (and hence the larger stress) is in the right hand segment.

$$\tau_{max} = \frac{T_B(d/2)}{I_P} = \frac{T_0\,ad}{2LI_P}$$

$$T_0 = \frac{2LI_P\tau_{max}}{ad} \qquad (T_0)_{max} = \frac{2L\,I_P\tau_{\,allow}}{ad}$$

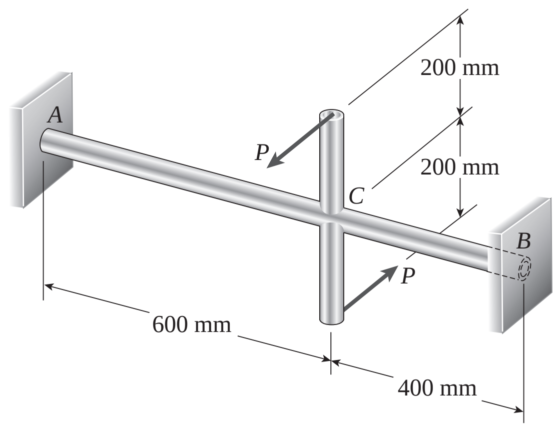
ANGLE OF ROTATION OF THE DISK (FROM Eq. 3-49)

$$\phi = \frac{T_0\,ab}{GLI_P}$$

$$\phi_{max} = \frac{(T_0)_{max}ab}{GLI_p} = \frac{2b\tau_{\,allow}}{Gd} \quad \leftarrow$$

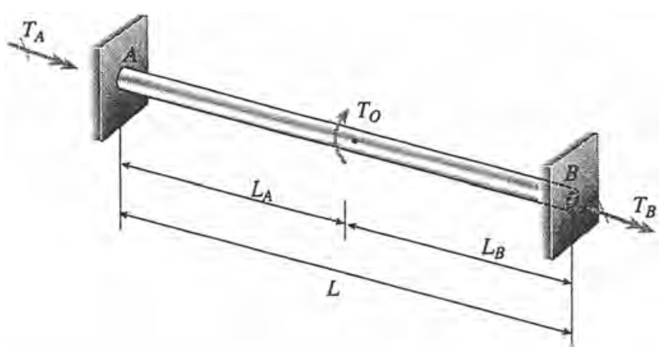
Problem 3.8-4 A hollow steel shaft *ACB* of outside diameter 50 mm and inside diameter 40 mm is held against rotation at ends *A* and *B* (see figure). Horizontal forces *P* are applied at the ends of a vertical arm that is welded to the shaft at point *C*.

Determine the allowable value of the forces *P* if the maximum permissible shear stress in the shaft is 45 MPa. (*Hint*: Use Eqs. 3-46a and b of Example 3-9 to obtain the reactive torques.)

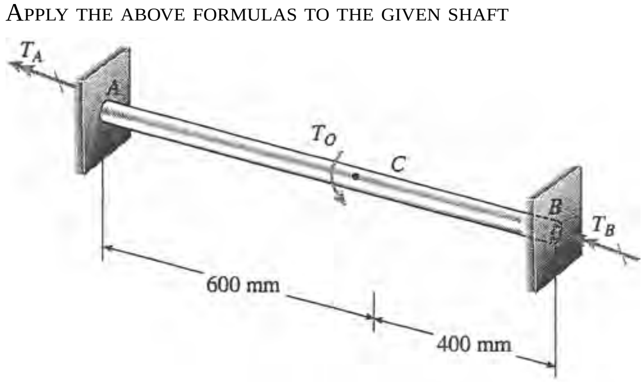


Solution 3.8-4 Hollow shaft with fixed ends

GENERAL FORMULAS:



APPLY THE ABOVE FORMULAS TO THE GIVEN SHAFT



$$T_0 = P(400\text{ mm})$$

$$L_B = 400\text{ mm}$$

$$L_A = 600\text{ mm}$$

$$L = L_A + L_B = 1000\text{ mm}$$

$$d_2 = 50\text{ mm} \quad d_1 = 40\text{ mm}$$

$$\tau_{allow} = 45\text{ MPa}$$

$$T_A = \frac{T_0\,L_B}{L} = \frac{P(0.4\text{ m})(400\text{ mm})}{1000\text{ mm}} = 0.16\,P$$

$$T_B = \frac{T_0\,L_A}{L} = \frac{P(0.4\text{ m})(600\text{ mm})}{1000\text{ mm}} = 0.24\,P$$

UNITS: P = Newtons T = Newton meters

From Eqs. (3-46a and b):

$$T_A = \frac{T_0\,L_B}{L}$$

$$T_B = \frac{T_0\,L_A}{L}$$

The larger torque, and hence the larger shear stress, occurs in part *CB* of the shaft.

$\therefore T_{max} = T_B = 0.24\,P$

SHEAR STRESS IN PART *CB*

$$\tau_{max} = \frac{T_{max}(d/2)}{I_P} \qquad T_{max} = \frac{2\tau_{max} I_P}{d} \qquad \text{(Eq. 1)}$$

UNITS: Newtons and meters

$\tau_{max} = 45 \times 10^6 \text{N/m}^2$

$$I_p = \frac{\pi}{32}(d_2^4 - d_1^4) = 362.26 \times 10^{-9}\text{m}^4$$

$d = d_2 = 0.05$ mm

Substitute numerical values into (Eq. 1):

$$0.24P = \frac{2(45 \times 10^6 \text{ N/m}^2)(362.26 \times 10^{-9} \text{ m}^4)}{0.05 \text{ m}}$$

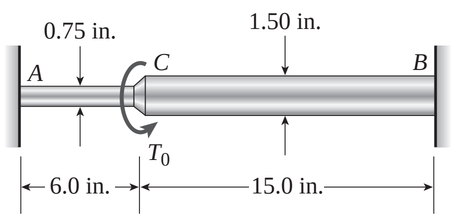
$$= 652.07 \text{ N} \cdot \text{m}$$

$$P = \frac{652.07 \text{ N} \cdot \text{m}}{0.24 \text{ m}} = 2717 \text{ N}$$

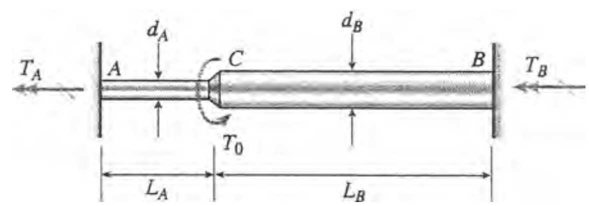
$P_{allow} = 2720$ N ←

Problem 3.8-5 A stepped shaft *ACB* having solid circular cross sections with two different diameters is held against rotation at the ends (see figure).

If the allowable shear stress in the shaft is 6000 psi, what is the maximum torque $(T_0)_{max}$ that may be applied at section *C*? (*Hint*: Use Eqs. 3-45a and b of Example 3-9 to obtain the reactive torques.)



Solution 3.8-5 Stepped shaft *ACB*



$d_A = 0.75$ in. $\quad d_B = 1.50$ in.

$L_A = 6.0$ in. $\quad L_B = 15.0$ in.

$\tau_{allow} = 6000$ psi

Find $(T_0)_{max}$

REACTIVE TORQUES (from Eqs. 3-45a and b)

$$T_A = T_0\left(\frac{L_B I_{PA}}{L_B I_{PA} + L_A I_{PB}}\right) \qquad (1)$$

$$T_B = T_0\left(\frac{L_A I_{PB}}{L_B I_{PA} + L_A I_{PB}}\right) \qquad (2)$$

ALLOWABLE TORQUE BASED UPON SHEAR STRESS IN SEGMENT *AC*

$$\tau_{AC} = \frac{16 T_A}{\pi d_A^3}$$

$$T_A = \frac{1}{16}\pi d_A^3 \tau_{AC} = \frac{1}{16}\pi d_A^3 \tau_{\text{allow}} \qquad (3)$$

Combine Eqs. (1) and (3) and solve for T_0:

$$(T_0)_{AC} = \frac{1}{16}\pi d_A^3 \tau_{\text{allow}}\left(1 + \frac{L_A I_{PB}}{L_B I_{PA}}\right)$$

$$= \frac{1}{16}\pi d_A^3 \tau_{\text{allow}}\left(1 + \frac{L_A d_B^4}{L_B d_A^4}\right) \qquad (4)$$

Substitute numerical values:

$(T_0)_{AC} = 3678$ lb-in.

ALLOWABLE TORQUE BASED UPON SHEAR STRESS IN SEGMENT *CB*

$$\tau_{CB} = \frac{16 T_B}{\pi d_B^3}$$

$$T_B = \frac{1}{16}\pi d_B^3 \tau_{CB} = \frac{1}{16}\pi d_B^3 \tau_{\text{allow}} \qquad (5)$$

Combine Eqs. (2) and (5) and solve for T_0:

$$(T_0)_{CB} = \frac{1}{16}\pi d_B^3 \tau_{\text{allow}}\left(1 + \frac{L_B I_{PA}}{L_A I_{PB}}\right)$$

$$= \frac{1}{16}\pi d_B^3 \tau_{\text{allow}}\left(1 + \frac{L_B d_A^4}{L_A d_B^4}\right) \qquad (6)$$

Substitute numerical values:

$(T_0)_{CB} = 4597$ lb-in.

SEGMENT AC GOVERNS

$(T_0)_{\max} = 3680$ lb-in. ←

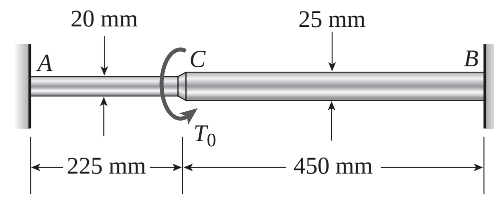
NOTE: From Eqs. (4) and (6) we find that

$$\frac{(T_0)_{AC}}{(T_0)_{CB}} = \left(\frac{L_A}{L_B}\right)\left(\frac{d_B}{d_A}\right)$$

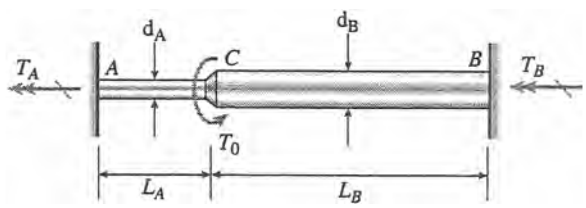
which can be used as a partial check on the results.

Problem 3.8-6 A stepped shaft ACB having solid circular cross sections with two different diameters is held against rotation at the ends (see figure).

If the allowable shear stress in the shaft is 43 MPa, what is the maximum torque $(T_0)_{\max}$ that may be applied at section C? (*Hint*: Use Eqs. 3-45a and b of Example 3-9 to obtain the reactive torques.)



Solution 3.8-6 Stepped shaft *ACB*



$d_A = 20$ mm

$d_B = 25$ mm

$L_A = 225$ mm

$L_B = 450$ mm

$\tau_{\text{allow}} = 43$ MPa

Find $(T_0)_{\max}$

REACTIVE TORQUES (from Eqs. 3-45a and b)

$$T_A = T_0\left(\frac{L_B I_{PA}}{L_B I_{PA} + L_A I_{PB}}\right) \qquad (1)$$

$$T_B = T_0\left(\frac{L_A I_{PB}}{L_B I_{PA} + L_A I_{PB}}\right) \qquad (2)$$

ALLOWABLE TORQUE BASED UPON SHEAR STRESS IN SEGMENT AC

$$\tau_{AC} = \frac{16 T_A}{\pi d_A^3}$$

$$T_A = \frac{1}{16}\pi d_A^3 \tau_{AC} = \frac{1}{16}\pi d_A^3 \tau_{\text{allow}} \qquad (3)$$

Combine Eqs. (1) and (3) and solve for T_0:

$$(T_0)_{AC} = \frac{1}{16}\pi d_A^3 \tau_{\text{allow}}\left(1 + \frac{L_A I_{PB}}{L_B I_{PA}}\right)$$

$$= \frac{1}{16}\pi d_A^3 \tau_{\text{allow}}\left(1 + \frac{L_A d_B^4}{L_B d_A^4}\right) \qquad (4)$$

Substitute numerical values:

$(T_0)_{AC} = 150.0\ \text{N}\cdot\text{m}$

ALLOWABLE TORQUE BASED UPON SHEAR STRESS IN SEGMENT CB

$$\tau_{CB} = \frac{16 T_B}{\pi d_B^3}$$

$$T_B = \frac{1}{16}\pi d_B^3 \tau_{CB} = \frac{1}{16}\pi d_B^3 \tau_{\text{allow}} \qquad (5)$$

Combine Eqs. (2) and (5) and solve for T_0:

$$(T_0)_{CB} = \frac{1}{16}\pi d_B^3 \tau_{\text{allow}}\left(1 + \frac{L_B I_{PA}}{L_A I_{PB}}\right)$$

$$= \frac{1}{16}\pi d_B^3 \tau_{\text{allow}}\left(1 + \frac{L_B d_A^4}{L_A d_B^4}\right) \qquad (6)$$

Substitute numerical values:

$(T_0)_{CB} = 240.0 \text{ N} \cdot \text{m}$

SEGMENT AC GOVERNS

$(T_0)_{max} = 150 \text{ N} \cdot \text{m}$ ←

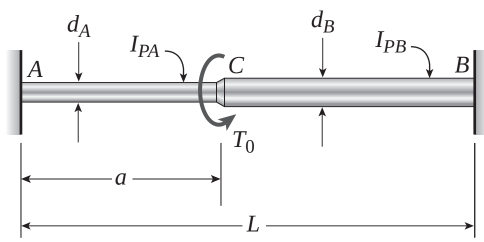
NOTE: From Eqs. (4) and (6) we find that

$$\frac{(T_0)_{AC}}{(T_0)_{CB}} = \left(\frac{L_A}{L_B}\right)\left(\frac{d_B}{d_A}\right)$$

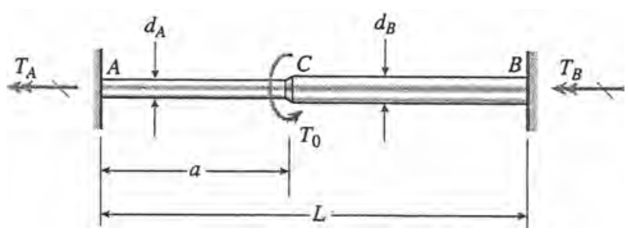
which can be used as a partial check on the results.

Problem 3.8-7 A stepped shaft ACB is held against rotation at ends A and B and subjected to a torque T_0 acting at section C (see figure). The two segments of the shaft (AC and CB) have diameters d_A and d_B, respectively, and polar moments of inertia I_{PA} and I_{PB}, respectively. The shaft has length L and segment AC has length a.

(a) For what ratio a/L will the maximum shear stresses be the same in both segments of the shaft?
(b) For what ratio a/L will the internal torques be the same in both segments of the shaft? (*Hint*: Use Eqs. 3-45a and b of Example 3-9 to obtain the reactive torques.)



Solution 3.8-7 Stepped shaft



SEGMENT AC: d_A, I_{PA} $\quad L_A = a$

SEGMENT CB: d_B, I_{PB} $\quad L_B = L - a$

REACTIVE TORQUES (from Eqs. 3-45a and b)

$$T_A = T_0\left(\frac{L_B I_{PA}}{L_B I_{PA} + L_A I_{PB}}\right); \quad T_B = T_0\left(\frac{L_A I_{PB}}{L_B I_{PA} + L_A I_{PB}}\right)$$

(a) EQUAL SHEAR STRESSES

$$\tau_{AC} = \frac{T_A(d_A/2)}{I_{PA}} \qquad \tau_{CB} = \frac{T_B(d_B/2)}{I_{PB}}$$

$$\tau_{AC} = \tau_{CB} \quad \text{or} \quad \frac{T_A d_A}{I_{PA}} = \frac{T_B d_B}{I_{PB}} \qquad \text{(Eq. 1)}$$

Substitute T_A and T_B into Eq. (1):

$$\frac{L_B I_{PA} d_A}{I_{PA}} = \frac{L_A I_{PB} d_B}{I_{PB}} \quad \text{or} \quad L_B d_A = L_A d_B$$

or $\quad (L - a)d_A = a d_B$

Solve for a/L: $\quad \dfrac{a}{L} = \dfrac{d_A}{d_A + d_B}$ ←

(b) EQUAL TORQUES

$T_A = T_B \quad$ or $\quad L_B I_{PA} = L_A I_{PB}$

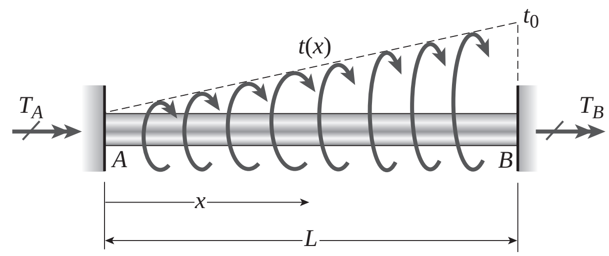
or $\quad (L - a)I_{PA} = a I_{PB}$

Solve for a/L: $\quad \dfrac{a}{L} = \dfrac{I_{PA}}{I_{PA} + I_{PB}}$

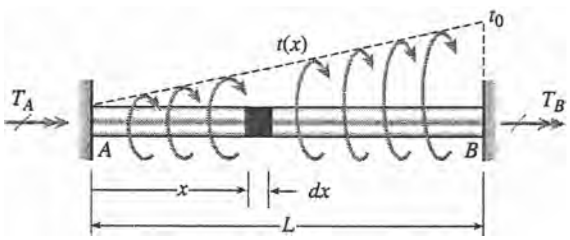
or $\quad \dfrac{a}{L} = \dfrac{d_A^4}{d_A^4 + d_B^4}$ ←

Problem 3.8-8 A circular bar AB of length L is fixed against rotation at the ends and loaded by a distributed torque $t(x)$ that varies linearly in intensity from zero at end A to t_0 at end B (see figure).

Obtain formulas for the fixed-end torques T_A and T_B.



Solution 3.8-8 Fixed-end bar with triangular load



$$t(x) = \frac{t_0 x}{L}$$

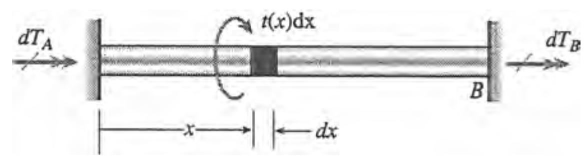
T_0 = Resultant of distributed torque

$$T_0 = \int_0^L t(x)dx = \int_0^L \frac{t_0 x}{L}\, dx = \frac{t_0 L}{2}$$

EQUILIBRIUM

$$T_A + T_B = T_0 = \frac{t_0 L}{2}$$

ELEMENT OF DISTRIBUTED LOAD



dT_A = Elemental reactive torque

dT_B = Elemental reactive torque

From Eqs. (3-46a and b):

$$dT_A = t(x)dx\left(\frac{L - x}{L}\right) \qquad dT_B = t(x)dx\left(\frac{x}{L}\right)$$

REACTIVE TORQUES (FIXED-END TORQUES)

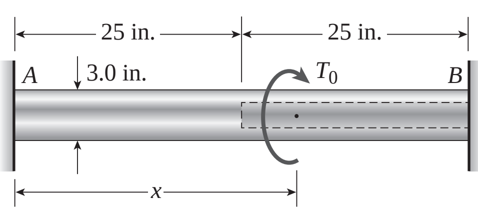
$$T_A = \int dT_A = \int_0^L \left(t_0 \frac{x}{L}\right)\left(\frac{L - x}{L}\right)dx = \frac{t_0 L}{6} \quad \longleftarrow$$

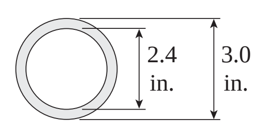
$$T_B = \int dT_B = \int_0^L \left(t_0 \frac{x}{L}\right)\left(\frac{x}{L}\right)dx = \frac{t_0 L}{3} \quad \longleftarrow$$

NOTE: $T_A + T_B = \dfrac{t_0 L}{2}$ (check)

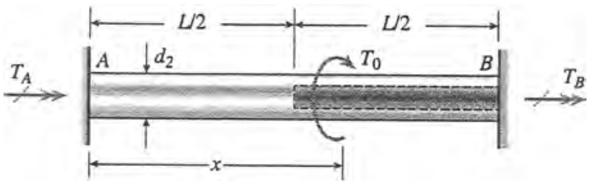
Problem 3.8-9 A circular bar AB with ends fixed against rotation has a hole extending for half of its length (see figure). The outer diameter of the bar is $d_2 = 3.0$ in. and the diameter of the hole is $d_1 = 2.4$ in. The total length of the bar is $L = 50$ in.

At what distance x from the left-hand end of the bar should a torque T_0 be applied so that the reactive torques at the supports will be equal?





Solution 3.8-9 Bar with a hole



$L = 50$ in.

$L/2 = 25$ in.

d_2 = outer diameter

$= 3.0$ in.

d_1 = diameter of hole

$= 2.4$ in.

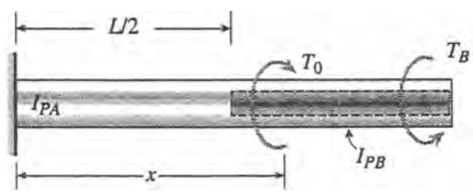
T_0 = Torque applied at distance x

Find x so that $T_A = T_B$

Equilibrium

$$T_A + T_B = T_0 \quad \therefore\ T_A = T_B = \frac{T_0}{2} \tag{1}$$

Remove the support at end B



ϕ_B = Angle of twist at B

I_{PA} = Polar moment of inertia at left-hand end

I_{PB} = Polar moment of inertia at right-hand end

$$\phi_B = \frac{T_B(L/2)}{GI_{PB}} + \frac{T_B(L/2)}{GI_{PA}} - \frac{T_0(x - L/2)}{GI_{PB}} - \frac{T_0(L/2)}{GI_{PA}} \tag{2}$$

Substitute Eq. (1) into Eq. (2) and simplify:

$$\phi_B = \frac{T_0}{G}\left[\frac{L}{4I_{PB}} + \frac{L}{4I_{PA}} - \frac{x}{I_{PB}} + \frac{L}{2I_{PB}} - \frac{L}{2I_{PA}}\right]$$

Compatibility $\phi_B = 0$

$$\therefore\ \frac{x}{I_{PB}} = \frac{3L}{4I_{PB}} - \frac{L}{4I_{PA}}$$

solve for x:

$$x = \frac{L}{4}\left(3 - \frac{I_{PB}}{I_{PA}}\right)$$

$$\frac{I_{PB}}{I_{PA}} = \frac{d_2^4 - d_1^4}{d_2^4} = 1 - \left(\frac{d_1}{d_2}\right)^4$$

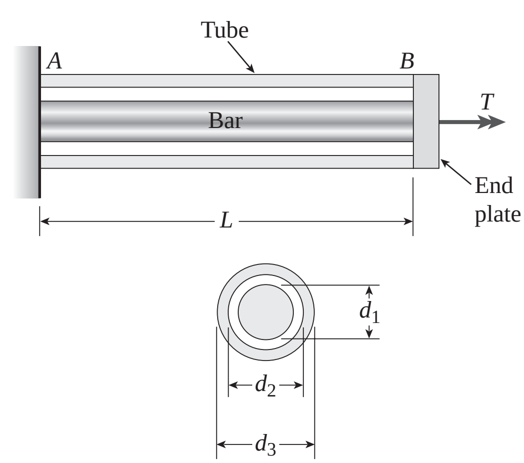
$$x = \frac{L}{4}\left[2 + \left(\frac{d_1}{d_2}\right)^4\right] \quad \longleftarrow$$

Substitute numerical values:

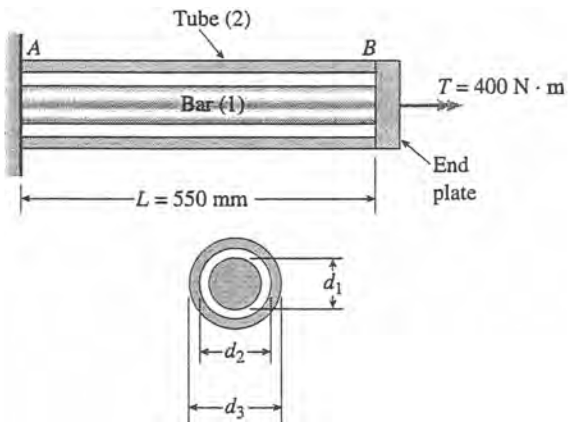
$$x = \frac{50\text{ in.}}{4}\left[2 + \left(\frac{2.4\text{ in.}}{3.0\text{ in.}}\right)^4\right] = 30.12\text{ in.} \quad \longleftarrow$$

Problem 3.8-10 A solid steel bar of diameter $d_1 = 25.0$ mm is enclosed by a steel tube of outer diameter $d_3 = 37.5$ mm and inner diameter $d_2 = 30.0$ mm (see figure). Both bar and tube are held rigidly by a support at end A and joined securely to a rigid plate at end B. The composite bar, which has a length $L = 550$ mm, is twisted by a torque $T = 400$ N · m acting on the end plate.

(a) Determine the maximum shear stresses τ_1 and τ_2 in the bar and tube, respectively.
(b) Determine the angle of rotation ϕ (in degrees) of the end plate, assuming that the shear modulus of the steel is $G = 80$ GPa.
(c) Determine the torsional stiffness k_T of the composite bar. (*Hint*: Use Eqs. 3-44a and b to find the torques in the bar and tube.)



Solution 3.8-10 Bar enclosed in a tube



$d_1 = 25.0$ mm $\quad d_2 = 30.0$ mm $\quad d_3 = 37.5$ mm

$G = 80$ GPa

POLAR MOMENTS OF INERTIA

$$\text{Bar: } I_{P1} = \frac{\pi}{32} d_1^4 = 38.3495 \times 10^{-9}\ \text{m}^4$$

$$\text{Tube: } I_{P2} = \frac{\pi}{32}(d_3^4 - d_2^4) = 114.6229 \times 10^{-9}\ \text{m}^4$$

TORQUES IN THE BAR (1) AND TUBE (2) FROM EQS. (3-44A AND B)

$$\text{Bar: } T_1 = T\left(\frac{I_{P1}}{I_{P1} + I_{P2}}\right) = 100.2783\ \text{N} \cdot \text{m}$$

$$\text{Tube: } T_2 = T\left(\frac{I_{P2}}{I_{P1} + I_{P2}}\right) = 299.7217\ \text{N} \cdot \text{m}$$

(a) MAXIMUM SHEAR STRESSES

$$\text{Bar: } \tau_1 = \frac{T_1(d_1/2)}{I_{P1}} = 32.7\ \text{MPa} \quad \longleftarrow$$

$$\text{Tube: } \tau_2 = \frac{T_2(d_3/2)}{I_{P2}} = 49.0\ \text{MPa} \quad \longleftarrow$$

(b) ANGLE OF ROTATION OF END PLATE

$$\phi = \frac{T_1 L}{G I_{P1}} = \frac{T_2 L}{G I_{P2}} = 0.017977\ \text{rad}$$

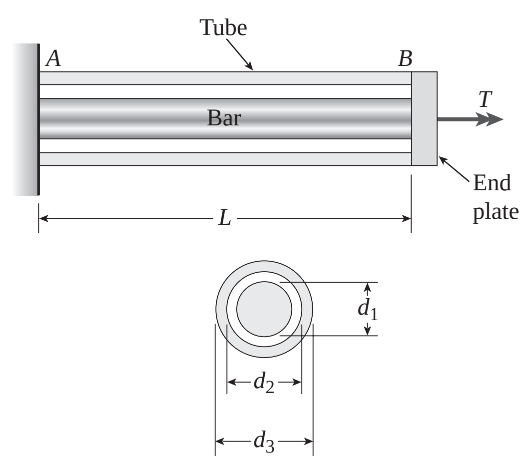
$$\phi = 1.03° \quad \longleftarrow$$

(c) TORSIONAL STIFFNESS

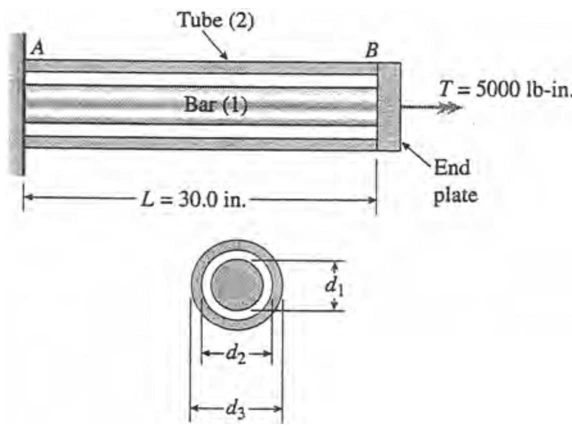
$$k_T = \frac{T}{\phi} = 22.3\ \text{kN} \cdot \text{m} \quad \longleftarrow$$

Problem 3.8-11 A solid steel bar of diameter $d_1 = 1.50$ in. is enclosed by a steel tube of outer diameter $d_3 = 2.25$ in. and inner diameter $d_2 = 1.75$ in. (see figure). Both bar and tube are held rigidly by a support at end A and joined securely to a rigid plate at end B. The composite bar, which has length $L = 30.0$ in., is twisted by a torque $T = 5000$ lb-in. acting on the end plate.

(a) Determine the maximum shear stresses τ_1 and τ_2 in the bar and tube, respectively.
(b) Determine the angle of rotation ϕ (in degrees) of the end plate, assuming that the shear modulus of the steel is $G = 11.6 \times 10^6$ psi.
(c) Determine the torsional stiffness k_T of the composite bar. (*Hint*: Use Eqs. 3-44a and b to find the torques in the bar and tube.)



Solution 3.8-11 Bar enclosed in a tube



$d_1 = 1.50$ in. $d_2 = 1.75$ in. $d_3 = 2.25$ in.

$G = 11.6 \times 10^6$ psi

POLAR MOMENTS OF INERTIA

$$\text{Bar: } I_{P1} = \frac{\pi}{32} d_1^4 = 0.497010 \text{ in.}^4$$

$$\text{Tube: } I_{P2} = \frac{\pi}{32}(d_3^4 - d_2^4) = 1.595340 \text{ in.}^4$$

TORQUES IN THE BAR (1) AND TUBE (2) FROM EQS. (3-44A AND B)

$$\text{Bar: } T_1 = T\left(\frac{I_{PI}}{I_{PI} + I_{PI}}\right) = 1187.68 \text{ lb-in.}$$

$$\text{Tube: } T_2 = T\left(\frac{I_{P2}}{I_{P1} + I_{P2}}\right) = 3812.32 \text{ lb-in.}$$

(a) MAXIMUM SHEAR STRESSES

$$\text{Bar: } \tau_1 = \frac{T_1(d_1/2)}{I_{P1}} = 1790 \text{ psi} \quad \longleftarrow$$

$$\text{Tube: } \tau_2 = \frac{T_2(d_3/2)}{I_{P2}} = 2690 \text{ psi} \quad \longleftarrow$$

(b) ANGLE OF ROTATION OF END PLATE

$$\phi = \frac{T_1 L}{G I_{P1}} = \frac{T_2 L}{G I_{P2}} = 0.006180015 \text{ rad}$$

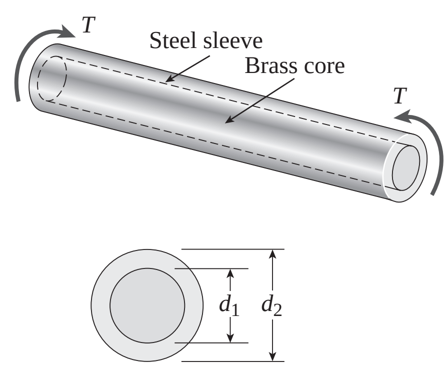
$$\phi = 0.354° \quad \longleftarrow$$

(c) TORSIONAL STIFFNESS

$$k_T = \frac{T}{\phi} = 809 \text{ k-in.} \quad \longleftarrow$$

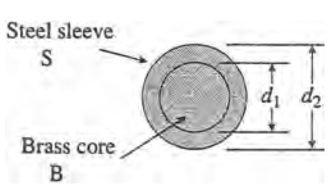
Problem 3.8-12 The composite shaft shown in the figure is manufactured by shrink-fitting a steel sleeve over a brass core so that the two parts act as a single solid bar in torsion. The outer diameters of the two parts are $d_1 = 40$ mm for the brass core and $d_2 = 50$ mm for the steel sleeve. The shear moduli of elasticity are $G_b = 36$ GPa for the brass and $G_s = 80$ GPa for the steel.

Assuming that the allowable shear stresses in the brass and steel are $\tau_b = 48$ MPa and $\tau_s = 80$ MPa, respectively, determine the maximum permissible torque T_{max} that may be applied to the shaft. (*Hint*: Use Eqs. 3-44a and b to find the torques.)



Probs. 3.8-12 and 3.8-13

Solution 3.8-12 Composite shaft shrink fit



$d_1 = 40$ mm

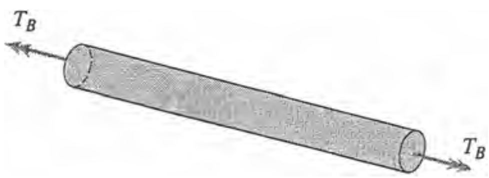
$d_2 = 50$ mm

$G_B = 36$ GPa $G_S = 80$ GPa

Allowable stresses:

$\tau_B = 48$ MPa $\tau_S = 80$ MPa

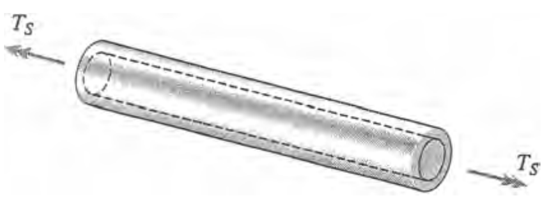
BRASS CORE (ONLY)



$$I_{PB} = \frac{\pi}{32} d_1^4 = 251.327 \times 10^{-9}\ \text{m}^4$$

$$G_B I_{PB} = 9047.79\ \text{N} \cdot \text{m}^2$$

STEEL SLEEVE (ONLY)



$$I_{PS} = \frac{\pi}{32}(d_2^4 - d_1^4) = 362.265 \times 10^{-9}\ \text{m}^4$$

$$G_S I_{PS} = 28{,}981.2\ \text{N} \cdot \text{m}^2$$

TORQUES

Total torque: $T = T_B + T_S$

$$\text{Eq. (3-44a): } T_B = T\left(\frac{G_B I_{PB}}{G_B I_{PB} + G_S I_{PS}}\right) = 0.237918\ T$$

$$\text{Eq. (3-44b): } T_S = T\left(\frac{G_S I_{PS}}{G_B I_{PB} + G_S I_{PS}}\right) = 0.762082\ T$$

$T = T_B + T_S$ (CHECK)

ALLOWABLE TORQUE T BASED UPON BRASS CORE

$$\tau_B = \frac{T_B(d_1/2)}{I_{PB}} \quad T_B = \frac{2\tau_B I_{PB}}{d_1}$$

Substitute numerical values:

$$T_B = 0.237918\ T = \frac{2(48\ \text{MPa})(251.327 \times 10^{-9}\ \text{m}^4)}{40\ \text{mm}}$$

$$T = 2535\ \text{N} \cdot \text{m}$$

ALLOWABLE TORQUE T BASED UPON STEEL SLEEVE

$$\tau_S = \frac{T_S(d_2/2)}{I_{PS}} \quad T_S = \frac{2\tau_S I_{PS}}{d_2}$$

SUBSTITUTE NUMERICAL VALUES:

$$T_S = 0.762082\ T = \frac{2(80\ \text{MPa})(362.265 \times 10^{-9}\ \text{m}^4)}{50\ \text{mm}}$$

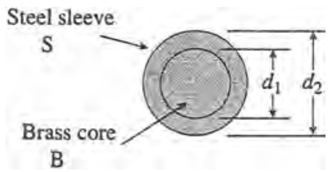
$$T = 1521\ \text{N} \cdot \text{m}$$

STEEL SLEEVE GOVERNS $T_{max} = 1520\ \text{N} \cdot \text{m}$ ←

Problem 3.8-13 The composite shaft shown in the figure is manufactured by shrink-fitting a steel sleeve over a brass core so that the two parts act as a single solid bar in torsion. The outer diameters of the two parts are $d_1 = 1.6$ in. for the brass core and $d_2 = 2.0$ in. for the steel sleeve. The shear moduli of elasticity are $G_b = 5400$ ksi for the brass and $G_s = 12{,}000$ ksi for the steel.

Assuming that the allowable shear stresses in the brass and steel are $\tau_b = 4500$ psi and $\tau_s = 7500$ psi, respectively, determine the maximum permissible torque T_{max} that may be applied to the shaft. (*Hint*: Use Eqs. 3-44a and b to find the torques.)

Solution 3.8-13 Composite shaft shrink fit



$d_1 = 1.6$ in.

$d_2 = 2.0$ in.

$G_B = 5{,}400$ psi $\quad G_S = 12{,}000$ psi

Allowable stresses:

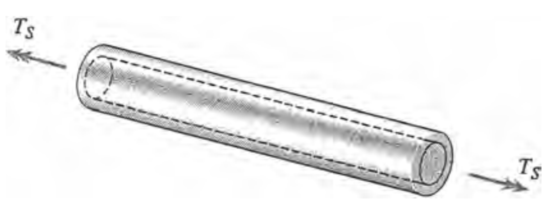
$\tau_B = 4500$ psi $\quad \tau_S = 7500$ psi

BRASS CORE (ONLY)

$$I_{PB} = \frac{\pi}{32} d_1^4 = 0.643398 \text{ in.}^4$$

$$G_B I_{PB} = 3.47435 \times 10^6 \text{ lb-in.}^2$$

STEEL SLEEVE (ONLY)



$$I_{PS} = \frac{\pi}{32}(d_2^4 - d_1^4) = 0.927398 \text{ in.}^4$$

$$G_S I_{PS} = 11.1288 \times 10^6 \text{ lb-in.}^2$$

TORQUES

Total torque: $T = T_B + T_S$

$$\text{Eq. (3-44 a): } T_B = T\left(\frac{G_B I_{PB}}{G_B I_{PB} + G_S I_{PS}}\right)$$

$$= 0.237918\ T$$

$$\text{Eq. (3-44 b): } T_S = T\left(\frac{G_S I_{PS}}{G_B I_{PB} + G_S I_{PS}}\right)$$

$$= 0.762082\ T$$

$T = T_B + T_S$ (CHECK)

ALLOWABLE TORQUE T BASED UPON BRASS CORE

$$\tau_B = \frac{T_B(d_1/2)}{I_{PB}} \quad T_B = \frac{2\tau_B I_{PB}}{d_1}$$

Substitute numerical values:

$$T_B = 0.237918\ T$$

$$= \frac{2(4500 \text{ psi})(0.643398 \text{ in.}^4)}{1.6 \text{ in.}}$$

$T = 15.21$ k-in.

ALLOWABLE TORQUE T BASED UPON STEEL SLEEVE

$$\tau_S = \frac{T_S(d_2/2)}{I_{PS}} \quad T_S = \frac{2\tau_S I_{PS}}{d_2}$$

Substitute numerical values:

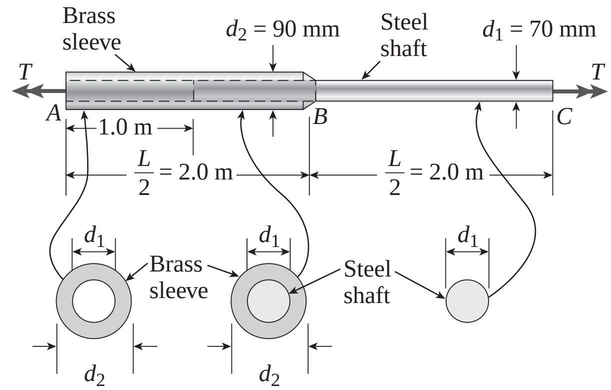
$$\tau_S = 0.762082\ T = \frac{2(7500 \text{ psi})(0.927398 \text{ in.}^4)}{2.0 \text{ in.}}$$

$T = 9.13$ k-in.

STEEL SLEEVE GOVERNS $\quad T_{max} = 9.13$ k-in. $\leftarrow$

Problem 3.8-14 A steel shaft ($G_s = 80$ GPa) of total length $L = 3.0$ m is encased for one-third of its length by a brass sleeve ($G_b = 40$ GPa) that is securely bonded to the steel (see figure). The outer diameters of the shaft and sleeve are $d_1 = 70$ mm and $d_2 = 90$ mm. respectively.

(a) Determine the allowable torque T_1 that may be applied to the ends of the shaft if the angle of twist between the ends is limited to 8.0°.
(b) Determine the allowable torque T_2 if the shear stress in the brass is limited to $\tau_b = 70$ MPa.
(c) Determine the allowable torque T_3 if the shear stress in the steel is limited to $\tau_s = 110$ MPa.
(d) What is the maximum allowable torque T_{max} if all three of the preceding conditions must be satisfied?



Solution 3.8-14

(a) ALLOWABLE TORQUE T_1 BASED ON TWIST AT ENDS OF 8 DEGREES

First find torques in steel (T_s) & brass (T_b) in segment in which they are joined - 1 degree stat-indet; use T_s as the internal redundant; see equ. 3-44a in text example

$$T_s = T_1\left(\frac{G_s I_{Ps}}{G_s I_{Ps} + G_b I_{Pb}}\right)$$

statics

$$T_b = T_1 - T_s \qquad T_b = T_1\left(\frac{G_b I_{Pb}}{G_s I_{Ps} + G_b I_{Pb}}\right)$$

now find twist of 3 segments:

$$\phi = \frac{T_1\frac{L}{4}}{G_b I_{Pb}} + \frac{T_s\frac{L}{4}}{G_s I_{Ps}} + \frac{T_1\frac{L}{2}}{G_s I_{Ps}}$$

For middle term, brass sleeve & steel shaft twist the same so could use $T_b(L/4)/(G_b I_{Pb})$ instead
Let $\phi_a = \phi_{allow}$; substitute expression for T_s then simplifiy; finally, solve for $T_{1,\,allow}$

$$\phi_a = \frac{T_1\frac{L}{4}}{G_b I_{Pb}} + \frac{T_1\left(\frac{G_s I_{Ps}}{G_s I_{Ps} + G_b I_{Pb}}\right)\frac{L}{4}}{G_s I_{Ps}} + \frac{T_1\frac{L}{2}}{G_s I_{Ps}}$$

$$\phi_a = \frac{T_1\frac{L}{4}}{G_b I_{Pb}} + \frac{T_1\frac{L}{4}}{G_s I_{Ps} + G_b I_{Pb}} + \frac{T_1\frac{L}{2}}{G_s I_{Ps}}$$

$$\phi_a = T_1\frac{L}{4}\left(\frac{1}{G_b I_{Pb}} + \frac{1}{G_s I_{Ps} + G_b I_{Pb}} + \frac{2}{G_s I_{Ps}}\right)$$

$$T_{1,\,allow} = \frac{4\phi_a}{L}\left[\frac{G_b I_{Pb}(G_s I_{Ps} + G_b I_{Pb})G_s I_{Ps}}{G_s^2 I_{Ps}^2 + 4 G_b I_{Pb} G_s I_{Ps} + 2 G_b^2 I_{Pb}^2}\right]$$

NUMERICAL VALUES $\phi_a = 8\left(\frac{\pi}{180}\right)$ rad

$G_s = 80$ GPa $G_b = 40$ GPa $L = 3.0$ m

$d_1 = 70$ mm $d_2 = 90$ mm

$I_{Ps} = \frac{\pi}{32} d_1^4$ $I_{Ps} = 2.357 \times 10^{-6}$ m^4

$I_{Pb} = \frac{\pi}{32}\left(d_2^4 - d_1^4\right)$ $I_{Pb} = 4.084 \times 10^{-6}$ m^4

$T_{1,\text{allow}} = 9.51$ kN·m $\leftarrow$

(b) ALLOWABLE TORQUE T_2 BASED ON ALLOWABLE SHEAR STRESS IN BRASS, τ_b

$\tau_b = 70$ MPa

First check hollow segment 1 (brass sleeve only)

$$\tau = \frac{T_2 \frac{d_2}{2}}{I_{Pb}} \quad T_{2,\text{allow}} = \frac{2\tau_b I_{Pb}}{d_2}$$

$T_{2,\text{allow}} = 6.35$ kN·m $\leftarrow$

controls over T_2 below also check segment 2 with brass sleeve over steel shaft

$$\tau = \frac{T_b \frac{d_2}{2}}{I_{Pb}} \quad \text{where from stat-indet analysis above}$$

$$T_b = T_2\left(\frac{G_b I_{Pb}}{G_s I_{PS} + G_b I_{Pb}}\right)$$

$$T_{2,\text{allow}} = \frac{2\tau_b(G_s I_{PS} + G_b I_{Pb})}{d_2 G_b}$$

$T_{2,\text{allow}} = 13.69$ kN·m

so T_2 for hollow segment controls

(c) ALLOWABLE TORQUE T_3 BASED ON ALLOWABLE SHEAR STRESS IN STEEL, τ_s

$\tau_s = 110$ MPa

First check segment 2 with brass sleeve over steel shaft

$$\tau = \frac{T_s \frac{d_1}{2}}{I_{PS}} \quad \text{where from stat-indet analysis above}$$

$$T_s = T_3\left(\frac{G_s I_{PS}}{G_s I_{PS} + G_b I_{Pb}}\right)$$

$$T_{3,\text{allow}} = \frac{2\tau_s(G_s I_Ps + G_b I_P b)}{d_1 G_s}$$

$T_{3,\text{allow}} = 13.83$ kN·m

also check segment 3 with steel shaft alone

$$\tau = \frac{T_3 \frac{d_1}{2}}{I_P s} \quad T_{3,\text{allow}} = \frac{2\tau_s I_P s}{d_1}$$

$T_{3,\text{allow}} = 7.41$ kN·m $\leftarrow$ controls over T_3 above

(d) $T_{\max}$ IF ALL PRECEDING CONDITIONS MUST BE CONSIDERED

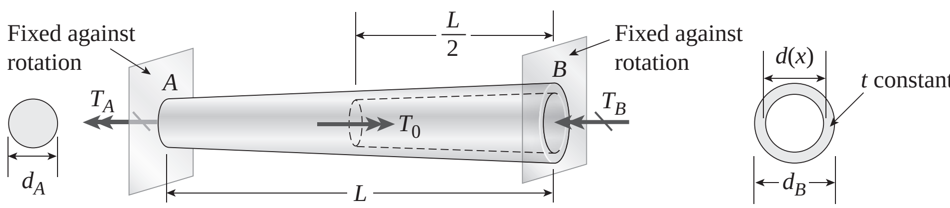
from (b) above

$T_{\max} = 6.35$ kN·m $\leftarrow$ max. shear stress in hollow brass sleeve in segment 1 controls overall

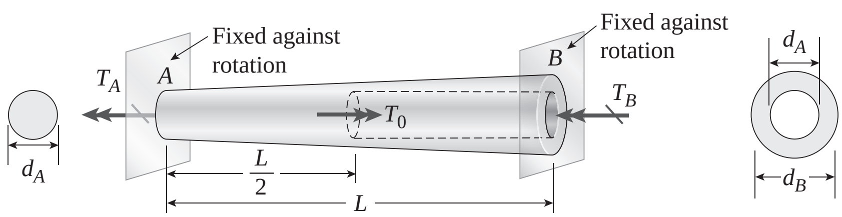
Problem 3.8-15 A uniformly tapered aluminum-alloy tube *AB* of circular cross section and length L is fixed against rotation at A and B, as shown in the figure. The outside diameters at the ends are d_A and $d_B = 2d_A$. A hollow section of length $L/2$ and constant thickness $t = d_A/10$ is cast into the tube and extends from B halfway toward A. Torque T_o is applied at $L/2$.

(a) Find the reactive torques at the supports, T_A and T_B. Use numerical values as follows: $d_A = 2.5$ in., $L = 48$ in., $G = 3.9 \times 10^6$ psi, $T_0 = 40{,}000$ in.-lb.

(b) Repeat (a) if the hollow section has constant diameter d_A.



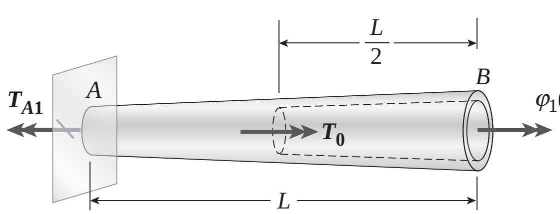
(a)



(b)

Solution 3.8-15

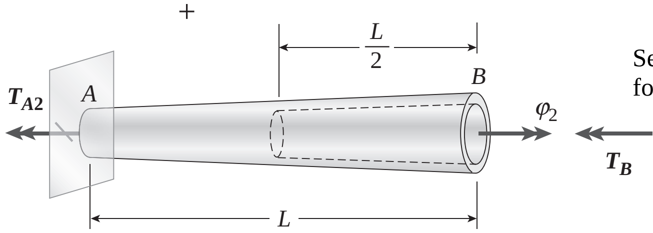
Solution approach-superposition: select T_B as the redundant (1° SI)



φ_1*(same results for parts a & b)*

$$\phi_1 = \frac{608 T_0 L}{81 G \pi d_A{}^4} \qquad \phi_1 = 2.389 \frac{T_0 L}{G d_A{}^4}$$

+



See ***Prob. 3.4-13*** for results for φ_2 for Parts *a* & *b*

$$\phi_{2a} = 3.868 \frac{T_0 L}{G d_A{}^4}$$

$$\phi_{2a} = 3.057 \frac{T_0 L}{G d_A{}^4}$$

(a) REACTIVE TORQUES, T_A & T_B, FOR CASE OF CONSTANT THICKNESS OF HOLLOW SECTION OF TUBE

compatibility equation: $\phi_1 - \phi_2 = 0$

T_B = redundant

T_0 = 40000 in.-lb

$$T_B = \left(\frac{608 T_0 L}{81 G \pi d_A{}^4}\right)\left(\frac{G d_A{}^4}{3.86804 L}\right)$$

$$T_B = 1.94056\frac{T_0}{\pi} \quad T_B = 24708 \text{ in-lb} \quad \leftarrow$$

$$T_A = T_0 - T_B \quad T_A = 15292 \text{ in-1b} \quad \leftarrow$$

$$T_A + T_B = 40{,}000 \text{ in.-lb (check)}$$

(b) REACTIVE TORQUES, T_A & T_B, FOR CASE OF CONSTANT DIAMETER OF HOLE

$$T_B = \left(\frac{608 T_0 L}{81 G \pi d_A{}^4}\right)\left(\frac{G d_A^4}{3.05676 L}\right)$$

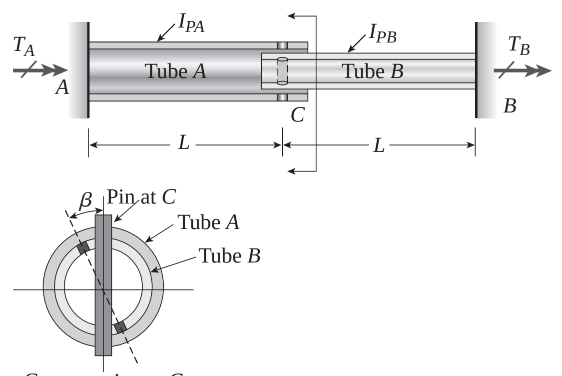
$$T_B = 2.45560\frac{T_0}{\pi} \quad T_B = 31266 \text{ in.-lb} \quad \leftarrow$$

$$T_A = T_0 - T_B \quad T_A = 8734 \text{ in.-lb} \quad \leftarrow$$

$$T_A + T_B = 40{,}000 \text{ in.-1b (check)}$$

Problem 3.8-16 A hollow circular tube A (outer diameter d_A, wall thickness t_A) fits over the end of a circular tube B (d_B, t_B), as shown in the figure. The far ends of both tubes are fixed. Initially, a hole through tube B makes an angle β with a line through two holes in tube A. Then tube B is twisted until the holes are aligned, and a pin (diameter d_p) is placed through the holes. When tube B is released, the system returns to equilibrium. Assume that G is constant.

(a) Use superposition to find the reactive torques T_A and T_B at the supports.
(b) Find an expression for the maximum value of β if the shear stress in the pin, τ_p, cannot exceed $\tau_{p,allow}$.
(c) Find an expression for the maximum value of β if the shear stress in the tubes, τ_t, cannot exceed $\tau_{t,allow}$.
(d) Find an expression for the maximum value of β if the bearing stress in the pin at C cannot exceed $\sigma_{b,allow}$.



Cross-section at C

Solution 3.8-16

(a) SUPERPOSITION TO FIND TORQUE REACTIONS - USE T_B AS THE REDUNDANT

compatibility: $\phi_{B1} + \phi_{B2} = 0$

$\phi_{B1} = -\beta$ < joint tubes by pin then release end B

$$\phi_{B2} = \frac{T_B L}{G}\left(\frac{1}{I_{PA}} + \frac{1}{I_{PB}}\right)$$

$$\phi_{B2} = \frac{T_B L}{G}\left(\frac{I_{PB} + I_{PA}}{I_{PA} I_{PB}}\right)$$

$$T_B = \frac{G\beta}{L}\left(\frac{I_{PA} I_{PB}}{I_{PA} + I_{PB}}\right) \quad \leftarrow$$

$$T_A = -T_B \quad \leftarrow \text{ statics}$$

(b) ALLOWABLE SHEAR IN PIN RESTRICTS MAGNITUDE OF β

TORQUE T_B = FORCE COUPLE $V d_B$ WITH V = SHEAR IN PIN AT C

$$V = \frac{T_B}{d_B} \quad \tau_p = \frac{V}{A_s}$$

$$\tau_{p,\text{ allow}} = \frac{\dfrac{T_B}{d_B}}{\dfrac{\pi}{4} d_P{}^2} \quad \tau_{p,\text{ allow}} = \frac{\dfrac{G\beta}{L}\left(\dfrac{I_{PA} I_{PB}}{I_{PA} + I_{PB}}\right)}{d_B \dfrac{\pi}{4} d_p^2}$$

$$\beta_{\max} = \tau_{p,\text{ allow}} \frac{L}{4G}$$

$$\left[\left(\frac{I_{PB} + I_{PA}}{I_{PA} I_{PB}}\right) d_B \pi d_P{}^2\right] \quad \leftarrow$$

(c) ALLOWABLE SHEAR IN TUBES RESTRICTS MAGNITUDE OF β

$$\tau_{max} = \frac{T_B\frac{d_A}{2}}{I_{PA}} \quad \text{or} \quad \tau_{max} = \frac{T_B\frac{d_B}{2}}{I_{PB}}$$

$$\tau_{max} = \frac{\frac{G\beta}{L}\left(\frac{I_{PA}I_{PB}}{I_{PA}+I_{PB}}\right)\frac{d_A}{2}}{I_{PA}}$$

or

$$\tau_{max} = \frac{\frac{G\beta}{L}\left(\frac{I_{PA}I_{PB}}{I_{PA}+I_{PB}}\right)\frac{d_B}{2}}{I_{PB}}$$

simplifying these two equ., then solving for β gives:

$$\tau_{max} = \frac{G\beta}{L}\left(\frac{I_{PB}}{I_{PA} + I_{PB}}\right)\frac{d_A}{2}$$

or

$$\tau_{max} = \frac{G\beta}{L}\left(\frac{I_{PA}}{I_{PA} + I_{PB}}\right)\frac{d_B}{2}$$

$$\beta_{max} = \tau_{t,\,allow}\left(\frac{2L}{Gd_A}\right)\left(\frac{I_{PA} + I_{PB}}{I_{PB}}\right) \quad \leftarrow$$

or

$$\beta_{max} = \tau_{t,\,allow}\left(\frac{2L}{Gd_B}\right)\left(\frac{I_{PA} + I_{PB}}{I_{PA}}\right) \quad \leftarrow$$

where lesser value of β controls

(d) ALLOWABLE BEARING STRESS IN PIN RESTRICTS MAGNITUDE OF β

Torque T_B = force couple $F_B(d_B - t_B)$ or $F_A(d_A - t_A)$, with F = ave. bearing force on pin at C

Bearing stresses from tubes A & B are:

$$\sigma_{bA} = \frac{F_A}{d_P t_A} \quad \sigma_{bB} = \frac{F_B}{d_P t_B}$$

$$\sigma_{bA} = \frac{\frac{T_B}{d_A - t_A}}{d_P t_A} \qquad \sigma_{bB} = \frac{\frac{T_B}{d_B - t_B}}{d_P t_B}$$

substitute T_B expression from part (a), then simplify ε solve for β

$$\sigma_{bA} = \frac{\dfrac{\frac{G\beta}{L}\left(\frac{I_{PA}I_{PB}}{I_{PA} + I_{PB}}\right)}{d_A - t_A}}{d_P t_A}$$

$$\sigma_{bB} = \frac{\dfrac{\frac{G\beta}{L}\left(\frac{I_{PA}I_{PB}}{I_{PA} + I_{PB}}\right)}{d_B - t_B}}{d_P t_B}$$

$$\sigma_{bA} = \beta\frac{GI_{PA}I_{PB}}{L(I_{PB} + I_{PA})(d_A - t_A)d_P t_A}$$

$$\sigma_{bB} = \beta\frac{G}{L}\frac{I_{PA}I_{PB}}{(I_{PB} + I_{PA})(d_B - t_B)d_P t_B}$$

$$\beta_{max} = \sigma_{b,\,allow}\frac{L}{G}\left[\frac{(I_{PB} + I_{PA})(d_A - t_A)d_P t_A}{I_{PA}I_{PB}}\right] \quad \leftarrow$$

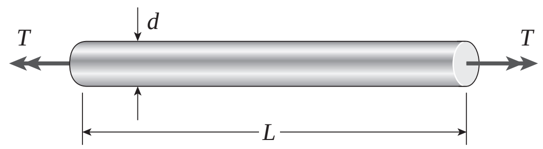
$$\beta_{max} = \sigma_{b,\,allow}\frac{L}{G}\left[\frac{(I_{PB} + I_{PA})(d_B - t_B)d_P t_B}{I_{PA}I_{PB}}\right] \quad \leftarrow$$

where lesser value controls

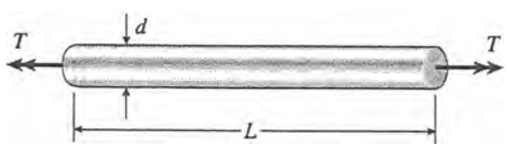
Strain Energy in Torsion

Problem 3.9-1 A solid circular bar of steel ($G = 11.4 \times 10^6$ psi) with length $L = 30$ in. and diameter $d = 1.75$ in. is subjected to pure torsion by torques T acting at the ends (see figure).

(a) Calculate the amount of strain energy U stored in the bar when the maximum shear stress is 4500 psi.
(b) From the strain energy, calculate the angle of twist ϕ (in degrees).



Solution 3.9-1 Steel bar



$G = 11.4 \times 10^6$ psi

$L = 30$ in.

$d = 1.75$ in.

$\tau_{max} = 4500$ psi

$$\tau_{max} = \frac{16\,T}{\pi d^3} \qquad T = \frac{\pi d^3 \tau_{max}}{16}$$

$$I_P = \frac{\pi d^4}{32} \qquad \text{(Eq. 1)}$$

(a) Strain energy

$$U = \frac{T^2 L}{2GI_P} = \left(\frac{\pi d^3 \tau_{max}}{16}\right)^2 \left(\frac{L}{2G}\right)\left(\frac{32}{\pi d^4}\right)$$

$$= \frac{\pi d^2 L \tau_{max}^2}{16G} \qquad \text{(Eq. 2)}$$

Substitute numerical values:

$U = 32.0$ in.-lb ←

(b) Angle of twist

$$U = \frac{T\phi}{2} \qquad \phi = \frac{2U}{T}$$

Substitute for T and U from Eqs. (1) and (2):

$$\phi = \frac{2L\tau_{max}}{Gd} \qquad \text{(Eq. 3)}$$

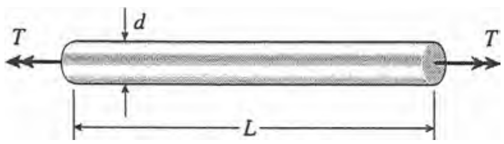
Substitute numerical values:

$\phi = 0.013534$ rad $= 0.775°$ ←

Problem 3.9-2 A solid circular bar of copper ($G = 45$ GPa) with length $L = 0.75$ m and diameter $d = 40$ mm is subjected to pure torsion by torques T acting at the ends (see figure).

(a) Calculate the amount of strain energy U stored in the bar when the maximum shear stress is 32 MPa.
(b) From the strain energy, calculate the angle of twist ϕ (in degrees)

Solution 3.9-2 Copper bar



$G = 45$ GPa

$L = 0.75$ m

$d = 40$ mm

$\tau_{max} = 32$ MPa

$$\tau_{max} = \frac{16T}{\pi d^3} \qquad T = \frac{\pi d^3 \tau_{max}}{16}$$

$$I_P = \frac{\pi d^4}{32} \qquad \text{(Eq. 1)}$$

(a) STRAIN ENERGY

$$U = \frac{T^2 L}{2GI_P} = \left(\frac{\pi d^3 \tau_{max}}{16}\right)^2 \left(\frac{L}{2G}\right)\left(\frac{32}{\pi d^4}\right)$$

$$= \frac{\pi d^2 L \tau_{max}^2}{16G} \qquad \text{(Eq. 2)}$$

Substitute numerical values:

$U = 5.36$ J $\longleftarrow$

(b) ANGLE OF TWIST

$$U = \frac{T\phi}{2} \qquad \phi = \frac{2U}{T}$$

Substitute for T and U from Eqs. (1) and (2):

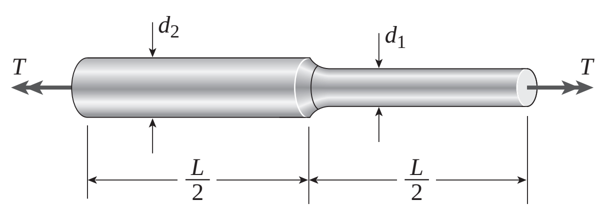
$$\phi = \frac{2L\tau_{max}}{Gd} \qquad \text{(Eq. 3)}$$

Substitute numerical values:

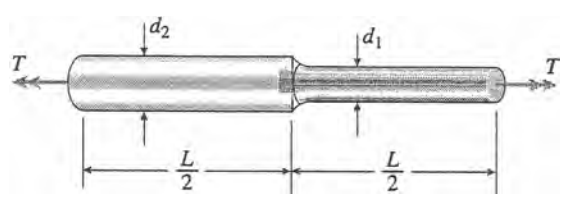
$\phi = 0.026667$ rad $= 1.53°$ $\longleftarrow$

Problem 3.9-3 A stepped shaft of solid circular cross sections (see figure) has length $L = 45$ in., diameter $d_2 = 1.2$ in., and diameter $d_1 = 1.0$ in. The material is brass with $G = 5.6 \times 10^6$ psi.

Determine the strain energy U of the shaft if the angle of twist is 3.0°.



Solution 3.9-3 Stepped shaft



$d_1 = 1.0$ in.

$d_2 = 1.2$ in.

$L = 45$ in.

$G = 5.6 \times 10^6$ psi (brass)

$\phi = 3.0° = 0.0523599$ rad

STRAIN ENERGY

$$U = \sum \frac{T^2 L}{2GI_P} = \frac{16\,T^2(L/2)}{\pi G d_2^4} + \frac{16\,T^2(L/2)}{\pi G d_1^4}$$

$$= \frac{8T^2 L}{\pi G}\left(\frac{1}{d_2^4} + \frac{1}{d_1^4}\right) \qquad \text{(Eq. 1)}$$

$$\text{Also, } U = \frac{T\phi}{2} \qquad \text{(Eq. 2)}$$

Equate U from Eqs. (1) and (2) and solve for T:

$$T = \frac{\pi G d_1^4\, d_2^4 \phi}{16L(d_1^4 + d_2^4)}$$

$$U = \frac{T\phi}{2} = \frac{\pi G \phi^2}{32L}\left(\frac{d_1^4\, d_2^4}{d_1^4 + d_2^4}\right) \qquad \phi = \text{radians}$$

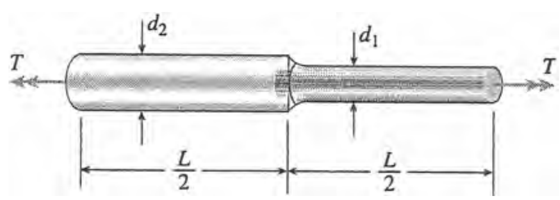
SUBSTITUTE NUMERICAL VALUES:

$U = 22.6$ in.-lb $\longleftarrow$

Problem 3.9-4 A stepped shaft of solid circular cross sections (see figure) has length $L = 0.80$ m, diameter $d_2 = 40$ mm, and diameter $d_1 = 30$ mm. The material is steel with $G = 80$ GPa.

Determine the strain energy U of the shaft if the angle of twist is 1.0°.

Solution 3.9-4 Stepped shaft



$d_1 = 30$ mm $\quad d_2 = 40$ mm

$L = 0.80$ m $\quad G = 80$ GPa (steel)

$\phi = 1.0° = 0.0174533$ rad

Strain energy

$$U = \sum \frac{T^2 L}{2GI_P} = \frac{16T^2(L/2)}{\pi G d_2^4} + \frac{16T^2(L/2)}{\pi G d_1^4}$$

$$= \frac{8T^2L}{\pi G}\left(\frac{1}{d_2^4} + \frac{1}{d_1^4}\right) \qquad \text{(Eq. 1)}$$

$$\text{Also, } U = \frac{T\phi}{2} \qquad \text{(Eq. 2)}$$

Equate U from Eqs. (1) and (2) and solve for T:

$$T = \frac{\pi G\, d_1^4\, d_2^4 \phi}{16L(d_1^4 + d_2^4)}$$

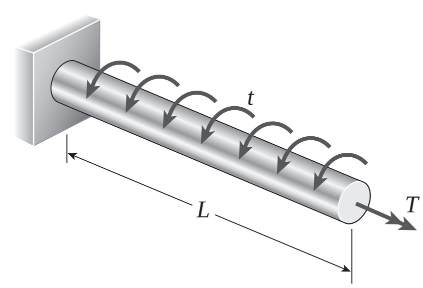
$$U = \frac{T\phi}{2} = \frac{\pi G \phi^2}{32L}\left(\frac{d_1^4\, d_2^4}{d_1^4 + d_2^4}\right) \qquad \phi = \text{radians}$$

Substitute numerical values:

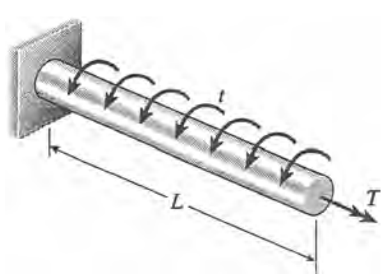
$U = 1.84$ J $\quad \longleftarrow$

Problem 3.9-5 A cantilever bar of circular cross section and length L is fixed at one end and free at the other (see figure). The bar is loaded by a torque T at the free end and by a distributed torque of constant intensity t per unit distance along the length of the bar.

(a) What is the strain energy U_1 of the bar when the load T acts alone?
(b) What is the strain energy U_2 when the load t acts alone?
(c) What is the strain energy U_3 when both loads act simultaneously?



Solution 3.9-5 Cantilever bar with distributed torque



G = shear modulus

I_P = polar moment of inertia

T = torque acting at free end

t = torque per unit distance

(a) LOAD T ACTS ALONE (Eq. 3-51a)

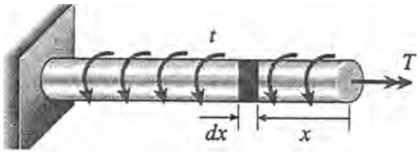
$$U_1 = \frac{T^2L}{2GI_P} \quad \leftarrow$$

(b) LOAD t ACTS ALONE

From Eq. (3-56) of Example 3-11:

$$U_2 = \frac{t^2L^3}{6GI_P} \quad \leftarrow$$

(c) BOTH LOADS ACT SIMULTANEOUSLY



At distance x from the free end:

$T(x) = T + tx$

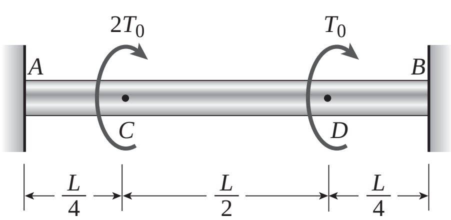
$$U_3 = \int_0^L \frac{[T(x)]^2}{2GI_P}dx = \frac{1}{2GI_P}\int_0^L (T + tx)^2 dx$$

$$= \frac{T^2L}{2GI_P} + \frac{TtL^2}{2GI_P} + \frac{t^2L^3}{6GI_P} \quad \leftarrow$$

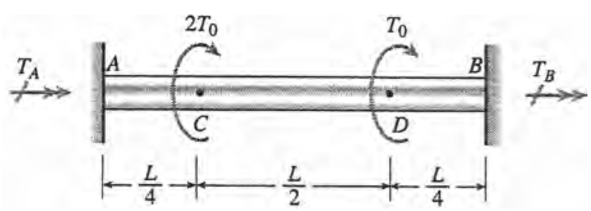
NOTE: U_3 is *not* the sum of U_1 and U_2.

Problem 3.9-6 Obtain a formula for the strain energy U of the statically indeterminate circular bar shown in the figure. The bar has fixed supports at ends A and B and is loaded by torques $2T_0$ and T_0 at points C and D, respectively.

Hint: Use Eqs. 3-46a and b of Example 3-9, Section 3.8, to obtain the reactive torques.



Solution 3.9-6 Statically indeterminate bar



REACTIVE TORQUES

From Eq. (3-46a):

$$T_A = \frac{(2T_0)\left(\frac{3L}{4}\right)}{L} + \frac{T_0\left(\frac{L}{4}\right)}{L} = \frac{7T_0}{4}$$

$$T_B = 3T_0 - T_A = \frac{5T_0}{4}$$

INTERNAL TORQUES

$$T_{AC} = -\frac{7T_0}{4} \quad T_{CD} = \frac{T_0}{4} \quad T_{DB} = \frac{5T_0}{4}$$

STRAIN ENERGY (from Eq. 3-53)

$$U = \sum_{i=1}^{n} \frac{T_i^2L_i}{2G_iI_{Pi}}$$

$$= \frac{1}{2GI_p}\left[T_{AC}^2\left(\frac{L}{4}\right) + T_{CD}^2\left(\frac{L}{2}\right) + T_{DB}^2\left(\frac{L}{4}\right)\right]$$

$$= \frac{1}{2GI_P}\left[\left(-\frac{7T_0}{4}\right)^2\left(\frac{L}{4}\right)\right.$$

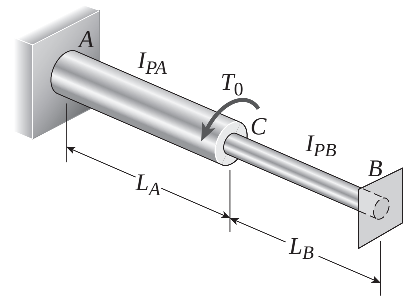
$$\left. + \left(\frac{T_0}{4}\right)^2\left(\frac{L}{2}\right) + \left(\frac{5T_0}{4}\right)^2\left(\frac{L}{4}\right)\right]$$

$$U = \frac{19T_0^2L}{32GI_P} \quad \leftarrow$$

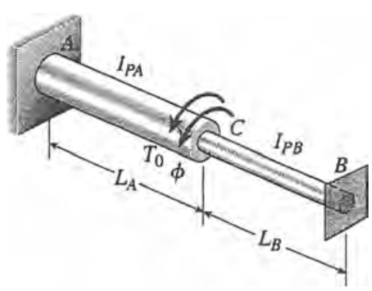
Problem 3.9-7 A statically indeterminate stepped shaft *ACB* is fixed at ends *A* and *B* and loaded by a torque T_0 at point *C* (see figure). The two segments of the bar are made of the same material, have lengths L_A and L_B, and have polar moments of inertia I_{PA} and I_{PB}.

Determine the angle of rotation ϕ of the cross section at *C* by using strain energy.

Hint: Use Eq. 3-51b to determine the strain energy *U* in terms of the angle ϕ. Then equate the strain energy to the work done by the torque T_0. Compare your result with Eq. 3-48 of Example 3-9, Section 3.8.



Solution 3.9-7 Statically indeterminate bar



STRAIN ENERGY (FROM EQ. 3-51B)

$$U = \sum_{i=1}^{n} \frac{GI_{Pi}\phi_i^2}{2L_i} = \frac{GI_{PA}\phi^2}{2L_A} + \frac{GI_{PB}\phi^2}{2L_B}$$

$$= \frac{G\phi^2}{2}\left(\frac{I_{PA}}{L_A} + \frac{I_{PB}}{L_B}\right)$$

WORK DONE BY THE TORQUE T_0

$$W = \frac{T_0\phi}{2}$$

EQUATE *U* AND *W* AND SOLVE FOR ϕ

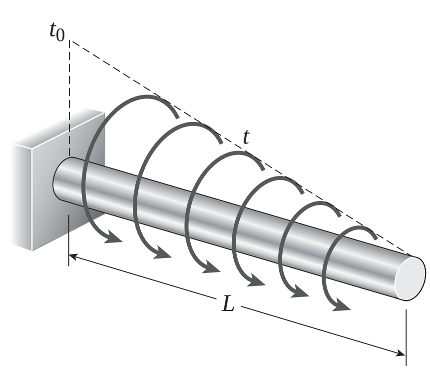
$$\frac{G\phi^2}{2}\left(\frac{I_{PA}}{L_A} + \frac{I_{PB}}{L_B}\right) = \frac{T_0\phi}{2}$$

$$\phi = \frac{T_0 L_A L_B}{G(L_B I_{PA} + L_A I_{PB})} \quad \leftarrow$$

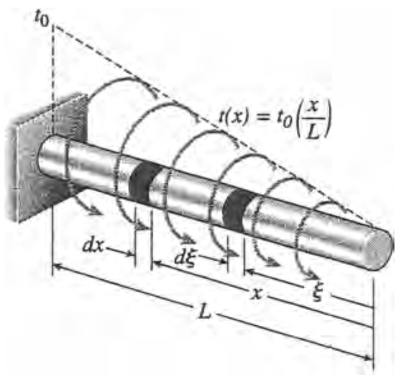
(This result agrees with Eq. (3-48) of Example 3-9, Section 3.8.)

Problem 3.9-8 Derive a formula for the strain energy *U* of the cantilever bar shown in the figure.

The bar has circular cross sections and length *L*. It is subjected to a distributed torque of intensity *t* per unit distance. The intensity varies linearly from $t = 0$ at the free end to a maximum value $t = t_0$ at the support.

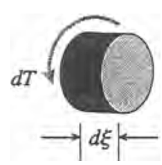


Solution 3.9-8 Cantilever bar with distributed torque



Element $d\xi$

Consider a differential element $d\xi$ at distance ξ from the right-hand end.

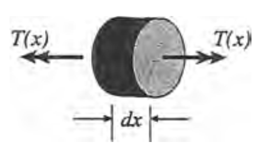


dT = external torque acting on this element

$$dT = t(\xi)d\xi$$

$$= t_0\left(\frac{\xi}{L}\right)d\xi$$

Element dx at distance x



$T(x)$ = internal torque acting on this element

$T(x)$ = total torque from $x = 0$ to $x = x$

$$T(x) = \int_0^x dT = \int_0^x t_0\left(\frac{\xi}{L}\right)d\xi$$

$$= \frac{t_0 x^2}{2L}$$

x = distance from right-hand end of the bar

Strain energy of element dx

$$dU = \frac{[T(x)]^2 dx}{2GI_P} = \frac{1}{2GI_P}\left(\frac{t_0}{2L}\right)^2 x^4 dx$$

$$= \frac{t_0^2}{8L^2 GI_P} x^4\, dx$$

Strain energy of entire bar

$$U = \int_0^L dU = \frac{t_0^2}{8L^2 GI_P}\int_0^L x^4\, dx$$

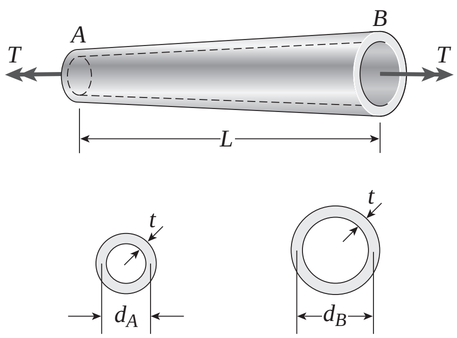
$$= \frac{t_0^2}{8L^2 GI_P}\left(\frac{L^5}{5}\right)$$

$$U = \frac{t_0^2 L^3}{40 GI_P} \quad \leftarrow$$

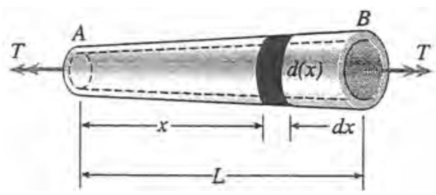
Problem 3.9-9 A thin-walled hollow tube AB of conical shape has constant thickness t and average diameters d_A and d_B at the ends (see figure).

(a) Determine the strain energy U of the tube when it is subjected to pure torsion by torques T.
(b) Determine the angle of twist ϕ of the tube.

Note: Use the approximate formula $I_P \approx \pi d^3 t/4$ for a thin circular ring; see Case 22 of Appendix D.



Solution 3.9-9 Thin-walled, hollow tube



t = thickness

d_A = average diameter at end A

d_B = average diameter at end B

$d(x)$ = average diameter at distance x from end A

$$d(x) = d_A + \left(\frac{d_B - d_A}{L}\right)x$$

POLAR MOMENT OF INERTIA

$$I_P = \frac{\pi d^3 t}{4}$$

$$I_P(x) = \frac{\pi[d(x)]^3 t}{4} = \frac{\pi t}{4}\left[d_A + \left(\frac{d_B - d_A}{L}\right)x\right]^3$$

(a) STRAIN ENERGY (FROM EQ. 3-54)

$$U = \int_0^L \frac{T^2 dx}{2GI_P(x)}$$

$$= \frac{2T^2}{\pi G t}\int_0^L \frac{dx}{\left[d_A + \left(\frac{d_B - d_A}{L}\right)x\right]^3} \qquad \text{(Eq. 1)}$$

From Appendix C:

$$\int \frac{dx}{(a + bx)^3} = -\frac{1}{2b(a + bx)^2}$$

Therefore,

$$\int_0^L \frac{dx}{\left[d_A + \left(\frac{d_B - d_A}{L}\right)x\right]^3}$$

$$= -\left.\frac{1}{\frac{2(d_B - d_A)}{L}\left[d_A + \left(\frac{d_B - d_A}{L}\right)x\right]^2}\right|_0^L$$

$$= -\frac{L}{2(d_B - d_A)(d_B)^2} + \frac{L}{2(d_B - d_A)(d_A)^2}$$

$$= \frac{L(d_A + d_B)}{2d_A^2 d_B^2}$$

Substitute this expression for the integral into the equation for U (Eq. 1):

$$U = \frac{2T^2}{\pi G t}\,\frac{L(d_A + d_B)}{2d_A^2 d_B^2} = \frac{T^2 L}{\pi G t}\left(\frac{d_A + d_B}{d_A^2 d_B^2}\right) \quad \leftarrow$$

(b) ANGLE OF TWIST

Work of the torque T: $W = \dfrac{T\phi}{2}$

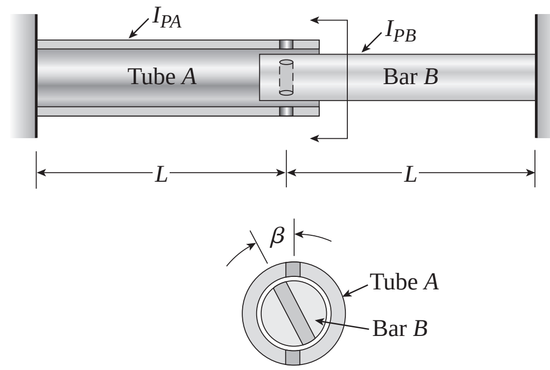
$$W = U \qquad \frac{T\phi}{2} = \frac{T^2 L(d_A + d_B)}{\pi G t\, d_A^2 d_B^2}$$

Solve for ϕ:

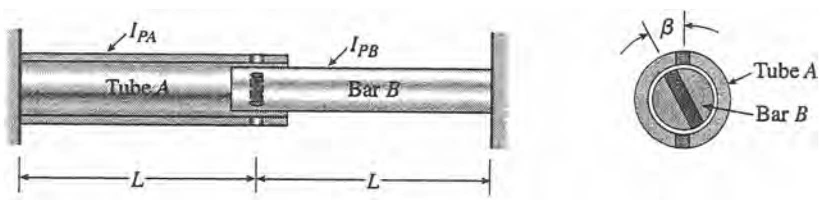
$$\phi = \frac{2TL(d_A + d_B)}{\pi G t\, d_A^2 d_B^2} \quad \leftarrow$$

Problem 3.9-10 A hollow circular tube A fits over the end of a solid circular bar B, as shown in the figure. The far ends of both bars are fixed. Initially, a hole through bar B makes an angle β with a line through two holes in tube A. Then bar B is twisted until the holes are aligned, and a pin is placed through the holes.

When bar B is released and the system returns to equilibrium, what is the total strain energy U of the two bars? (Let I_{PA} and I_{PB} represent the polar moments of inertia of bars A and B, respectively. The length L and shear modulus of elasticity G are the same for both bars.)



Solution 3.9-10 Circular tube and bar

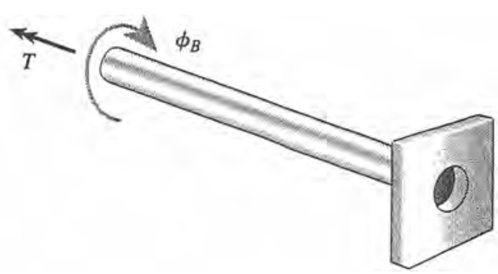


TUBE A

T = torque acting on the tube

ϕ_A = angle of twist

BAR B



T = torque acting on the bar

ϕ_B = angle of twist

COMPATIBILITY

$$\phi_A + \phi_B = \beta$$

FORCE-DISPLACEMENT RELATIONS

$$\phi_A = \frac{TL}{GI_{PA}} \qquad \phi_B = \frac{TL}{GI_{PB}}$$

Substitute into the equation of compatibility and solve for T:

$$T = \frac{\beta G}{L}\left(\frac{I_{PA}I_{PB}}{I_{PA} + I_{PB}}\right)$$

STRAIN ENERGY

$$U = \Sigma\frac{T^2L}{2GI_P} = \frac{T^2L}{2GI_{PA}} + \frac{T^2L}{2GI_{PB}}$$

$$= \frac{T^2L}{2G}\left(\frac{1}{I_{PA}} + \frac{1}{I_{PB}}\right)$$

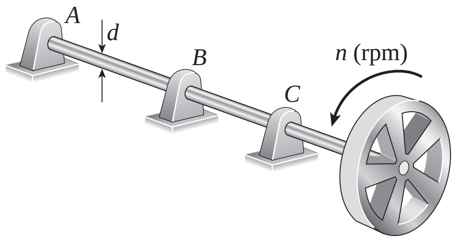
Substitute for T and simplify:

$$U = \frac{\beta^2 G}{2L}\left(\frac{I_{PA}I_{PB}}{I_{PA} + I_{PB}}\right) \quad \leftarrow$$

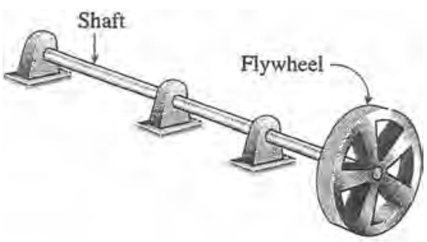
Problem 3.9-11 A heavy flywheel rotating at n revolutions per minute is rigidly attached to the end of a shaft of diameter d (see figure). If the bearing at A suddenly freezes, what will be the maximum angle of twist ϕ of the shaft? What is the corresponding maximum shear stress in the shaft?

(Let L = length of the shaft, G = shear modulus of elasticity, and I_m = mass moment of inertia of the flywheel about the axis of the shaft. Also, disregard friction in the bearings at B and C and disregard the mass of the shaft.)

Hint: Equate the kinetic energy of the rotating flywheel to the strain energy of the shaft.



Solution 3.9-11 Rotating flywheel



d = diameter

n = rpm

KINETIC ENERGY OF FLYWHEEL

$$\text{K.E.} = \frac{1}{2} I_m \text{v}^2$$

$$\text{v} = \frac{2\pi n}{60}$$

$$n = \text{rpm}$$

$$\text{K.E.} = \frac{1}{2} I_m \left(\frac{2\pi n}{60}\right)^2$$

$$= \frac{\pi^2 n^2 I_m}{1800}$$

UNITS:

I_m = (force)(length)(second)^2

ω = radians per second

K.E. = (length)(force)

STRAIN ENERGY OF SHAFT (FROM EQ. 3-51b)

$$U = \frac{G I_P \phi^2}{2L}$$

$$I_P = \frac{\pi}{32} d^4$$

d = diameter of shaft

$$U = \frac{\pi G d^4 \phi^2}{64L}$$

UNITS:

G = (force)/(length)^2

I_P = (length)^4

ϕ = radians

L = length

U = (length)(force)

EQUATE KINETIC ENERGY AND STRAIN ENERGY

$$\text{K.E.} = U \qquad \frac{\pi^2 n^2 I_m}{1800} = \frac{\pi G d^4 \phi^2}{64\,L}$$

Solve for ϕ:

$$\phi = \frac{2n}{15d^2}\sqrt{\frac{2\pi I_m L}{G}} \qquad \longleftarrow$$

MAXIMUM SHEAR STRESS

$$\tau = \frac{T(d/2)}{I_P} \qquad \phi = \frac{TL}{GI_P}$$

Eliminate T:

$$\tau = \frac{Gd\phi}{2L}$$

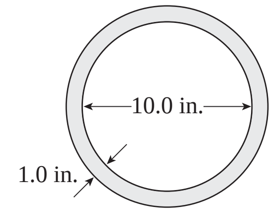
$$\tau_{\max} = \frac{Gd2n}{2L15d^2}\sqrt{\frac{2\pi I_m L}{G}}$$

$$\tau_{\max} = \frac{n}{15d}\sqrt{\frac{2\pi G I_m}{L}} \qquad \longleftarrow$$

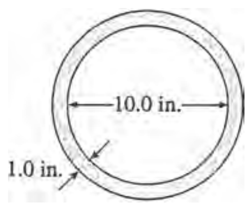
Thin-Walled Tubes

Problem 3.10-1 A hollow circular tube having an inside diameter of 10.0 in. and a wall thickness of 1.0 in. (see figure) is subjected to a torque $T = 1200$ k-in.

Determine the maximum shear stress in the tube using (a) the approximate theory of thin-walled tubes, and (b) the exact torsion theory. Does the approximate theory give conservative or nonconservative results?



Solution 3.10-1 Hollow circular tube



$T = 1200$ k-in.

$t = 1.0$ in.

$r =$ radius to median line

$r = 5.5$ in.

$d_2 =$ outside diameter $= 12.0$ in.

$d_1 =$ inside diameter $= 10.0$ in.

Approximate theory (Eq. 3-63)

$$\tau_1 = \frac{T}{2\pi r^2 t} = \frac{1200 \text{ k-in.}}{2\pi(5.5 \text{ in.})^2(1.0 \text{ in.})} = 6314 \text{ psi}$$

$$\tau_{\text{approx}} = 6310 \text{ psi} \quad \longleftarrow$$

Exact theory (Eq. 3-11)

$$\tau_2 = \frac{T(d_2/2)}{I_P} = \frac{Td_2}{2\left(\frac{\pi}{32}\right)(d_2^4 - d_1^4)}$$

$$= \frac{16(1200\text{k-in.})(12.0 \text{ in.})}{\pi[(12.0 \text{ in.})^4 - (10.0 \text{ in.})^4]}$$

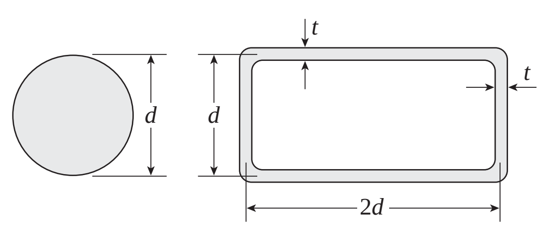
$$= 6831 \text{ psi}$$

$$\tau_{\text{exact}} = 6830 \text{ psi} \quad \longleftarrow$$

Because the approximate theory gives stresses that are too low, it is nonconservative. Therefore, the approximate theory should only be used for very thin tubes.

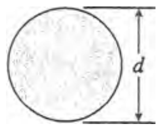
Problem 3.10-2 A solid circular bar having diameter d is to be replaced by a rectangular tube having cross-sectional dimensions $d \times 2d$ to the median line of the cross section (see figure).

Determine the required thickness t_{min} of the tube so that the maximum shear stress in the tube will not exceed the maximum shear stress in the solid bar.



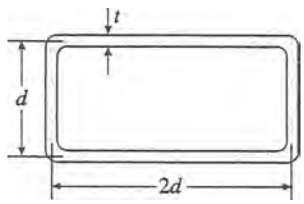
Solution 3.10-2 Bar and tube

SOLID BAR



$$\tau_{max} = \frac{16T}{\pi d^3} \qquad \text{(Eq. 3-12)}$$

RECTANGULAR TUBE



$$A_m = (d)(2d) = 2d^2 \qquad \text{(Eq. 3-64)}$$

$$\tau_{max} = \frac{T}{2tA_m} = \frac{T}{4td^2} \qquad \text{(Eq. 3-61)}$$

EQUATE THE MAXIMUM SHEAR STRESSES AND SOLVE FOR t

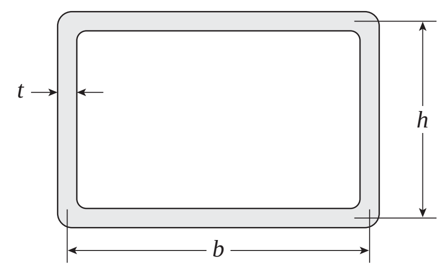
$$\frac{16T}{\pi d^3} = \frac{T}{4td^2}$$

$$t_{min} = \frac{\pi d}{64} \quad \longleftarrow$$

If $t > t_{min}$, the shear stress in the tube is less than the shear stress in the bar.

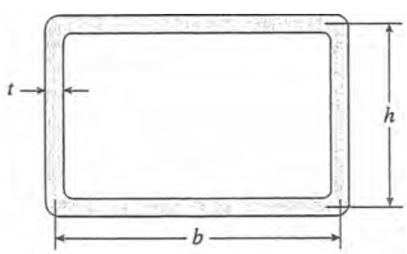
Problem 3.10-3 A thin-walled aluminum tube of rectangular cross section (see figure) has a centerline dimensions $b = 6.0$ in. and $h = 4.0$ in. The wall thickness t is constant and equal to 0.25 in.

(a) Determine the shear stress in the tube due to a torque $T = 15$ k-in.
(b) Determine the angle of twist (in degrees) if the length L of the tube is 50 in. and the shear modulus G is 4.0×10^6 psi.



Probs. 3.10-3 and 3.10-4

Solution 3.10-3 Thin-walled tube



$b = 6.0$ in.

$h = 4.0$ in.

$t = 0.25$ in.

$T = 15$ k-in.

$L = 50$ in.

$G = 4.0 \times 10^6$ psi

Eq. (3-64): $A_m = bh = 24.0 \text{ in.}^2$

Eq. (3-71) with $t_1 = t_2 = t$: $\quad J = \dfrac{2b^2h^2t}{b + h}$

$J = 28.8 \text{ in.}^4$

(a) SHEAR STRESS (EQ. 3-61)

$$\tau = \frac{T}{2tA_m} = 1250 \text{ psi} \quad \longleftarrow$$

(b) ANGLE OF TWIST (EQ. 3-72)

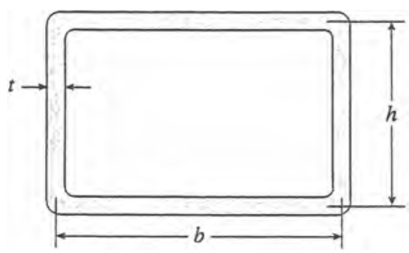
$$\phi = \frac{TL}{GJ} = 0.0065104 \text{ rad}$$

$$= 0.373^\circ \quad \longleftarrow$$

Problem 3.10-4 A thin-walled steel tube of rectangular cross section (see figure) has centerline dimensions $b = 150$ mm and $h = 100$ mm. The wall thickness t is constant and equal to 6.0 mm.

(a) Determine the shear stress in the tube due to a torque $T = 1650$ N · m.
(b) Determine the angle of twist (in degrees) if the length L of the tube is 1.2 m and the shear modulus G is 75 GPa.

Solution 3.10-4 Thin-walled tube



$b = 150$ mm
$h = 100$ mm
$t = 6.0$ mm
$T = 1650$ N · m
$L = 1.2$ m
$G = 75$ GPa

Eq. (3-64): $A_m = bh = 0.015 \text{ m}^2$

Eq. (3-71) with $t_1 = t_2 = t$: $\quad J = \dfrac{2b^2h^2t}{b + h}$

$J = 10.8 \times 10^{-6} \text{ m}^4$

(a) Shear stress (Eq. 3-61)

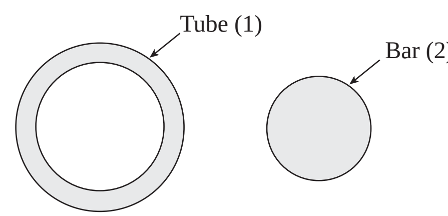
$$\tau = \frac{T}{2tA_m} = 9.17 \text{ MPa} \quad \leftarrow$$

(b) Angle of twist (Eq. 3-72)

$$\phi = \frac{TL}{GJ} = 0.002444 \text{ rad}$$
$$= 0.140^\circ \quad \leftarrow$$

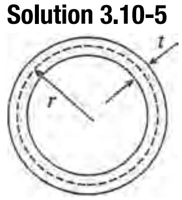
Problem 3.10-5 A thin-walled circular tube and a solid circular bar of the same material (see figure) are subjected to torsion. The tube and bar have the same cross-sectional area and the same length.

What is the ratio of the strain energy U_1 in the tube to the strain energy U_2 in the solid bar if the maximum shear stresses are the same in both cases? (For the tube, use the approximate theory for thin-walled bars.)



Solution 3.10-5

Thin-walled tube (1)



$$A_m = \pi r^2 \quad J = 2\pi r^3 t \quad A = 2\pi r t$$

$$\tau_{max} = \frac{T}{2tA_m} = \frac{T}{2\pi r^2 t}$$

$$T = 2\pi r^2 t \tau_{max}$$

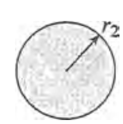
$$U_1 = \frac{T^2L}{2GJ} = \frac{(2\pi r^2 t\tau_{max})^2 L}{2G(2\pi r^3 t)}$$

$$= \frac{\pi r t \tau_{max}^2 L}{G}$$

But $rt = \dfrac{A}{2\pi}$

$$\therefore U_1 = \frac{A\tau_{max}^2 L}{2G}$$

Solid bar (2)



$$A = \pi r_2^2 \qquad I_P = \frac{\pi}{2} r_2^4$$

$$\tau_{max} = \frac{Tr_2}{I_P} = \frac{2T}{\pi r_2^3} \qquad T = \frac{\pi r_2^3 \tau_{max}}{2}$$

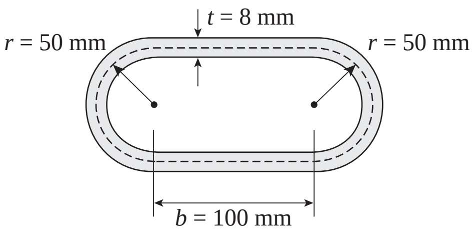
$$U_2 = \frac{T^2L}{2GI_P} = \frac{(\pi r_2^3 \tau_{max})^2 L}{8G\left(\dfrac{\pi}{2} r_2^4\right)} = \frac{\pi r_2^2 \tau_{max}^2 L}{4G}$$

But $\pi r_2^2 = A \qquad \therefore U_2 = \dfrac{A\tau_{max}^2 L}{4G}$

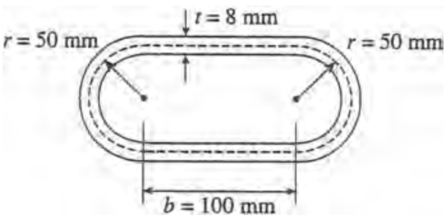
Ratio

$$\frac{U_1}{U_2} = 2 \quad \leftarrow$$

Problem 3.10-6 Calculate the shear stress τ and the angle of twist ϕ (in degrees) for a steel tube (G = 76 GPa) having the cross section shown in the figure. The tube has length L = 1.5 m and is subjected to a torque T = 10 kN · m.



Solution 3.10-6 Steel tube



$G = 76$ GPa.

$L = 1.5$ m

$T = 10$ kN · m

$$A_m = \pi r^2 + 2br$$

$$A_m = \pi(50\text{ mm})^2 + 2(100\text{ mm})(50\text{ mm})$$

$$= 17{,}850\text{ mm}^2$$

$$L_m = 2b + 2\pi r$$

$$= 2(100\text{ mm}) + 2\pi(50\text{ mm})$$

$$= 514.2\text{ mm}$$

$$J = \frac{4tA_m^2}{L_m} = \frac{4(8\text{ mm})(17{,}850\text{ mm}^2)^2}{514.2\text{ mm}}$$

$$= 19.83 \times 10^6\text{ mm}^4$$

SHEAR STRESS

$$\tau = \frac{T}{2tA_m} = \frac{10\text{ kN}\cdot\text{m}}{2(8\text{ mm})(17{,}850\text{ mm}^2)}$$

$$= 35.0\text{ MPa} \quad \leftarrow$$

ANGLE OF TWIST

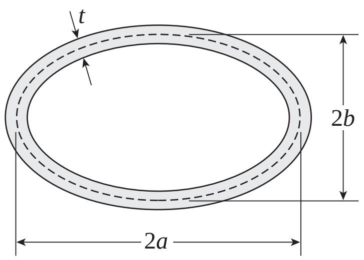
$$\phi = \frac{TL}{GJ} = \frac{(10\text{ kN}\cdot\text{ m})(1.5\text{ m})}{(76\text{ GPa})(19.83 \times 10^6\text{ mm}^4)}$$

$$= 0.00995\text{ rad}$$

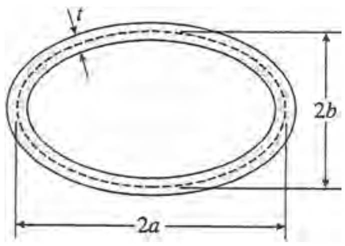
$$= 0.570^\circ \quad \leftarrow$$

Problem 3.10-7 A thin-walled steel tube having an elliptical cross section with constant thickness t (see figure) is subjected to a torque T = 18 k-in.

Determine the shear stress τ and the rate of twist θ (in degrees per inch) if $G = 12 \times 10^6$ psi, t = 0.2 in., a = 3 in., and b = 2 in. (*Note:* See Appendix D, Case 16, for the properties of an ellipse.)



Solution 3.10-7 Elliptical tube



$T = 18$ k-in.

$G = 12 \times 10^6$ psi

$t =$ constant

$t = 0.2$ in $\quad a = 3.0$ in. $\quad b = 2.0$ in.

FROM APPENDIX D, CASE 16:

$$A_m = \pi ab = \pi(3.0 \text{ in.})(2.0 \text{ in.}) = 18.850 \text{ in.}^2$$

$$L_m \approx \pi[1.5(a + b) - \sqrt{ab}]$$

$$= \pi[1.5(5.0 \text{ in.}) - \sqrt{6.0 \text{ in.}^2}] = 15.867 \text{ in.}$$

$$J = \frac{4tA_m^2}{L_m} = \frac{4(0.2 \text{ in.})(18.850 \text{ in.}^2)^2}{15.867 \text{ in.}}$$

$$= 17.92 \text{ in.}^4$$

SHEAR STRESS

$$\tau = \frac{T}{2tA_m} = \frac{18 \text{ k-in.}}{2(0.2 \text{ in.})(18.850 \text{ in.}^2)}$$

$$= 2390 \text{ psi} \quad \leftarrow$$

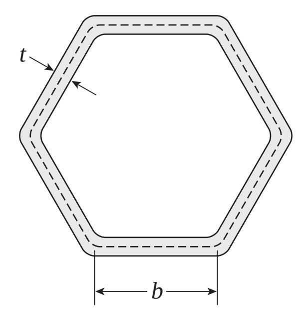
ANGLE OF TWIST PER UNIT LENGTH (RATE OF TWIST)

$$\theta = \frac{\phi}{L} = \frac{T}{GJ} = \frac{18 \text{ k-in.}}{(12 \times 10^6 \text{ psi})(17.92 \text{ in.})^4}$$

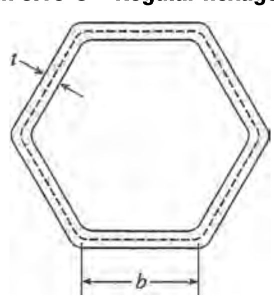
$$\theta = 83.73 \times 10^{-6} \text{ rad/in.} = 0.0048°/\text{in.} \quad \leftarrow$$

Problem 3.10-8 A torque T is applied to a thin-walled tube having a cross section in the shape of a regular hexagon with constant wall thickness t and side length b (see figure).

Obtain formulas for the shear stress τ and the rate of twist θ.



Solution 3.10-8 Regular hexagon



$b =$ Length of side

$t =$ Thickness

$L_m = 6b$

FROM APPENDIX D, CASE 25:

$\beta = 60° \quad n = 6$

$$A_m = \frac{nb^2}{4} \cot \frac{\beta}{2} = \frac{6b^2}{4} \cot 30°$$

$$= \frac{3\sqrt{3}b^2}{2}$$

SHEAR STRESS

$$\tau = \frac{T}{2tA_m} = \frac{T\sqrt{3}}{9b^2t} \quad \longleftarrow$$

ANGLE OF TWIST PER UNIT LENGTH (RATE OF TWIST)

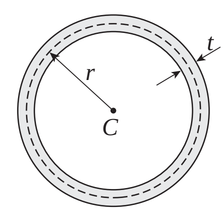
$$J = \frac{4A_m^2 t}{\int_0^{L_m} \frac{d_s}{t}} = \frac{4A_m^2 t}{L_m} = \frac{9b^3 t}{2}$$

$$\theta = \frac{T}{GJ} = \frac{2T}{G(9b^3 t)} = \frac{2T}{9Gb^3 t} \quad \longleftarrow$$

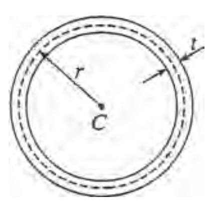
(radians per unit length)

Problem 3.10-9 Compare the angle of twist ϕ_1 for a thin-walled circular tube (see figure) calculated from the approximate theory for thin-walled bars with the angle of twist ϕ_2 calculated from the exact theory of torsion for circular bars.

(a) Express the ratio ϕ_1/ϕ_2 in terms of the nondimensional ratio $\beta = r/t$.
(b) Calculate the ratio of angles of twist for $\beta = 5$, 10, and 20. What conclusion about the accuracy of the approximate theory do you draw from these results?



Solution 3.10-9 Thin-walled tube



APPROXIMATE THEORY

$$\phi_1 = \frac{TL}{GJ} \quad J = 2\pi r^3 t \quad \phi_1 = \frac{TL}{2\pi G r^3 t}$$

EXACT THEORY

$$\phi_2 = \frac{TL}{GI_P} \quad \text{From Eq. (3-17): } I_p = \frac{\pi r t}{2}(4r^2 + t^2)$$

$$\phi_2 = \frac{TL}{GI_P} = \frac{2TL}{\pi G r t(4r^2 + t^2)}$$

(a) RATIO

$$\frac{\phi_1}{\phi_2} = \frac{4r^2 + t^2}{4r^2} = 1 + \frac{t^2}{4r^2}$$

$$\text{Let } \beta = \frac{r}{t} \qquad \frac{\phi_1}{\phi_2} = 1 + \frac{1}{4\beta^2} \quad \longleftarrow$$

(b)

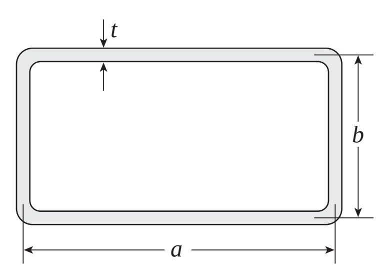
β	ϕ_1/ϕ_2
5	1.0100
10	1.0025
20	1.0006

As the tube becomes thinner and β becomes larger, the ratio ϕ_1/ϕ_2 approaches unity. Thus, the thinner the tube, the more accurate the approximate theory becomes.

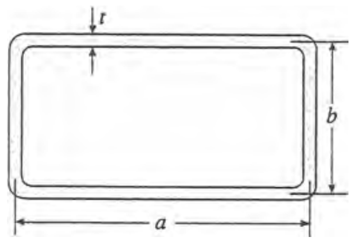
Problem 3.10-10 A thin-walled rectangular tube has uniform thickness t and dimensions $a \times b$ to the median line of the cross section (see figure).

How does the shear stress in the tube vary with the ratio $\beta = a/b$ if the total length L_m of the median line of the cross section and the torque T remain constant?

From your results, show that the shear stress is smallest when the tube is square ($\beta = 1$).



Solution 3.10-10 Rectangular tube



t = thickness (constant)

a, b = dimensions of the tube

$$\beta = \frac{a}{b}$$

$$L_m = 2(a + b) = \text{constant}$$

$$T = \text{constant}$$

SHEAR STRESS

$$\tau = \frac{T}{2tA_m} \qquad A_m = ab = \beta b^2$$

$$L_m = 2b(1 + \beta) = \text{constant}$$

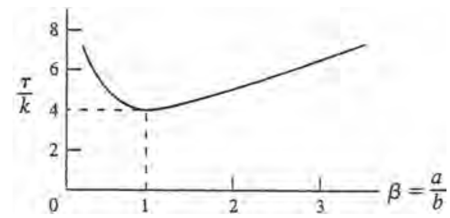
$$b = \frac{L_m}{2(1 + \beta)} \qquad A_m = \beta\left[\frac{L_m}{2(1 + \beta)}\right]^2$$

$$A_m = \frac{\beta L_m^2}{4(1 + \beta)^2}$$

$$\tau = \frac{T}{2tA_m} = \frac{T(4)(1 + \beta)^2}{2t\beta L_m^2} = \frac{2T(1 + \beta)^2}{tL_m^2\beta} \quad \leftarrow$$

T, t, and L_m are constants.

$$\text{Let } k = \frac{2T}{tL_m^2} = \text{constant} \qquad \tau = k\frac{(1 + \beta)^2}{\beta}$$



$$\left(\frac{\tau}{k}\right)_{min} = 4 \qquad \tau_{\min} = \frac{8T}{tL_m^2}$$

From the graph, we see that τ is minimum when $\beta = 1$ and the tube is square.

ALTERNATE SOLUTION

$$\tau = \frac{2T}{tL_m^2}\left[\frac{(1 + \beta)^2}{\beta}\right]$$

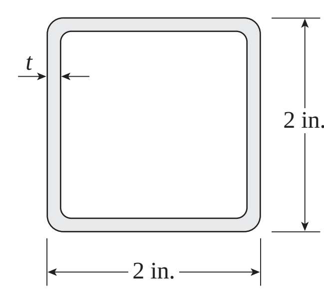
$$\frac{d\tau}{d\beta} = \frac{2T}{tL_m^2}\left[\frac{\beta(2)(1 + \beta) - (1 + \beta)^2(1)}{\beta^2}\right] = 0$$

or $2\beta\,(1 + \beta) - (1 + \beta)^2 = 0 \quad \therefore \beta = 1$

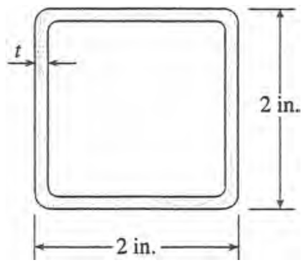
Thus, the tube is square and τ is either a minimum or a maximum. From the graph, we see that τ is a minimum.

Problem 3.10-11 A tubular aluminum bar ($G = 4 \times 10^6$ psi) of square cross section (see figure) with outer dimensions 2 in. $\times$ 2 in. must resist a torque $T = 3000$ lb-in.

Calculate the minimum required wall thickness $t_{\min}$ if the allowable shear stress is 4500 psi and the allowable rate of twist is 0.01 rad/ft.



Solution 3.10-11 Square aluminum tube



Outer dimensions:

2.0 in. $\times$ 2.0 in.

$G = 4 \times 10^6$ psi

$T = 3000$ lb-in.

$\tau_{\text{allow}} = 4500$ psi

$\theta_{\text{allow}} = 0.01 \text{ rad/ft} = \dfrac{0.01}{12} \text{ rad/in.}$

Let b = outer dimension

$= 2.0$ in.

Centerline dimension $= b - t$

$A_m = (b - t)^2 \quad L_m = 4(b - t)$

$$J = \frac{4tA_m^2}{L_m} = \frac{4t(b - t)^4}{4(b - t)} = t(b - t)^3$$

THICKNESS t BASED UPON SHEAR STRESS

$$\tau = \frac{T}{2tA_m} \quad tA_m = \frac{T}{2\tau} \quad t(b - t)^2 = \frac{T}{2\tau}$$

UNITS: t = in. b = in. T = lb-in. τ = psi

$$t(2.0 \text{ in.} - t)^2 = \frac{3000 \text{ lb-in.}}{2(4500 \text{ psi})} = \frac{1}{3} \text{ in.}^3$$

$3t(2 - t)^2 - 1 = 0$

Solve for t: $t = 0.0915$ in.

THICKNESS t BASED UPON RATE OF TWIST

$$\theta = \frac{T}{GJ} = \frac{T}{Gt(b - t)^3} \quad t(b - t)^3 = \frac{T}{G\theta}$$

UNITS: t = in. G = psi θ = rad/in.

$$t(2.0 \text{ in.} - t)^3 = \frac{3000 \text{ lb-in}}{(4 \times 10^6 \text{ psi})(0.01/12 \text{ rad/in.})} = \frac{9}{10}$$

$10t(2 - t)^3 - 9 = 0$

Solve for t:

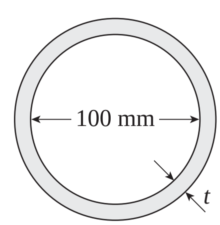
$t = 0.140$ in.

ANGLE OF TWIST GOVERNS

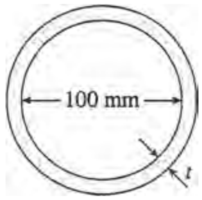
$t_{\min} = 0.140 \text{ in.} \quad \leftarrow$

Problem 3.10-12 A thin tubular shaft of circular cross section (see figure) with inside diameter 100 mm is subjected to a torque of 5000 N · m.

If the allowable shear stress is 42 MPa, determine the required wall thickness t by using (a) the approximate theory for a thin-walled tube, and (b) the exact torsion theory for a circular bar.



Solution 3.10-12 Thin tube



$T = 5{,}000\ \text{N} \cdot \text{m} \quad d_1 = \text{inner diameter} = 100\ \text{mm}$

$\tau_{\text{allow}} = 42\ \text{MPa}$

t is in millimeters.

$r = \text{Average radius}$

$\quad = 50\ \text{mm} + \dfrac{t}{2}$

$r_1 = \text{Inner radius}$

$\quad = 50\ \text{mm}$

$r_2 = \text{Outer radius}$

$\quad = 50\ \text{mm} + t \quad A_m = \pi r^2$

(a) Approximate theory

$$\tau = \frac{T}{2tA_m} = \frac{T}{2t(\pi r^2)} = \frac{T}{2\pi r^2\, t}$$

$$42\ \text{MPa} = \frac{5{,}000\ \text{N} \cdot \text{m}}{2\pi\left(50 + \dfrac{t}{2}\right)^2 t}$$

or

$$t\left(50 + \frac{t}{2}\right)^2 = \frac{5{,}000\ \text{N} \cdot \text{m}}{2\pi(42\ \text{MPa})} = \frac{5 \times 10^6}{84\pi}\ \text{mm}^3$$

Solve for t:

$t = 6.66\ \text{mm} \quad \leftarrow$

(b) Exact theory

$$\tau = \frac{Tr_2}{I_p}$$

$$I_p = \frac{\pi}{2}(r_2^4 - r_1^4) = \frac{\pi}{2}[(50 + t)^4 - (50)^4]$$

$$42\ \text{MPa} = \frac{(5{,}000\ \text{N} \cdot \text{m})(50 + t)}{\dfrac{\pi}{2}[(50 + t)^4 - (50)^4]}$$

$$\frac{(50 + t)^4 - (50)^4}{50 + t} = \frac{(5000\ \text{N} \cdot \text{m})(2)}{(\pi)(42\ \text{MPa})}$$

$$= \frac{5 \times 10^6}{21\pi}\ \text{mm}^3$$

Solve for t:

$t = 7.02\ \text{mm} \quad \leftarrow$

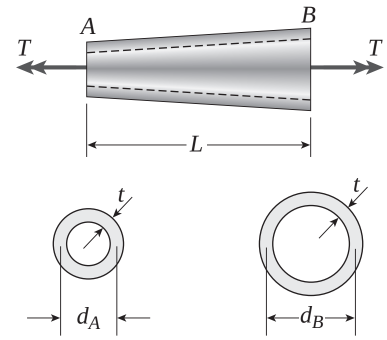
The approximate result is 5% less than the exact result. Thus, the approximate theory is nonconservative and should only be used for thin-walled tubes.

Problem 3.10-13 A long, thin-walled tapered tube AB of circular cross section (see figure) is subjected to a torque T. The tube has length L and constant wall thickness t. The diameter to the median lines of the cross sections at the ends A and B are d_A and d_B, respectively.

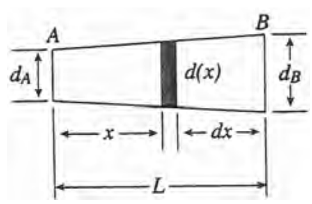
Derive the following formula for the angle of twist of the tube:

$$\phi = \frac{2TL}{\pi G t}\left(\frac{d_A + d_B}{d_A^2 d_B^2}\right)$$

Hint: If the angle of taper is small, we may obtain approximate results by applying the formulas for a thin-walled prismatic tube to a differential element of the tapered tube and then integrating along the axis of the tube.



Solution 3.10-13 Thin-walled tapered tube



t = thickness

d_A = average diameter at end A

d_B = average diameter at end B

T = torque

$d(x)$ = average diameter at distance x from end A.

$$d(x) = d_A + \left(\frac{d_B - d_A}{L}\right)x$$

$$J = 2\pi r^3 t = \frac{\pi d^3 t}{4}$$

$$J(x) = \frac{\pi t}{4}[d(x)]^3 = \frac{\pi t}{4}\left[d_A + \left(\frac{d_B - d_A}{L}\right)x\right]^3$$

For element of length dx:

$$d\phi = \frac{Tdx}{GJ(x)} = \frac{4Tdx}{G\pi t\left[d_A + \left(\frac{d_B - d_A}{L}\right)x\right]^3}$$

For entire tube:

$$\phi = \frac{4T}{\pi GT}\int_0^L \frac{dx}{\left[d_A + \left(\frac{d_B - d_A}{L}\right)x\right]^3}$$

From table of integrals (see Appendix C):

$$\int \frac{dx}{(a + bx)^3} = -\frac{1}{2b(a + bx)^2}$$

$$\phi = \frac{4T}{\pi G t}\left[-\frac{1}{2\left(\frac{d_B - d_A}{L}\right)\left(d_A + \frac{d_B - d_A}{L}\cdot x\right)^2}\right]_0^L$$

$$= \frac{4T}{\pi G t}\left[-\frac{L}{2(d_B - d_A)d_B^2} + \frac{L}{2(d_B - d_A)d_A^2}\right]$$

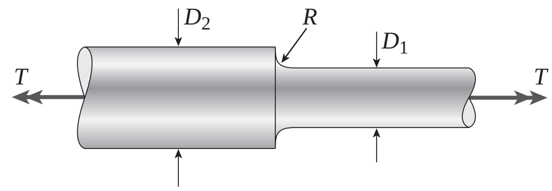
$$\phi = \frac{2TL}{\pi G t}\left(\frac{d_A + d_B}{d_A^2 d_B^2}\right) \quad \leftarrow$$

Stress Concentrations in Torsion

The problems for Section 3.11 are to be solved by considering the stress-concentration factors.

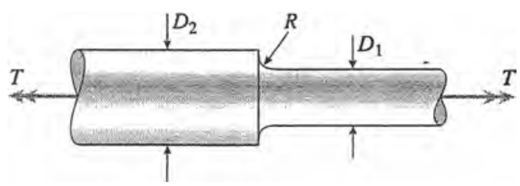
Problem 3.11-1 A stepped shaft consisting of solid circular segments having diameters $D_1 = 2.0$ in. and $D_2 = 2.4$ in. (see figure) is subjected to torques T. The radius of the fillet is $R = 0.1$ in.

If the allowable shear stress at the stress concentration is 6000 psi, what is the maximum permissible torque T_{max}?



Probs. 3.11-1 through 3.11-5

Solution 3.11-1 Stepped shaft in torsion



$D_1 = 2.0$ in.

$D_2 = 2.4$ in.

$R = 0.1$ in.

$\tau_{allow} = 6000$ psi

USE FIG. 3-48 FOR THE STRESS-CONCENTRATION FACTOR

$$\frac{R}{D_1} = \frac{0.1 \text{ in.}}{2.0 \text{ in.}} = 0.05 \qquad \frac{D_2}{D_1} = \frac{2.4 \text{ in.}}{2.0 \text{ in.}} = 1.2$$

$$K \approx 1.52 \qquad \tau_{max} = K\tau_{nom} = K\left(\frac{16\,T_{max}}{\pi D_1^3}\right)$$

$$T_{max} = \frac{\pi D_1^3 \tau_{max}}{16K}$$

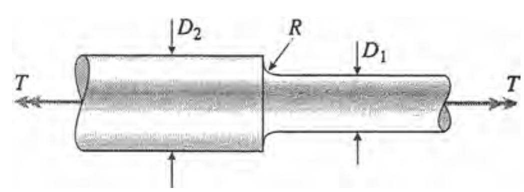
$$= \frac{\pi(2.0 \text{ in.})^3(6000 \text{ psi})}{16(1.52)} = 6200 \text{ lb-in.}$$

$$\therefore T_{max} \approx 6200 \text{ lb-in.} \quad \leftarrow$$

Problem 3.11-2 A stepped shaft with diameters $D_1 = 40$ mm and $D_2 = 60$ mm is loaded by torques $T = 1100$ N · m (see figure).

If the allowable shear stress at the stress concentration is 120 MPa, what is the smallest radius R_{min} that may be used for the fillet?

Solution 3.11-2 Stepped shaft in torsion



$D_1 = 40$ mm

$D_2 = 60$ mm

$T = 1100$ N · m

$\tau_{allow} = 120$ MPa

USE FIG. 3-48 FOR THE STRESS-CONCENTRATION FACTOR

$$\tau_{max} = K\tau_{nom} = K\left(\frac{16T}{\pi D_1^3}\right)$$

$$K = \frac{\pi D_1^3 \tau_{max}}{16\,T} = \frac{\pi(40 \text{ mm})^3(120 \text{ MPa})}{16(1100 \text{ N}\cdot\text{ m})} = 1.37$$

$$\frac{D_2}{D_1} = \frac{60 \text{ mm}}{40 \text{ mm}} = 1.5$$

From Fig. (3-48) with $\frac{D_2}{D_1} = 1.5$ and $K = 1.37$,

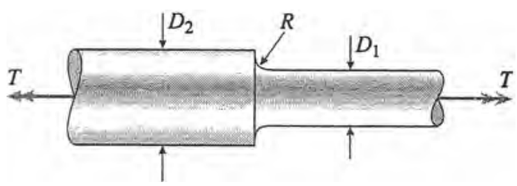
we get $\frac{R}{D_1} \approx 0.10$

$$\therefore R_{min} \approx 0.10(40 \text{ mm}) = 4.0 \text{ mm} \quad \leftarrow$$

Problem 3.11-3 A full quarter-circular fillet is used at the shoulder of a stepped shaft having diameter $D_2 = 1.0$ in. (see figure). A torque $T = 500$ lb-in. acts on the shaft.

Determine the shear stress τ_{max} at the stress concentration for values as follows: D_1 5 0.7, 0.8, and 0.9 in. Plot a graph showing τ_{max} versus D_1.

Solution 3.11-3 Stepped shaft in torsion



D_1 (in.)	D_2/D_1	R(in.)	R/D_1	K	τ_{max}(psi)
0.7	1.43	0.15	0.214	1.20	8900
0.8	1.25	0.10	0.125	1.29	6400
0.9	1.11	0.05	0.056	1.41	4900

$D_2 = 1.0$ in.

$T = 500$ lb-in.

$D_1 = 0.7$, 0.8, and 0.9 in.

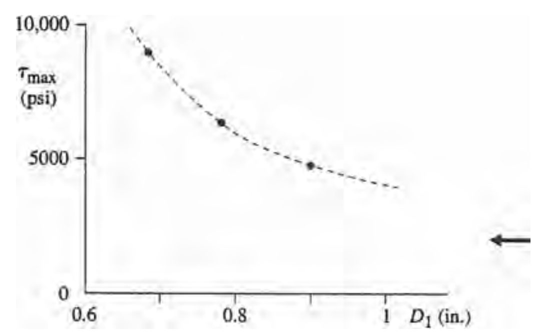
Full quarter-circular fillet ($D_2 = D_1 + 2R$)

$$R = \frac{D_2 - D_1}{2} = 0.5 \text{ in.} - \frac{D_1}{2}$$

USE FIG. 3-48 FOR THE STRESS-CONCENTRATION FACTOR

$$\tau_{max} = K\tau_{nom} = K\left(\frac{16\,T}{\pi D_1^3}\right)$$

$$= K\frac{16(500 \text{ lb-in.})}{\pi D_1^3} = 2546\frac{K}{D_1^3}$$

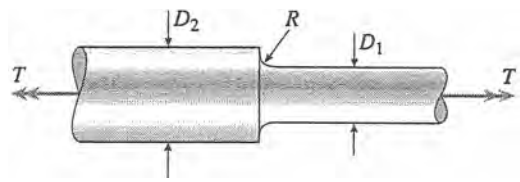


NOTE that τ_{max} gets smaller as D_1 gets larger, even though K is increasing.

Problem 3.11-4 The stepped shaft shown in the figure is required to transmit 600 kW of power at 400 rpm. The shaft has a full quarter-circular fillet, and the smaller diameter $D_1 = 100$ mm.

If the allowable shear stress at the stress concentration is 100 MPa, at what diameter D_2 will this stress be reached? Is this diameter an upper or a lower limit on the value of D_2?

Solution 3.11-4 Stepped shaft in torsion



$P = 600$ kW $\quad D_1 = 100$ mm

$n = 400$ rpm $\quad \tau_{allow} = 100$ MPa

Full quarter-circular fillet

POWER $P = \dfrac{2\pi nT}{60}$ (Eq. 3-42 of Section 3.7)

$P =$ watts $\quad n =$ rpm $\quad T =$ Newton meters

$$T = \frac{60P}{2\pi n} = \frac{60(600 \times 10^3 \text{ W})}{2\pi(400 \text{ rpm})} = 14{,}320 \text{ N}\cdot\text{m}$$

USE FIG. 3-48 FOR THE STRESS-CONCENTRATION FACTOR

$$\tau_{max} = K\tau_{nom} = K\left(\frac{16T}{\pi D_1^3}\right)$$

$$K = \frac{\tau_{max}(\pi D_1^3)}{16T}$$

$$= \frac{(100 \text{ MPa})(\pi)(100 \text{ mm})^3}{16(14{,}320 \text{ N}\cdot\text{m})} = 1.37$$

Use the dashed line for a full quarter-circular fillet.

$$\frac{R}{D_1} \approx 0.075 \qquad R \approx 0.075\, D_1 = 0.075\,(100 \text{ mm})$$

$$= 7.5 \text{ mm}$$

$$D_2 = D_1 + 2R = 100 \text{ mm} + 2(7.5 \text{ mm}) = 115 \text{ mm}$$

$\therefore D_2 \approx 115$ mm $\quad\leftarrow$

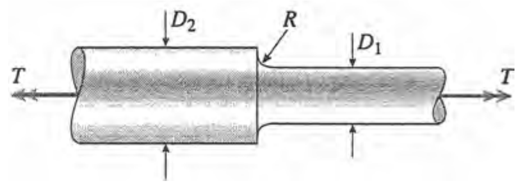
This value of D_2 is a *lower limit* $\quad\leftarrow$

(If D_2 is less than 115 mm, R/D_1 is smaller, K is larger, and τ_{max} is larger, which means that the allowable stress is exceeded.)

Problem 3.11-5 A stepped shaft (see figure) has diameter $D_2 = 1.5$ in. and a full quarter-circular fillet. The allowable shear stress is 15,000 psi and the load $T = 4800$ lb-in.

What is the smallest permissible diameter D_1?

Solution 3.11-5 Stepped shaft in torsion



$D_2 = 1.5$ in.

$\tau_{allow} = 15{,}000$ psi

$T = 4800$ lb-in.

Full quarter-circular fillet $D_2 = D_1 + 2R$

$$R = \frac{D_2 - D_1}{2} = 0.75 \text{ in.} - \frac{D_1}{2}$$

USE FIG. 3-48 FOR THE STRESS-CONCENTRATION FACTOR

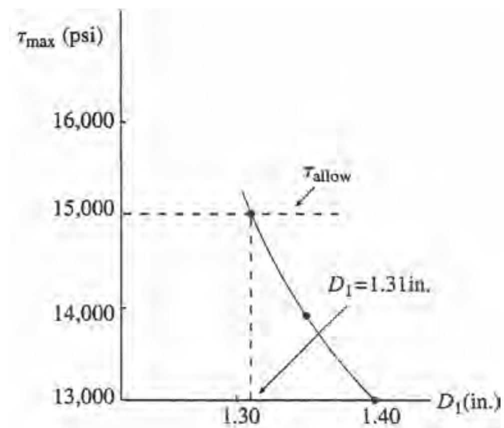
$$\tau_{max} = K\tau_{nom} = K\left(\frac{16T}{\pi D_1^3}\right)$$

$$= \frac{K}{D_1^3}\left[\frac{16(4800 \text{ lb-in.})}{\pi}\right]$$

$$= 24{,}450\frac{K}{D_1^3}$$

Use trial-and-error. Select trial values of D_1

D_1 (in.)	R (in.)	R/D_1	K	τ_{max}(psi)
1.30	0.100	0.077	1.38	15,400
1.35	0.075	0.056	1.41	14,000
1.40	0.050	0.036	1.46	13,000



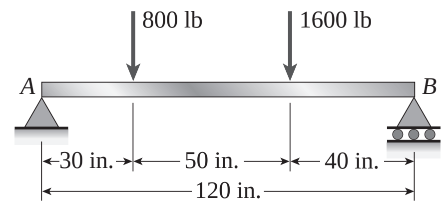
From the graph, minimum $D_1 \approx 1.31$ in. ←

4

Shear Forces and Bending Moments

Shear Forces and Bending Moments

Problem 4.3-1 Calculate the shear force V and bending moment M at a cross section just to the left of the 1600-1b load acting on the simple beam AB shown in the figure.

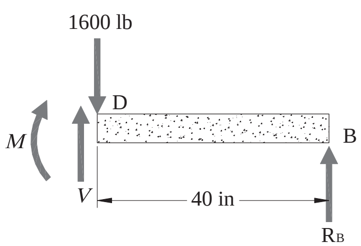


Solution 4.3-1

$$\Sigma M_A = 0: R_B = \frac{3800}{3} = 1267 \text{ lb}$$

$$\Sigma M_B = 0: R_A = \frac{3400}{3} = 1133 \text{ lb}$$

FREE-BODY DIAGRAM OF SEGMENT *DB*



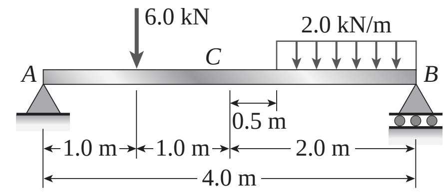
$$\Sigma F_{\text{VERT}} = 0: V = 1600 \text{ lb} - 1267 \text{ lb}$$

$$= 333 \text{ lb} \quad \leftarrow$$

$$\Sigma M_D = 0: M = (1267 \text{ lb})(40 \text{ in.})$$

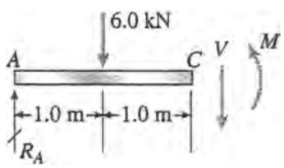
$$= \frac{152000}{3} \text{ lb} \cdot \text{in} = 50667 \text{ lb} \cdot \text{in.} \quad \leftarrow$$

Problem 4.3-2 Determine the shear force V and bending moment M at the midpoint C of the simple beam AB shown in the figure.



Solution 4.3-2

FREE-BODY DIAGRAM OF SEGMENT AC



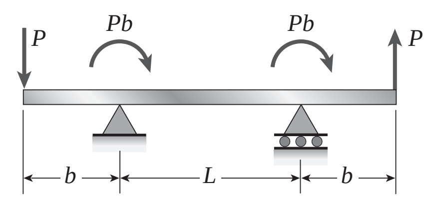
$$\sum M_A = 0: \quad R_B = 3.9375 \text{ kN}$$

$$\sum M_B = 0: \quad R_A = 5.0625 \text{ kN}$$

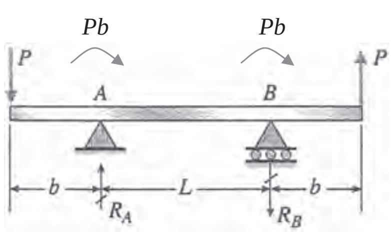
$$\sum F_{\text{VERT}} = 0: \quad V = R_A - 6 = -0.938 \text{ kN} \quad \leftarrow$$

$$\sum M_C = 0: \quad M = R_A \cdot 2 \text{ m} - 6 \text{ kN} \cdot 1 \text{ m} = 4.12 \text{ kN·m} \quad \leftarrow$$

Problem 4.3-3 Determine the shear force V and bending moment M at the midpoint of the beam with overhangs (see figure). Note that one load acts downward and the other upward. Also clockwise moments Pb are applied at each support.

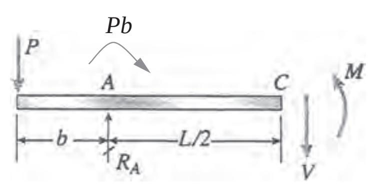


Solution 4.3-3



$$\sum M_B = 0: \quad R_A = \frac{1}{L}(2Pb - (b + L)P - Pb) = P \text{ (upward)}$$

$$\sum M_A = 0: \quad R_B = P \text{ (downward)}$$

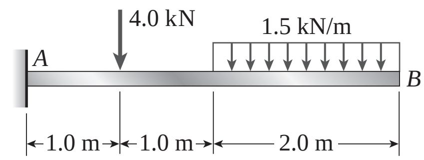


FREE-BODY DIAGRAM (C IS THE MIDPOINT)

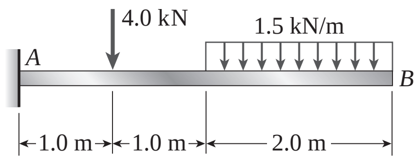
$$\sum F_{\text{VERT}} = 0: \quad V = R_A - P = 0 \quad \leftarrow$$

$$\sum M_C = 0: \quad M = -P\left(b + \frac{L}{2}\right) + R_A\frac{L}{2} + Pb = 0 \quad \leftarrow$$

Problem 4.3-4 Calculate the shear force V and bending moment M at a cross section located 0.5 m from the fixed support of the cantilever beam AB shown in the figure.

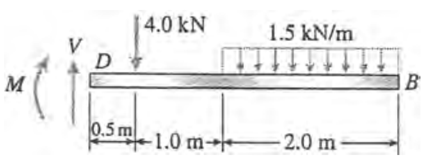


Solution 4.3-4 Cantilever beam



FREE-BODY DIAGRAM OF SEGMENT DB

Point D is 0.5 m from support A.



$\Sigma F_{VERT} = 0$:

$V = 4.0 \text{ kN} + (1.5 \text{ kN/m})(2.0 \text{ m})$

$= 4.0 \text{ kN} + 3.0 \text{ kN} = 7.0 \text{ kN} \quad \leftarrow$

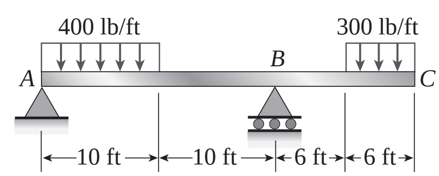
$\Sigma M_D = 0$: $\quad M = -(4.0 \text{ kN})(0.5 \text{ m})$

$- (1.5 \text{ kN/m})(2.0 \text{ m})(2.5 \text{ m})$

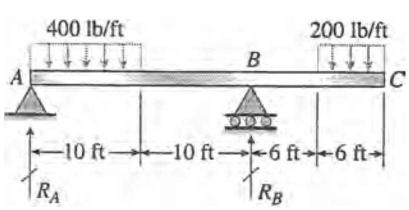
$= -2.0 \text{ kN} \cdot \text{m} - 7.5 \text{ kN} \cdot \text{m}$

$= -9.5 \text{ kN} \cdot \text{m} \quad \leftarrow$

Problem 4.3-5 Determine the shear force V and bending moment M at a cross section located 18 ft from the left-hand end A of the beam with an overhang shown in the figure.



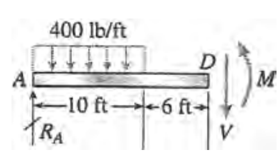
Solution 4.3-5



$\Sigma M_B = 0$: $\quad R_A = 2190 \text{ lb}$

$\Sigma M_A = 0$: $\quad R_B = 3610 \text{ lb}$

FREE-BODY DIAGRAM OF SEGMENT AD



Point D is 18ft from support A.

$\Sigma F_{VERT} = 0$: $\quad V = 2190 \text{ lb} - (400 \text{ lb/ft})(10 \text{ ft})$

$= -1810 \text{ lb} \quad \leftarrow$

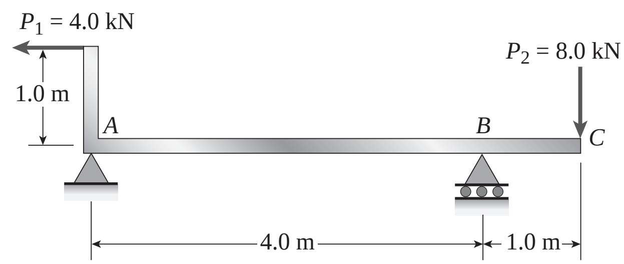
$\Sigma M_c = 0$: $\quad M = (2190 \text{ lb})(18 \text{ ft})$

$- (400 \text{ lb/ft})(10 \text{ ft})(13 \text{ ft})$

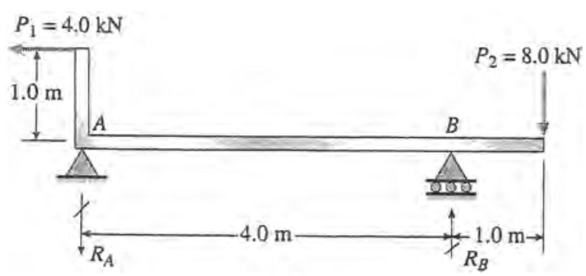
$= -12580 \text{ lb} \cdot \text{ft} \quad \leftarrow$

Problem 4.3-6 The beam *ABC* shown in the figure is simply supported at *A* and *B* and has an overhang from *B* to *C*. The loads consist of a horizontal force $P_1 = 4.0$ kN acting at the end of a vertical arm and a vertical force $P_2 = 8.0$ kN acting at the end of the overhang.

Determine the shear force *V* and bending moment *M* at a cross section located 3.0 m from the left-hand support. (*Note*: Disregard the widths of the beam and vertical arm and use centerline dimensions when making calculations.)



Solution 4.3-6 Beam with vertical arm

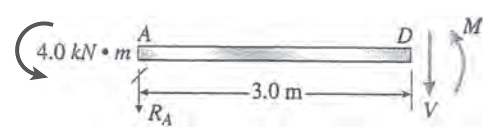


$\sum M_B = 0$: $R_A = 1.0$ kN (downward)

$\sum M_A = 0$: $R_B = 9.0$ kN (upward)

FREE-BODY DIAGRAM OF SEGMENT *AD*

Point *D* is 3.0 m from support *A*.

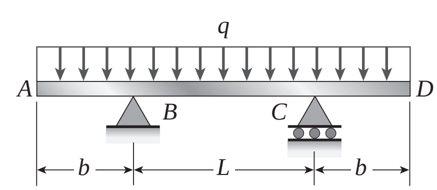


$\sum F_{\text{VERT}} = 0$: $V = -R_A = -1.0 \text{ kN}$ ←

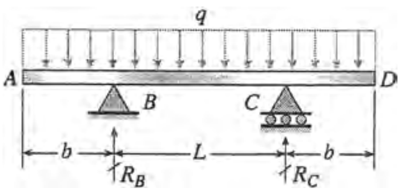
$\sum M_D = 0$: $M = -R_A(3.0 \text{ m}) - 4.0 \text{ kN} \cdot \text{m}$
$= -7.0 \text{ kN} \cdot \text{m}$ ←

Problem 4.3-7 The beam *ABCD* shown in the figure has overhangs at each end and carries a uniform load of intensity *q*.

For what ratio *b/L* will the bending moment at the midpoint of the beam be zero?



Solution 4.3-7 Beam with overhangs

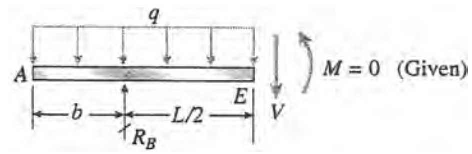


From symmetry and equilibrium of vertical forces:

$$R_B = R_C = q\left(b + \frac{L}{2}\right)$$

FREE-BODY DIAGRAM OF LEFT-HAND HALF OF BEAM:

Point E is at the midpoint of the beam.



$\Sigma M_E = 0$

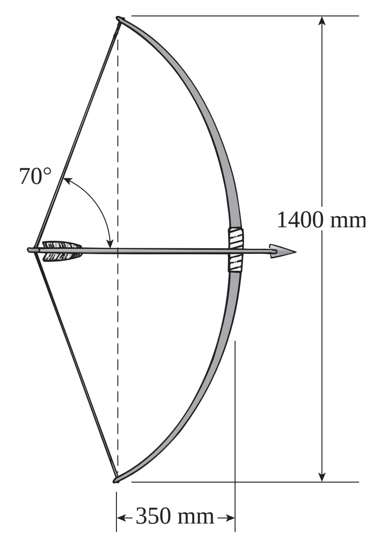
$$-R_B\left(\frac{L}{2}\right) + q\left(\frac{1}{2}\right)\left(b + \frac{L}{2}\right)^2 = 0$$

$$-q\left(b + \frac{L}{2}\right)\left(\frac{L}{2}\right) + q\left(\frac{1}{2}\right)\left(b + \frac{L}{2}\right)^2 = 0$$

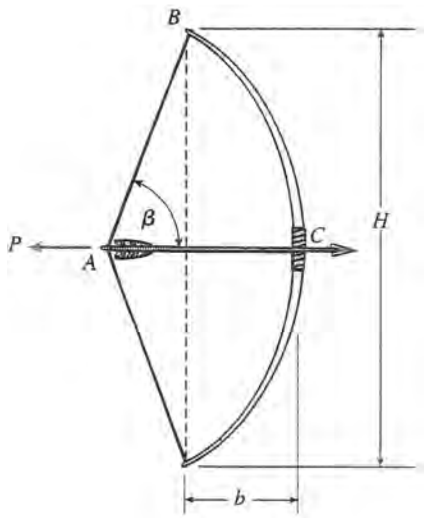
Solve for b/L:

$$\frac{b}{L} = \frac{1}{2} \quad \leftarrow$$

Problem 4.3-8 At full draw, an archer applies a pull of 130 N to the bowstring of the bow shown in the figure. Determine the bending moment at the midpoint of the bow.



Solution 4.3-8 Archer's bow



$P = 130$ N

$\beta = 70°$

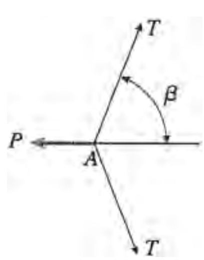
$H = 1400$ mm

$= 1.4$ m

$b = 350$ mm

$= 0.35$ m

Free-body diagram of point A

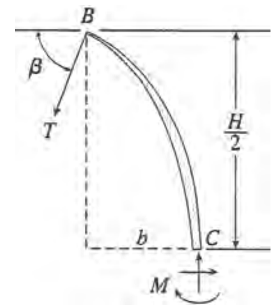


T = tensile force in the bowstring

$\sum F_{\text{HORIZ}} = 0$: $\quad 2T \cos \beta - P = 0$

$$T = \frac{P}{2 \cos \beta}$$

Free-body diagram of segment BC



$\sum M_C = 0$

$$T(\cos \beta)\left(\frac{H}{2}\right) + T(\sin \beta)(b) - M = 0$$

$$M = T\left(\frac{H}{2}\cos \beta + b \sin \beta\right)$$

$$= \frac{P}{2}\left(\frac{H}{2} + b \tan \beta\right)$$

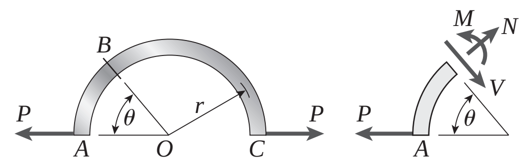
Substitute numerical values:

$$M = \frac{130\text{ N}}{2}\left[\frac{1.4\text{ m}}{2} + (0.35\text{ m})(\tan 70°)\right]$$

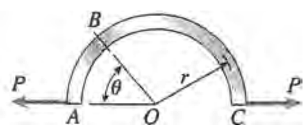
$M = 108$ N · m $\quad\longleftarrow$

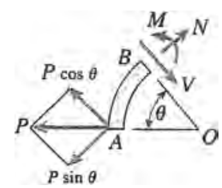
Problem 4.3-9 A curved bar *ABC* is subjected to loads in the form of two equal and opposite forces *P*, as shown in the figure. The axis of the bar forms a semicircle of radius *r*.

Determine the axial force *N*, shear force *V*, and bending moment *M* acting at a cross section defined by the angle θ.



Solution 4.3-9 Curved bar





$\Sigma F_N = 0 \quad \nearrow_{+} \swarrow^{-} \quad N - P\sin\theta = 0$

$N = P\sin\theta \quad \leftarrow$

$\Sigma F_V = 0 \quad \searrow^{+} \nwarrow^{-} \quad V - P\cos\theta = 0$

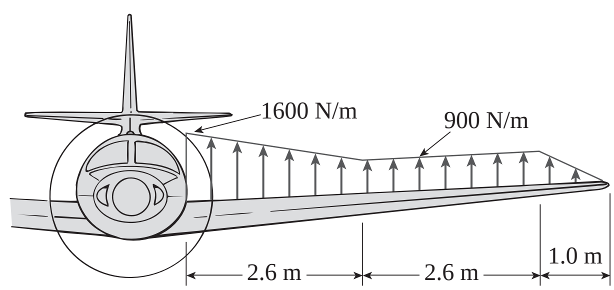
$V = P\cos\theta \quad \leftarrow$

$\Sigma M_O = 0 \quad \circlearrowleft^{+} \circlearrowright^{-} \quad M - Nr = 0$

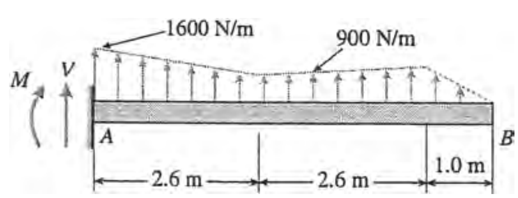
$M = Nr = Pr\sin\theta \quad \leftarrow$

Problem 4.3-10 Under cruising conditions the distributed load acting on the wing of a small airplane has the idealized variation shown in the figure.

Calculate the shear force *V* and bending moment *M* at the inboard end of the wing.



Solution 4.3-10 Airplane wing



SHEAR FORCE

$\Sigma F_{\text{VERT}} = 0 \quad \uparrow_{+} \downarrow^{-}$

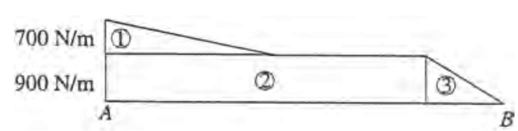
$$V + \frac{1}{2}(700\text{ N/m})(2.6\text{ m}) + (900\text{ N/m})(5.2\text{ m})$$

$$+ \frac{1}{2}(900\text{ N/m})(1.0\text{ m}) = 0$$

$V = -6040\text{ N} = -6.04\text{ kN} \quad \leftarrow$

(Minus means the shear force acts opposite to the direction shown in the figure.)

LOADING (IN THREE PARTS)



BENDING MOMENT

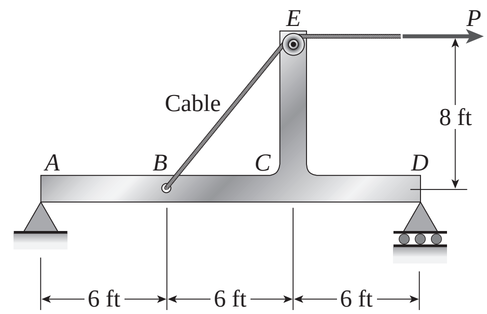
$\sum M_A = 0$ ↺+

$$-M + \frac{1}{2}(700 \text{ N/m})(2.6 \text{ m})\left(\frac{2.6 \text{ m}}{3}\right) + (900 \text{ N/m})(5.2 \text{ m})(2.6 \text{ m}) + \frac{1}{2}(900 \text{ N/m})(1.0 \text{ m})\left(5.2 \text{ m} + \frac{1.0 \text{ m}}{3}\right) = 0$$

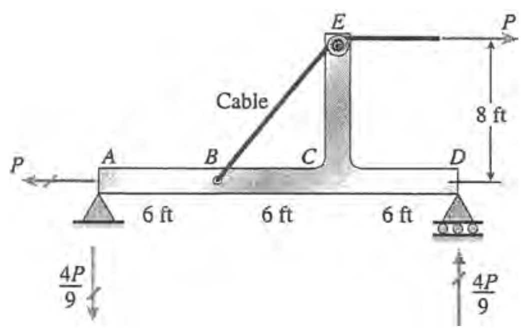
$$M = 788.67 \text{ N}\cdot\text{m} + 12{,}168 \text{ N}\cdot\text{m} + 2490 \text{ N}\cdot\text{m}$$
$$= 15{,}450 \text{ N}\cdot\text{m}$$
$$= 15.45 \text{ kN}\cdot\text{m} \quad \leftarrow$$

Problem 4.3-11 A beam *ABCD* with a vertical arm *CE* is supported as a simple beam at *A* and *D* (see figure). A cable passes over a small pulley that is attached to the arm at *E*. One end of the cable is attached to the beam at point *B*.

What is the force *P* in the cable if the bending moment in the beam just to the left of point *C* is equal numerically to 640 lb-ft? (*Note*: Disregard the widths of the beam and vertical arm and use centerline dimensions when making calculations.)

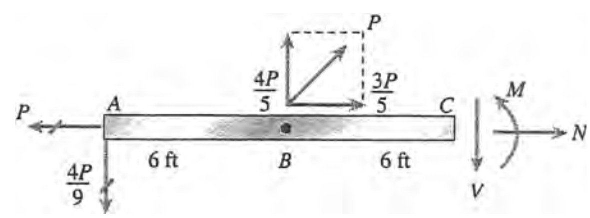


Solution 4.3-11 Beam with a cable



UNITS:
P in lb
M in lb-ft

FREE-BODY DIAGRAM OF SECTION *AC*



$\sum M_C = 0$ ↺+

$$M - \frac{4P}{5}(6 \text{ ft}) + \frac{4P}{9}(12 \text{ ft}) = 0$$

$$M = -\frac{8P}{15} \text{ lb-ft}$$

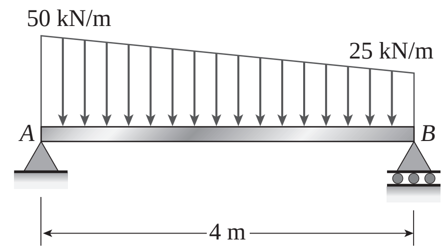
Numerical value of *M* equals 640 lb-ft.

$$\therefore\ 640 \text{ lb-ft} = \frac{8P}{15} \text{ lb-ft}$$

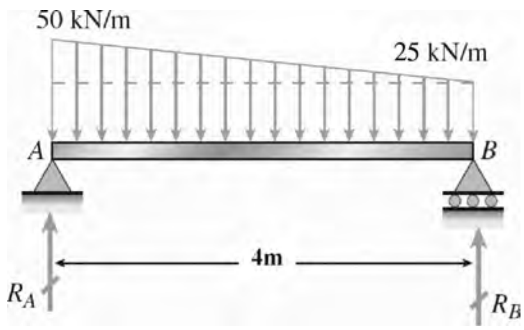
and $P = 1200$ lb $\leftarrow$

Problem 4.3-12 A simply supported beam AB supports a trapezoidally distributed load (see figure). The intensity of the load varies linearly from 50 kN/m at support A to 25 kN/m at support B.

Calculate the shear force V and bending moment M at the midpoint of the beam.



Solution 4.3-12



$$\Sigma M_B = 0: \quad -R_A(4\text{m}) + (25 \text{ kN/m})(4\text{m})(2\text{m}) + (25 \text{ kN/m})(4 \text{ m})\left(\frac{1}{2}\right)\left(4 \text{ m}\frac{2}{3}\right) = 0$$

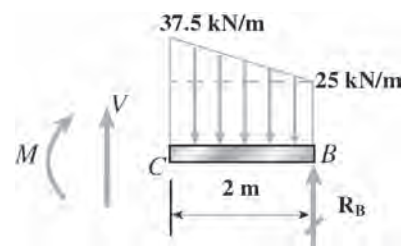
$R_A = 83.33$ kN

$$\Sigma F_{\text{VERT}} = 0: \quad R_A + R_B - \frac{1}{2}(50 \text{ kN/m} + 25 \text{ kN/m})(4 \text{ m}) = 0$$

$R_B = 66.67$ kN

Free-body diagram of section CB

Point C is at the midpoint of the beam.



$$\Sigma F_{\text{VERT}} = 0: \quad V - (25 \text{ kN/m})(2 \text{ m}) - (12.5 \text{ kN/m})(2 \text{ m})\frac{1}{2} + R_B = 0$$

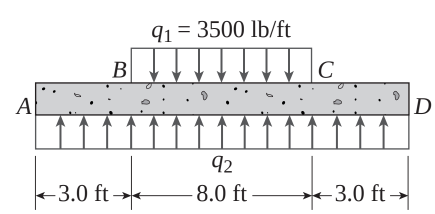
$V = -4.17$ kN $\longleftarrow$

$$\Sigma M_C = 0: \quad -M - (25 \text{ kN/m})(2 \text{ m})(1 \text{ m}) - (12.5 \text{ kN/m})(2 \text{ m})\frac{1}{2}\left(2 \text{ m}\frac{1}{3}\right) + R_B(2 \text{ m}) = 0$$

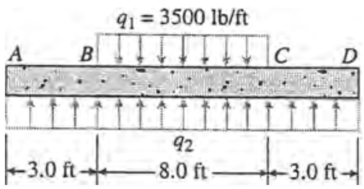
$M = 75 \text{ kN} \cdot \text{m}$ $\longleftarrow$

Problem 4.3-13 Beam $ABCD$ represents a reinforced-concrete foundation beam that supports a uniform load of intensity $q_1 = 3500$ lb/ft (see figure). Assume that the soil pressure on the underside of the beam is uniformly distributed with intensity q_2.

(a) Find the shear force V_B and bending moment M_B at point B.
(b) Find the shear force V_m and bending moment M_m at the midpoint of the beam.



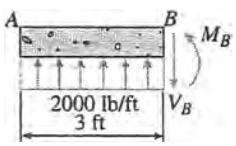
Solution 4.3-13 Foundation beam



$\sum F_{\text{VERT}} = 0$: $q_2(14\text{ ft}) = q_1(8\text{ ft})$

$$\therefore\ q_2 = \frac{8}{14} q_1 = 2000\text{ lb/ft}$$

(a) V AND M AT POINT B

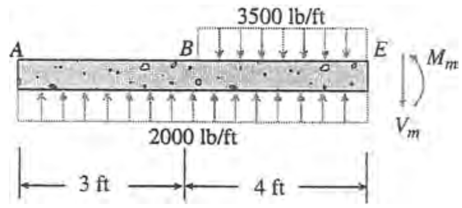


$\sum F_{\text{VERT}} = 0$:

$V_B = 6000\text{ lb}$ ←

$\sum M_B = 0$: $M_B = 9000\text{ lb-ft}$ ←

(b) V AND M AT MIDPOINT E



$\sum F_{\text{VERT}} = 0$: $V_m = (2000\text{ lb/ft})(7\text{ ft}) - (3500\text{ lb/ft})(4\text{ ft})$

$V_m = 0$ ←

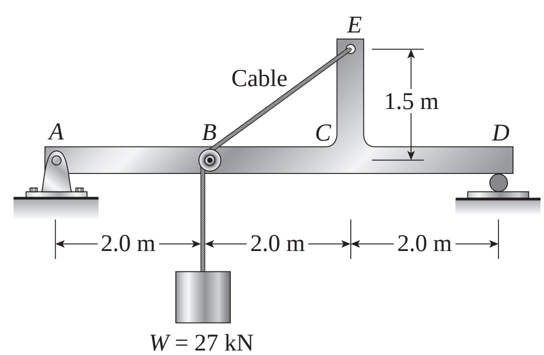
$\sum M_E = 0$:

$M_m = (2000\text{ lb/ft})(7\text{ ft})(3.5\text{ ft}) - (3500\text{ lb/ft})(4\text{ ft})(2\text{ ft})$

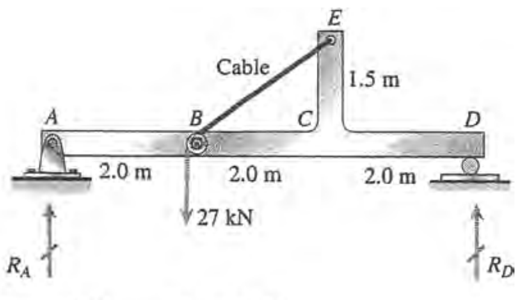
$M_m = 21{,}000\text{ lb-ft}$ ←

Problem 4.3-14 The simply-supported beam *ABCD* is loaded by a weight W = 27 kN through the arrangement shown in the figure. The cable passes over a small frictionless pulley at *B* and is attached at *E* to the end of the vertical arm.

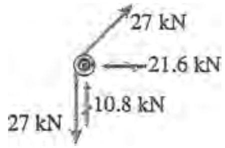
Calculate the axial force *N*, shear force *V*, and bending moment M at section C, which is just to the left of the vertical arm. (*Note*: Disregard the widths of the beam and vertical arm and use centerline dimensions when making calculations.)



Solution 4.3-14 Beam with cable and weight

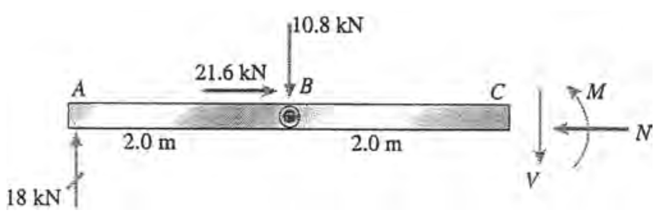


FREE-BODY DIAGRAM OF PULLEY AT B



$R_A = 18$ kN $\quad R_D = 9$ kN

FREE-BODY DIAGRAM OF SEGMENT ABC OF BEAM



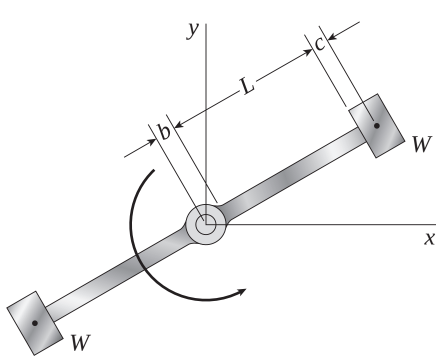
$\Sigma F_{\text{HORIZ}} = 0$: $\quad N = 21.6$ kN (compression) $\quad\leftarrow$

$\Sigma F_{\text{VERT}} = 0$: $\quad V = 7.2$ kN $\quad\leftarrow$

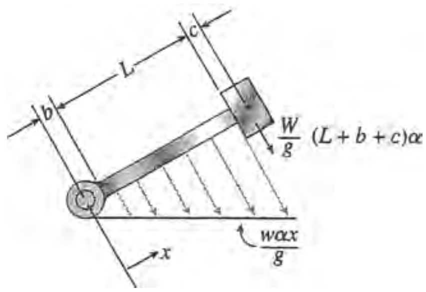
$\Sigma M_C = 0$: $\quad M = 50.4$ kN $\cdot$ m $\quad\leftarrow$

Problem 4.3-15 The centrifuge shown in the figure rotates in a horizontal plane (the xy plane) on a smooth surface about the z axis (which is vertical) with an angular acceleration α. Each of the two arms has weight w per unit length and supports a weight $W = 2.0\ wL$ at its end.

Derive formulas for the maximum shear force and maximum bending moment in the arms, assuming $b = L/9$ and $c = L/10$.



Solution 4.3-15 Rotating centrifuge



Tangential acceleration $= r\alpha$

Inertial force $Mr\,\alpha = \frac{W}{g} r\,\alpha$

Maximum V and M occur at $x = b$.

$$V_{max} = \frac{W}{g}(L + b + c)\alpha + \int_b^{L+b} \frac{w\alpha}{g} x\,dx$$

$$= \frac{W\alpha}{g}(L + b + c) + \frac{wL\alpha}{2g}(L + 2b) \quad \leftarrow$$

$$M_{max} = \frac{W\alpha}{g}(L + b + c)(L + c) + \int_b^{L+b} \frac{w\alpha}{g} x(x - b)dx$$

$$= \frac{W\alpha}{g}(L + b + c)(L + c) + \frac{wL^2\alpha}{6g}(2L + 3b) \quad \leftarrow$$

Substitute numerical data:

$W = 2.0\,wL \quad b = \frac{L}{9} \quad c = \frac{L}{10}$

$$V_{max} = \frac{91wL^2\alpha}{30g} \quad \leftarrow$$

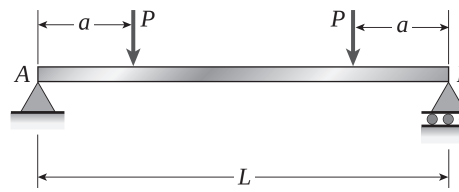
$$M_{max} = \frac{229wL^3\alpha}{75g} \quad \leftarrow$$

Shear-Force and Bending-Moment Diagrams

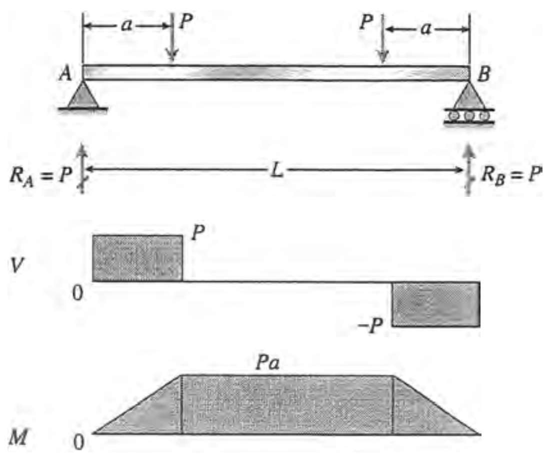
When solving the problems for Section 4.5, draw the shear-force and bending-moment diagrams approximately to scale and label all critical ordinates, including the maximum and minimum values.

Probs. 4.5-1 through 4.5-10 are symbolic problems and Probs. 4.5-11 through 4.5-24 are numerical problems. The remaining problems (4.5-25 through 4.5-40) involve specialized topics, such as optimization, beams with hinges, and moving loads.

Problem 4.5-1 Draw the shear-force and bending-moment diagrams for a simple beam AB supporting two equal concentrated loads P (see figure).

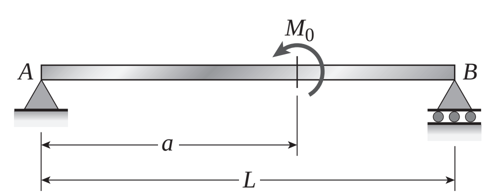


Solution 4.5-1 Simple beam

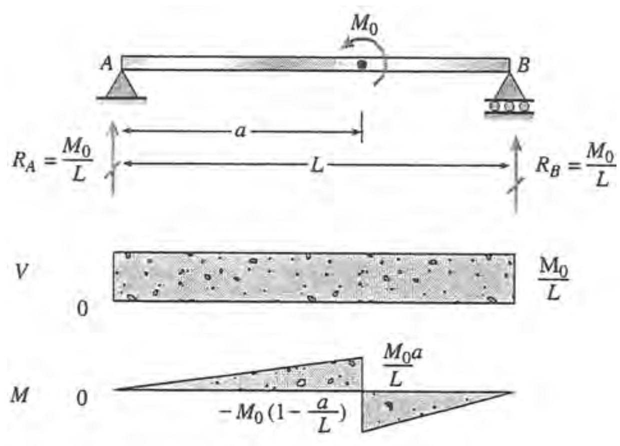


Problem 4.5-2 A simple beam AB is subjected to a counterclockwise couple of moment M_0 acting at distance a from the left-hand support (see figure).

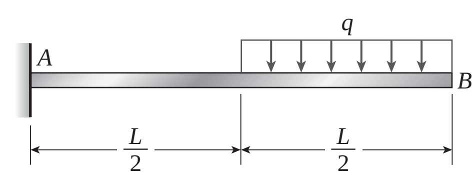
Draw the shear-force and bending-moment diagrams for this beam.



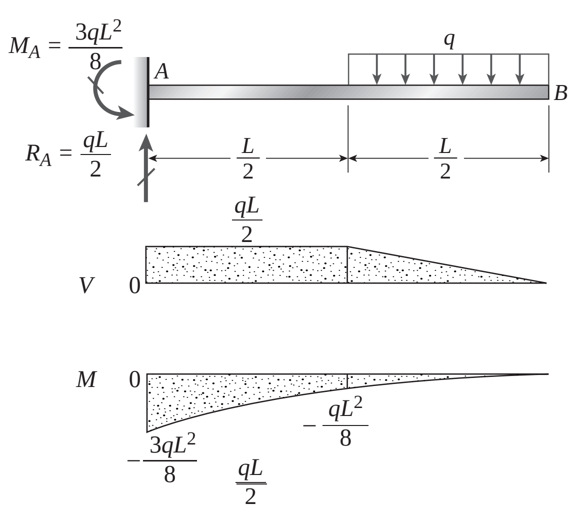
Solution 4.5-2 Simple beam



Problem 4.5-3 Draw the shear-force and bending-moment diagrams for a cantilever beam AB carrying a uniform load of intensity q over one-half of its length (see figure).

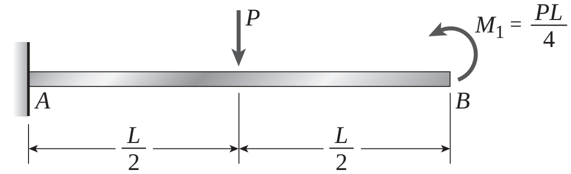


Solution 4.5-3 Cantilever beam

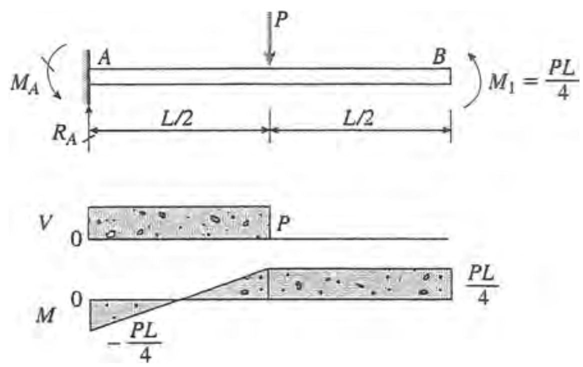


Problem 4.5-4 The cantilever beam AB shown in the figure is subjected to a concentrated load P at the midpoint and a counterclockwise couple of moment $M_1 = PL/4$ at the free end.

Draw the shear-force and bending-moment diagrams for this beam.



Solution 4.5-4 Cantilever beam

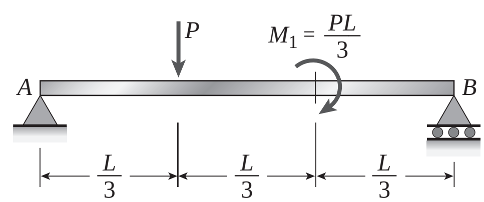


$$R_A = P$$

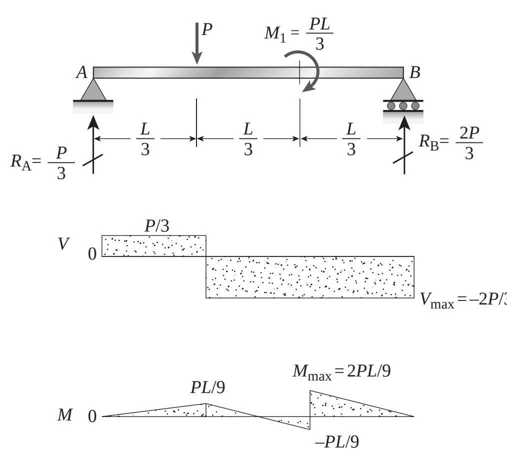
$$M_A = \frac{PL}{4}$$

Problem 4.5-5 The simple beam AB shown in the figure is subjected to a concentrated load P and a clockwise couple $M_1 = PL/3$ acting at the third points.

Draw the shear-force and bending-moment diagrams for this beam.

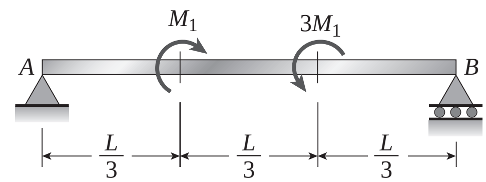


Solution 4.5-5

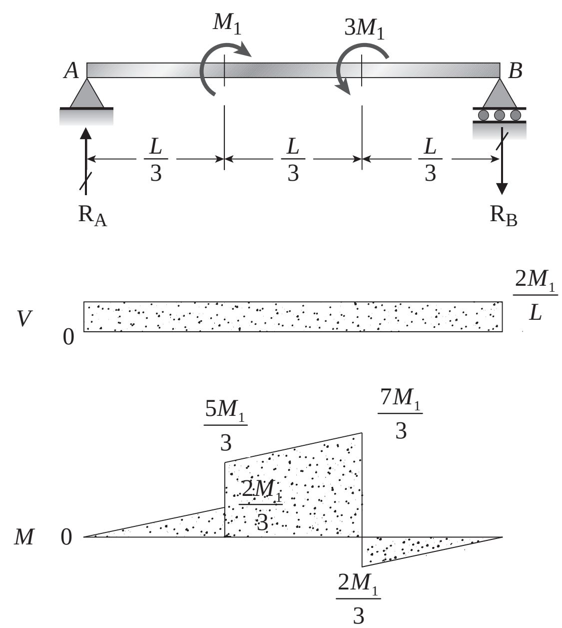


Problem 4.5-6 A simple beam AB subjected to couples M_1 and $3M_1$ acting at the third points is shown in the figure.

Draw the shear-force and bending-moment diagrams for this beam.

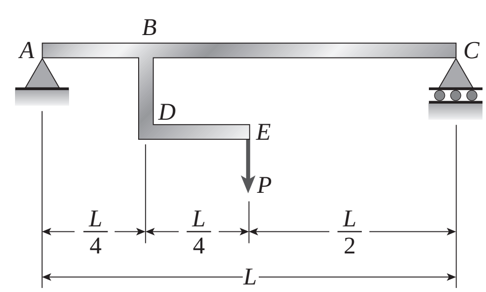


Solution 4.5-6

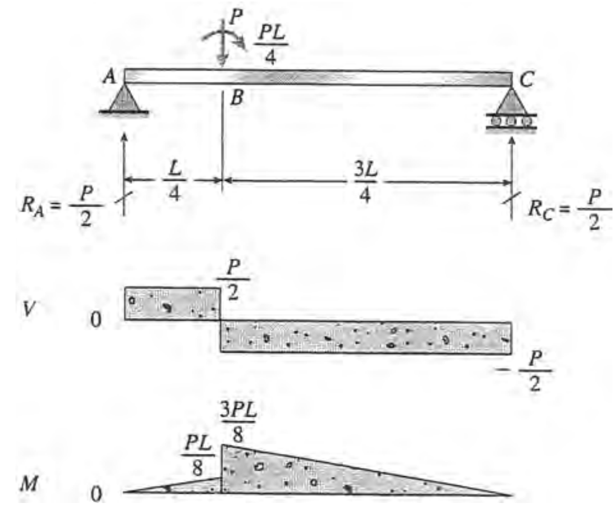


Problem 4.5-7 A simply supported beam ABC is loaded by a vertical load P acting at the end of a bracket BDE (see figure).

Draw the shear-force and bending-moment diagrams for beam ABC.

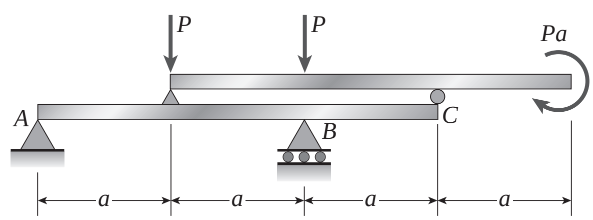


Solution 4.5-7 Beam with bracket

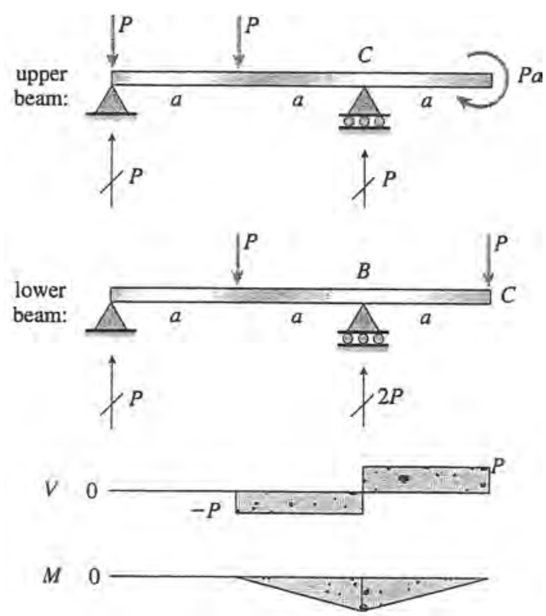


Problem 4.5-8 A beam *ABC* is simply supported at *A* and *B* and has an overhang *BC* (see figure). The beam is loaded by two forces *P* and a clockwise couple of moment *Pa* that act through the arrangement shown.

Draw the shear-force and bending-moment diagrams for beam *ABC*.

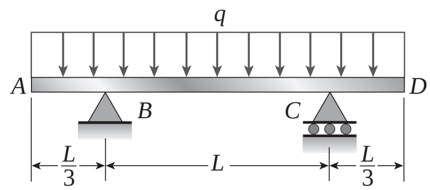


Solution 4.5-8 Beam with overhang

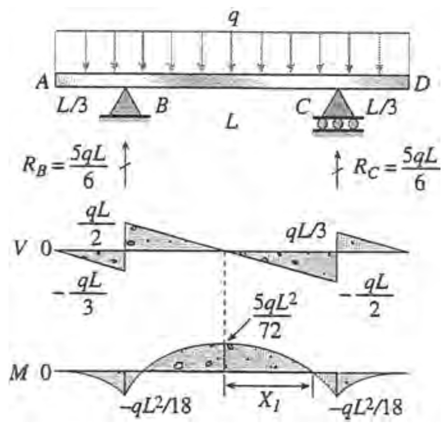


Problem 4.5-9 Beam *ABCD* is simply supported at *B* and *C* and has overhangs at each end (see figure). The span length is L and each overhang has length $L/3$. A uniform load of intensity q acts along the entire length of the beam.

Draw the shear-force and bending-moment diagrams for this beam.

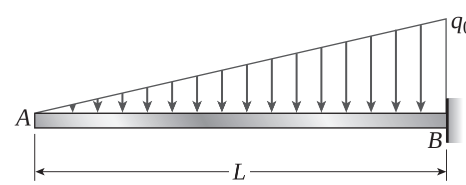


Solution 4.5-9 Beam with overhangs

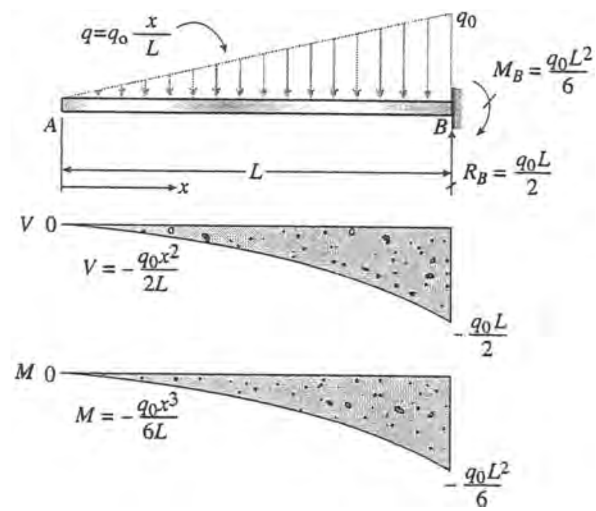


$$x_1 = L\frac{\sqrt{5}}{6} = 0.3727L$$

Problem 4.5-10 Draw the shear-force and bending-moment diagrams for a cantilever beam *AB* supporting a linearly varying load of maximum intensity q_0 (see figure).

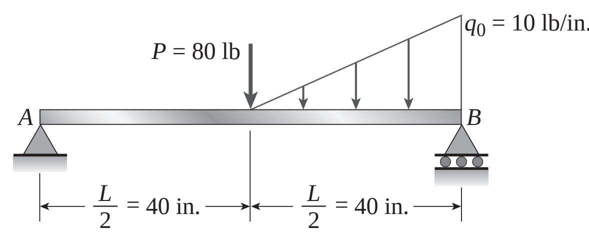


Solution 4.5-10 Cantilever beam



Problem 4.5-11 The simple beam AB supports a triangular load of maximum intensity $q_0 = 10$ lb/in. acting over one-half of the span and a concentrated load $P = 80$ lb acting at midspan (see figure).

Draw the shear-force and bending-moment diagrams for this beam.



Solution 4.5-11 Simple beam

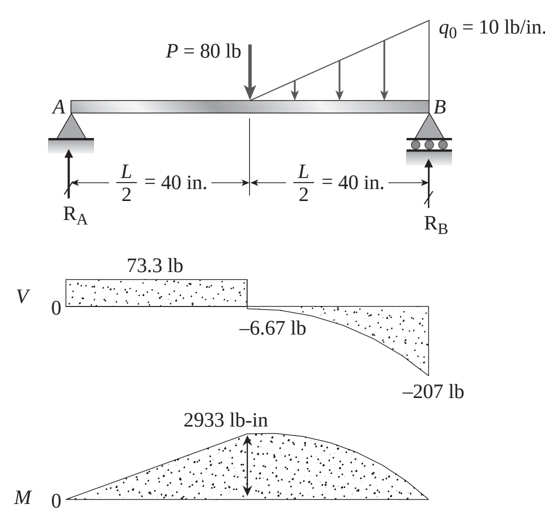
$$\Sigma M_A = 0: R_B\,(80 \text{ in.}) - (80 \text{ lb})(40 \text{ in.})$$

$$- \left(10 \text{ lb/in.} \frac{1}{2}\right)(40 \text{ in.})\left(40 + 40\frac{2}{3} \text{ in.}\right) = 0$$

$$R_B = 206.7 \text{ lb}$$

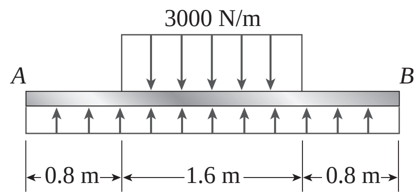
$$\Sigma F_{VERT} = 0: R_A + R_B - 80 \text{ lb} - \left(10 \text{ lb/in.} \frac{1}{2}\right)(40 \text{ in.}) = 0$$

$$R_A = 73.3 \text{ lb}$$

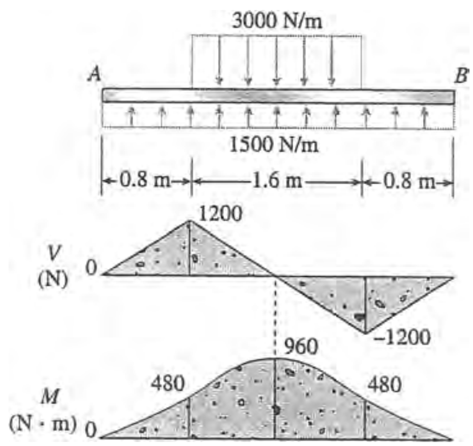


Problem 4.5-12 The beam *AB* shown in the figure supports a uniform load of intensity 3000 N/m acting over half the length of the beam. The beam rests on a foundation that produces a uniformly distributed load over the entire length.

Draw the shear-force and bending-moment diagrams for this beam.

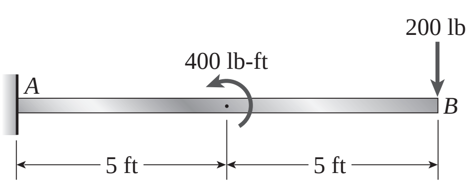


Solution 4.5-12 Beam with distributed loads

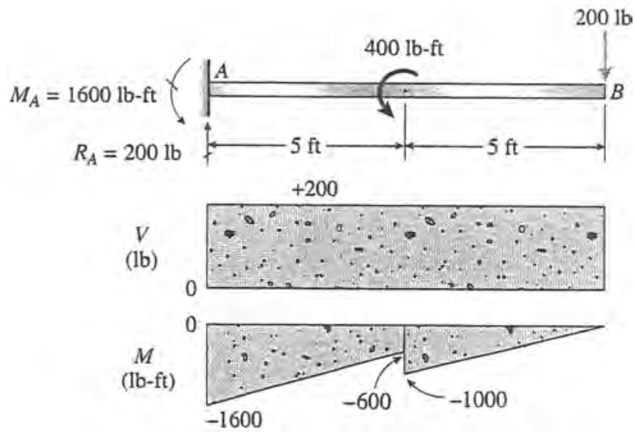


Problem 4.5-13 A cantilever beam *AB* supports a couple and a concentrated load, as shown in the figure.

Draw the shear-force and bending-moment diagrams for this beam.

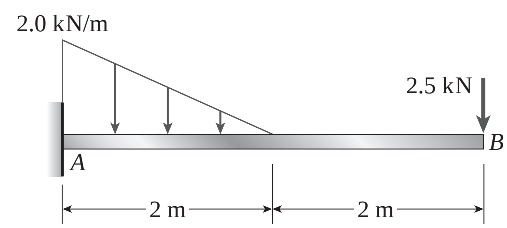


Solution 4.5-13 Cantilever beam

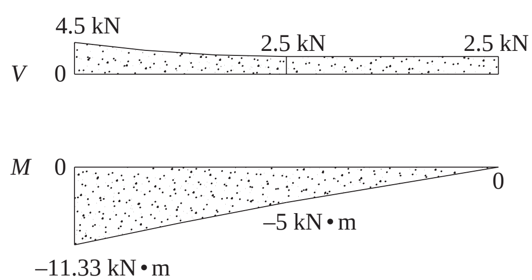


Problem 4.5-14 The cantilever beam AB shown in the figure is subjected to a triangular load acting throughout one-half of its length and a concentrated load acting at the free end.

Draw the shear-force and bending-moment diagrams for this beam.

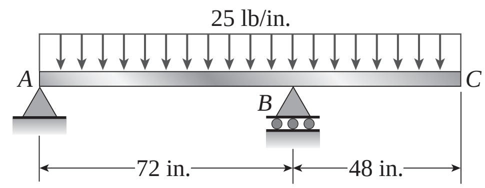


Solution 4.5-14

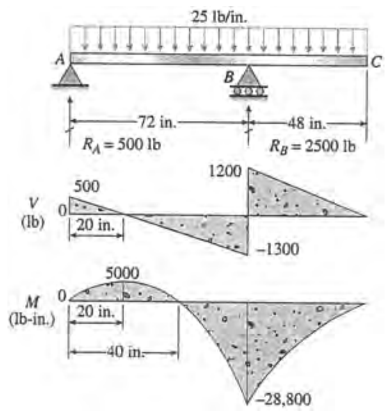


Problem 4.5-15 The uniformly loaded beam *ABC* has simple supports at *A* and *B* and an overhang *BC* (see figure).

Draw the shear-force and bending-moment diagrams for this beam.

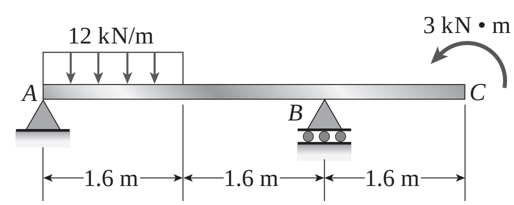


Solution 4.5-15 Beam with an overhang

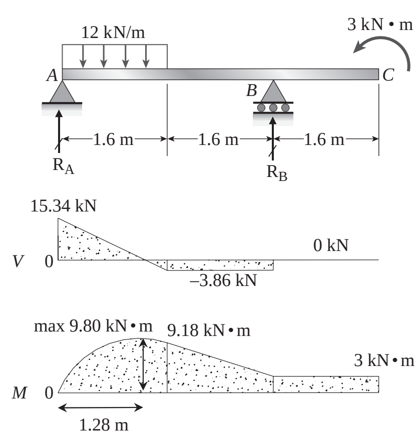


Problem 4.5-16 A beam *ABC* with an overhang at one end supports a uniform load of intensity 12 kN/m and a concentrated moment of magnitude 3 kN · m at *C* (see figure).

Draw the shear-force and bending-moment diagrams for this beam.

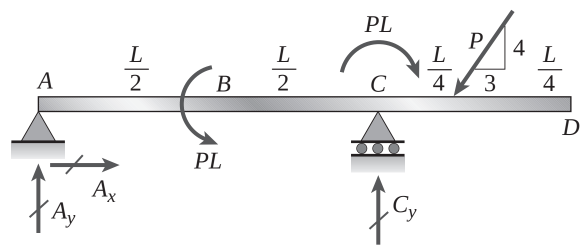


Solution 4.5-16 Beam with an overhang

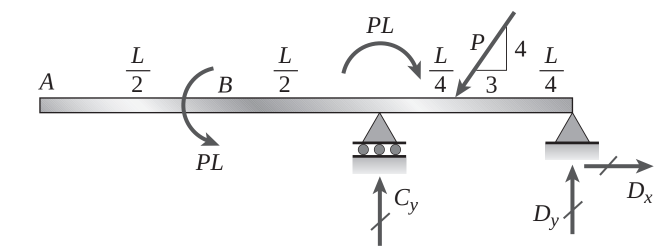


Problem 4.5-17 Consider the two beams below; they are loaded the same but have different support conditions. Which beam has the larger maximum moment?

First, find support reactions, then plot axial force (N), shear (V) and moment (M) diagrams for both beams. *Label* all critical N, V & M values and also the *distance* to points where N, V &/or M is zero.



(a)



(b)

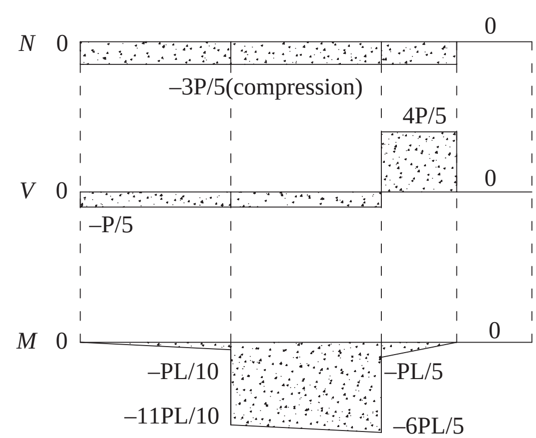
Solution 4.5-17

BEAM (a):

$$\sum M_A = 0: C_y = \frac{1}{L}\left(\frac{4}{5}P\frac{5}{4}L\right) = P \text{ (upward)}$$

$$\sum F_V = 0: \ A_y = \frac{4}{5}P - C_y = -\frac{P}{5} \text{ (downward)}$$

$$\sum F_H = 0: \ A_x = \frac{3}{5}P \text{ (right)}$$



BEAM (b):

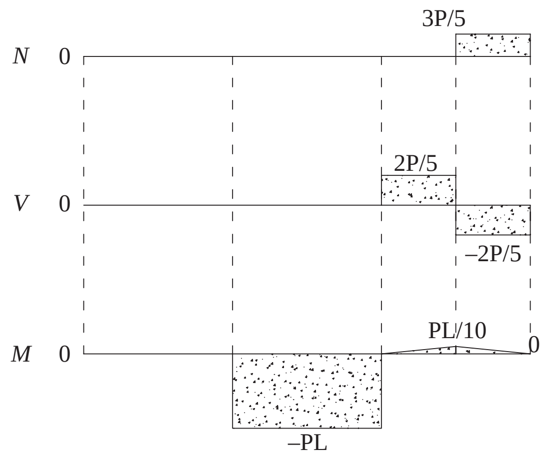
$$\sum M_D = 0\colon C_y = \frac{2}{L}\left(\frac{4}{5}P\,\frac{1}{4}L\right) = \frac{2}{5}P \text{ (upward)}$$

$$\sum F_V = 0\colon D_y = \frac{4}{5}P - C_y = \frac{2}{5}P \text{ (upward)}$$

$$\sum F_H = 0\colon D_x = \frac{3}{5}P \text{ (right)}$$

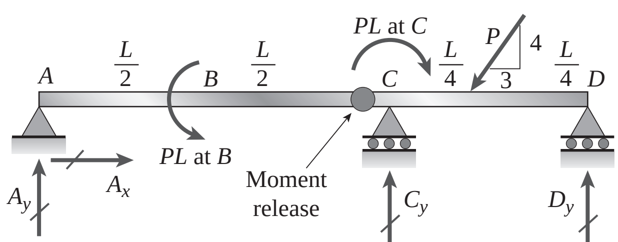
$\therefore$ The first case has the larger maximum moment

$$\left(\frac{6}{5}PL\right) \quad \leftarrow$$

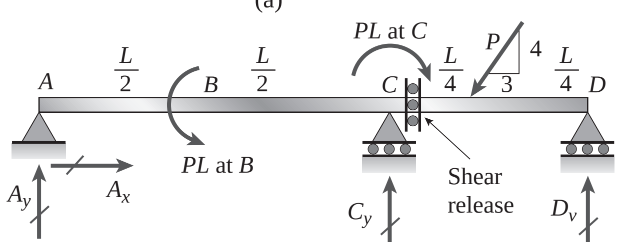


Problem 4.5-18 The three beams below are loaded the same and have the same support conditions. However, one has a *moment release* just to the left of *C*, the second has a *shear release* just to the right of *C*, and the third has an axial release just to the left of *C*. Which beam has the largest maximum moment?

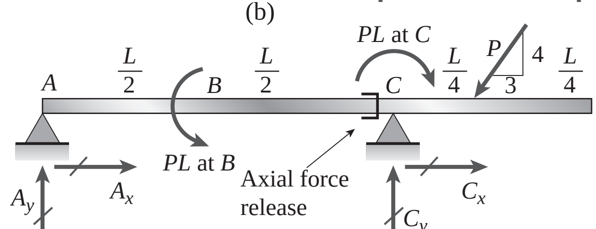
First, find support reactions, then plot axial force (N), shear (V) and moment (M) diagrams for all three beams. *Label* all critical N, V & M values and also the *distance* to points where N, V &/or M is zero.



(a)



(b)



(c)

Solution 4.5-18

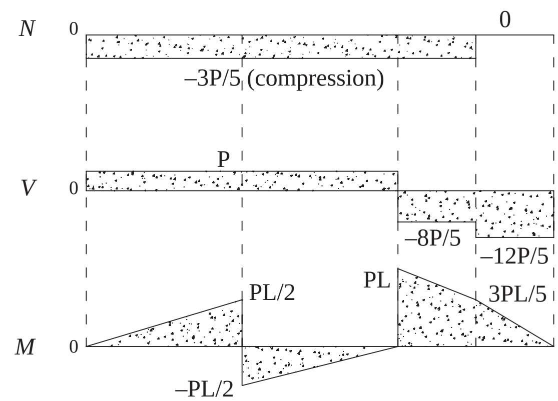
BEAM (a): MOMENT RELEASE

$A_y = P$ (upward)

$C_y = -\dfrac{13}{5}P$ (downward)

$D_y = \dfrac{12}{5}P$ (upward)

$A_x = \dfrac{3}{5}P$ (right)



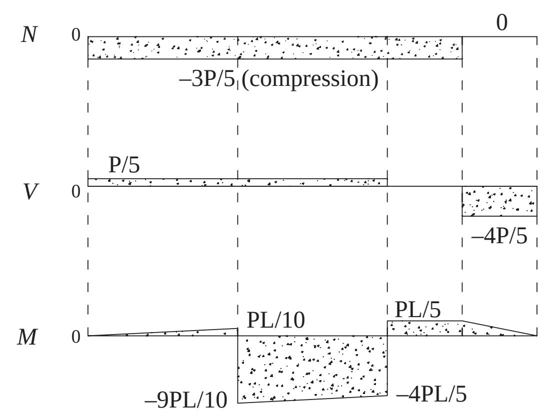
BEAM (b): SHEAR RELEASE

$A_y = \dfrac{1}{5}P$ (upward)

$C_y = -\dfrac{1}{5}P$ (downward)

$D_y = \dfrac{4}{5}P$ (upward)

$A_x = \dfrac{3}{5}P$ (right)



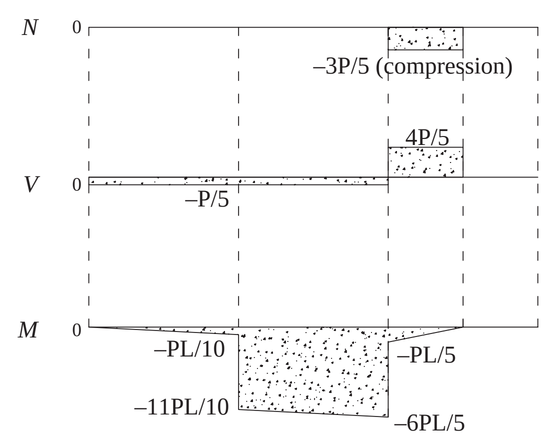
BEAM (c): AXIAL RELEASE

$A_y = -\dfrac{1}{5}P$ (downward)

$C_y = P$ (upward)

$A_x = 0$

$C_x = \dfrac{3}{5}P$ (right)

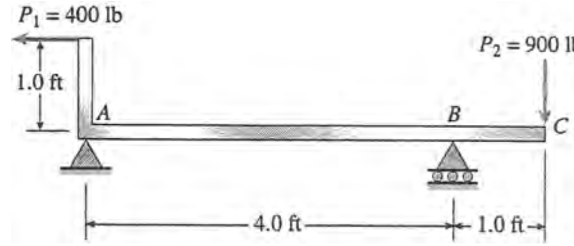


$\therefore$ The third case has the largest maximum moment

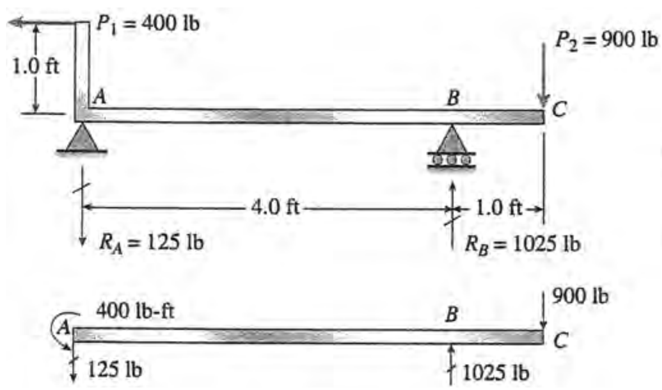
$\left(\dfrac{6}{5}PL\right) \quad \leftarrow$

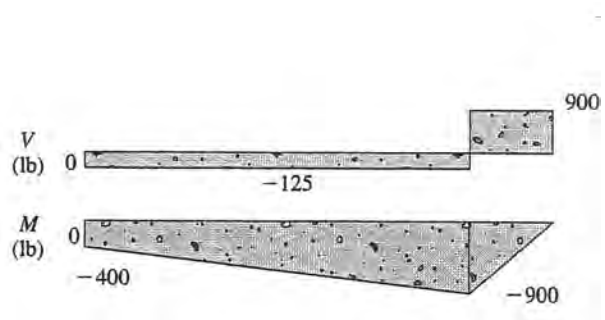
Problem 4.5-19 A beam *ABCD* shown in the figure is simply supported at *A* and *B* and has an overhang from *B* to *C*. The loads consist of a horizonatal force $P_1 = 400$ lb acting at the end of the vertical arm and a vertical force $P_2 = 900$ lb acting at the end of the overhang.

Draw the shear-force and bending-moment diagrams for this beam. (*Note:* Disregard the widths of the beam and vertical arm and use centerline dimensions when making calculations.)



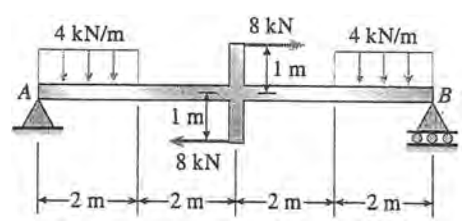
Solution 4.5-19 Beam with vertical arm



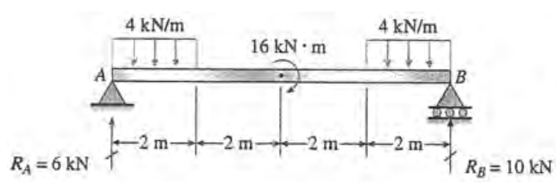


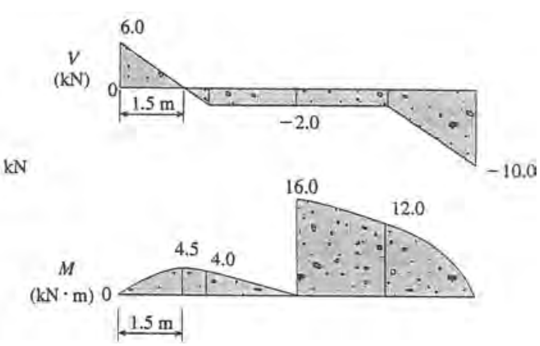
Problem 4.5-20 A simple beam *AB* is loaded by two segments of uniform load and two horizontal forces acting at the ends of a vertical arm (see figure).

Draw the shear-force and bending-moment diagrams for this beam.



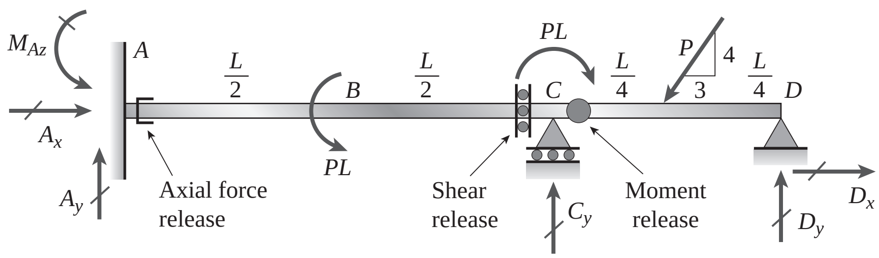
Solution 4.5-20 Simple beam



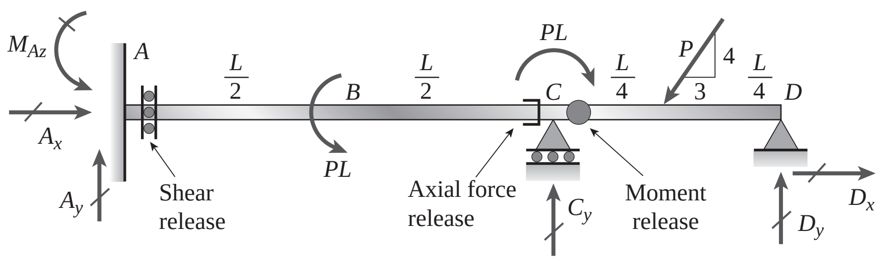


Problem 4.5-21 The two beams below are loaded the same and have the same support conditions. However, the location of internal *axial*, *shear* and *moment releases* is different for each beam (see figures). Which beam has the larger maximum moment?

First, find support reactions, then plot axial force (N), shear (V) and moment (M) diagrams for both beams. *Label* all critical N, V & M values and also the *distance* to points where N, V &/or M is zero.



(a)



(b)

Solution 4.5-21

Support reactions for both beams:

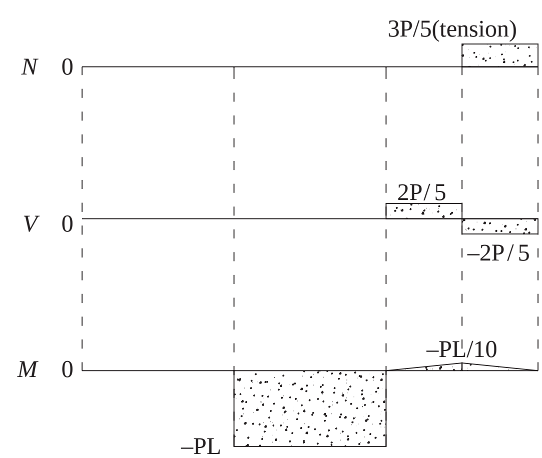
$M_{Az} = 0, A_x = 0, A_y = 0$

$C_y = \frac{2}{5} P$ (upward), $D_y = \frac{2}{5} P$ (upward)

$D_x = \frac{3}{5} P$ (rightward)

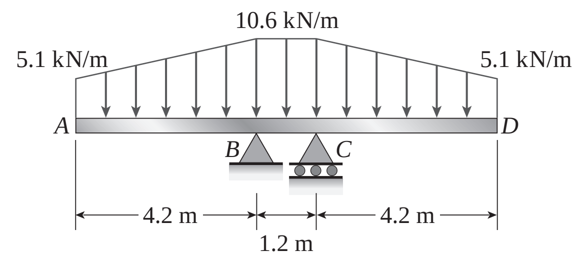
$\therefore$ These two cases have the same maximum moment (PL) $\leftarrow$

(Both beams have the same N, V and M diagrams)

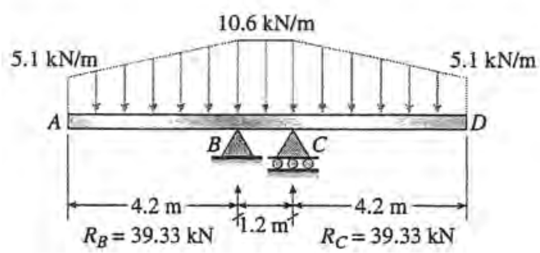


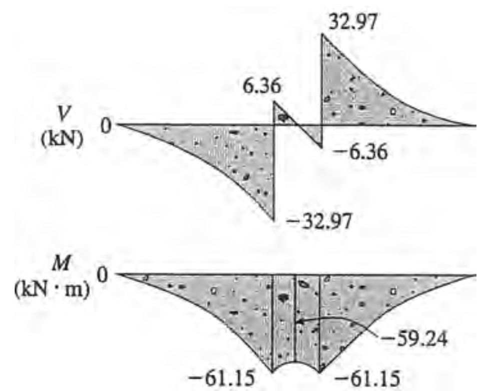
Problem 4.5-22 The beam *ABCD* shown in the figure has overhangs that extend in both directions for a distance of 4.2 m from the supports at *B* and *C*, whiich are 1.2 m apart.

Draw the shear-force and bending-moment diagrams for this overhanging beam.



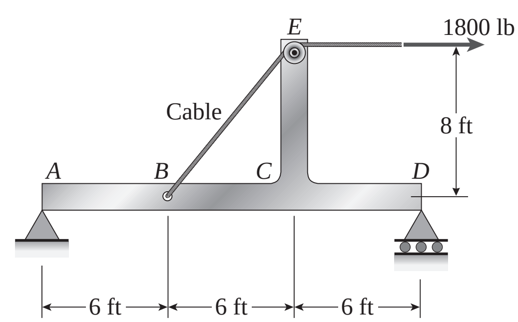
Solution 4.5-22 Beam with overhangs



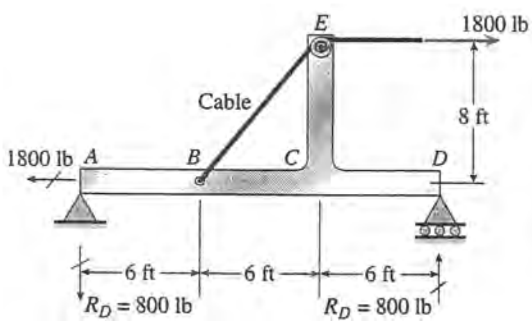


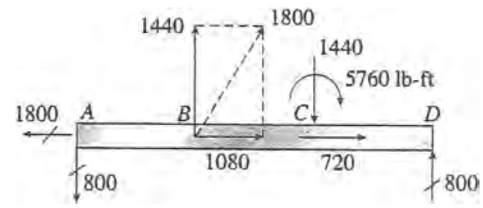
Problem 4.5-23 A beam *ABCD* with a vertical arm *CE* is supported as a simple beam at *A* and *B* (see figure). A cable passes over a small pulley that is attached to the arm at *E*. One end of the cable is attached to the beam at point *B*. The tensile force in the cable is 1800 lb.

Draw the shear-force and bending-moment diagrams for beam *ABCD*. (*Note:* Disregard the widths of the beam and vertical arm and use center-line dimensions when making calculations.)

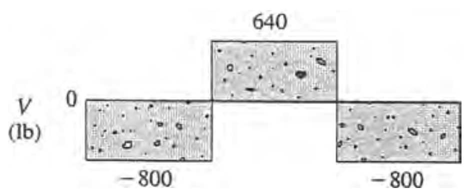


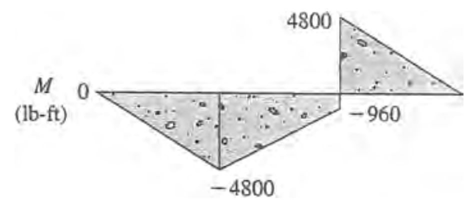
Solution 4.5-23 Beam with a cable



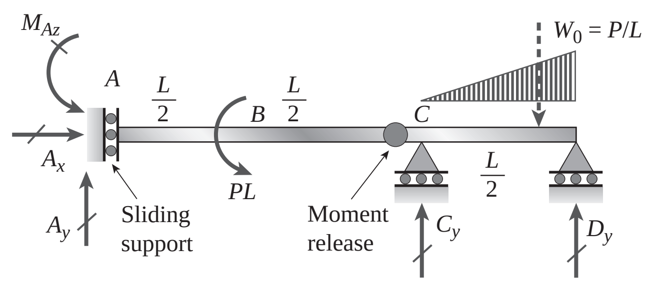


Note: All forces have units of pounds.





Problem 4.5-24 Beams *ABC* and *CD* are supported at *A*, *C* and *D*, and are joined by a hinge (or *moment release*) just to the left of *C* and a *shear release* just to the right of *C*. The support at *A* is a sliding support (hence reaction $A_y = 0$ for the loading shown below). Find all support reactions then plot shear (V) and moment (M) diagrams. *Label* all critical V & M values and also the *distance* to points where either V &/or M is zero.

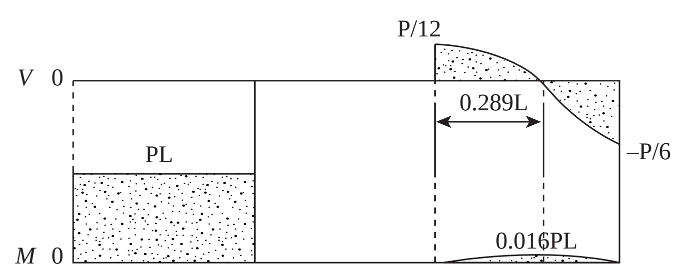


Solution 4.5-24

$M_{Az} = -PL$ (clockwise), $A_x = 0$, $A_y = 0$ $\longleftarrow$

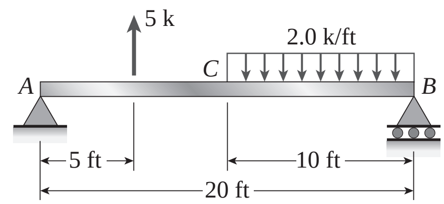
$C_y = \dfrac{1}{12} P$ (upward), $D_y = \dfrac{1}{6} P$ (upward) $\longleftarrow$

$V_{MAX} = \dfrac{P}{6}$ $M_{MAX} = PL$ $\longleftarrow$

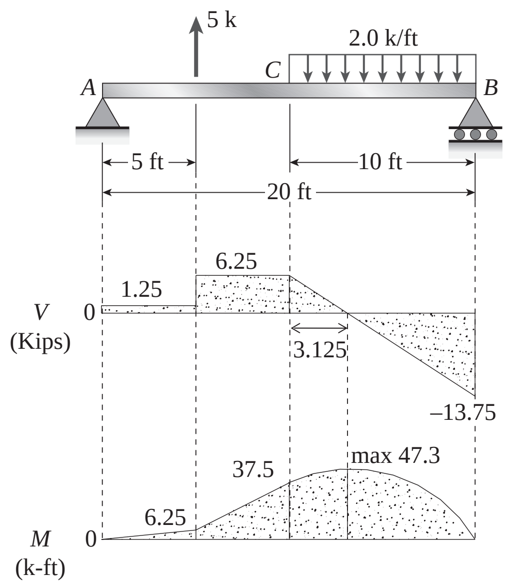


Problem 4.5-25 The simple beam AB shown in the figure supports a concentrated load and a segment of uniform load.

Draw the shear-force and bending-moment diagrams for this beam.

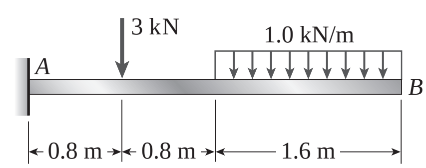


Solution 4.5-25 Simple beam

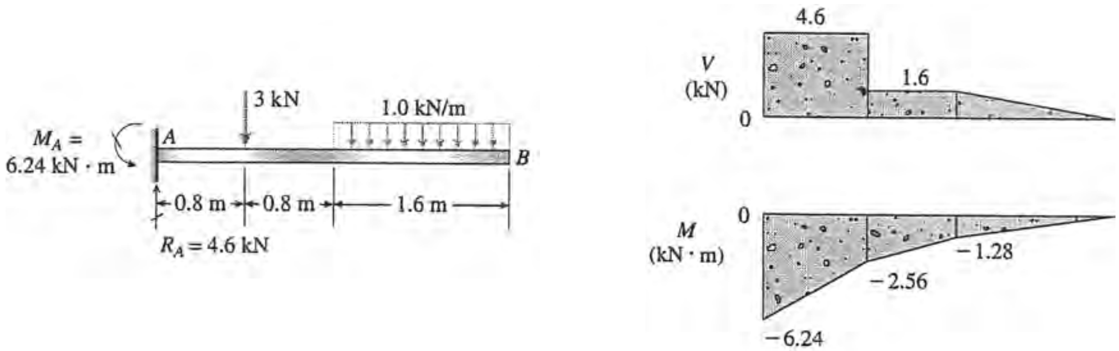


Problem 4.5-26 The cantilever beam shown in the figure supports a concentrated load and a segment of uniform load.

Draw the shear-force and bending-moment diagrams for this cantilever beam.

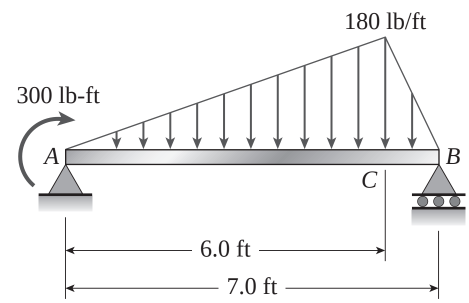


Solution 4.5-26 Cantilever beam

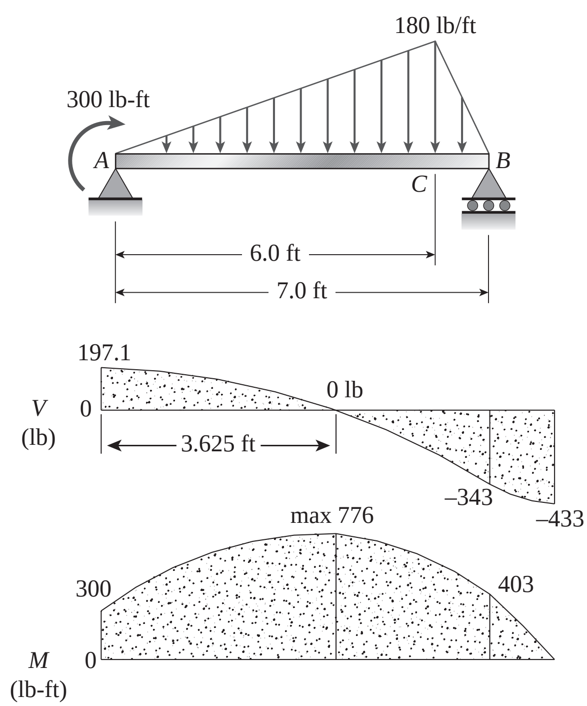


Problem 4.5-27 The simple beam *ACB* shown in the figure is subjected to a triangular load of maximum intensity 180 lb/ft and a concentrated moment of 300 lb-ft at *A*.

Draw the shear-force and bending-moment diagrams for this beam.

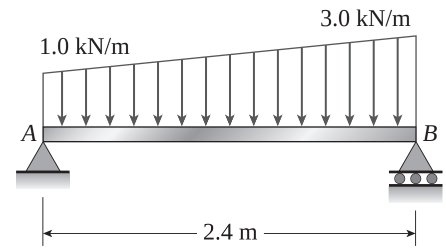


Solution 4.5-27 Simple beam

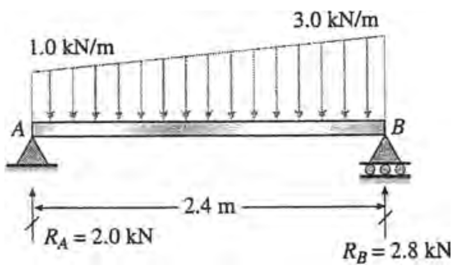


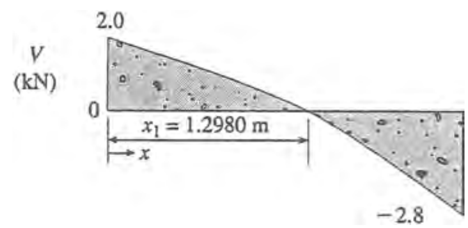
Problem 4.5-28 A beam with simple supports is subjected to a trapezoidally distributed load (see figure). The intensity of the load varies from 1.0 kN/m at support A to 3.0 kN/m at support B.

Draw the shear-force and bending-moment diagrams for this beam.



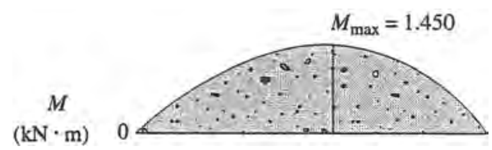
Solution 4.5-28 Simple beam





$$V = 2.0 - x - \frac{x^2}{2.4} \quad (x = \text{meters};\ V = \text{kN})$$

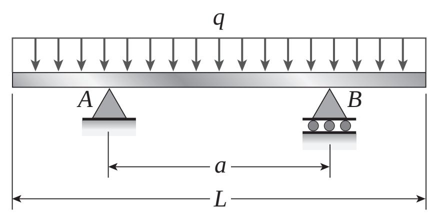
Set $V = 0$: $x_1 = 1.2980$ m



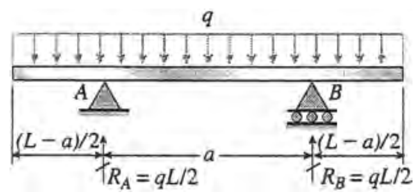
Problem 4.5-29 A beam of length L is being designed to support a uniform load of intensity q (see figure). If the supports of the beam are placed at the ends, creating a simple beam, the maximum bending moment in the beam is $ql^2/8$. However, if the supports of the beam are moved symmetrically toward the middle of the beam (as pictured), the maximum bending moment is reduced.

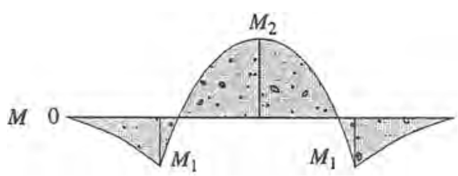
Determine the distance a between the supports so that the maximum bending moment in the beam has the smallest possible numerical value.

Draw the shear-force and bending-moment diagrams for this condition.



Solution 4.5-29 Beam with overhangs





The maximum bending moment is smallest when $M_1 = M_2$ (numerically).

$$M_1 = \frac{q(L - a)^2}{8}$$

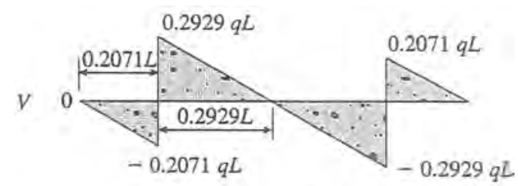
$$M_2 = R_A\left(\frac{a}{2}\right) - \frac{qL^2}{8} = \frac{qL}{8}(2a - L)$$

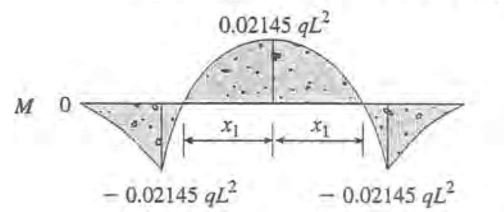
$$M_1 = M_2 \qquad (L - a)^2 = L(2a - L)$$

Solve for a: $a = (2 - \sqrt{2})L = 0.5858L \quad \leftarrow$

$$M_1 = M_2 = \frac{q}{8}(L - a)^2$$

$$= \frac{qL^2}{8}(3 - 2\sqrt{2}) = 0.02145qL^2 \quad \leftarrow$$

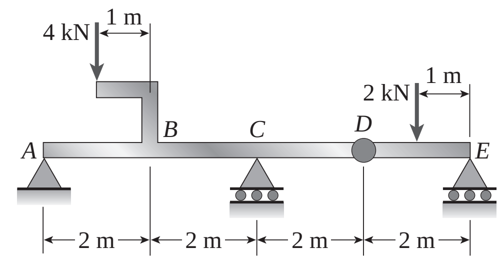




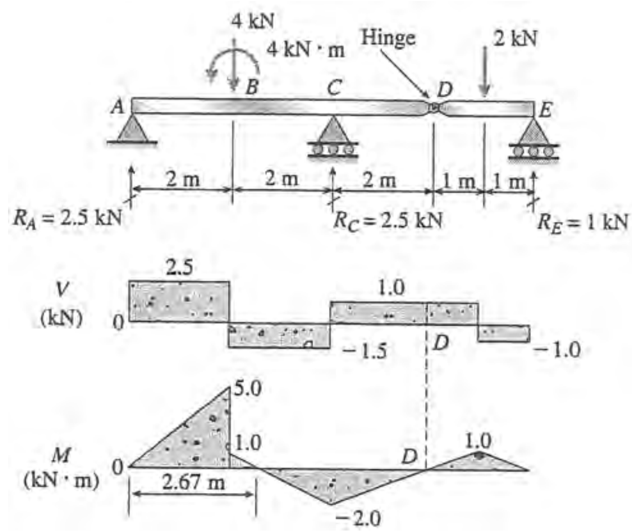
$$x_1 = 0.3536\,a$$
$$= 0.2071\,L$$

Problem 4.5-30 The compound beam *ABCDE* shown in the figure consists of two beams (*AD* and *DE*) joined by a hinged connection at *D*. The hinge can transmit a shear force but not a bending moment. The loads on the beam consist of a 4-kN force at the end of a bracket attached at point *B* and a 2-kN force at the midpoint of beam *DE*.

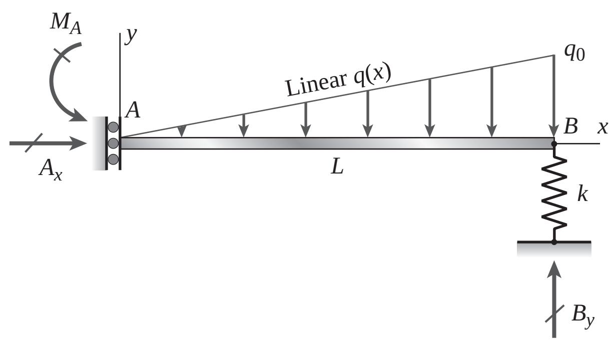
Draw the shear-force and bending-moment diagrams for this compound beam.



Solution 4.5-30 Compound beam



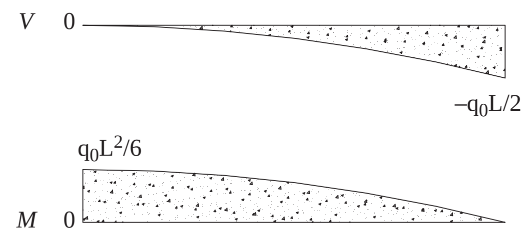
Problem 4.5-31 The beam shown below has a sliding support at *A* and an elastic support with spring constant *k* at *B*. A distributed load *q*(x) is applied over the entire beam. Find all support reactions, then plot shear (V) and moment (*M*) diagrams for beam *AB*; *label* all critical *V* & *M* values and also the *distance* to points where any critical ordinates are zero.



Solution 4.5-31

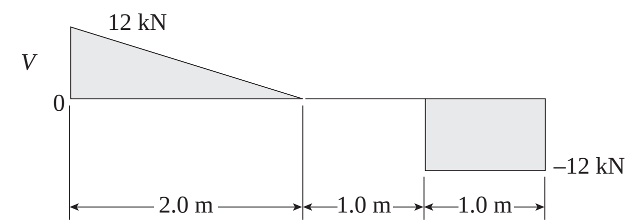
$$M_A = -\frac{q_0}{6}L^2 \text{ (clockwise)}, \quad A_x = 0 \quad \longleftarrow$$

$$B_y = \frac{q_0}{2}L \text{ (upward)} \quad \longleftarrow$$

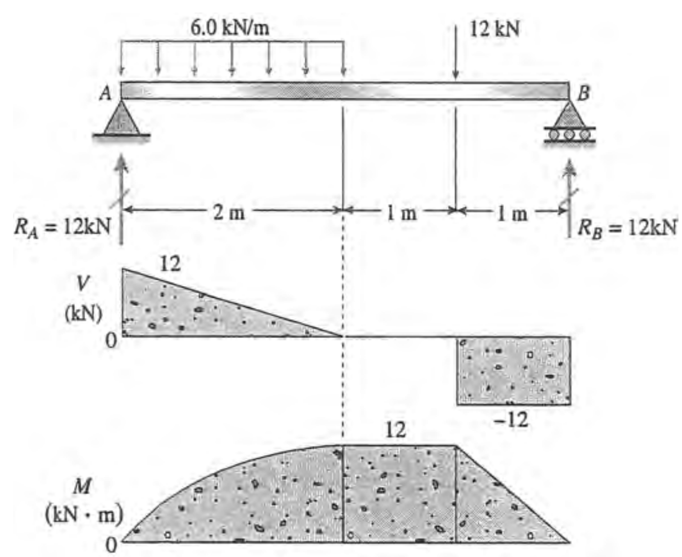


Problem 4.5-32 The shear-force diagram for a simple beam is shown in the figure.

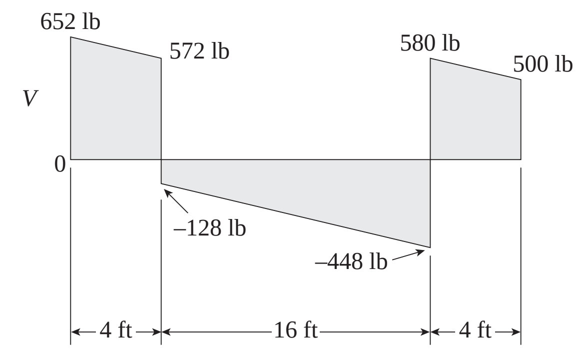
Determine the loading on the beam and draw the bending-moment diagram, assuming that no couples act as loads on the beam.



Solution 4.5-32 Simple beam (*V* is given)

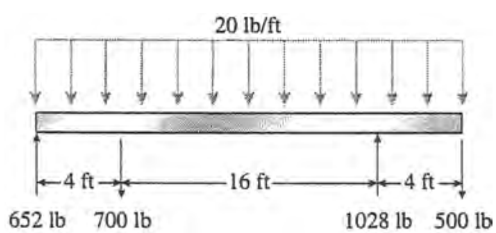


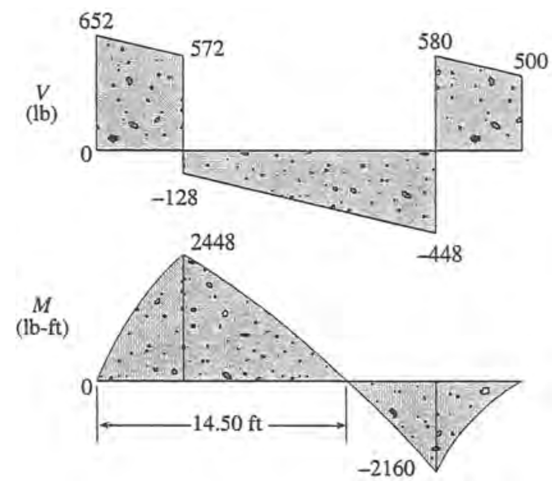
Problem 4.5-33 The shear-force diagram for a beam is shown in the figure. Assuming that no couples act as loads on the beam, determine the forces acting on the beam and draw the bending-moment diagram.



Solution 4.5-33 Forces on a beam (*V* is given)

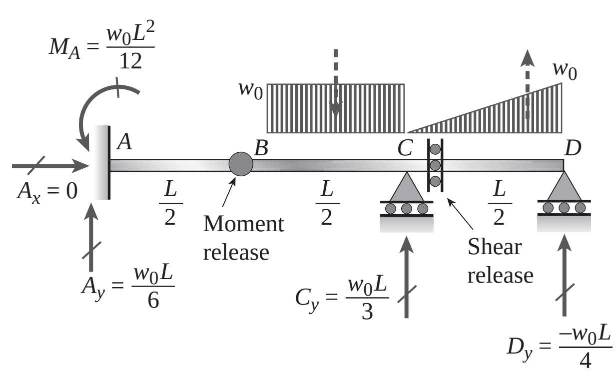
FORCE DIAGRAM





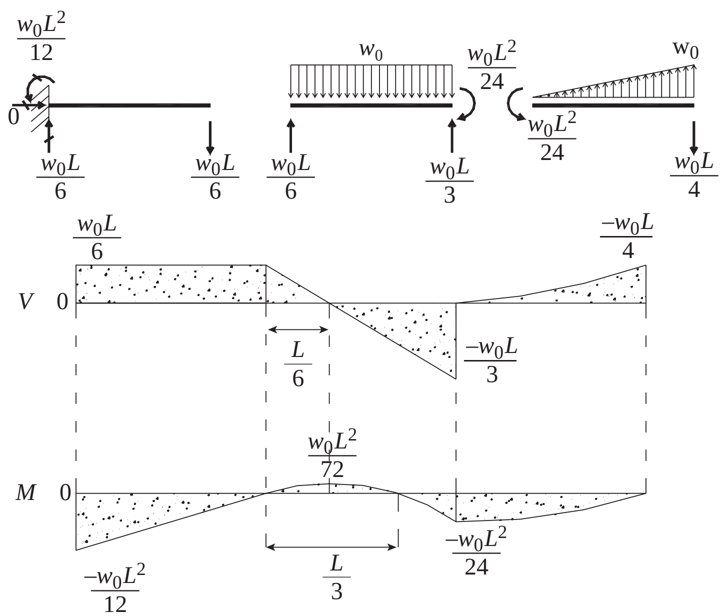
Problem 4.5-34 The compound beam below has an internal *moment release* just to the left of *B* and a *shear release* just to the right of *C*. Reactions have been computed at *A*, *C* and *D* and are shown in the figure.

First, confirm the reaction expressions using statics, then plot shear (V) and moment (M) diagrams. *Label* all critical V and M values and also the *distance* to points where either V and/or M is zero.



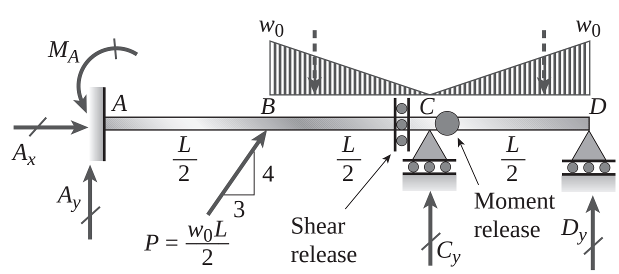
Solution 4.5-34

FREE-BODY DIAGRAM



Problem 4.5-35 The compound beam below has an *shear release* just to the left of C and a *moment release* just to the right of C. A plot of the moment diagram is provided below for applied load P at B and triangular distributed loads $w(x)$ on segments BC and CD.

First, solve for reactions using statics, then plot axial force (N) and shear (V) diagrams. Confirm that the moment diagram is that shown below. *Label* all critical N and V & M values and also the *distance* to points where N, V &/or M is zero.



Solution 4.5-35

Solve for reactions using statics

$$M_A = -\frac{7w_0}{30}L^2 \text{ (clockwise)}, \quad \leftarrow$$

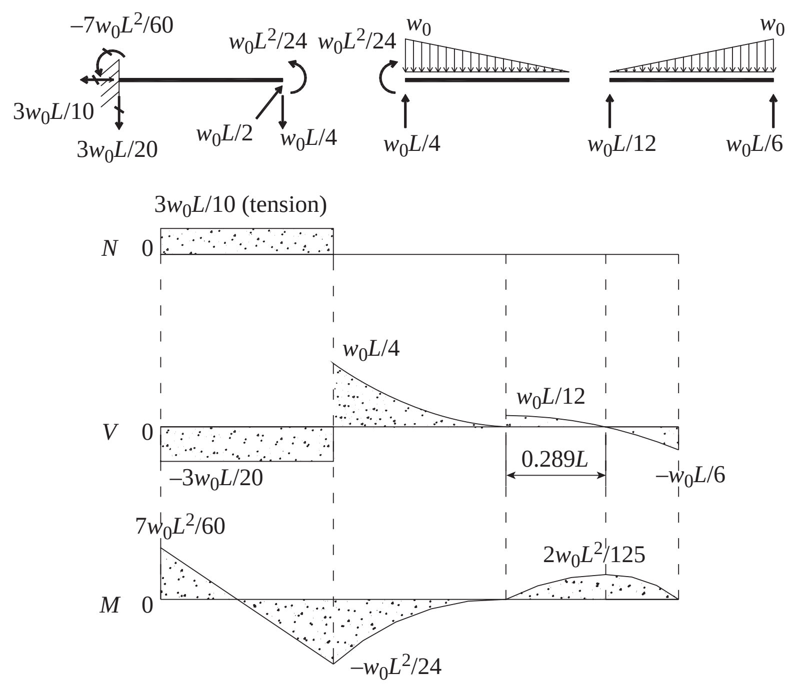
$$A_x = -\frac{3}{10}w_0L \text{ (left)} \quad \leftarrow$$

$$A_y = -\frac{3}{20}w_0L \text{ (downward)} \quad \leftarrow$$

$$C_y = \frac{w_0}{12}L \text{ (upward)} \quad \leftarrow$$

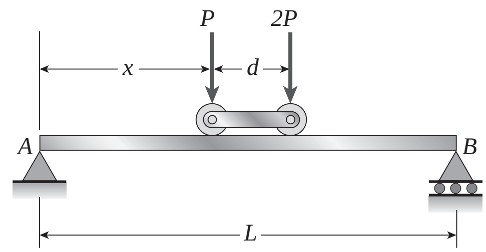
$$D_y = \frac{w_0}{6}L \text{ (upward)} \quad \leftarrow$$

FREE-BODY DIAGRAM

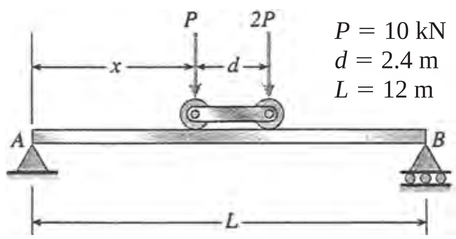


Problem 4.5-36 A simple beam AB supports two connected wheel loads P and $2P$ that are distance d apart (see figure). The wheels may be placed at any distance x from the left-hand support of the beam.

(a) Determine the distance x that will produce the maximum shear force in the beam, and also determine the maximum shear force V_{max}.
(b) Determine the distance x that will produce the maximum bending moment in the beam, and also draw the corresponding bending-moment diagram. (Assume $P = 10$ kN, $d = 2.4$ m, and $L = 12$ m.)



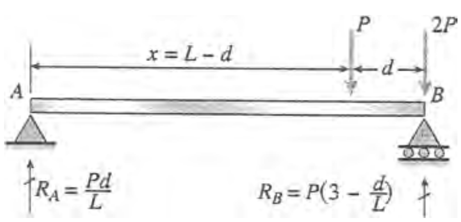
Solution 4.5-36 Moving loads on a beam



$P = 10$ kN
$d = 2.4$ m
$L = 12$ m

(a) Maximum shear force

By inspection, the maximum shear force occurs at support B when the larger load is placed close to, but not directly over, that support.

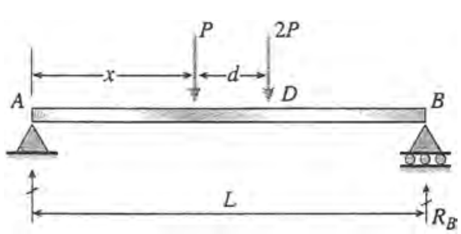


$$x = L - d = 9.6 \text{ m} \quad \leftarrow$$

$$V_{max} = R_B = P\left(3 - \frac{d}{L}\right) = 28 \text{ kN} \quad \leftarrow$$

(b) Maximum bending moment

By inspection, the maximum bending moment occurs at point D, under the larger load $2P$.



Reaction at support B:

$$R_B = \frac{P}{L}x + \frac{2P}{L}(x + d) = \frac{P}{L}(2d + 3x)$$

Bending moment at D:

$$M_D = R_B(L - x - d)$$

$$= \frac{P}{L}(2d + 3x)(L - x - d)$$

$$= \frac{P}{L}[-3x^2 + (3L - 5d)x + 2d(L - d)] \quad \text{Eq.(1)}$$

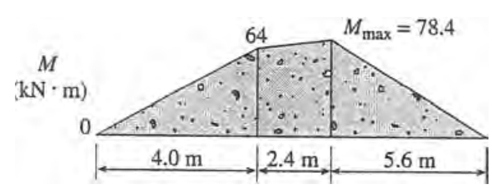
$$\frac{dM_D}{dx} = \frac{P}{L}(-6x + 3L - 5d) = 0$$

Solve for x: $x = \frac{L}{6}\left(3 - \frac{5d}{L}\right) = 4.0 \text{ m} \quad \leftarrow$

Substitute x into Eq (1):

$$M_{max} = \frac{P}{L}\left[-3\left(\frac{L}{6}\right)^2\left(3 - \frac{5d}{L}\right)^2 + (3L - 5d) \times \left(\frac{L}{6}\right)\left(3 - \frac{5d}{L}\right) + 2d(L - d)\right]$$

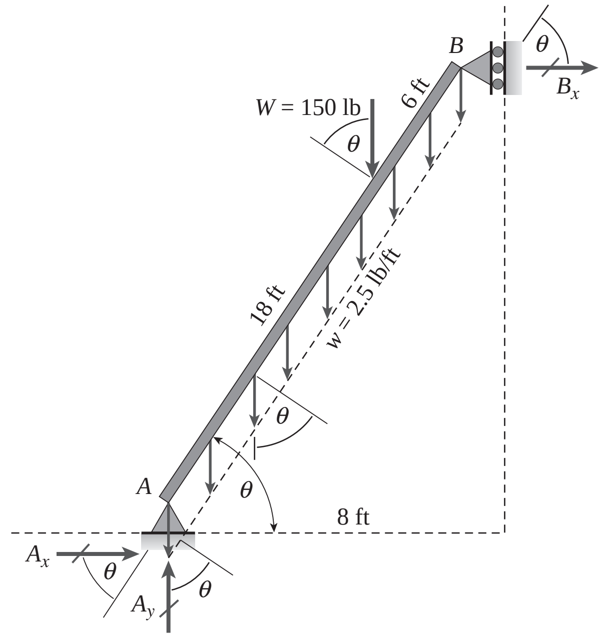
$$= \frac{PL}{12}\left(3 - \frac{d}{L}\right)^2 = 78.4 \text{ kN} \cdot \text{m} \quad \leftarrow$$



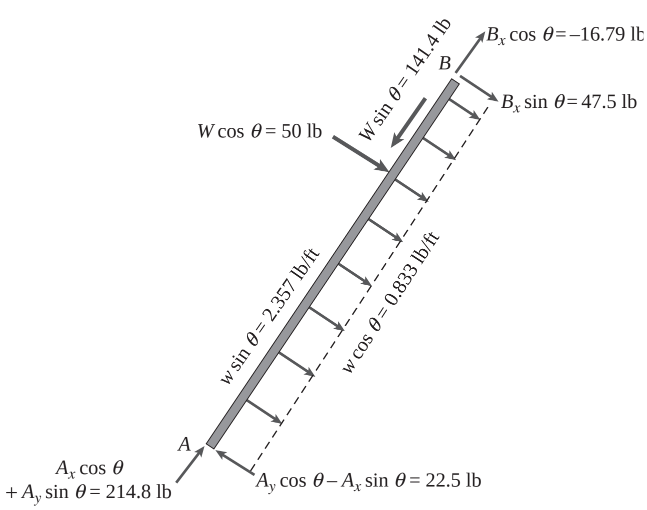
Note: $R_A = \frac{P}{2}\left(3 + \frac{d}{L}\right) = 16$ kN

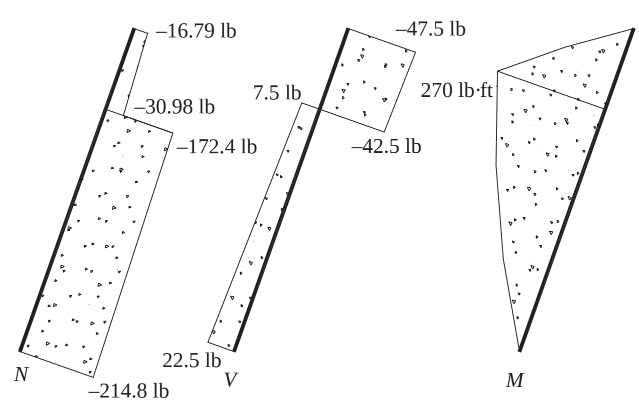
$R_B = \frac{P}{2}\left(3 - \frac{d}{L}\right) = 14$ kN

Problem 4.5-37 The inclined beam below represents the loads applied to a ladder: the weight (W) of the house painter and the self weight (w) of the ladder itself. Find support reactions at A and B, then plot axial force (N), shear (V) and moment (M) diagrams. *Label* all critical N, V & M values and also the *distance* to points where any critical ordinates are zero. Plot N, V & M diagrams normal to the inclined ladder.



Solution 4.5-37





$$\cos\theta = \frac{8}{18+6} = \frac{1}{3}, \ \sin\theta = \frac{2\sqrt{2}}{3}$$

Solution procedure:

(1) Use statics to find reaction forces at A & B

$$\Sigma F_V = 0: A_y = 150 + 2.5\,(18+6) = 210 \text{ lb}$$

$$A_y = 210 \text{ lb (upward)} \quad \leftarrow$$

$$\Sigma M_A = 0: B_x \cdot 24\sin\theta + 150 \cdot 6 + 2.5 \cdot 24 \cdot 4 = 0$$

$$B_x = -50.38 \text{ lb (left)} \quad \leftarrow$$

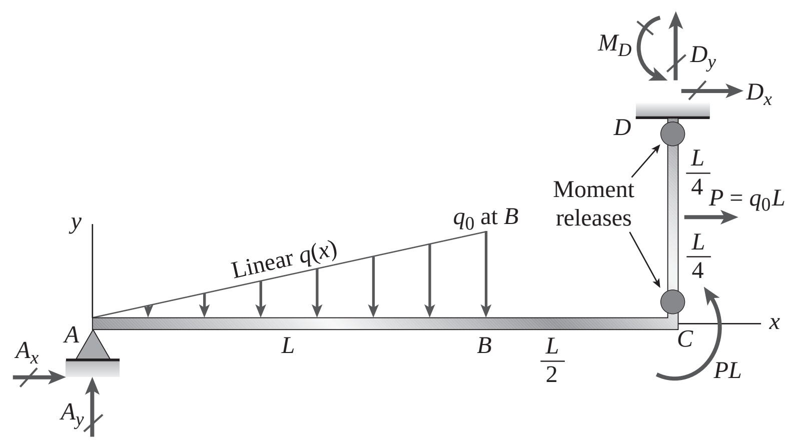
$$\Sigma F_H = 0;\ A_x = 50.38 \text{ lb (right)} \quad \leftarrow$$

(2) Use θ to find forces at ends A & B which are along and perpendicular to member AB (see free-body diagram); also resolve forces W and w into components along & perpendicular to member AB

(3) Starting at end A, plot N, V and M diagrams (see plots)

Problem 4.5-38 Beam ABC is supported by a tie rod CD as shown (see Prob. 10.4-9). Two configurations are possible: pin support at A and downward triangular load on AB, or pin at B and upward load on AB. Which has the larger maximum moment?

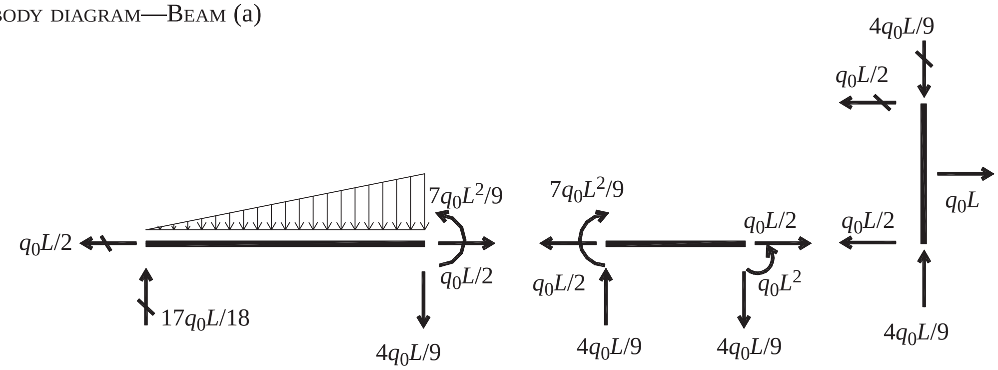
First, find all support reactions, then plot axial force (N), shear (V) and moment (M) diagrams for ABC only and *label* all critical N, V & M values. Label the *distance* to points where any critical ordinates are zero.



(a)

Solution 4.5-38

FREE-BODY DIAGRAM—BEAM (a)



Use statics to find reactions at A and D for Beam (a)

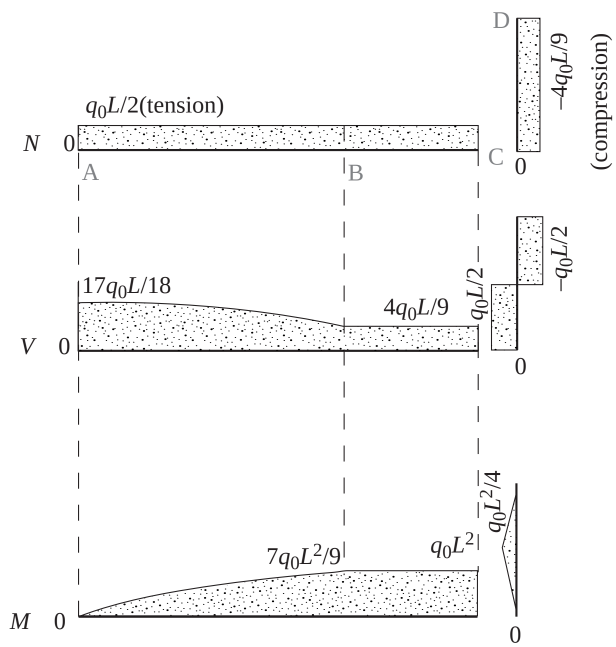
$$A_x = -\frac{1}{2}\,q_0L \text{ (left)} \quad \leftarrow$$

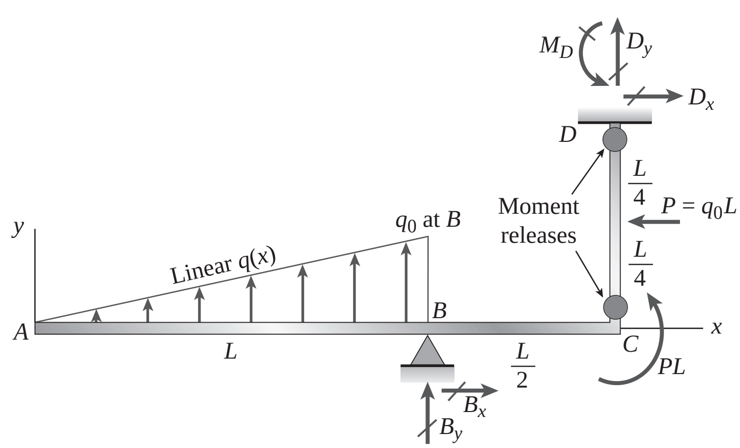
$$A_y = \frac{17}{18}\,q_0L \text{ (upward)} \quad \leftarrow$$

$$D_x = -\frac{1}{2}\,q_0L \text{ (left)} \quad \leftarrow$$

$$D_y = -\frac{4}{9}\,q_0L \text{ (downward)} \quad \leftarrow$$

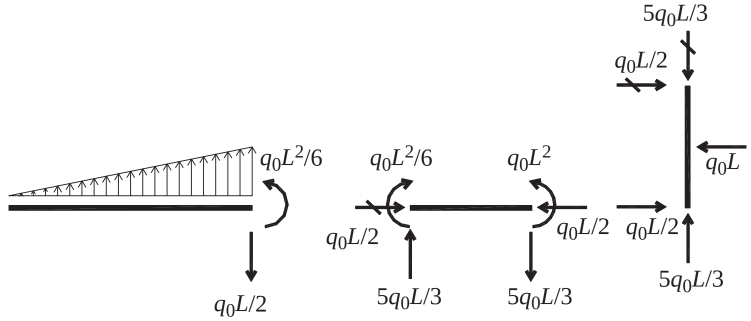
$$M_D = 0 \quad \leftarrow$$





(b)

FREE-BODY DIAGRAM—BEAM (b)



Use statics to find reactions at B and D for Beam (b)

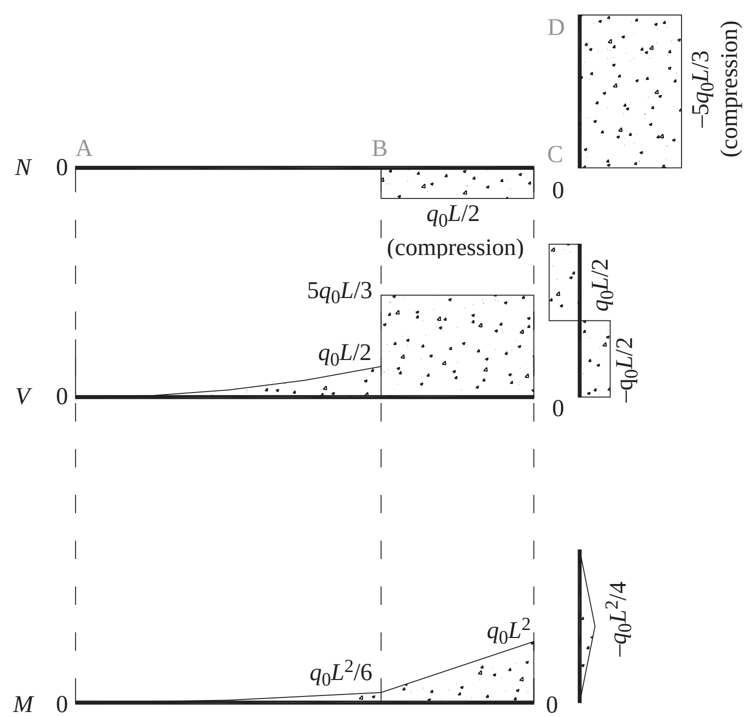
$B_x = \dfrac{1}{2}q_0L$ (right) $\leftarrow$

$B_y = -\dfrac{1}{2}q_0L + \dfrac{5}{3}q_0L = \dfrac{7}{6}q_0L$ (upward) $\leftarrow$

$D_x = \dfrac{1}{2}q_0L$ (right) $\leftarrow$

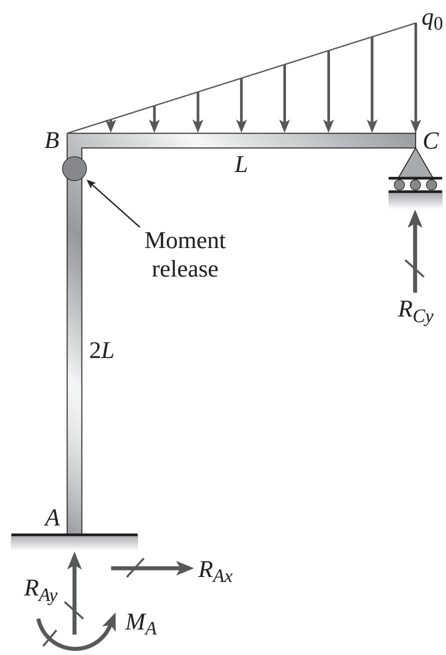
$D_y = -\dfrac{5}{3}q_0L$ (downward) $\leftarrow$

$M_D = 0$ $\leftarrow$



Problem 4.5-39 The plane frame below consists of column AB and beam BC which carries a triangular distributed load. Support A is fixed and there is a roller support at C. Column AB has a *moment release* just below joint B.

Find support reactions at A and C, then plot axial force (N), shear (V) and moment (M) diagrams for both members. *Label* all critical N, V & M values and also the *distance* to points where any critical ordinates are zero.



Solution 4.5-39

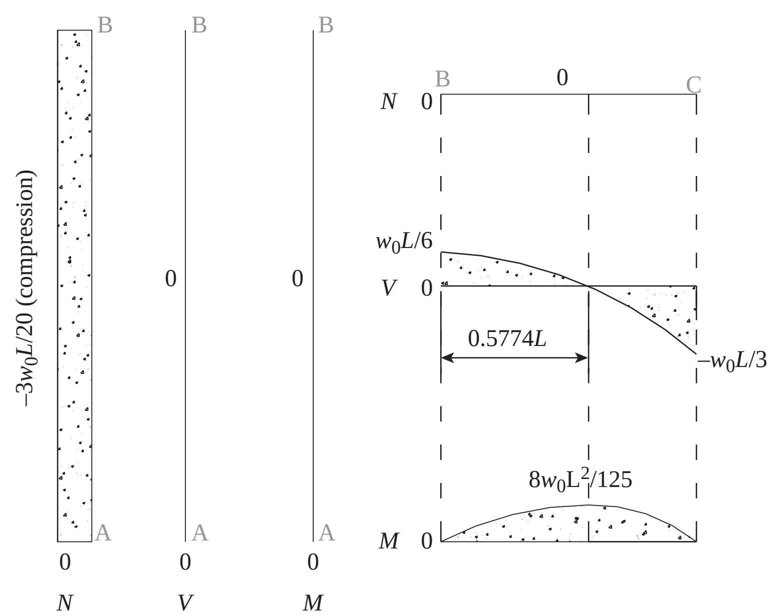
Use statics to find reactions at A and C

$M_A = 0$ ←

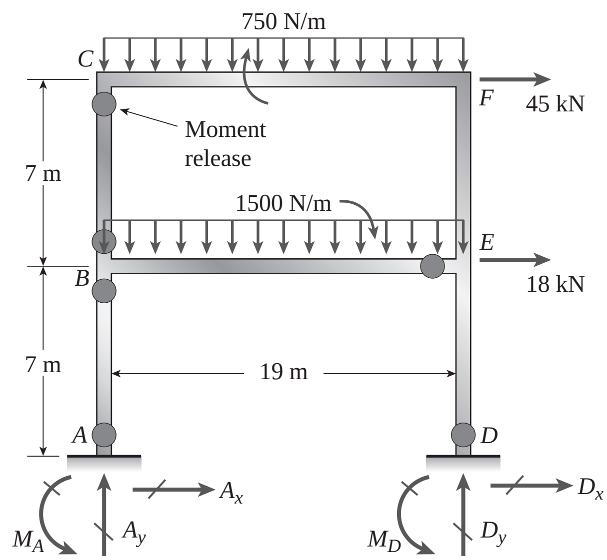
$R_{Ay} = \dfrac{q_0}{6} L$ (upward) ←

$R_{Cy} = \dfrac{q_0}{3} L$ (upward) ←

$R_{Ax} = 0$ ←



Problem 4.5-40 The plane frame shown below is part of an elevated freeway system. Supports at A and D are fixed but there are *moment releases* at the base of both columns (AB and DE), as well near in column BC and at the end of beam BE. Find all support reactions, then plot axial force (N), shear (V) and moment (M) diagrams for all beam and column members. *Label* all critical N, V & M values and also the *distance* to points where any critical ordinates are zero.



Solution 4.5-40

Solution procedure:

(1) $M_A = M_D = 0$ due to moment releases

(2) $\sum M_A = 0$: $D_y = 61{,}164 \text{ N} = 61.2 \text{ kN}$

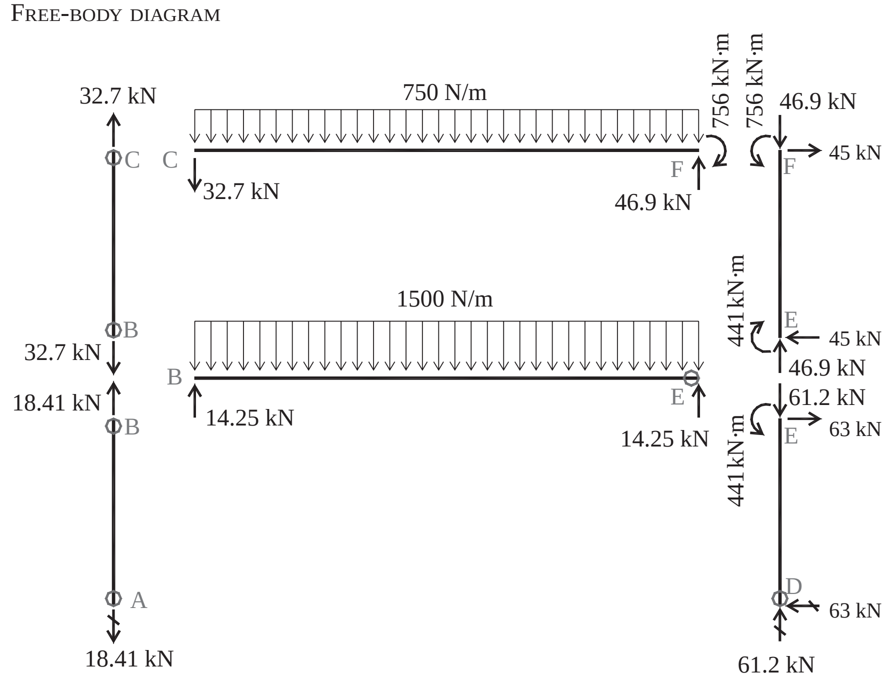
(3) $\sum F_y = 0$: $A_y = -18{,}414 \text{ N} = -18.41\text{kN}$

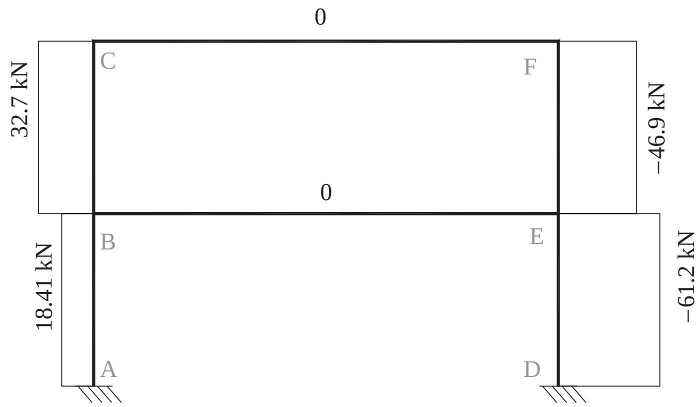
(4) $\sum M_B = 0$ for AB: $A_x = 0$

(5) $\sum F_H = 0$: $D_x = -63 \text{ kN}$

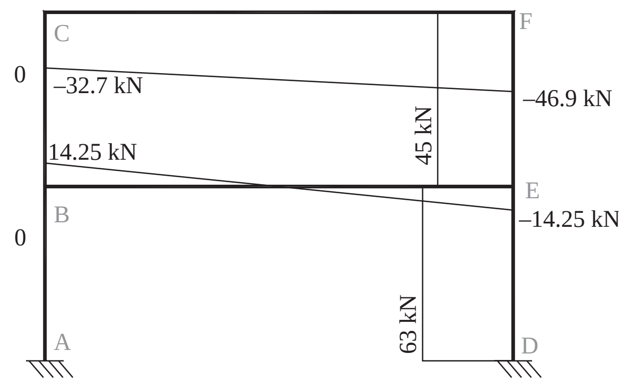
(6) Draw separate FBD's of each member (see below) to find N, V and M for each member; plot diagrams (see below)

FREE-BODY DIAGRAM

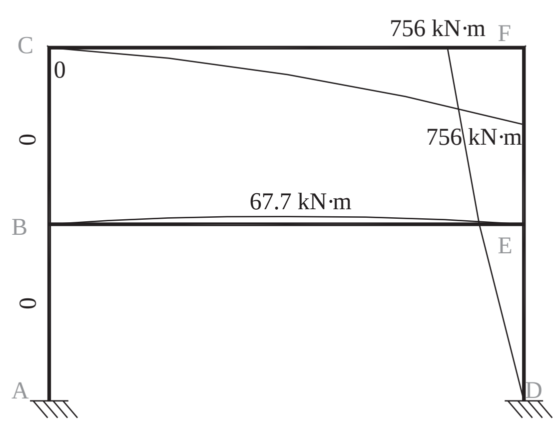




AXIAL FORCE DIAGRAM.
(−) COMPRESSION



SHEAR FORCE DIAGRAM.



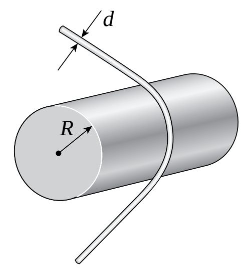
BENDING MOMENT DIAGRAM

5

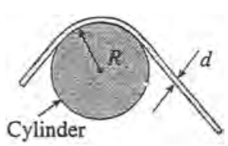
Stresses in Beams (Basic Topics)

Longitudinal Strains in Beams

Problem 5.4-1 Determine the maximum normal strain ε_{max} produced in a steel wire of diameter $d = 1/16$ in. when it is bent around a cylindrical drum of radius $R = 24$ in. (see figure).



Solution 5.4-1 Steel wire



$R = 24 \text{ in.} \qquad d = \frac{1}{16} \text{ in.}$

From Eq. (5-4):

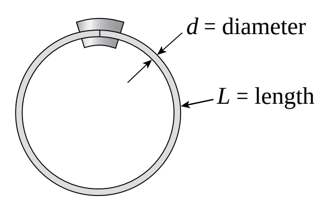
$$\varepsilon_{max} = \frac{y}{\rho}$$

$$= \frac{d/2}{R + d/2} = \frac{d}{2R + d}$$

Substitute numerical values:

$$\varepsilon_{max} = \frac{1/16 \text{ in.}}{2(24 \text{ in.}) + 1/16 \text{ in.}} = 1300 \times 10^{-6} \quad \leftarrow$$

Problem 5.4-2 A copper wire having diameter $d = 3$ mm is bent into a circle and held with the ends just touching (see figure). If the maximum permissible strain in the copper is $\varepsilon_{max} = 0.0024$, what is the shortest length L of wire that can be used?



Solution 5.4-2 Copper wire

$d = 3 \text{ mm} \qquad \varepsilon_{max} = 0.0024$

$L = 2\pi\rho \qquad \rho = \frac{L}{2\pi}$

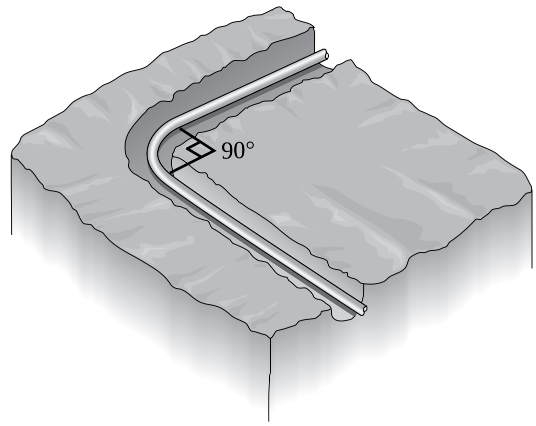
From Eq. (5-4):

$$\varepsilon_{max} = \frac{y}{\rho} = \frac{d/2}{L/2\pi} = \frac{\pi d}{L}$$

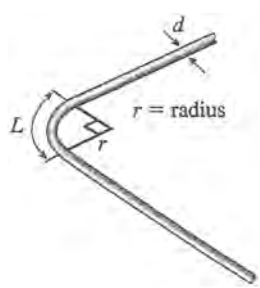
$$L_{min} = \frac{\pi d}{\varepsilon_{max}} = \frac{\pi(3 \text{ mm})}{0.0024} = 3.93 \text{ m} \quad \leftarrow$$

Problem 5.4-3 A 4.5 in. outside diameter polyethylene pipe designed to carry chemical wastes is placed in a trench and bent around a quarter-circular 90° bend (see figure). The bent section of the pipe is 46 ft long.

Determine the maximum compressive strain ε_{max} in the pipe.



Solution 5.4-3 Polyethylene pipe



L = length of 90° bend

$L = 46 \text{ ft} = 552 \text{ in.}$

$d = 4.5 \text{ in.}$

$$L = \frac{2\pi r}{4} = \frac{\pi r}{2}$$

Angle equals 90° or $\pi/2$ radians,

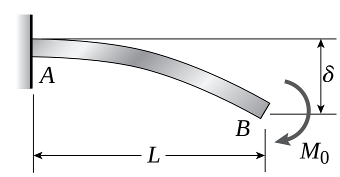
$r = \rho$ = radius of curvature

$$\rho = \frac{L}{\pi/2} = \frac{2L}{\pi} \qquad \varepsilon_{max} = \frac{y}{\rho} = \frac{d/2}{2L/\pi}$$

$$\varepsilon_{max} = \frac{\pi d}{4L} = \frac{\pi}{4}\left(\frac{4.5 \text{ in.}}{552 \text{ in.}}\right) = 6400 \times 10^{-6} \quad \longleftarrow$$

Problem 5.4-4 A cantilever beam AB is loaded by a couple M_0 at its free end (see figure.) The length of the beam is $L = 1.5$ m and the longitudinal normal strain at the top surface is 0.001. The distance from the top surface of the beam to the neutral surface is 75 mm.

Calculate the radius of curvature ρ, the curvature κ, and the vertical deflection δ at the end of the beam.



Solution 5.4-4

NUMERICAL DATA

$L = 2.0 \text{ m} \qquad \varepsilon_{max} = 0.0012$

$c = 82.5 \text{ mm}$

RADIUS OF CURVATURE

$$\rho = \frac{c}{\varepsilon_{max}} \qquad \rho = 68.8 \text{ m} \quad \longleftarrow$$

CURVATURE

$$\kappa = \frac{1}{\rho} \qquad \kappa = 1.455 \times 10^{-5} \text{ m}^{-1} \quad \longleftarrow$$

Deflection: constant curvature for pure bending so gives a circular arc; assume flat deflection curve (small defl.) so $BC = L$

$$\sin(u) = \frac{L}{\rho} \qquad u = \operatorname{asin}\left(\frac{L}{\rho}\right)$$

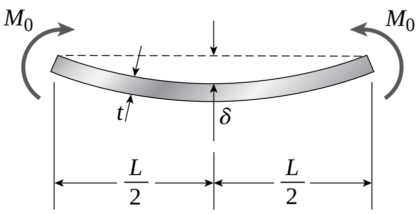
$$u = 0.029 \quad \text{radians} \qquad \frac{L}{\rho} = 0.029$$

$$1 - \cos(u) = 4.232 \times 10^{-4} \qquad \rho = 6.875 \times 10^{4} \text{ mm}$$

$$\delta = \rho(1 - \cos(u)) \qquad \delta = 29.1 \text{ mm} \quad \longleftarrow$$

Problem 5.4-5 A thin strip of steel of length $L = 20$ in. and thickness $t = 0.2$ in. is bent by couples M_0 (see figure). The deflection at the midpoint of the strip (measured from a line joining its end points) is found to be 0.20 in.

Determine the longitudinal normal strain ε at the top surface of the strip.



Solution 5.4-5

NUMERICAL DATA

$L = 28$ inches $\quad t = 0.25$ inches

$\delta = 0.20$ inches

LONGITUDINAL NORMAL STRAIN AT TOP SURFACE

$$\varepsilon = \frac{\frac{-t}{2}}{\rho} \qquad \varepsilon = \frac{-t}{2\rho}$$

$$\delta = \rho\,(1 - \cos(\theta)) \qquad \sin(\theta) = \frac{\frac{L}{2}}{\rho} \qquad \sin(\theta) = \frac{L}{2\rho}$$

assume angle is small so that

$$\theta = \frac{L}{2\rho} \qquad \delta = \rho\left(1 - \cos\left(\frac{L}{2\rho}\right)\right)$$

solving for ρ: $\quad \rho = \dfrac{\delta}{1 - \cos\left(\dfrac{L}{2\rho}\right)}$

insert numerical data: $\quad \rho = \dfrac{0.20}{1 - \cos\left(\dfrac{14}{\rho}\right)}$

numerical solution for radius of curvature ρ gives
$\rho = 489.719$ inches

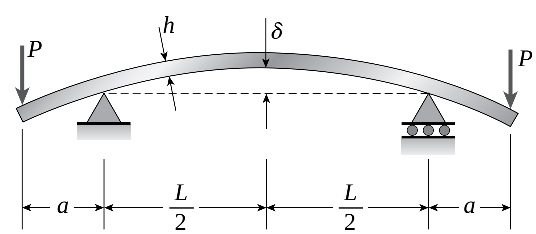
strain at top (compressive):

$$\varepsilon = \frac{t}{2\rho} \qquad \varepsilon = 2.552 \times 10^{-4}$$

$$\varepsilon = 255\,(10^{-6}) \qquad \leftarrow$$

Problem 5.4-6 A bar of rectangular cross section is loaded and supported as shown in the figure. The distance between supports is $L = 1.5$ m and the height of the bar is $h = 120$ mm. The deflection at the midpoint is measured as 3.0 mm.

What is the maximum normal strain ε at the top and bottom of the bar?



Solution 5.4-6

NUMERICAL DATA

$L = 1.5 \text{ m} \qquad h = 120 \text{ mm}$

$\delta = 3.0 \text{ mm}$

NORMAL STRAIN AT TOP OF BAR:

$$\varepsilon = \frac{\frac{h}{2}}{\rho} \qquad \varepsilon = \frac{h}{2\rho}$$ tensile strain, $\rho =$ radius of curvature

SMALL DEFLECTION SO SMALL ANGLE θ

$$\sin(\theta) = \frac{\frac{L}{2}}{\rho} \qquad \theta = \frac{L}{2\rho}$$

$$\delta = \rho\left(1 - \cos\left(\frac{L}{2\rho}\right)\right)$$

$$\therefore \rho\left(1 - \cos\left(\frac{L}{2\rho}\right)\right) - \delta = 0$$

numerical solution for radius of curvature ρ gives $\rho = 93.749$ m

strain at top (compressive):

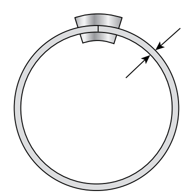
$$\varepsilon = \frac{h}{2\rho} \qquad \varepsilon = 640 \times 10^{-6} \quad \leftarrow$$

Normal Stresses in Beams

Problem 5.5-1 A thin strip of hard copper ($E = 16{,}000$ ksi) having length $L = 90$ in. and thickness $t = 3/32$ in. is bent into a circle and held with the ends just touching (see figure).

(a) Calculate the maximum bending stress σ_{max} in the strip.

(b) By what percent does the stress increase or decrease if the thickness of the strip is increased by 1/32 in.?



Solution 5.5-1

(a) MAXIMUM BENDING STREES

$$E = 16000 \text{ ksi} \qquad L = 90 \text{ inches} \qquad t = \frac{3}{32} \text{ inches}$$

$$\sigma = E\left(\frac{\frac{t}{2}}{\rho}\right) \qquad \rho = \frac{L}{2\pi}$$

$$\rho = 14.324 \text{ inches} \qquad \sigma_{max} = \frac{Et}{2\rho}$$

$$\sigma_{max} = 52.4 \text{ ksi} \quad \leftarrow$$

(b) % CHANGE IN STRESS

$$t_{new} = \frac{4}{32} \qquad \sigma_{maxnew} = \frac{Et_{new}}{2\rho}$$

$$\sigma_{maxnew} = 69.813 \text{ ksi}$$

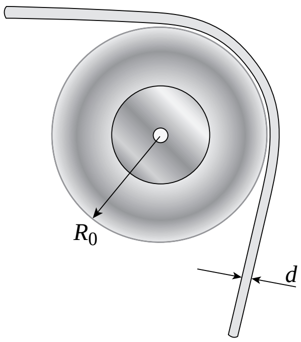
$$\frac{\sigma_{maxnew} - \sigma_{max}}{\sigma_{max}}(100) = 33.3 \quad \leftarrow$$

33% increase (linear) in max.stress due to increase in t; same as % increase in thickness t

$$\frac{\frac{4}{32} - \frac{3}{32}}{\frac{3}{32}}(100) = 33.3$$

Problem 5.5-2 A steel wire (E = 200 GPa) of diameter d = 1.25 mm is bent around a pulley of radius R_0 = 500 mm (see figure).

(a) What is the maximum stress σ_{max} in the wire?
(b) By what percent does the stress increase or decrease if the radius of the pulley is increased by 25%?



Solution 5.5-2

(a) MAX. NORMAL STRESS IN WIRE

E = 200 GPa d = 1.25 mm R_0 = 500 mm

$$\sigma = \frac{E\frac{d}{2}}{\rho} \qquad \sigma_{max} = \frac{E\frac{d}{2}}{R_0 + \frac{d}{2}}$$

$$\sigma_{max} = 250 \text{ MPa} \quad \longleftarrow$$

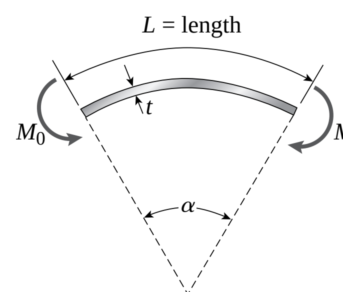
(b) % CHANGE IN MAX. STRESS DUE TO INCREASE IN PULLEY RADIUS BY 25%

$$\sigma_{new} = \frac{E\frac{d}{2}}{1.25\,R_0 + \frac{d}{2}} \qquad \sigma_{new} = 199.8 \text{ MPa}$$

$$\frac{\sigma_{new} - \sigma_{max}}{\sigma_{max}}(100) = -20\% \quad \longleftarrow$$

Problem 5.5-3 A thin, high-strength steel rule ($E = 30 \times 10^6$ psi) having thickness t = 0.175 in. and length L = 48 in. is bent by couples M_0 into a circular are subtending a central angle α = 40° (see figure).

(a) What is the maximum bending stress σ_{max} in the rule?
(b) By what percent does the stress increase or decrease if the central angle is increased by 10%?



Solution 5.5-3

(a) MAX. BENDING STRESS

$$\alpha = 40\left(\frac{\pi}{180}\right) \qquad \alpha = 0.698 \text{ radians}$$

L = 48 inches t = 0.175 in. $E = 30\,(10^6)$ psi

$$\rho = \frac{L}{\alpha} \qquad \rho = 68.755 \text{ inches}$$

$$\sigma_{max} = \frac{E\frac{t}{2}}{\rho} \qquad \sigma_{max} = \frac{Et}{2\rho} \qquad \sigma_{max} = \frac{Et\alpha}{2L}$$

$$\sigma_{max} = 38.2 \text{ ksi} \quad \longleftarrow$$

(b) % CHANGE IN STRESS DUE TO 10% INCREASE IN ANGLE α

$$\sigma_{new} = \frac{E\,t(1.1\alpha)}{2L} \qquad \sigma_{new} = 41997 \text{ psi}$$

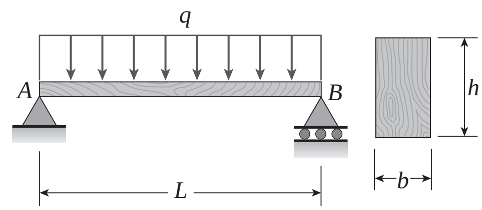
$$\frac{\sigma_{new} - \sigma_{max}}{\sigma_{max}}(100) = 10\% \quad \longleftarrow$$

linear increase (%)

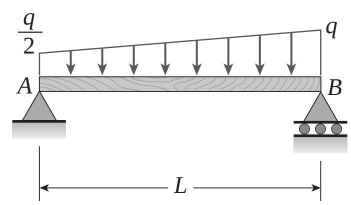
Problem 5.5-4 A simply supported wood beam AB with span length $L = 4$ m carries a uniform load of intensity $q = 5.8$ kN/m (see figure).

(a) Calculate the maximum bending stress σ_{max} due to the load q if the beam has a rectangular cross section with width $b = 140$ mm and height $h = 240$ mm.

(b) Repeat (a) but use the trapezoidal distuibuted load shown in the figure part (b).



(a)



(b)

Solution 5.5-4

(a) MAX. BENDING STRESS DUE TO UNIFORM LOAD q

$$M_{max} = \frac{qL^2}{8} \qquad S = \frac{I}{\frac{h}{2}}$$

$$S = \frac{\frac{bh^3}{12}}{\frac{h}{2}} \qquad S = \frac{1}{6}bh^2$$

$$\sigma_{max} = \frac{M_{max}}{S} \qquad \sigma_{max} = \frac{\frac{qL^2}{8}}{\left(\frac{1}{6}bh^2\right)}$$

$$\sigma_{max} = \frac{3}{4}q\frac{L^2}{bh^2}$$

$$q = 5.8\,\frac{\text{kN}}{\text{m}} \qquad L = 4\text{ m} \qquad b = 140\text{ mm}$$

$$h = 240\text{ mm}$$

$$M_{max} = \frac{qL^2}{8}$$

$$M_{max} = 11.6\text{ kN}\cdot\text{m}$$

$$\sigma_{max} = 8.63\text{ MPa} \quad \leftarrow$$

(b) MAX. BENDING STRESS DUE TO TRAPEZOIDAL LOAD q

$$R_A = \left[\frac{1}{2}\left(\frac{q}{2}\right)L + \frac{1}{3}\left(\frac{q}{2}\frac{1}{2}\right)L\right]$$

uniform load ($q/2$) & triang. load ($q/2$)

$$R_A = \frac{1}{3}qL$$

find x = location of zero shear

$$R_A - \frac{q}{2}x - \frac{1}{2}\left(\frac{x}{L}\frac{q}{2}\right)x = 0$$

$$3x^2 + 6Lx - 4L^2 = 0$$

$$x = \frac{-6L - \sqrt{(84L^2)}}{2(3)}$$

$$\frac{x}{L} = \left(-1 + \frac{1}{6}\sqrt{84}\right)$$

$$x_{max} = 0.52753\,L$$

$$M_{max} = R_A x_{max} - \frac{q}{2}\frac{x_{max}^2}{2} - \frac{1}{2}\left(\frac{x_{max}}{L}\frac{q}{2}\right)\frac{x_{max}^2}{3}$$

$$M_{max} = 9.40376 \times 10^{-2} qL^2$$

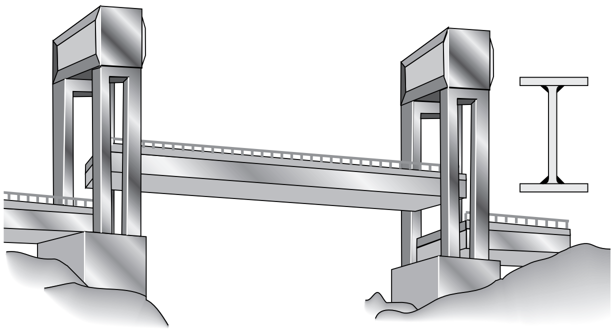
$$M_{max} = 8.727 \text{ kN} \cdot \text{m}$$

$$\sigma_{max} = \frac{M_{max}}{S}$$

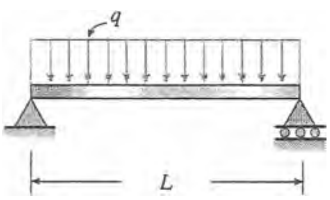
$$\sigma_{max} = 6.493 \times 10^3 \frac{\text{N}}{\text{m}^2} \qquad \sigma_{max} = 6.49 \text{ MPa} \quad \leftarrow$$

Problem 5.5-5 Each girder of the lift bridge (see figure) is 180 ft long and simply supported at the ends. The design load for each girder is a uniform load of intensity 1.6 k/ft. The girders are fabricated by welding tree steel plates so as to form an I-shaped cross section (see figure) having section modulus $S = 3600 \text{ in.}^3$.

What is the maximum bending stress σ_{max} in a girder due to the uniform load?



Solution 5.5-5 Bridge girder



$$L = 180 \text{ ft} \qquad q = 1.6 \text{ k/ft}$$

$$S = 3600 \text{ in.}^3$$

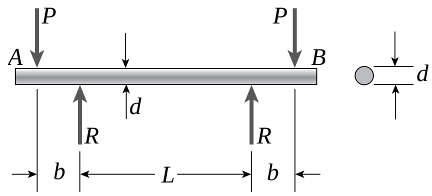
$$M_{max} = \frac{qL^2}{8}$$

$$\sigma_{max} = \frac{M_{max}}{S} = \frac{qL^2}{8S}$$

$$\sigma_{max} = \frac{(1.6 \text{ k/ft})(180 \text{ ft})^2(12 \text{ in./ft})}{8(3600 \text{ in.}^3)} = 21.6 \text{ ksi} \quad \leftarrow$$

Problem 5.5-6 A freight-car axle AB is loaded approximately as shown in the figure, with the forces P representing the car loads (transmitted to the axle through the axle boxes) and the forces R representing the rail loads (transmitted to the axle through the wheels). The diameter of the axle is $d = 80$ mm, the distance between centers of the rails is L, and the distance between the forces P and is R is $b = 200$ mm.

Calculate the maximum bending stress σ_{max} in the axle if $P = 47$ kN.



Solution 5.5-6

NUMERICAL DATA

$d = 82$ mm $\qquad b = 220$ mm

$P = 50$ kN

$I = \dfrac{\pi d^4}{64} \qquad I = 2.219 \times 10^{-6}$ m^4

$M_{max} = Pb \qquad M_{max} = 11$ kN·m

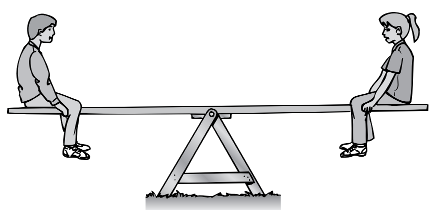
MAX. BENDING STRESS

$$\sigma_{max} = \frac{Md}{2I}$$

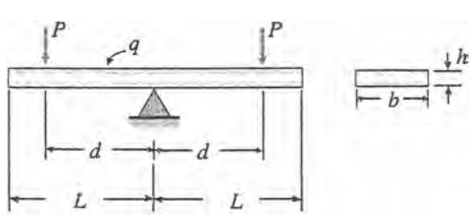
$$\sigma_{max} = 203 \text{ MPa} \qquad \leftarrow$$

Problem 5.5-7 A seesaw weighing 3 lb/ft of length is occupied by two children, each weighing 90 lb (see figure). The center of gravity of each child is 8 ft from the fulcrum. The board is 19 ft long, 8 in. wide, and 1.5 in. thick.

What is the maximum bending stress in the board?



Solution 5.5-7 Seesaw



$b = 8$ in. $\qquad h = 1.5$ in.

$q = 3$ lb/ft $\qquad P = 90$ lb $\qquad d = 8.0$ ft $\qquad L = 9.5$ ft

$$M_{max} = Pd + \frac{qL^2}{2} = 720 \text{ lb-ft} + 135.4 \text{ lb-ft}$$

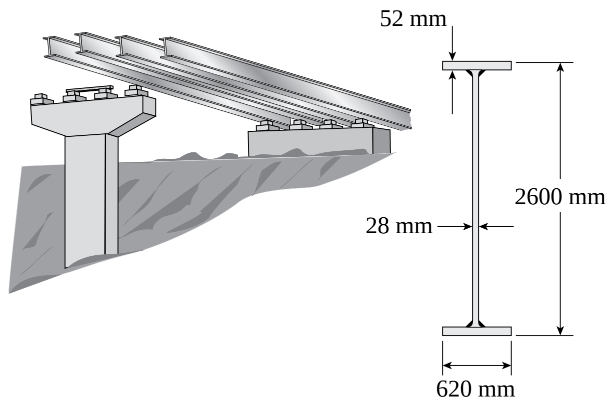
$$= 855.4 \text{ lb-ft} = 10{,}264 \text{ lb-in.}$$

$$S = \frac{bh^2}{6} = 3.0 \text{ in.}^3.$$

$$\sigma_{max} = \frac{M}{S} = \frac{10{,}264 \text{ lb-in.}}{3.0 \text{ in.}^3} = 3420 \text{ psi} \qquad \leftarrow$$

Problem 5.5-8 During construction of a highway bridge, the main girders are cantilevered outward from one pier toward the next (see figure). Each girder has a cantilever length of 48 m and an I-shaped cross section with dimensions shown in the figure. The load on each girder (during construction) is assumed to be 9.5 kN/m, which includes the weight of the girder.

Determine the maximum bending stress in a girder due to this load.



Solution 5.5-8

NUMERICAL DATA

$t_f = 52 \text{ mm} \qquad t_w = 28 \text{ mm}$

$h = 2600 \text{ mm} \qquad b_f = 620 \text{ mm}$

$L = 48 \text{ m} \qquad q = 9.5 \dfrac{\text{kN}}{\text{m}}$

$$I = \frac{1}{12}(b_f)\,h^3 - \frac{1}{12}(b_f - t_w)\,[\,h - 2(t_f)]^3$$

$I = 1.41 \times 10^{11} \text{ mm}^4$

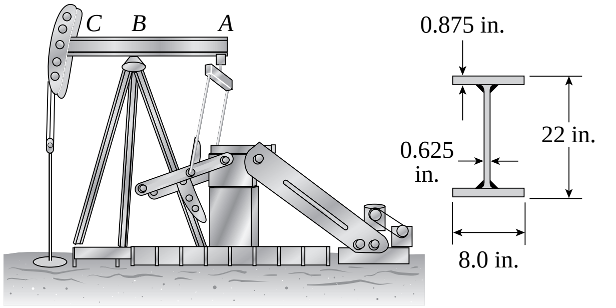
$M_{max} = qL\left(\dfrac{L}{2}\right) \qquad M_{max} = 1.094 \times 10^4 \text{ kN m}$

$\sigma_{max} = \dfrac{M_{max}\,h}{2I}$

$\sigma_{max} = 101 \text{ MPa} \qquad \leftarrow$

Problem 5.5-9 The horizontal beam *ABC* of an oil-well pump has the cross section shown in the figure. If the vertical pumping force acting at end *C* is 9 k and if the distance from the line of action of that force to point *B* is 16 ft, what is the maximum bending stress in the beam due to the pumping force?

Horizontal beam transfers loads as part of oil well pump



Solution 5.5-9

NUMERICAL DATA

$F_C = 9$ k $\qquad BC = 16$ ft

$M_{max} = F_C(BC) \qquad M_{max} = 144$ k-ft

$$I = \frac{1}{12}(8)(22)^3 - \frac{1}{12}(8 - 0.625)$$
$$\times [22 - 2(0.875)]^3 \qquad I = 1.995 \times 10^3 \text{ in.}^4$$

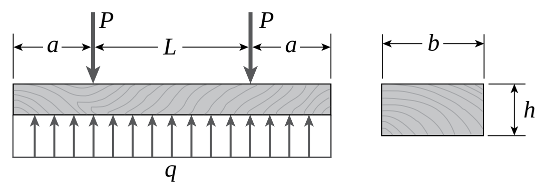
MAX. BENDING STRESS AT B

$$\sigma_{max} = \frac{M_{max}(12)\left(\frac{22}{2}\right)}{I}$$

$$\sigma_{max} = 9.53 \text{ ksi} \quad \leftarrow$$

Problem 5.5-10 A railroad tie (or *sleeper*) is subjected to two rail loads, each of magnitude $P = 175$ kN, acting as shown in the figure. The reaction q of the ballast is assumed to be uniformly distributed over the length of the tie, which has cross-sectional dimensions $b = 300$ mm and $h = 250$ mm.

Calculate the maximum bending stress σ_{max} in the tie due to the loads P, assuming the distance $L = 1500$ mm and the overhang length $a = 500$ mm.



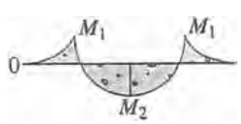
Solution 5.5-10 Railroad tie (or sleeper)

DATA $\quad P = 175$ kN $\quad b = 300$ mm $\quad h = 250$ mm

$L = 1500$ mm $\quad a = 500$ mm

$$q = \frac{2P}{L + 2a} \qquad S = \frac{bh^2}{6} = 3.125 \times 10^{-3} \text{ m}^3$$

BENDING-MOMENT DIAGRAM



$$M_1 = \frac{qa^2}{2} = \frac{Pa^2}{L + 2a}$$

$$M_2 = \frac{q}{2}\left(\frac{L}{2} + a\right)^2 - \frac{PL}{2}$$

$$= \frac{P}{L + 2a}\left(\frac{L}{2} + a\right)^2 - \frac{PL}{2}$$

$$= \frac{P}{4}(2a - L)$$

Substitute numerical values:

$M_1 = 17{,}500$ N · m $\qquad M_2 = -21{,}875$ N · m

$M_{max} = 21{,}875$ N · m

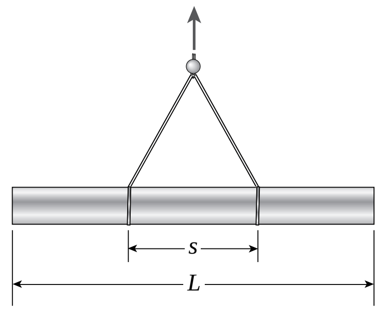
MAXIMUM BENDING STRESS

$$\sigma_{max} = \frac{M_{max}}{5} = \frac{21{,}875 \text{ N} \cdot \text{m}}{3.125 \times 10^{-3} \text{ m}^3} = 7.0 \text{ MPa} \quad \leftarrow$$

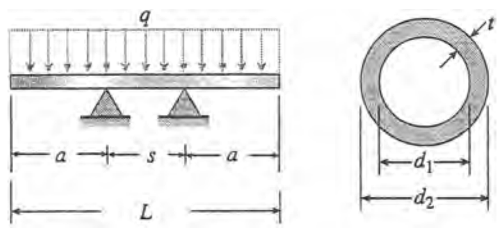
(Tension on top; compression on bottom)

Problem 5.5-11 A fiberglass pipe is lifted by a sling, as shown in the figure. The outer diameter of the pipe is 6.0 in., its thickness is 0.25 in., and its weight density is 0.053 lb/in.^{3} The length of the pipe is $L = 36$ ft and the distance between lifting points is $s = 11$ ft.

Determine the maximum bending stress in the pipe due to its own weight.



Solution 5.5-11 Pipe lifted by a sling



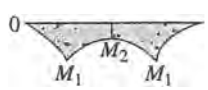
$L = 36 \text{ ft} = 432 \text{ in.} \qquad d_2 = 6.0 \text{ in.} \qquad t = 0.25 \text{ in.}$

$s = 11 \text{ ft} = 132 \text{ in.} \qquad d_1 = d_2 - 2t = 5.5 \text{ in.}$

$\gamma = 0.053 \text{ lb/in.}^3 \qquad A = \dfrac{\pi}{4}(d_2^2 - d_1^2) = 4.5160 \text{ in.}^2$

$a = (L - s)/2 = 150 \text{ in.}$

BENDING-MOMENT DIAGRAM



$$M_1 = -\frac{qa^2}{2} = -2{,}692.7 \text{ lb-in.}$$

$$M_2 = -\frac{qL}{4}\left(\frac{L}{2} - s\right) = -2{,}171.4 \text{ lb-in.}$$

$$M_{max} = 2{,}692.7 \text{ lb-in.}$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 18.699 \text{ in.}^4$$

$$q = \gamma A = (0.053 \text{ lb/in.}^3)(4.5160 \text{ in.}^2) = 0.23935 \text{ lb/in.}$$

MAXIMUM BENDING STRESS

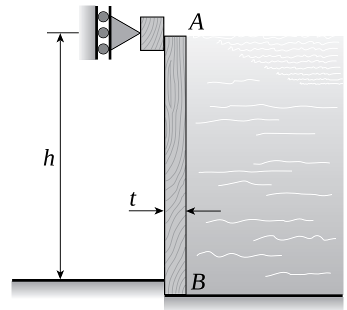
$$\sigma_{max} = \frac{M_{max}\, c}{I} \qquad c = \frac{d_2}{2} = 3.0 \text{ in.}$$

$$\sigma_{max} = \frac{(2{,}692.7 \text{ lb-in.})(3.0 \text{ in.})}{18.699 \text{ in.}^4} = 432 \text{ psi} \qquad \leftarrow$$

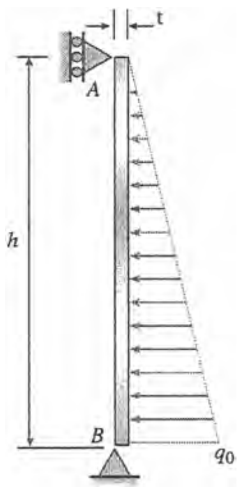
(Tension on top)

Problem 5.5-12 A small dam of height $h = 2.0$ m is constructed of vertical wood beams AB of thickness $t = 120$ mm, as shown in the figure. Consider the beams to be simply supported at the top and bottom.

Determine the maximum bending stress σ_{max} in the beams, assuming that the weight density of water is $\gamma = 9.81$ kN.m^3



Solution 5.5-12 Vertical wood beam



$h = 2.0$ m

$t = 120$ mm

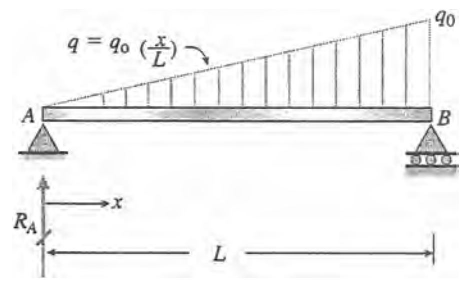
$\gamma = 9.81\text{kN/m}^3\text{(water)}$

Let b = width of beam perpendicular to the plane of the figure

Let q_0 = maximum intensity of distributed load

$$q_0 = \gamma b h \quad S = \frac{bt^2}{6}$$

MAXIMUM BENDING MOMENT



$$R_A = \frac{q_0 L}{6}$$

$$M = R_A x - \frac{q_0 x^3}{6L}$$

$$= \frac{q_0 Lx}{6} - \frac{q_0 x^3}{6L}$$

$$\frac{dM}{dx} = \frac{q_0 L}{6} - \frac{q_0 x^2}{2L} = 0 \quad x = \frac{L}{\sqrt{3}}$$

Substitute $x = L/\sqrt{3}$ into the equation for M:

$$M_{max} = \frac{q_0 L}{6}\left(\frac{L}{\sqrt{3}}\right) - \frac{q_0}{6L}\left(\frac{L^3}{3\sqrt{3}}\right) = \frac{q_0 L^2}{9\sqrt{3}}$$

For the vertical wood beam: $L = h$; $M_{max} = \dfrac{q_0 h^2}{9\sqrt{3}}$

Maximum bending stress

$$\sigma_{max} = \frac{M_{max}}{S} = \frac{2q_0 h^2}{3\sqrt{3}\, bt^2} = \frac{2\gamma h^3}{3\sqrt{3}\, t^2}$$

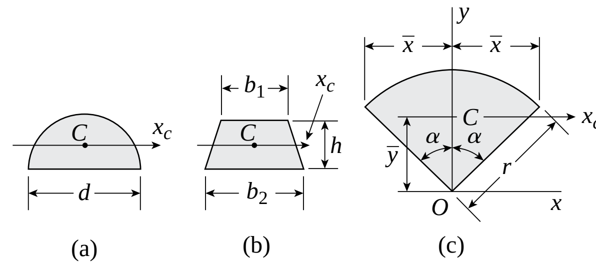
SUBSTITUTE NUMERICAL VALUES:

$\sigma_{max} = 2.10$ MPa $\quad\leftarrow$

NOTE: For $b = 1.0$ m, we obtain $q_0 = 19{,}620$ N/m, $S = 0.0024\text{ m}^3$,
$M_{max} = 5{,}034.5\text{ N}\cdot\text{m}$, and $\sigma_{max} = M_{max}/S = 2.10$ MPa

Problem 5.5-13 Determine the maximum tensile stress σ_t (due to pure bending about a horizontal axis through C by positive bending moments M) for beams having cross sections as follows (see figure).

(a) A semicircle of diameter d
(b) An isosceles trapezoid with bases $b_1 = b$ and $b_2 = 4b/3$, and altitude h
(c) A circular sector with $\alpha = \pi/3$ and $r = d/2$



Solution 5.5-13

MAX. TENSILE STRESS DUE TO POSITIVE BENDING MOMENT IS ON BOTTOM OF BEAM CROSS-SECTION

(a) SEMICIRCLE

From Appendix D, Case 10:

$$I_c = \frac{(9\pi^2 - 64)r^4}{72\pi} = \frac{(9\pi^2 - 64)d^4}{1152\pi}$$

$$c = \frac{4r}{3\pi} = \frac{2d}{3\pi}$$

$$\sigma_t = \frac{Mc}{I_c} = \frac{768M}{(9\pi^2 - 64)d^3} = 30.93\frac{M}{d^3} \quad \leftarrow$$

(b) ISOSCELES TRAPEZOID

From Appendix D, Case 8:

$$I_C = \frac{h^3(b_1^2 + 4b_1b_2 + b_2^2)}{36(b_1 + b_2)}$$

$$= \frac{73bh^3}{756}$$

$$c = \frac{h(2b_1 + b_2)}{3(b_1 + b_2)} = \frac{10h}{21}$$

$$\sigma_t = \frac{Mc}{I_c} = \frac{360M}{73bh^2} \quad \leftarrow$$

(c) CIRCULAR **SECTOR** WITH $\alpha = \pi/3$, $r = d/2$

From Appendix D, Case 13:

$$A = r^2(\alpha)$$

$$I_x = \frac{r^4}{4}(\alpha + \sin(\alpha)\cos(\alpha))$$

$$y_{\text{bar}} = \frac{2r}{3}\left(\frac{\sin(\alpha)}{\alpha}\right) \qquad c = y_{\text{bar}}$$

$$d = 1$$

For $\alpha = \pi/3$, $r = d/2$: $\quad A = \left(\frac{d}{2}\right)^2\left(\frac{\pi}{3}\right)$

$$A = d^2\left(\frac{\pi}{12}\right) \qquad A = 0.2618\, d^2$$

$$c = \frac{2\left(\frac{d}{2}\right)}{3}\left(\frac{\sin\left(\frac{\pi}{3}\right)}{\frac{\pi}{3}}\right) \qquad c = 0.276\, d$$

$$I_x = \frac{\left(\frac{d}{2}\right)^4}{4}\left(\frac{\pi}{3} + \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)\right)$$

$$I_x = 0.02313\, d^4$$

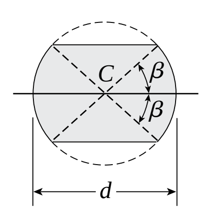
$$I_C = I_x - A y_{\text{bar}}^2$$

$$I_C = \left[d^4\frac{(4\pi + 3\sqrt{3})}{768} - d^2\left(\frac{\pi}{12}\right)\left[\frac{d}{2}\left(\frac{\sqrt{3}}{\pi}\right)\right]^2\right]$$

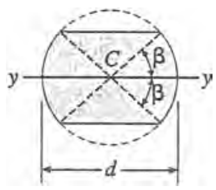
$$I_C = 3.234 \times 10^{-3}\, d^4$$

max. tensile stress $\sigma_t = \dfrac{Mc}{I_C} \qquad \sigma_t = 85.24\dfrac{M}{d^3} \quad \leftarrow$

Problem 5.5-14 Determine the maximum bending stress σ_{max} (due to pure bending by a moment M) for a beam having a cross section in the form of a circular core (see figure). The circle has diameter d and the angle $\beta = 60°$. (*Hint*: Use the formulas given in Appendix D, Cases 9 and 15.)



Solution 5.5-14 Circular core



From Appendix D, Cases 9 and 15:

$$I_y = \frac{\pi r^4}{4} - \frac{r^4}{2}\left(\alpha - \frac{ab}{r^2} + \frac{2ab^3}{r^4}\right)$$

$$r = \frac{d}{2} \quad \alpha = \frac{\pi}{2} - \beta$$

β = radians $\quad \alpha$ = radians $\quad a = r\sin\beta \quad b = r\cos\beta$

$$I_y = \frac{\pi d^4}{64} - \frac{d^4}{32}\left(\frac{\pi}{2} - \beta - \sin\beta\cos\beta + 2\sin\beta\cos^3\beta\right)$$

$$= \frac{\pi d^4}{64} - \frac{d^4}{32}\left(\frac{\pi}{2} - \beta - (\sin\beta\cos\beta)(1 - 2\cos^2\beta)\right)$$

$$= \frac{\pi d^4}{64} - \frac{d^4}{32}\left(\frac{\pi}{2} - \beta - \left(\frac{1}{2}\sin 2\beta\right)(-\cos 2\beta)\right)$$

$$= \frac{\pi d^4}{64} - \frac{d^4}{32}\left(\frac{\pi}{2} - \beta + \frac{1}{4}\sin 4\beta\right)$$

$$= \frac{d^4}{128}(4\beta - \sin 4\beta)$$

Maximum bending stress

$$\sigma_{max} = \frac{Mc}{I_y} \quad c = r\sin\beta = \frac{d}{2}\sin\beta$$

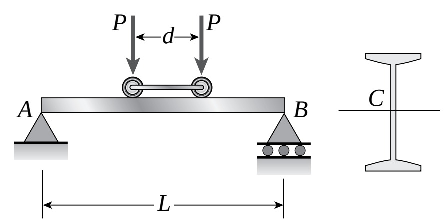
$$\sigma_{max} = \frac{64M\sin\beta}{d^3(4\beta - \sin 4\beta)} \quad \leftarrow$$

For $\beta = 60° = \pi/3$ rad:

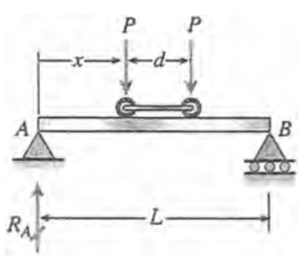
$$\sigma_{max} = \frac{576M}{(8\pi\sqrt{3} + 9)d^3} = 10.96\frac{M}{d^3} \quad \leftarrow$$

Problem 5.5-15 A simple beam AB of span length $L = 24$ ft is subjected to two wheel loads acting at distance $d = 5$ ft apart (see figure). Each wheel transmits a load $P = 3.0$ k, and the carriage may occupy any position on the beam.

Determine the maximum bending stress σ_{max} due to the wheel loads if the beam is an I-beam having section modulus $S = 16.2$ in.^{3}



Solution 5.5-15 Wheel loads on a beam



$L = 24$ ft $= 288$ in.

$d = 5$ ft $= 60$ in.

$P = 3$ k

$S = 16.2$ in.^{3}

Maximum bending moment

$$R_A = \frac{P}{L}L - x + \frac{P}{L}(L - x - d) = \frac{P}{L}(2L - d - 2x)$$

$$M = R_A x = \frac{P}{L}(2Lx - dx - 2x^2)$$

$$\frac{dM}{dx} = \frac{P}{L}(2L - d - 4x) = 0 \quad x = \frac{L}{2} - \frac{d}{4}$$

Substitute x into the equation for M:

$$M_{max} = \frac{P}{2L}\left(L - \frac{d}{2}\right)^2$$

Maximum bending stress

$$\sigma_{max} = \frac{M_{max}}{S} = \frac{P}{2LS}\left(L - \frac{d}{2}\right)^2 \quad \leftarrow$$

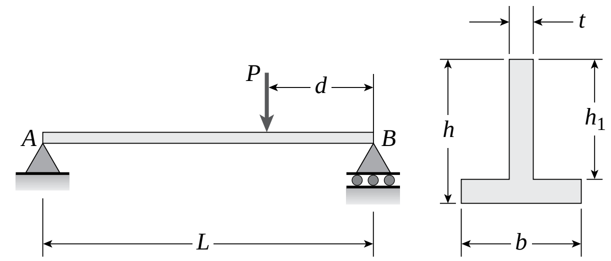
Substitute numerical values:

$$\sigma_{max} = \frac{3\text{k}}{2(288\text{ in.})(16.2\text{ in.}^3)}(288\text{ in.} - 30\text{ in.})^2$$

$$= 21.4\text{ ksi} \quad \leftarrow$$

Problem 5.5-16 Determine the maximum tensile stress σ_t and maximum compressive stress σ_c due to the load P acting on the simple beam AB (see figure).

Data are as follows: $P = 6.2$ kN, $L = 3.2$ m, $d = 1.25$ m, $b = 80$ mm, $t = 25$ mm, $h = 120$ mm, and $h_1 = 90$ mm.



Solution 5.5-16

Numerical data

$P = 6.2$ kN L = 3.2 m

$d = 1.25$ m b = 80 mm

$t = 25$ mm h = 120 mm

$h_1 = 90$ mm

Beam cross section properties: centroid and moment of inertia

$A_f = b(h - h_1) \qquad A_w = th_1$

$$c_1 = \frac{A_w \dfrac{h_1}{2} + A_f\left[h - \dfrac{(h - h_1)}{2}\right]}{A_f + A_w} \qquad c_1 = 76 \text{ mm}$$

$c_2 = h - c_1 \qquad c_2 = 44$ mm dist. to C from bottom

$$I = \frac{1}{12}th_1^3 + \frac{1}{12}b(h - h_1)^3 + A_f\left[c_2 - \frac{(h - h_1)}{2}\right]^2 + A_w\left(c_1 - \frac{h_1}{2}\right)^2$$

$I = 5879395.2 \text{ mm}^4$

Max. moment & normal stresses

$$M_{max} = \frac{Pd(L - d)}{L} \qquad M_{max} = 4.7 \text{ kN}\cdot\text{m}$$

Max. compressive stress at top ($c = c_1$)

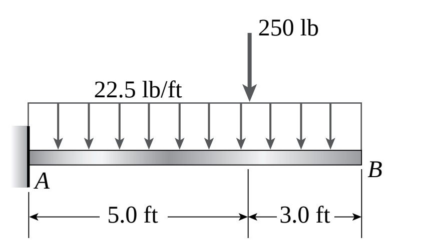
$$\sigma_c = \frac{M_{max}c_1}{I} \qquad \sigma_c = 61.0 \text{ MPa} \quad \leftarrow$$

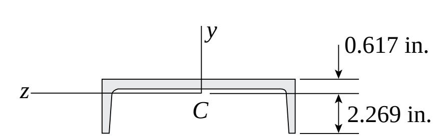
Max. tensile stress at bottom ($c = c_2$)

$$\sigma_t = \frac{M_{max}c_2}{I} \qquad \sigma_t = 35.4 \text{ MPa} \quad \leftarrow$$

Problem 5.5-17 A cantilever beam AB, loaded by a uniform load and a concentrated load (see figure), is constructed of a channel section.

Find the maximum tensile stress σ_t and maximum compressive stress σ_c if the cross section has the dimensions indicated and the moment of inertia about the z axis (the neutal axis) is $I = 3.36 \text{ in.}^4$ (*Note*: The uniform load represents the weight of the beam.)





Solution 5.5-17

NUMERICAL DATA

$I = 3.36 \text{ in.}^4 \qquad c_1 = 0.617 \text{ in.}$

$c_2 = 2.269 \text{ in.}$

$$M_{Amax} = \frac{22.5\,(8)^2}{2} + 250\,(5) \text{ ft-lb}$$

$M_{Amax} = 1970 \text{ ft-lb}$

$M_{Amax}\,(12) = 23640 \text{ in.-lb}$

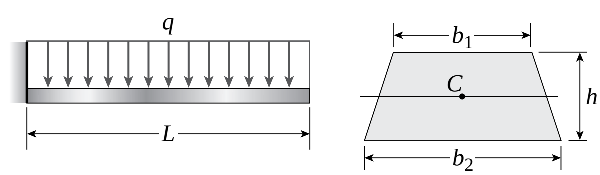
MAXIMUM STRESSES

$$\sigma_t = \frac{M_{Amax}\,c_1}{I} \qquad \sigma_t = 4341 \text{ psi} \quad \longleftarrow$$

$$\sigma_c = \frac{M_{Amax}\,c_2}{I} \qquad \sigma_c = 15964 \text{ psi} \quad \longleftarrow$$

Problem 5.5-18 A cantilever beam AB of isosceles trapezoidal cross section has length $L = 0.8$ m, dimensions $b_1 = 80$ mm, $b_2 = 90$ mm, and height $h = 110$ mm (see figure). The beam is made of brass weighing 85 kN/m^3.

(a) Determine the maximum tensile stress σ_t and maximum compressive stress σ_c due to the beam's own weight.
(b) If the width b_1 is doubled, what happens to the stresses?
(c) If the height h is doubled, what happens to the stresses?



Solution 5.5-18

NUMERICAL DATA

$$L = 0.8 \text{ m} \qquad \gamma = 85\,\frac{\text{kN}}{\text{m}^3}$$

$b_1 = 80 \text{ mm} \qquad b_2 = 90 \text{ mm}$

$h = 110 \text{ mm}$

(a) MAX. STRESSES DUE TO BEAM'S OWN WEIGHT

$$M_{max} = \frac{qL^2}{2} \qquad q = \gamma A \qquad A = \frac{1}{2}(b_1 + b_2)h$$

$A = 9.35 \times 10^3 \text{ mm}^2$

$$q = 7.9475 \times 10^2\,\frac{\text{N}}{\text{m}}$$

$M_{max} = 254.32 \text{ N} \cdot \text{m}$

$$y_{bar} = \frac{h(2b_1 + b_2)}{3(b_1 + b_2)} \qquad y_{bar} = 53.922 \text{ mm}$$

$$I = h^3\,\frac{(b_1^2 + 4b_1b_2 + b_2^2)}{36(b_1 + b_2)}$$

$I = 9.417 \times 10^6 \text{ mm}^4$

MAX. TENSILE STRESS AT SUPPORT (TOP)

$$\sigma_t = \frac{M_{max}(h - y_{bar})}{I} \qquad \sigma_t = 1.514 \text{ MPa} \quad \longleftarrow$$

MAX. COMPRESSIVE STRESS AT SUPPORT (BOTTOM)

$$\sigma_c = \frac{M_{max}y_{bar}}{I} \qquad \sigma_c = 1.456 \text{ MPa} \quad \longleftarrow$$

(b) DOUBLE b_1& RECOMPUTE STRESSES

$b_1 = 160 \text{ mm}$

$$A = \frac{1}{2}(b_1 + b_2)h \qquad A = 1.375 \times 10^4 \text{ mm}^2$$

$$q = \gamma A \qquad q = 1.169 \times 10^3\,\frac{\text{N}}{\text{m}}$$

$$M_{max} = \frac{qL^2}{2}$$

$M_{max} = 374 \text{ N} \cdot \text{m}$

$$y_{bar} = \frac{h(2b_1 + b_2)}{3(b_1 + b_2)} \qquad y_{bar} = 60.133 \text{ mm}$$

$$I = h^3 \frac{(b_1^2 + 4b_1b_2 + b_2^2)}{36(b_1 + b_2)}$$

$$I = 1.35 \times 10^7 \text{ mm}^4$$

MAX. TENSILE STRESS AT SUPPORT (TOP)

$$\sigma_t = \frac{M_{max}(h - y_{bar})}{I} \qquad \sigma_t = 1.381 \text{ MPa} \quad \leftarrow$$

MAX. COMPRESSIVE STRESS AT SUPPORT (BOTTOM)

$$\sigma_c = \frac{M_{max}\, y_{bar}}{2} \qquad \sigma_c = 1.666 \text{ MPa} \quad \leftarrow$$

(c) DOUBLE h & RECOMPUTE STRESSES

$$b_1 = 80 \text{ mm} \quad h = 220 \text{ mm}$$

$$A = \frac{1}{2}(b_1 + b_2)\, h \qquad A = 1.87 \times 10^4 \text{ mm}^2$$

$$q = \gamma A \qquad q = 1.589 \times 10^3 \frac{\text{N}}{\text{m}}$$

$$M_{max} = \frac{qL^2}{2} \qquad M_{max} = 508.64 \text{ N}\cdot\text{m}$$

$$y_{bar} = \frac{h(2b_1 + b_2)}{3(b_1 + b_2)} \qquad y_{bar} = 107.843 \text{ mm}$$

$$I = h^3 \frac{(b_1{}^2 + 4b_1b_2 + b_2{}^2)}{36(b_1 + b_2)}$$

$$I = 7.534 \times 10^7 \text{ mm}^4$$

MAX. TENSILE STRESS AT SUPPORT (TOP)

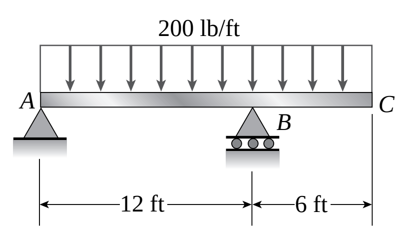
$$\sigma_t = \frac{M_{max}(h - y_{bar})}{I} \qquad \sigma_t = 0.757 \text{ MPa} \quad \leftarrow$$

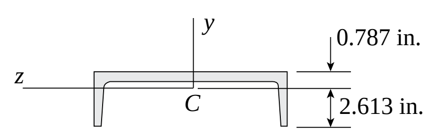
MAX. COMPRESSIVE STRESS AT SUPPORT (BOTTOM)

$$\sigma_c = \frac{M_{max}\, y_{bar}}{I} \qquad \sigma_c = 0.728 \text{ MPa} \quad \leftarrow$$

Problem 5.5-19 A beam ABC with an overhang from B to C supports a uniform load of 200 lb/ft throughout its length (see figure). The beam is a channel section with dimensions as shown in the figure. The moment of inertia about the z axis (the neutral axis) equals 8.13 in.^{4}

Calculate the maximum tensile stress σ_t and maximum compressive stress σ_c due to the uniform load.





Solution 5.5-19

NUMERICAL DATA

$$q = 200 \frac{\text{lb}}{\text{ft}} \qquad I = 8.13 \text{ in.}^4$$

$$c_1 = 0.787 \text{ in.} \qquad c_2 = 2.613 \text{ in.}$$

COMPUTE SUPPORT REACTIONS

$$\sum M_A = 0 \qquad R_B = \frac{q\frac{(18)^2}{2}}{12} \qquad R_B = 2700 \text{ lb}$$

$$\sum F_v = 0 \qquad R_A = q(18) - R_B \qquad R_A = 900 \text{ lb}$$

LOCATON OF ZERO SHEAR IN SPAN AB & MAX. (+) MOMENT IN SPAN AB

$$x_{max} = \frac{R_A}{q} \qquad x_{max} = 4.5 \text{ ft}$$

$$M_{maxAB} = R_A x_{max} - q\frac{x_{max}{}^2}{2}$$

$$M_{maxAB} = 2025 \text{ ft-lb}$$

max. $(-)$ moment at B

$$M_B = q\frac{(6)^2}{2} \qquad M_B = 3600 \text{ ft-lb}$$

MAX. STRESSES IN SPAN *AB*

$$\sigma_C = \frac{M_{maxAB}(12)\,c_1}{I} \qquad \sigma_c = 2352 \text{ psi}$$

$$\sigma_t = \frac{M_{maxAB}(12)\,c_2}{I}$$

$$\sigma_t = 7810 \text{ psi} \quad \longleftarrow \quad \text{max. tensile stress}$$

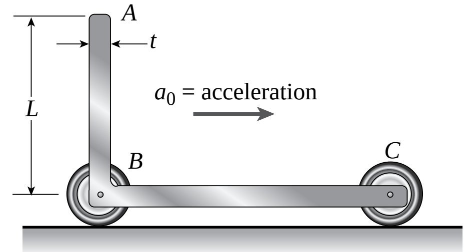
MAX. STRESSES IN SPAN *BC*

$$\sigma_c = \frac{M_B(12)\,c_2}{I}$$

$$\sigma_c = 13885 \text{ psi} \quad \longleftarrow \quad \text{max. compressive stress}$$

$$\sigma_t = \frac{M_B(12)\,c_1}{I} \qquad \sigma_t = 4182 \text{ psi}$$

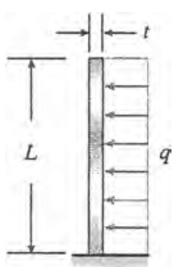
Problem 5.5-20 A frame *ABC* travels horizontally with an acceleration a_0 (see figure). Obtain a formula for the maximum stress σ_{max} in the vertical arm *AB*, which had length *L*, thickness *t* , and mass density ρ.



Solution 5.5-20 Accelerating frame

L = length of vertical arm
t = thickness of vertical arm
ρ = mass density
a_0 = acceleration
Let b = width of arm perpendicular to the plane of the figure
Let q = inertia force per unit distance along vertical arm

VERTICAL ARM



$$q = \rho b t a_0 \qquad M_{max} = \frac{qL^2}{2} = \frac{\rho b t a_0 L^2}{2}$$

$$S = \frac{bt^2}{6} \qquad \sigma_{max} = \frac{M_{max}}{S} = \frac{3\rho L^2 a_0}{t} \quad \longleftarrow$$

TYPICAL UNITS FOR USE IN THE PRECEDING EQUATION

SI units: $\rho = \text{kg/m}^3 = \text{N} \cdot \text{s}^2/\text{m}^4$

L = meters (m)

$a_0 = \text{m/s}^2$

t = meters (m)

$\sigma_{max} = \text{N/m}^2$ (pascals)

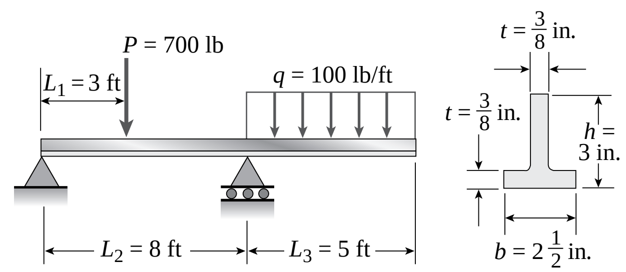
USCS units: $\rho = \text{slug/ft}^3 = \text{lb-s}^2/\text{ft}^4$

L = ft $\quad a_0 = \text{ft/s}^2 \quad t$ = ft

$\sigma_{max} = \text{lb/ft}^2$ (Divide by 144 to obtain psi)

Problem 5.5-21 A beam of T-section is supported and loaded as shown in the figure. The cross section has width $b = 2\ 1/2$ in., height $h = 3$ in., and thickness $t = 3/8$ in.

Determine the maximum tensile and compressive stresses in the beam.



Solution 5.5-21

Numerical data

$L_1 = 3\text{ ft} \qquad L_2 = 8\text{ ft} \qquad L_3 = 5\text{ ft}$

$P = 700\text{ lb} \qquad q = 100\,\dfrac{\text{lb}}{\text{ft}}$

$t = \dfrac{3}{8}\text{ in.} \qquad h = 3\text{ in.} \qquad b = 2.5\text{ in.}$

Find centroid of cross section (c_2 from bottom, c_1 from top) $\quad A_w = t\,(h - t) \qquad A_f = tb$

$$c_2 = \frac{A_f\dfrac{t}{2} + A_w\left(t + \dfrac{h-t}{2}\right)}{A_f + A_w} \qquad c_2 = 1\text{ in.}$$

$c_1 = h - c_2 \qquad c_1 = 2\text{ in.}$

$$\text{check} \quad c_1 = \frac{A_w\left(\dfrac{h-t}{2}\right) + A_f\left(h - \dfrac{t}{2}\right)}{A_f + A_w}$$

$c_1 = 2 \qquad c_1 + c_2 = 3 \quad$ equals h

Moment of inertia

$$I = \frac{1}{12}\,t(h - t)^3 + \frac{1}{12}\,b\,t^3 + A_f\left(c_2 - \frac{t}{2}\right)^2$$

$$+ A_w\left[c_1 - \frac{(h-t)}{2}\right]^2 \qquad I = 2\text{ in.}^4$$

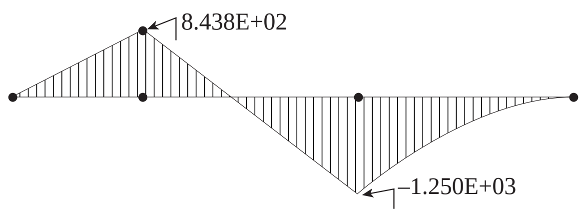
Find support reactions-Sum moments about left support

$$\sum M_{lf} = 0 \qquad R_{rt} = \frac{PL_1 + qL_3\left(L_2 + \dfrac{L_3}{2}\right)}{L_2}$$

$R_{rt} = 919\text{ lb}$

$\sum F_v = 0 \qquad R_{lf} = P + qL_3 - R_{rt} \qquad R_{lf} = 281\text{ lb}$

Moment diagram (843.75 ft-lb at load P, -1250 ft-lb at right support)



$M_P = 843.75\text{ ft-lb}$

$M_{rt} = 1250\text{ ft-lb}$

Max. stresses in beam

at load P

$\sigma_c = \dfrac{M_P\,(12)\,c_1}{I} \qquad \sigma_c = 12494\text{ psi} \quad \leftarrow$

(max. compressive stress)

$\sigma_t = \dfrac{M_P\,(12)\,c_2}{I} \qquad \sigma_t = 5842\text{ psi}$

at right support

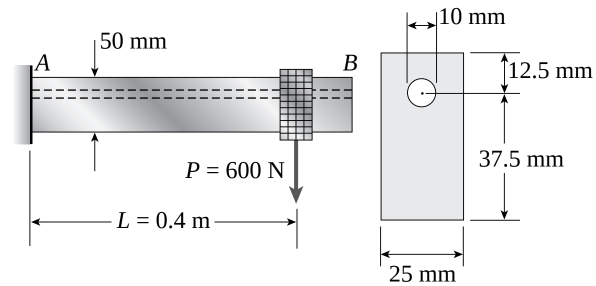
$\sigma_c = \dfrac{M_{rt}\,(12)\,c_2}{I} \qquad \sigma_c = 8654\text{ psi}$

$\sigma_t = \dfrac{M_{rt}\,(12)\,c_1}{I} \qquad \sigma_t = 18509\text{ psi} \quad \leftarrow$

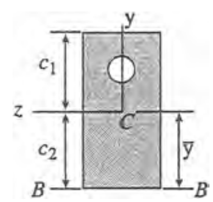
(max. tensile stress)

Problem 5.5-22 A cantilever beam AB with a rectangular cross section has a longitudinal hole drilled throughout its length (see figure). The beam supports a load $P = 600$ N. The cross section is 25 mm wide and 50 mm high, and the hole has a diameter of 10 mm.

Find the bending stresses at the top of the beam, at the top of the hole, and at the bottom of the beam.



Solution 5.5-22 Rectangular beam with a hole



MAXIMUM BENDING MOMENT

$M = PL = (600 \text{ N})(0.4 \text{ m}) = 240 \text{ N} \cdot \text{m}$

PROPERTIES OF THE CROSS SECTION

A_1 = area of rectangle

$= (25 \text{ mm})(50 \text{ mm}) = 1250 \text{ mm}^2$

A_2 = area of hole

$= \dfrac{\pi}{4}(10 \text{ mm})^2 = 78.54 \text{ mm}^2$

A = area of cross section

$= A_1 - A_2 = 1171.5$ mm

Using line $B - B$ as reference axis:

$\Sigma A_i y_i = A_1(25 \text{ mm}) - A_2(37.5 \text{ mm}) = 28{,}305 \text{ mm}^3$

$$\bar{y} = \frac{\sum A_i y_i}{A} = \frac{28{,}305 \text{ mm}^3}{1171.5 \text{ mm}^2} = 24.162 \text{ mm}$$

Distances to the centroid C:

$c_2 = \bar{y} = 24.162$ mm

$c_1 = 50 \text{ mm} - c_2 = 25.838$ mm

MOMENT OF INERTIA ABOUT THE NEUTRAL AXIS (THE Z AXIS)

All dimensions in millimeters.

Rectangle:

$I_z = I_c + Ad^2$

$$= \frac{1}{12}(25)(50)^3 + (25)(50)(25 - 24.162)^2$$

$= 260{,}420 + 878 = 261{,}300 \text{ mm}^4$

Hole:

$$I_z = I_c + Ad^2 = \frac{\pi}{64}(10)^4 + (78.54)(37.5 - 24.162)^2$$

$= 490.87 + 13{,}972 = 14{,}460 \text{ mm}^4$

Cross-section:

$I = 261{,}300 - 14{,}460 = 246{,}800 \text{ mm}^4$

STRESS AT THE TOP OF THE BEAM

$$\sigma_1 = \frac{Mc_1}{I} = \frac{(240 \text{ N} \cdot \text{m})(25.838 \text{ mm})}{246{,}800 \text{ mm}^4}$$

$= 25.1$ MPa ←
(tension)

STRESS AT THE TOP OF THE HOLE

$$\sigma_2 = \frac{My}{I} \qquad y = c_1 - 7.5 \text{ mm} = 18.338 \text{ mm}$$

$$\sigma_2 = \frac{(240 \text{ N} \cdot \text{m})(18.338 \text{ mm})}{246{,}800 \text{ mm}^4} = 17.8 \text{ MPa} \quad \leftarrow$$

(tension)

STRESS AT THE BOTTOM OF THE BEAM

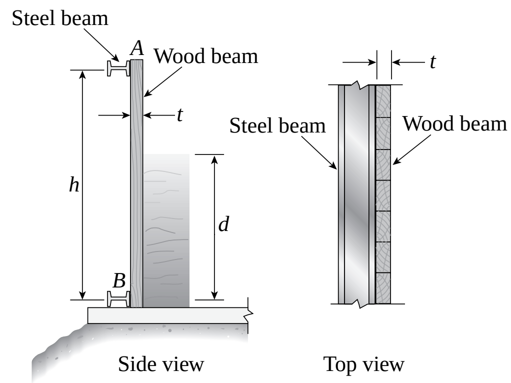
$$\sigma_3 = -\frac{Mc_2}{I} = -\frac{(240 \text{ N} \cdot \text{m})(24.162 \text{ mm})}{246{,}800 \text{ mm}^4}$$

$= -23.5$ MPa ←

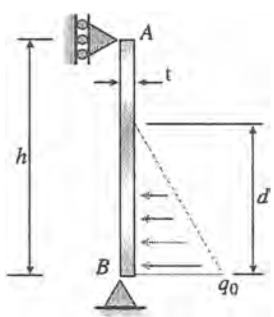
(compression)

Problem 5.5-23 A small dam of height $h = 6$ ft is constructed of vertical wood beams AB, as shown in the figure. The wood beams, which have thickness $t = 2.5$ in., are simply supported by horizontal steel beams at A and B.

Construct a graph showing the maximum bending stress σ_{max} in the wood beams versus the depth d of the water above the lower support at B. Plot the stress σ_{max} (psi) as the ordinate and the depth d (ft) as the abscissa. (*Note*: The weight density γ of water equals 62.4 lb/ft^3.)



Solution 5.5-23 Vertical wood beam in a dam



$h = 6$ ft

$t = 2.5$ in.

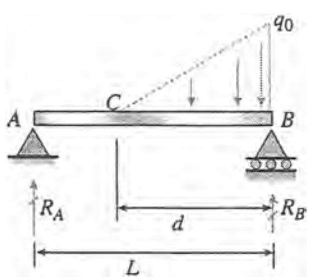
$\gamma = 62.4$ lb/ft^3

Let $b =$ width of beam (perpendicular to the figure)

Let $q_0 =$ intensity of load at depth d

$q_0 = \gamma b d$

ANALYSIS OF BEAM



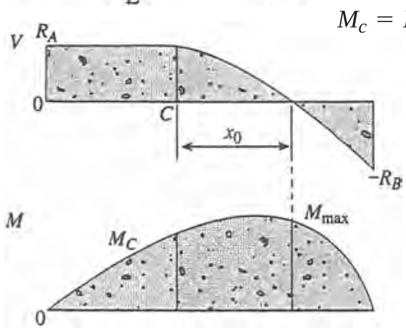
$L = h = 6$ ft

$$R_A = \frac{q_0 d^2}{6L}$$

$$R_B = \frac{q_0 d}{6}\left(3 - \frac{d}{L}\right)$$

$$x_0 = d\sqrt{\frac{d}{3L}}$$

$$M_c = R_A(L - d) = \frac{q_0 d^2}{6}\left(1 - \frac{d}{L}\right)$$



$$M_{max} = \frac{q_0 d^2}{6}\left(1 - \frac{d}{L} + \frac{2d}{3L}\sqrt{\frac{d}{3L}}\right)$$

MAXIMUM BENDING STRESS

Section modulus: $S = \frac{1}{6}bt^2$

$$\sigma_{max} = \frac{M_{max}}{S} = \frac{6}{bt^2}\left[\frac{q_0 d^2}{6}\left(1 - \frac{d}{L} + \frac{2d}{3L}\sqrt{\frac{d}{3L}}\right)\right]$$

$q_0 = \gamma b d$

$$\sigma_{max} = \frac{\gamma d^3}{t^2}\left(1 - \frac{d}{L} + \frac{2d}{3L}\sqrt{\frac{d}{3L}}\right) \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$d =$ depth of water (ft) (Max. $d = h = 6$ ft)

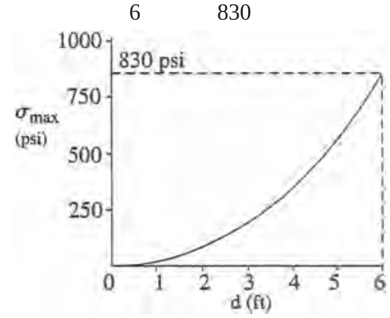
$L = h = 6$ ft $\quad \gamma = 62.4$ lb/ ft^3 $\quad t = 2.5$ in.

$\sigma_{max} =$ psi

$$\sigma_{max} = \frac{(62.4)d^3}{(2.5)^2}\left(1 - \frac{d}{6} + \frac{d}{9}\sqrt{\frac{d}{18}}\right)$$

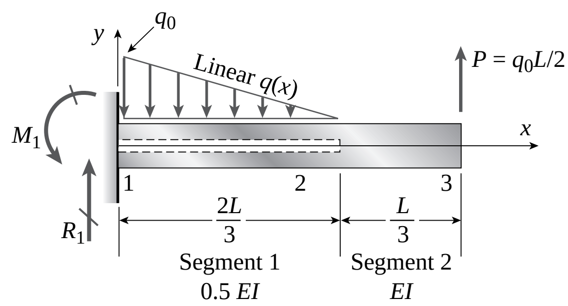
$$= 0.1849d^3(54 - 9d + d\sqrt{2d}) \quad \leftarrow$$

d(ft)	σ_{max}(psi)
0	0
1	9
2	59
3	171
4	347
5	573
6	830



Problem 5.5-24 Consider the nonprismatic *cantilever beam* of circular cross section shown. The beam has an internal cylindrical hole in segment 1; the bar is solid (radius r) in segment 2. The beam is loaded by a downward triangular load with maximum intensity q_0 as shown.

Find expressions for maximum tensile and compressive flexural stresses at joint 1.



Solution 5.5-24

STATICS

$$\sum F_v = 0 \qquad R_1 = \frac{1}{2}q_0\left(\frac{2L}{3}\right) - \frac{q_0L}{2}$$

$$R_1 = \frac{-1}{6}q_0L$$

$$\sum M_1 = 0$$

$$M_1 = \left[\frac{1}{2}q_0\left(\frac{2L}{3}\right)\left(\frac{1}{3}\frac{2L}{3}\right) - \frac{q_0L}{2}L\right]$$

$$M_1 = \frac{-23}{54}q_0L^2 \qquad \frac{23}{54} = 0.426$$

MAX. STRESSES AT JOINT 1

MAX. COMPRESSION AT TOP (RADIUS r)

$$\sigma_c = \frac{M_1 r}{0.5\,EI} \qquad \sigma_c = \frac{\frac{23}{54}q_0L^2(r)}{\frac{EI}{2}}$$

$$\sigma_c = \frac{23}{27}\frac{q_0L^2r}{EI} \qquad \leftarrow \qquad \frac{23}{27} = 0.852$$

Max. tensile stress at bottom = same magnitude as compressive stress at top

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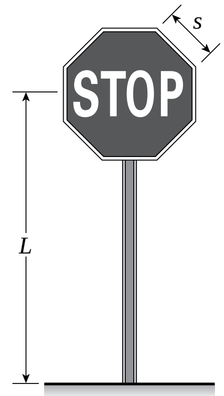
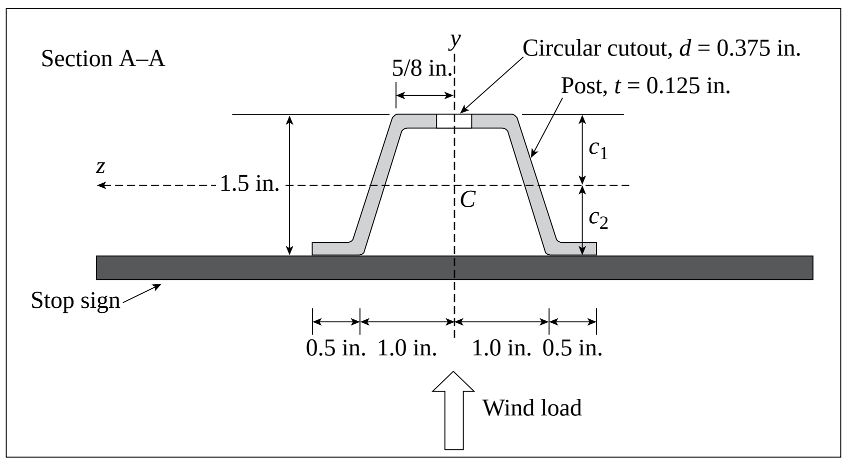
Problem 5.5-25 A steel post ($E = 30 \times 10^6$ psi) having thickness $t = 1/8$ in. and height $L = 72$ in. supports a stop sign (see figure: $s = 12.5$ in.). The height of the post L is measured from the base to the centroid of the sign. The stop sign is subjected to wind pressure $p = 20$ lb/ft^2 normal to its surface. Assume that the post is fixed at its base.

(a) What is the resultant load on the sign? [See Appendix D, Case 25, for properties of an octagon, $n = 8$].

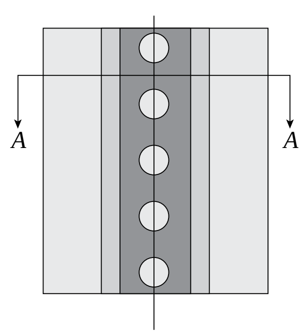
(b) What is the maximum bending stress σ_{max} in the post?





Numerical properties of post

$A = 0.578$ in.^{2}, $c_1 = 0.769$ in., $c_2 = 0.731$ in.,
$I_y = 0.44867$ in.^{4}, $I_z = 0.16101$ in.^{4}



Elevation
view of post

Solution 5.5-25

(a) RESULTANT LOAD F ON SIGN

$p = 20$ psf $\quad s = 12.5$ in. $\quad n = 8$

$\beta = \dfrac{360}{n}\left(\dfrac{\pi}{180}\right) \qquad \beta = 0.785$ rad

$A = \dfrac{ns^2}{4}\cot\left(\dfrac{\beta}{2}\right) \qquad A = 754.442$ in.^{2}

or $A = 5.239$ ft^2

$F = pA \qquad F = 104.8$ lb $\quad \leftarrow$

(b) MAX. BENDING STRESS IN POST

$L = 72$ in. $\quad I_Z = 0.16101$ in.^{4}

$c_1 = 0.769$ in. $\quad c_2 = 0.731$ in.

$M_{max} = FL \qquad \dfrac{M_{max}}{12} = 628.701$ ft-lb

$\sigma_c = \dfrac{M_{max}\,c_1}{I_z} \qquad \sigma_c = 36.0$ ksi $\quad \leftarrow$

(max. bending stress at base of post)

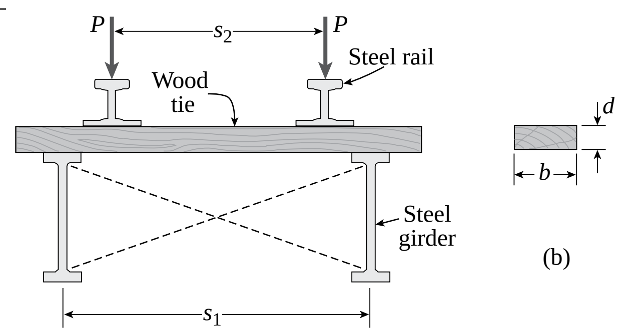
$\sigma_t = \dfrac{M_{max}\,c_2}{I_z} \qquad \sigma_t = 34.2$ ksi

Design of Beams

Problem 5.6-1 The cross section of a narrow-gage railway bridge is shown in part (a) of the figure. The bridge is constructed with longitudinal steel girders that support the wood cross ties. The girders are restrained against lateral buckling by diagonal bracing, as indicated by the dashed lines.

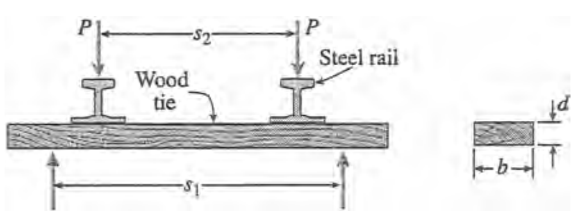
The spacing of the girders is $s_1 = 50$ in. and the spacing of the rails is $s_2 = 30$ in. The load transmitted by each rail to a single tie is $P = 1500$ lb. The cross section of a tie, shown in part (b) of the figure, has width $b = 5.0$ in. and depth d.

Determine the minimum value of d based upon an allowable bending stress of 1125 psi in the wood tie. (Disregard the weight of the tie itself.)



(a)

Solution 5.6-1 Railway cross tie



$s_1 = 50$ in. $\quad b = 5.0$ in. $\quad s_2 = 30$ in.

d = depth of tie $\quad P = 1500$ lb $\quad \sigma_{allow} = 1125$ psi

$M_{max} = \dfrac{P(s_1 - s_2)}{2} = 15{,}000$ lb-in.

$S = \dfrac{bd^2}{6} = \dfrac{1}{6}(50 \text{ in.})(d^2) = \dfrac{5d^2}{6} \qquad d = \text{inches}$

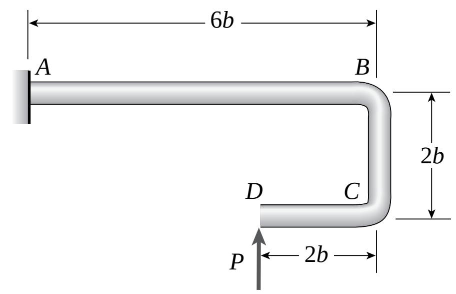
$M_{max} = \sigma_{allow}S \quad 15{,}000 = (1125)\left(\dfrac{5d^2}{6}\right)$

Solving, $\quad d^2 = 16.0$ in. $\quad d_{min} = 4.0$ in. $\quad \leftarrow$

NOTE: Symbolic solution: $d^2 = \dfrac{3P(s_1 - s_2)}{b\sigma_{allow}}$

Problem 5.6-2 A fiberglass bracket *ABCD* of solid circular cross section has the shape and dimensions shown in the figure. A vertical load $p = 40$ N acts at the free end *D*.

Determine the minimum permissible diameter d_{min} of the bracket if the allowable bending stress in the material is 30 MPa and $b = 37$ mm. (*Note:* Disregard the weight of the bracket itself.)



Solution 5.6-2

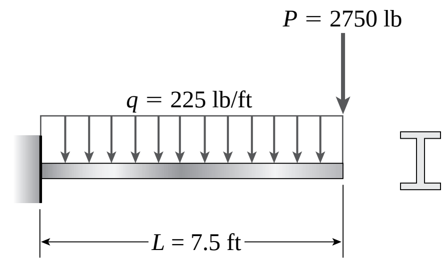
$$\sigma_a = \frac{(3Pb)\left(\frac{d_{min}}{2}\right)}{\left(\frac{\pi d_{min}{}^4}{64}\right)} \qquad d_{min}{}^3 = \frac{96Pb}{\pi\sigma_a}$$

$$d_{min} = \left(\frac{96Pb}{\pi\sigma_a}\right)^{\frac{1}{3}} \qquad d_{min} = \left[\frac{96\,(40)\,(37)}{\pi\,(30)}\right]^{\frac{1}{3}}$$

$$d_{min} = 11.47 \text{ mm} \quad \leftarrow$$

Problem 5.6-3 A cantilever beam of length $L = 7.5$ ft supports a uniform load of intensity $q = 225$ lb/ft and a concentrated load $P = 2750$ lb (see figure).

Calculate the required section modulus S if $\sigma_{allow} = 17{,}000$ psi. Then select a suitable wide-flange beam (W shape) from Table E-1(a), Appendix E, and recalculate S taking into account the weight of beam. Select a new beam size if necessary.



Solution 5.6-3

$\sigma_a = 17000$ psi $\qquad P = 2750$ lb

$q = 225\,\frac{\text{lb}}{\text{ft}} \qquad L = 7.5$ ft

$$M_{max1} = PL + \frac{qL^2}{2} \qquad M_{max1} = 2.695 \times 10^4 \text{ lb-ft}$$

Find S_{reqd} without beam weight

$$S_{reqd} = \frac{M_{max1}\,(12)}{\sigma_a} \qquad S_{reqd} = 19.026 \text{ in.}^3$$

try $W\,8 \times 28$ ($S = 24.3$ in.^{3})

Check - add weight per ft for beam

$W = 28\,\frac{\text{lb}}{\text{ft}} \qquad S_{act} = 24.3$ in.^{3}

$$M_{max2} = PL + \frac{(q + w)\,L^2}{2}$$

$$M_{max2} = 2.774 \times 10^4 \text{ lb-ft}$$

$$\sigma_{max} = \frac{M_{max2}\,(12)}{S_{act}} \qquad \sigma_{max} = 13699 \text{ psi}$$

below allowable -OK

Repeat for W14 × 26 which is lighter than W8 × 28

$w = 26\,\frac{\text{lb}}{\text{ft}} \qquad S_{act} = 35.3$ in.^{3}

$$M_{max3} = PL + \frac{(q + w)\,L^2}{2}$$

$$M_{max3} = 2.768 \times 10^4 \text{ lb-ft}$$

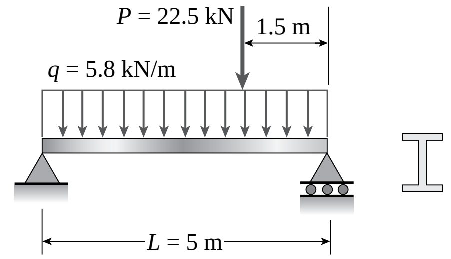
$$\sigma_{max} = \frac{M_{max3}\,(12)}{S_{act}} \qquad \sigma_{max} = 9411 \text{ psi}$$

well below allowable - OK

use W 14 × 26 $\quad\leftarrow$

Problem 5.6-4 A simple beam of length $L = 5$ m carries a uniform load of intensity $q = 5.8\,\dfrac{\text{kN}}{\text{m}}$ and a concentrated load 22.5 kN (see figure).

Assuming $\sigma_{allow} = 110$ MPa, calculate the required section modulus S. Then select an 200 mm wide-flange beam (W shape) from Table E-1(b) Appendix E, and recalculate S taking into account the weight of beam. Select a new 200 mm beam if necessary.



Solution 5.6-4

NUMERICAL DATA

$L = 5\text{ m} \quad q = 5.8\,\dfrac{\text{kN}}{\text{m}}$

$P = 22.5\text{ kN} \qquad b = 1.5\text{ m}$

$a = L - b \quad a = 3.5\text{ m}$

$\sigma_{allow} = 110\text{ MPa}$

statics $\quad R_A = \dfrac{qL}{2} + \dfrac{Pb}{L} \qquad R_A = 21.25\text{ kN}$

$R_B = \dfrac{qL}{2} + \dfrac{Pa}{L} \qquad R_B = 30.25\text{ kN}$

$qL + P = 51.5\text{ kN} \qquad R_A + R_B = 51.5\text{ kN}$

LOCATE POINT OF ZERO SHEAR

$x_m = \dfrac{R_A}{q} \qquad x_m = 3.664\text{ m}$

greater than dist. a to load P so zero shear is at load point

$M_{max} = R_A\,a - \dfrac{q\,a^2}{2} \qquad M_{max} = 38.85\text{ kN}\cdot\text{m}$

FIND REQUIRED SECTION MODULUS

$S_{reqd} = \dfrac{M_{max}}{\sigma_{allow}} \qquad S_{reqd} = 353.182 \times 10^3\text{ mm}^3$

select $W\,200 \times 41.7 \quad \leftarrow \quad (S_{act} = 398 \times 10^3\text{ mm}^3)$

RECOMPUTE MAX. MOMENT WITH BEAM MASS INCLUDED & THEN CHECK ALLOWABLE STRESS

$w = \left(41.7\,\dfrac{\text{kg}}{\text{m}}\right)\left(9.81\,\dfrac{\text{M}}{\text{s}^2}\right)$

$w = 409.077\,\dfrac{\text{N}}{\text{m}} \qquad S_{act} = 398 \times 10^3\text{ mm}^3$

$R_A = \dfrac{\left(q + \dfrac{W}{1000}\right)L}{2} + \dfrac{Pd}{L}$

$R_A = 22.273\text{ kN} \qquad x_m = \dfrac{R_A}{q + W}$

$x_m = 3.587\text{ m}$ greater than a so max. moment at load pt

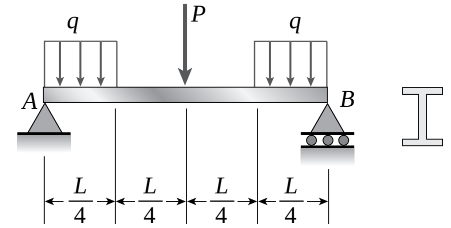
$M_{max} = R_A\,a - \dfrac{(q + W)\,a^2}{2}$

$M_{max} = 39.924\text{ kN}\cdot\text{m}$

$\sigma_{max} = \dfrac{M_{max}}{S_{act}}$

$\sigma_{max} = 100.311\text{ MPa} \quad$ OK, less than 110 MPa

Problem 5.6-5 A simple beam AB is loaded as shown in the figure. Calculate the required section modulus S if $\sigma_{allow} = 17{,}000$ psi, $L = 28$ ft, $P = 2200$ lb, and $q = 425$ lb/ft. Then select a suitable I-beam (S shape) from Table E-2(a), Appendix E, and recalculate S taking into account the weight of the beam. Select a new beam size if necessary.



Solution 5.6-5

NUMERICAL DATA

$\sigma_a = 17000 \text{ psi} \qquad L = 28 \text{ ft}$

$P = 2200 \text{ lb} \qquad q = 425 \dfrac{\text{lb}}{\text{ft}}$

FIND REACTIONS (EQUAL DUE TO SYMMETRY) THEN MAX. MOMENT AT CENTER OF BEAM

$$R_A = \frac{P}{2} + q\frac{L}{4} \qquad R_A = 4.075 \times 10^3 \text{ lb}$$

$$M_{max} = R_A\frac{L}{2} - \frac{qL}{4}\left(\frac{L}{4} + \frac{1}{2}\frac{L}{4}\right)$$

$$M_{max} = 2.581 \times 10^4 \text{ ft-lb}$$

Compute S_{reqd} & then select S shape

$$S_{reqd} = \frac{M_{max}(12)}{\sigma_a} \qquad S_{reqd} = 18.221 \text{ in.}^3$$

select S 10 × 25.4 ←

($S_{act} = 24.6 \text{ in.}^3$, $w = 25.4$ lb/ft)

RECOMPUTE REACTIONS AND MAX. MOMENT THEN CHECK MAX. STRESS $\quad w = 25.4 \dfrac{\text{lb}}{\text{ft}}$

$$R_A = \frac{P}{2} + q\frac{L}{4} + w\frac{L}{2} \qquad R_A = 4.431 \times 10^3 \text{ lb}$$

$$M_{max} = R_A\frac{L}{2} - \frac{qL}{4}\left(\frac{L}{4} + \frac{1}{2}\frac{L}{4}\right) - w\frac{L}{2}\left(\frac{1}{2}\frac{L}{2}\right)$$

$$M_{max} = 2.83 \times 10^4 \text{ ft-lb}$$

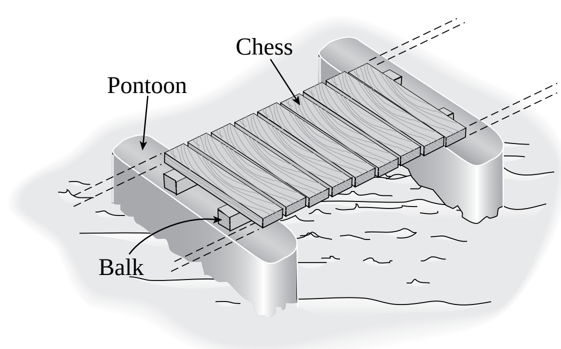
$$\sigma_{max} = \frac{M_{max}(12)}{S_{act}}$$

$\sigma_{max} = 13{,}806$ psi $\quad$ less than allowable so OK

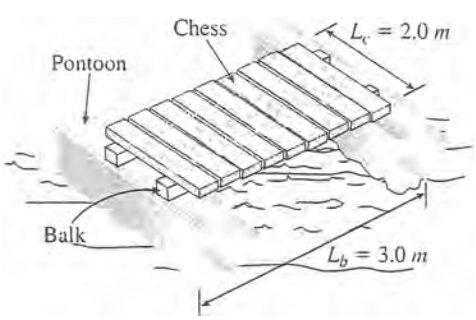
Problem 5.6-6 A pontoon bridge (see figure) is constructed of two longitudinal wood beams, known an *balks*, that span between adjacent pontoons and support the transverse floor beams, which are called *chesses*.

For purposes of design, assume that a uniform floor load of 8.0 kPa acts over the chesses. (This load includes an allowance for the weights of the chesses and balks.) Also, assume that the chesses are 2.0 m long and that the balks are simply supported with a span of 3.0 m. The allowable bending stress in the wood is 16 MPa.

If the balks have a square cross section, what is their minimum required width b_{min}?



Solution 5.6-6 Pontoon bridge

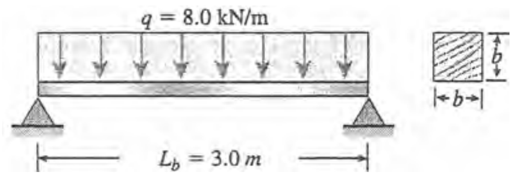


FLOOR LOAD: $W = 8.0$ kPa

ALLOWABLE STRESS: $\sigma_{allow} = 16$ MPa

L_c = length of chesses $= 2.0$ m $\qquad L_b$ = length of balks $= 3.0$ m

LOADING DIAGRAM FOR ONE BALK



$W =$ total load

$= wL_bL_c$

$$q = \frac{W}{2L_b} = \frac{wL_c}{2}$$

$$= \frac{(8.0\ \text{kPa})(2.0\ \text{m})}{2}$$

$= 8.0$ kN/m

Section modulus $S = \dfrac{b^3}{6}$

$$M_{\max} = \frac{qL_b^2}{8} = \frac{(8.0\ \text{kN/m})(3.0\ \text{m})^2}{8} = 9{,}000\ \text{N}\cdot\text{m}$$

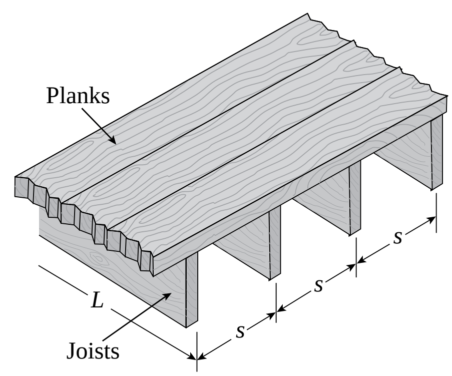
$$S = \frac{M_{\max}}{\sigma_{\text{allow}}} = \frac{9{,}000\ \text{N}\cdot\text{m}}{16\ \text{MPa}} = 562.5 \times 10^{-6}\ \text{m}^3$$

$$\therefore\ \frac{b^3}{6} = 562.5 \times 10^{-6}\ \text{m}^3 \quad \text{and} \quad b^3 = 3375 \times 10^{-6}\ \text{m}^3$$

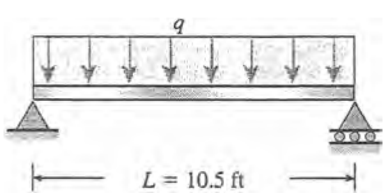
Solving, $b_{\min} = 0.150$ m $= 150$ mm $\leftarrow$

Problem 5.6-7 A floor system in a small building consists of wood planks supported by 2 in. (nominal width) joists spaced at distance s, measured from center to center (see figure). The span length L of each joist is 10.5 ft, the spacing s of the joists is 16 in., and the allowable bending stress in the wood is 1350 psi. The uniform floor load is 120 lb/ft^2, which includes an allowance for the weight of the floor system itself.

Calculate the required section modulus S for the joists, and then select a suitable joist size (surfaced lumber) from Appendix F, assuming that each joist may be represented as a simple beam carrying a uniform load.



Solution 5.6-7 Floor joists



$\sigma_{\text{allow}} = 1350$ psi

$L = 10.5$ ft $= 126$ in.

$w =$ floor load $= 120$ lb/ft^2 $= 0.8333$ lb/in.^{2}

$s =$ spacing of joists $= 16$ in.

$q = ws = 13.333$ lb/in.

$$M_{\max} = \frac{qL^2}{8} = \frac{1}{8}(13.333\ \text{lb/in.})(126\ \text{in.})^2 = 26{,}460\ \text{lb-in.}$$

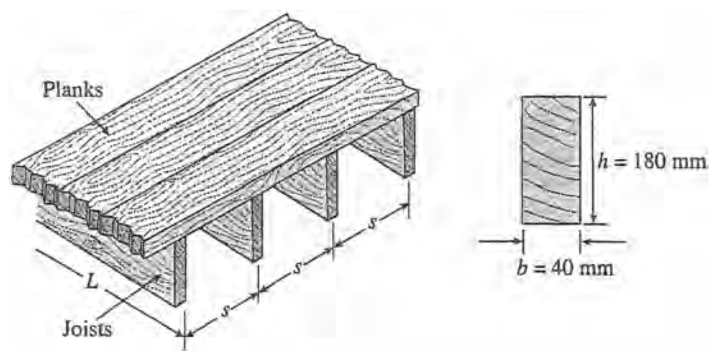
$$\text{Required } S = \frac{M_{\max}}{\sigma_{\text{allow}}} = \frac{26{,}460\ \text{lb/in.}}{1350\ \text{psi}} = 19.6\ \text{in.}^3 \quad \leftarrow$$

From Appendix F: Select 2×10 in. joists $\leftarrow$

Problem 5.6-8 The wood joists supporting a plank floor (see figure) are 40 mm × 180 mm in cross section (actual dimensions) and have a span length $L = 4.0$ m. The floor load is 3.6 kPa, which includes the weight of the joists and the floor.

Calculate the maximum permissible spacing s of the joists if the allowable bending stress is 15 MPa. (Assume that each joist may be represented as a simple beam carrying a uniform load.)

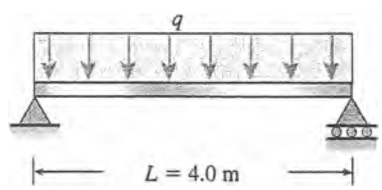
Solution 5.6-8 Spacing of floor joists



$L = 4.0$ m

w = floor load = 3.6 kPa $\qquad \sigma_{allow} = 15$ MPa

s = spacing of joists



$$q = ws$$

$$S = \frac{bh^2}{6}$$

$$M_{max} = \frac{qL^2}{8} = \frac{wsL^2}{8}$$

$$S = \frac{M_{max}}{\sigma_{allow}} = \frac{wsL^2}{8\sigma_{allow}} = \frac{bh^2}{6}$$

SPACING OF JOISTS $\quad s_{max} = \dfrac{4\,bh^2\sigma_{allow}}{3wL^2} \quad \leftarrow$

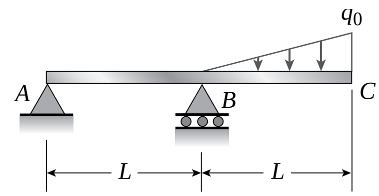
Substitute numerical values:

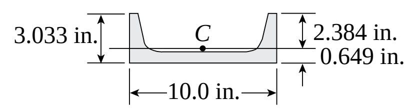
$$s_{max} = \frac{4(40\text{ mm})(180\text{ mm})^2(15\text{ MPa})}{3(3.6\text{ kPa})(4.0\text{ m})^2}$$

$$= 0.450\text{ m} = 450\text{ mm} \quad \leftarrow$$

Problem 5.6-9 A beam *ABC* with an overhang from *B* to *C* is constructed of a *C* 10 × 30 channel section (see figure). The beam supports its own weight (30 lb/ft) plus a *triangular* load of maximum intensity q_0 acting on the overhang. The allowable stresses in tension and compression are 20 ksi and 11 ksi, respectively.

Determine the allowable *triangular* load intensity $q_{0,allow}$ if the distance *L* equals 3.5 ft.





Solution 5.6-9

NUMERICAL DATA

$$w = 30\,\frac{\text{lb}}{\text{ft}} \qquad \sigma_{at} = 20\text{ ksi} \qquad \sigma_{ac} = 11\text{ ksi}$$

$$L = 3.5\text{ ft}$$

$$c_1 = 2.384\text{ in.} \qquad c_2 = 0.649\text{ in.}$$

from Table E-3(a) $\quad I_{22} = 3.93\text{ in.}^4$

MAX. MOMENT IS AT *B* (TENSION TOP, COMPRESSION BOTTOM)

$$M_B = wL\,\frac{L}{2} + \frac{1}{2}\,q_0 L\left(\frac{2}{3}\,L\right)$$

$$M_B = \frac{1}{2}\,wL^2 + \frac{1}{3}\,q_0 L^2$$

check tension on top

$$\sigma_t = \frac{M_B\,c_1}{I_{22}} \qquad M_B = \sigma_{at}\,\frac{I_{22}}{c_1}$$

$$q_{0allow} = \frac{3}{L^2}\left[\sigma_{at}\left(\frac{I_{22}}{c_1}\right) - \frac{1}{2}\,wL^2\right]$$

$$q_{0allow} = 628\text{ lb/ft} \quad \leftarrow \quad \text{governs}$$

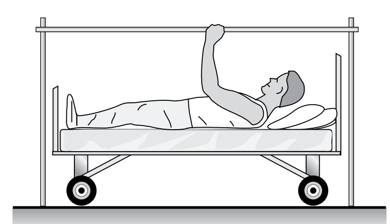
check compression on bottom

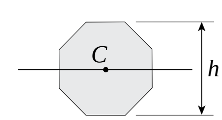
$$q_{0allow} = \frac{3}{L^2}\left[\sigma_{ac}\left(\frac{I_{22}}{c_2}\right) - \frac{1}{2}\,wL^2\right]$$

$$q_{0allow} = 1314\,\frac{\text{lb}}{\text{ft}}$$

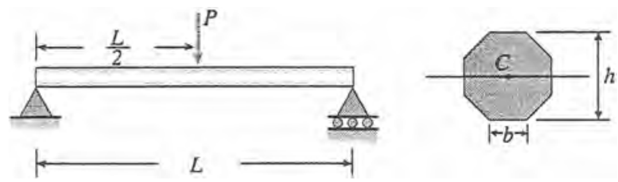
Problem 5.6-10 A so-called "trapeze bar" in a hospital room provides a means for patients to exercise while in bed (see figure). The bar is 2.1 m long and has a cross section in the shape of a regular octagon. The design load is 1.2 kN applied at the midpoint of the bar, and the allowable bending stress is 200 MPa.

Determine the minimum height *h* of the bar. (Assume that the ends of the bar are simply supported and that the weight of the bar is negligible.)





Solution 5.6-10 Trapeze bar (regular octagon)



$P = 1.2\text{ kN} \quad L = 2.1\text{ m} \qquad \sigma_{allow} = 200\text{ MPa}$

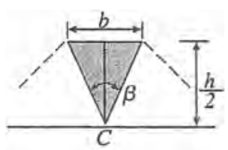
Determine minimum height h.

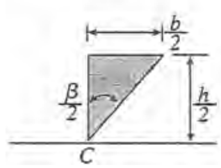
MAXIMUM BENDING MOMENT

$$M_{max} = \frac{PL}{4} = \frac{(1.2\text{ kN})(2.1\text{ m})}{4} = 630\text{ N}\cdot\text{m}$$

PROPERTIES OF THE CROSS SECTION

Use Appendix D, Case 25, with $n = 8$





b = length of one side

$$\beta = \frac{360°}{n} = \frac{360°}{8} = 45°$$

$$\tan\frac{\beta}{2} = \frac{b}{h} \text{ (from triangle)}$$

$$\cot\frac{\beta}{2} = \frac{h}{b}$$

For $\beta = 45°$: $\dfrac{b}{h} = \tan\dfrac{45°}{2} = 0.41421$

$$\frac{h}{b} = \cot\frac{45°}{2} = 2.41421$$

MOMENT OF INERTIA

$$I_c = \frac{nb^4}{192}\left(\cot\frac{\beta}{2}\right)\left(3\cot^2\frac{\beta}{2} + 1\right)$$

$$I_c = \frac{8b^4}{192}(2.41421)[3(2.41421)^2 + 1] = 1.85948b^4$$

$b = 0.41421h \quad \therefore I_c = 1.85948(0.41421h)^4 = 0.054738h^4$

SECTION MODULUS

$$S = \frac{I_c}{h/2} = \frac{0.054738h^4}{h/2} = 0.109476h^3$$

MINIMUM HEIGHT h

$$\sigma = \frac{M}{S} \quad S = \frac{M}{\sigma}$$

$$0.109476h^3 = \frac{630\text{ N}\cdot\text{m}}{200\text{ MPa}} = 3.15 \times 10^{-6}\text{ m}^3$$

$h^3 = 28.7735 \times 10^{-6}\text{ m}^3 \quad h = 0.030643\text{ m}$

$\therefore h_{min} = 30.6\text{ mm} \qquad \leftarrow$

ALTERNATIVE SOLUTION ($n = 8$)

$$M = \frac{PL}{4} \quad \beta = 45° \quad \tan\frac{\beta}{2} = \sqrt{2} - 1 \quad \cot\frac{\beta}{2} = \sqrt{2} + 1$$

$b = (\sqrt{2} - 1)h \quad h = (\sqrt{2} + 1)b$

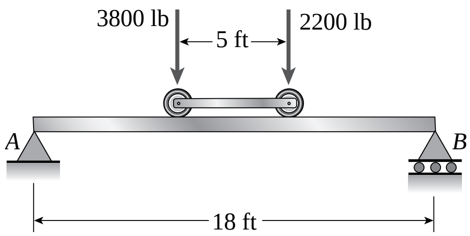
$$I_c = \left(\frac{11 + 8\sqrt{2}}{12}\right)b^4 = \left(\frac{4\sqrt{2} - 5}{12}\right)h^4$$

$$S = \left(\frac{4\sqrt{2} - 5}{6}\right)h^3 \quad h^3 = \frac{3PL}{2(4\sqrt{2} - 5)\sigma_{allow}} \qquad \leftarrow$$

Substitute numerical values:

$h^3 = 28.7735 \times 10^{-6}\text{ m}^3 \quad h_{min} = 30.643\text{ mm} \qquad \leftarrow$

Problem 5.6-11 A two-axle carriage that is part of an overhead traveling crane in a testing laboratory moves slowly across a simple beam *AB* (see figure). The load transmitted to the beam from the front axle is 2200 lb and from the rear axle is 3800 lb. The weight of the beam itself may be disregarded.



(a) Determine the minimum required section modulus *S* for the beam if the allowable bending stress is 17.0 ksi, the length of the beam is 18 ft, and the wheelbase of the carriage is 5 ft.
(b) Select the most economical I-beam (*S* shape) from Table E-2(a), Appendix E.

Solution 5.6-11

NUMERICAL DATA

$L = 18$ ft $\quad P_1 = 2200$ lb

$P_2 = 3800$ lb $\quad d = 5$ ft

$\sigma_a = 17$ ksi

(a) FIND REACTION R_A THEN AN EXPRESSION FOR MOMENT UNDER LARGER LOAD P_2; LET x = DIST. FROM A TO LOAD P_2

$$R_A = P_2\left(\frac{L - x}{L}\right) + P_1\left[\frac{L - (x + d)}{L}\right]$$

$$M_2 = R_A x$$

$$M_2 = x\left[P_2\left(\frac{L - x}{L}\right) + P_1\left[\frac{L - (x + d)}{L}\right]\right]$$

$$M_2 = \frac{xP_2L - P_2x^2 + xP_1L - P_1x^2 - xP_1d}{L}$$

Take derivative of M_A & set to zero to find max. bending moment at $x = x_m$

$$\frac{d}{dx}\left(\frac{xP_2L - P_2x^2 + xP_1L - P_1x^2 - xP_1d}{L}\right)$$

$$= \frac{P_2L - 2P_2x + P_1L - 2P_1x - P_1d}{L}$$

$$P_2L - 2P_2x + P_1L - 2P_1x - P_1d = 0$$

$$x_m = \frac{(P_1 + P_2)L - P_1d}{2(P_1 + P_2)} \qquad x_m = 8.083 \text{ ft}$$

$$R_A = P_2\left(\frac{L - x_m}{L}\right) + P_1\left[\frac{L - (x_m + d)}{L}\right]$$

$$R_A = 2694 \text{ lb}$$

$$M_{max} = x_m\left[P_2\left(\frac{L - x_m}{L}\right) + P_1\left[\frac{L - (x_m + d)}{L}\right]\right]$$

$$M_{max} = 21780 \text{ ft-lb}$$

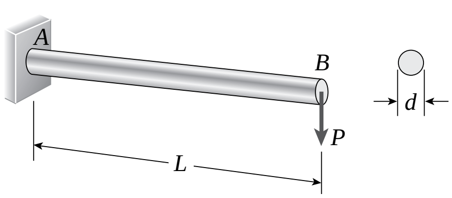
$$S_{reqd} = \frac{M_{max}}{\sigma_a} \qquad S_{reqd} = 15.37 \text{ in.}^3 \quad \leftarrow$$

(b) SELECT MOST ECONOMICAL S SHAPE FROM TABLE E-2(A)

select $S8 \times 23$ $\quad\leftarrow\quad$ $S_{act} = 16.2$ in.^{3}

Problem 5.6-12 A cantilever beam *AB* of circular cross section and length $L = 450$ mm supports a load $P = 400$ N acting at the free end (see figure). The beam is made of steel with an allowable bending stress of 60 MPa.

Determine the required diameter d_{min} of the beam, considering the effect of the beam's own weight.



Solution 5.6-12 Cantilever beam

DATA $L = 450$ mm $P = 400$ N

$\sigma_{allow} = 60$ MPa

$\gamma =$ weight density of steel

$= 77.0\ \text{kN/m}^3$

WEIGHT OF BEAM PER UNIT LENGTH

$$q = \gamma\left(\frac{\pi d^2}{4}\right)$$

MAXIMUM BENDING MOMENT

$$M_{max} = PL + \frac{qL^2}{2} = PL + \frac{\pi\gamma d^3 L^2}{8}$$

SECTION MODULUS $S = \dfrac{\pi d^3}{32}$

MINIMUM DIAMETER

$M_{max} = \sigma_{allow} S$

$$PL + \frac{\pi\gamma d^2 L^2}{8} = \sigma_{allow}\left(\frac{\pi d^3}{32}\right)$$

Rearrange the equation:

$$\sigma_{allow}\, d^3 - 4\gamma L^2\, d^2 - \frac{32\,PL}{\pi} = 0$$

(Cubic equation with diameter d as unknown.)

Substitute numerical values ($d =$ meters):

$$(60 \times 10^6\ \text{N/m}^2)d^3 - 4(77{,}000\ \text{N/m}^3)(0.45\ \text{m})^2 d^2$$
$$-\frac{32}{\pi}(400\ \text{N})(0.45\ \text{m}) = 0$$

$$60{,}000d^3 - 62.37d^2 - 1.833465 = 0$$

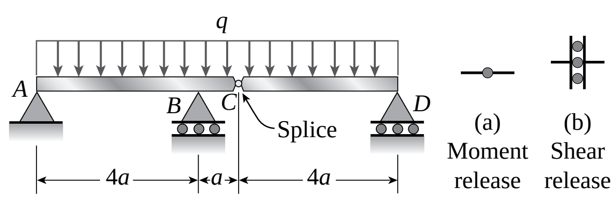
Solve the equation numerically:

$d = 0.031614$ m $\quad d_{min} = 31.61$ mm $\leftarrow$

Problem 5.6-13 A compound beam *ABCD* (see figure) is supported at points *A*, *B*, and *D* and has a splice at point *C*. The distance $a = 6.25$ ft, and the beam is a *S* 18 × 70 wide-flange shape with an allowable bending stress of 12,800 psi.

(a) If the splice is a *moment release*, find the allowable uniform load q_{allow} that may be placed on top of the beam, taking into account the weight of the beam itself. [See figure part (a).]

(b) Repeat assuming now that the splice is a *shear release*, as in figure part (b).



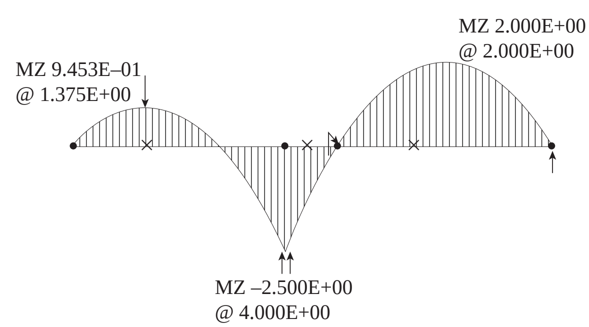
Solution 5.6-13

NUMERICAL DATA

$w = 70\ \dfrac{\text{lb}}{\text{ft}}$ $\quad S = 103\ \text{in.}^3$

$a = 6.25$ ft $\quad \sigma_a = 12800$ psi

(a) MOMENT RELEASE AT *C*-GIVES MAX. MOMENT AT *B* (SEE MOMENT DIAGRAM) $= -2.5\,q\,a^2$



$$\sigma_a = \frac{M_{max}}{S} \qquad M_{max} = [(q_{allow} + w)\,a^2(2.5)]$$

and $\quad M_{max} = \sigma_a S$

$w = 70\ \dfrac{\text{lb}}{\text{ft}}$ $\quad S = 103\ \text{in.}^3$

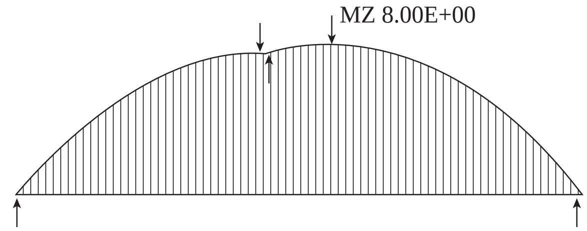
$a = 6.25$ ft $\quad \sigma_a = 12800$ psi

$$q_{\text{allow}} = \frac{\dfrac{\sigma_a S}{12\ \text{in./ft}}}{2.5\, a^2} - w$$

$$q_{\text{allow}} = 1055\ \frac{\text{lb}}{\text{ft}} \quad \leftarrow \quad \text{for moment release}$$

(b) Shear release at C-gives max. moment at C (see moment diagram) $= 8\, q\, a^2$

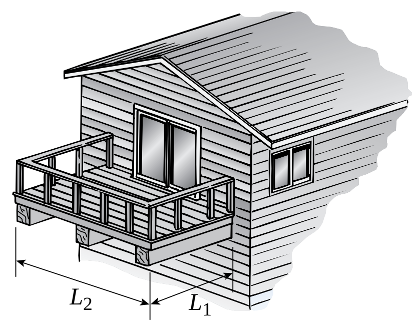
$$q_{\text{allow}} = \frac{\dfrac{\sigma_a S}{12\ \text{in./ft}}}{8\, a^2} - w$$

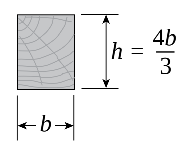


$$q_{\text{allow}} = 282\ \frac{\text{lb}}{\text{ft}} \quad \leftarrow \quad \text{for shear release}$$

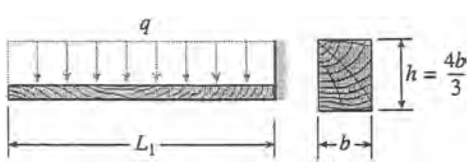
Problem 5.6-14 A small balcony constructed of wood is supported by three identical cantilever beams (see figure). Each beam has length $L_1 = 2.1$ m, width b, and height $h = 4b/3$. The dimensions of the balcon floor are $L_1 \times L_2$, with $L_2 = 2.5$ m. The design load is 5.5 kPa acting over the entire floor area. (This load accounts for all loads except the weights of the cantilever beams, which have a weight density $\gamma = 5.5$ kN/m^3.) The allowable bending stress in the cantilevers is 15 MPa.

Assuming that the middle cantilever supports 50% of the load and each outer cantilever supports 25% of the load, determine the required dimensions b and h.





Solution 5.6-14 Compound beam



$L_1 = 2.1$ m $\quad L_2 = 2.5$ m $\quad$ Floor dimensions: $L_1 \times L_2$

Design load $= w = 5.5$ kPa

$\gamma = 5.5$ kN/m^3 (weight density of wood beam)

$\sigma_{\text{allow}} = 15$ MPa

Middle beam supports 50% of the load.

$$\therefore\ q = w\left(\frac{L_2}{2}\right) = (5.5\ \text{kPA})\left(\frac{2.5\ \text{m}}{2}\right) = 6875\ \text{N/m}$$

Weight of beam

$$q_0 = \gamma bh = \frac{4\gamma b^2}{3} = \frac{4}{3}(5.5\ \text{kN/m}^2)\, b^2$$

$$= 7333 b^2\ (\text{N/m}) \qquad (b = \text{meters})$$

Maximum bending moment

$$M_{\max} = \frac{(q + q_0)L_1^2}{2} = \frac{1}{2}(6875\ \text{N/m} + 7333 b^2)(2.1\ \text{m})^2$$

$$= 15{,}159 + 16{,}170 b^2\ (\text{N} \cdot \text{m})$$

$$S = \frac{bh^2}{6} = \frac{8b^3}{27}$$

$$M_{\max} = \sigma_{\text{allow}}\, S$$

$$15{,}159 + 16{,}170 b^2 = (15 \times 10^6\ \text{N/m}^2)\left(\frac{8b^3}{27}\right)$$

Rearrange the equation:

$$(120 \times 10^6) b^3 - 436{,}590 b^2 - 409{,}300 = 0$$

Solve numerically for dimension b

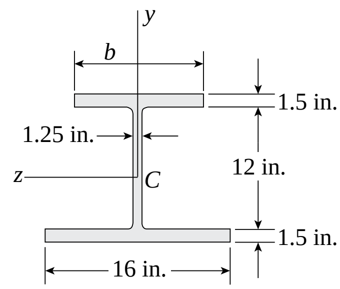
$$b = 0.1517\ \text{m} \qquad h = \frac{4b}{3} = 0.2023\ \text{m}$$

Required dimensions

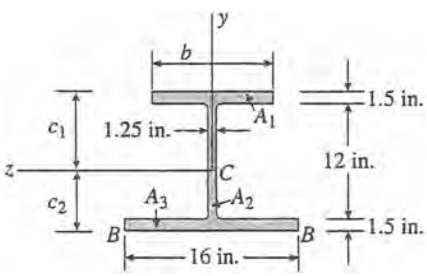
$$b = 152\ \text{mm} \qquad h = 202\ \text{mm} \qquad \leftarrow$$

Problem 5.6-15 A beam having a cross section in the form of an unsymmetric wide-flange shape (see figure) is subjected to a negative bending moment acting about the z axis.

Determine the width b of the top flange in order that the stresses at the top and bottom of the beam will be in the ratio 4:3, respectively.



Solution 5.6-15 Unsymmetric wide-flange beam



Stresses at top and bottom are in the ratio 4:3.
Find b (inches)

$h =$ height of beam $= 15$ in.

Locate centroid

$$\frac{\sigma_{\text{top}}}{\sigma_{\text{bottom}}} = \frac{c_1}{c_2} = \frac{4}{3}$$

$$c_1 = \frac{4}{7}h = \frac{60}{7} = 8.57143 \text{ in.}$$

$$c_2 = \frac{3}{7}h = \frac{45}{7} = 6.42857 \text{ in.}$$

Areas of the cross section (in.^{2})

$$A_1 = 1.5b \quad A_2 = (12)(1.25) = 15 \text{ in.}^2$$

$$A_3 = (16)(1.5) = 24 \text{ in.}^2$$

$$A = A_1 + A_2 + A_3 = 39 + 1.5b \text{ (in.}^2)$$

First moment of the cross-sectional area about the lower edge B-B

$$Q_{BB} = \Sigma \bar{y}_i A_i = (14.25)(1.5b) + (7.5)(15) + (0.75)(24)$$

$$= 130.5 + 21.375b \text{ (in.}^3)$$

Distance c_2 from line B-B to the centroid C

$$c_2 = \frac{Q_{BB}}{A} = \frac{130.5 + 21.375b}{39 + 1.5b} = \frac{45}{7} \text{ in.}$$

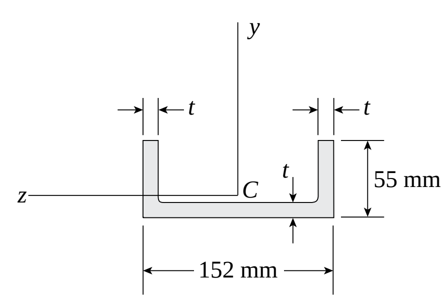
Solve for b

$$(39 + 1.5b)(45) = (130.5 + 21.375b)(7)$$

$$82.125b = 841.5 \quad b = 10.25 \text{ in.} \quad \leftarrow$$

Problem 5.6-16 A beam having a cross section in the form of a channel (see figure) is subjected to a bending moment acting about the z axis.

Calculate the thickness t of the channel in order that the bending stresses at the top and bottom of the beam will be in the ratio 7:3, respectively.



Solution 5.6-16

NUMERICAL DATA

$h = 152$ mm $\qquad b = 55$ mm

take 1st moments to find distances c_1 & c_2

1st moments about base

$$c_2 = \frac{\frac{t}{2}(h - 2t)(t) + 2bt\left(\frac{b}{2}\right)}{2bt + t(h - 2t)}$$

$$c_1 = b - c_2$$

$$c_2 = \frac{\frac{t}{2}(152 - 2t)(t) + 2.55t\left(\frac{55}{2}\right)}{2.55t + t(152 - 2t)}$$

$$c_1 = 55 - \frac{\frac{t}{2}(152 - 2t)(t) + 2.55t\left(\frac{55}{2}\right)}{2.55t + t(152 - 2t)}$$

$$c_1 = \frac{-1}{2}\,\frac{11385 - 186t + t^2}{-131 + t}$$

ratio of top to bottom stresses $= c_1/c_2 = 7/3$

$$\frac{\frac{-1}{2}\,\frac{11385 - 186t + t^2}{-131 + t}}{\left[\frac{\frac{t}{2}(152 - 2t)(t) + 2.55t\left(\frac{55}{2}\right)}{2.55t + t(152 - 2t)}\right]}$$

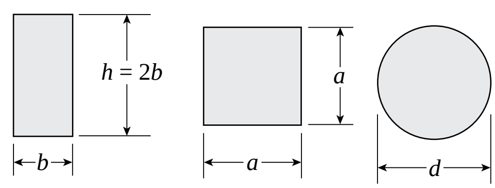
$$= \frac{-(11385 - 186t + t^2)}{-76t + t^2 - 3025} = 7/3$$

$$\left[3\left[-\left(11385 - 186t + t^2\right)\right] - 7\left(-76t + t^2 - 3025\right)\right] = 0$$

$$t^2 - 109t + 1298 = 0$$

$$t = \frac{109 - \sqrt{109^2 - 4(1298)}}{2} \qquad t = 13.61 \text{ mm} \quad \leftarrow$$

Problem 5.6-17 Determine the ratios of the weights of three beams that have the same length, are made of the same material, are subjected to the same maximum bending moment, and have the same maximum bending stress if their cross sections are (1) a rectangle with height equal to twice the width, (2) a square, and (3) a circle (see figures).



Solution 5.6-17 Ratio of weights of three beams

Beam 1: Rectangle ($h = 2b$)
Beam 2: Square (a = side dimension)
Beam 3: Circle (d = diameter)
L, γ, M_{max}, and σ_{max} are the same in all three beams.

S = section modulus $\quad S = \dfrac{M}{\sigma}$

Since M and σ are the same, the section moduli must be the same.

(1) RECTANGLE: $S = \dfrac{bh^2}{6} = \dfrac{2b^3}{3} \quad b = \left(\dfrac{3S}{2}\right)^{1/3}$

$$A_1 = 2b^2 = 2\left(\frac{3S}{2}\right)^{2/3} = 2.6207S^{2/3}$$

(2) SQUARE: $S = \dfrac{a^3}{6} \quad a = (6S)^{1/3}$

$$A_2 = a^2 = (6S)^{2/3} = 3.3019\,S^{2/3}$$

(3) CIRCLE: $S = \dfrac{\pi d^3}{32} \quad d = \left(\dfrac{32S}{\pi}\right)^{1/3}$

$$A_3 = \frac{\pi d^2}{4} = \frac{\pi}{4}\left(\frac{32S}{\pi}\right)^{2/3} = 3.6905\,S^{2/3}$$

Weights are proportional to the cross-sectional areas (since L and γ are the same in all 3 cases).

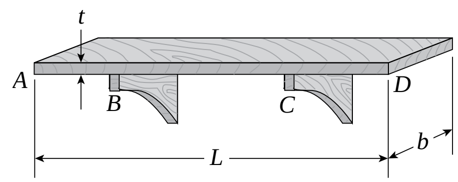
$W_1 : W_2 : W_3 = A_1 : A_2 : A_3$

$A_1 : A_2 : A_3 = 2.6207 : 3.3019 : 3.6905$

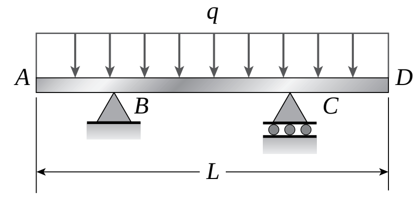
$W_1 : W_2 : W_3 = 1 : 1.260 : 1.408 \quad \leftarrow$

Problem 5.6-18 A horizontal shelf AD of length $L = 915$ mm, width $b = 305$ mm, and thickness $t = 22$ mm is supported by brackets at B and C [see part (a) of the figure]. The brackets are adjustable and may be placed in any desired positions between the ends of the shelf. A uniform load of intensity q, which includes the weight of the shelf itself, acts on the shelf [see part (b) of the figure].

Determine the maximum permissible value of the load q if the allowable bending stress in the shelf is $\sigma_{allow} = 7.5$ MPa and the position of the supports is adjusted for maximum load-carrying capacity.



(a)



(b)

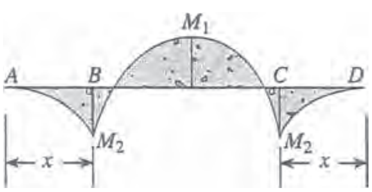
Solution 5.6-18

Numerical data

$L = 915$ mm $\quad b = 305$ mm $\quad t = 22$ mm

$\sigma_{allow} = 7.5$ MPa

Moment diagram



For maximum load-carrying capacity, place the supports so that $M_1 = |M_2|$.

Let x = length of overhang

$$M_1 = \frac{qL}{8}(L - 4x) \qquad |M_2| = \frac{qx^2}{2}$$

$$\therefore \frac{qL}{8}(L - 4x) = \frac{qx^2}{2}$$

Solve for x: $x = \frac{L}{2}(\sqrt{2} - 1)$

Substitute x into the equation for either M_1 or $|M_2|$:

$$M_{max} = \frac{qL^2}{8}(3 - 2\sqrt{2}) \qquad \text{Eq. (1)}$$

$$M_{max} = \sigma_{allow} S = \sigma_{allow}\left(\frac{bt^2}{6}\right) \qquad \text{Eq. (2)}$$

Equate M_{max} from Eqs. (1) and (2) and solve for q:

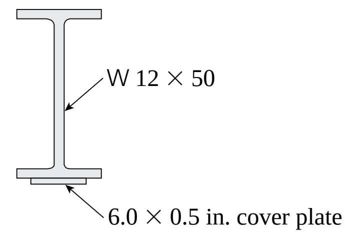
$$q_{max} = \frac{4bt^2\sigma_{allow}}{3L^2(3 - 2\sqrt{2})}$$

Substitute numerical values:

$q_{max} = 10.28$ kN/m $\quad \leftarrow$

Problem 5.6-19 A steel plate (called a *cover plate*) having cross-sectional dimensions 6.0 in. $\times$ 0.5 in. is welded along the full length of the bottom flange of a W 12 $\times$ 50 wide-flange beam (see figure, which shows the beam cross section).

What is the percent increase in the smaller section modulus (as compared to the wide-flange beam alone)?



Solution 5.6-19

NUMERICAL PROPERTIES FOR W 12 $\times$ 50 (FROM TABEL E-1(a))

$A = 14.6 \text{ in.}^2 \qquad d = 12.2 \text{ in.}$

$c_1 = c_2 \qquad c_1 = \dfrac{d}{2}$

$I = 391 \text{ in.}^4 \qquad S = 64.2 \text{ in.}^3$

FIND CENTROID OF BEAM WITH COVER PLATE (TAKE 1ST MOMENTS ABOUT TOP TO FIND $c_1 > c_2$)

$$c_1 = \frac{A\dfrac{d}{2} + (6)(0.5)\left(d + \dfrac{0.5}{2}\right)}{A + (6)(0.5)} \qquad c_1 = 7.182 \text{ in.}$$

$c_2 = (d + 0.5) - c_1 \qquad c_2 = 5.518 \text{ in.}$

FIND I ABOUT HORIZ. CENTROIDAL AXIS

$$I_h = I + A\left(c_1 - \frac{d}{2}\right)^2 + \frac{1}{12}(6)(0.5)^3 + (6)(0.5)\left(c_2 - \frac{0.5}{2}\right)^2$$

$I_h = 491.411 \text{in.}^4$

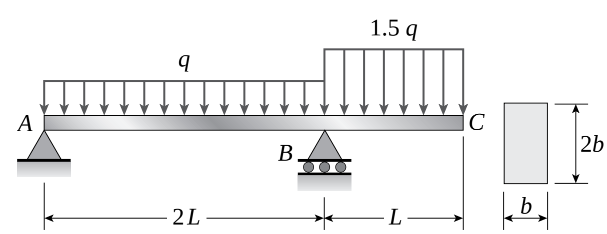
FIND SMALLER SECTION MODULUS

$S_{top} = \dfrac{I_h}{c_1} \qquad S_{top} = 68.419 \text{ in.}^3$

% increase in smaller section modulus

$$\frac{S_{top} - S}{S}(100) = 6.57\% \qquad \leftarrow$$

Problem 5.6-20 A steel beam ABC is simply supported at A and B and has an overhang BC of length $L = 150$ mm (see figure). The beam supports a uniform load of intensity $q = 4.0$ kN/m over its entire span AB and $1.5q$ over BC. The cross section of the beam is rectangular with width b and height $2b$. The allowable bending stress in the steel is $\sigma_{allow} = 60$ MPa, and its weight density is $\gamma = 77.0 \text{ kN/m}^3$.



(a) Disregarding the weight of the beam, calculate the required width b of the rectangular cross section.
(b) Taking into account the weight of the beam, calculate the required width b.

Solution 5.6-20

NUMERICAL DATA

$L = 150 \text{ mm} \qquad q = 4 \dfrac{\text{kN}}{\text{m}}$

$\sigma_a = 60 \text{ MPa} \qquad \gamma = 77 \dfrac{\text{kN}}{\text{m}^3}$

(a) IGNORE BEAM SELF WEIGHT-FIND $b_{\min}$

$$M_{\max 1} = 1.5\, q \frac{L^2}{2} \qquad \text{at } B$$

$$\text{and} \qquad M_{\max 2} = \sigma_a S \qquad S = \frac{2}{3} b^3$$

Equate $M_{\max 1}$ to $M_{\max 2}$ & solve for $b_{\min}$

$$b_{\min} = \left(\frac{9}{8} \frac{qL^2}{\sigma_a}\right)^{\frac{1}{3}}$$

$$b_{\min} = 11.91 \text{ mm} \qquad \leftarrow$$

(b) NOW MODIFY-INCLUDE BEAM WEIGHT

$$w = \gamma A \qquad w = \gamma\,(2b^2)$$

$$M_{\max} = (1.5q + w)\frac{L^2}{2}$$

$$\text{and} \qquad M_{\max} = \sigma_a\left(\frac{2}{3} b^3\right)$$

Equate $M_{\max 1}$ to $M_{\max 2}$ & solve for $b_{\min}$

$$\left(\frac{2}{3}\sigma_a\right) b^3 - (\gamma L^2)\, b^2 - \frac{3}{4} qL^2 = 0$$

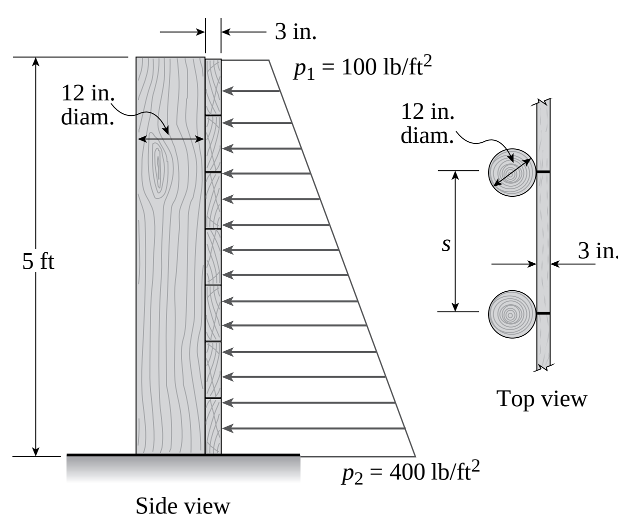
Insert numerical values, then solve for b

$$b_{\min} = 11.92 \text{ mm} \qquad \leftarrow$$

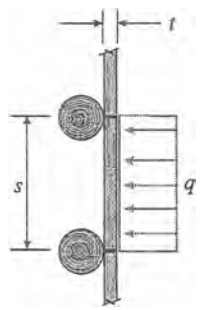
Problem 5.6-21 A retaining wall 5 ft high is constructed of horizontal wood planks 3 in. thick (actual dimension) that are supported by vertical wood piles of 12 in. diameter (actual dimension), as shown in the figure. The lateral earth pressure is $p_1 = 100 \text{ lb/ft}^2$ at the top of the wall and $p_2 = 400 \text{ lb/ft}^2$ at the bottom.

Assuming that the allowable stress in the wood is 1200 psi, calculate the maximum permissible spacing s of the piles.

(*Hint:* Observe that the spacing of the piles may be governed by the load-carrying capacity of either the planks or the piles. Consider the piles to act as cantilever beams subjected to a trapezoidal distribution of load, and consider the planks to act as simple beams between the piles. To be on the safe side, assume that the pressure on the bottom plank is uniform and equal to the maximum pressure.)



Solution 5.6-21 Retaining wall



(1) Plank at the bottom of the dam

t = thickness of plank = 3 in.

b = width of plank (perpendicular to the plane of the figure)

p_2 = maximum soil pressure = 400 lb/ft^2 = 2.778 lb/in.^{2}

s = spacing of piles

$q = p_2 b \quad \sigma_{\text{allow}} = 1200$ psi

S = section modulus

$$M_{\max} = \frac{qs^2}{8} = \frac{p_2 bs^2}{8} \qquad S = \frac{bt^2}{6}$$

$$M_{\max} = \sigma_{\text{allow}} S \quad \text{or} \quad \frac{p_2 bs^2}{8} = \sigma_{\text{allow}}\left(\frac{bt^2}{6}\right)$$

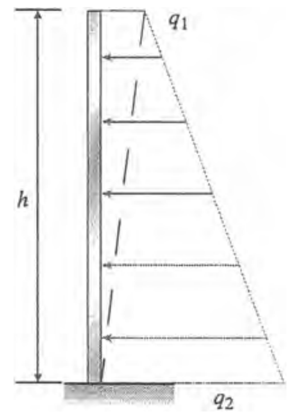
Solve for s:

$$s = \sqrt{\frac{4\sigma_{\text{allow}}\, t^2}{3p^2}} = 72.0 \text{ in.}$$

(2) Vertical pile

h = 5 ft = 60 in.

p_1 = soil pressure at the top = 100 lb/ft^2 = 0.6944 lb/in.^{2}



$q_1 = p_1 s$

$q_2 = p_2 s$

d = diameter of pile = 12 in.

Divide the trapezoidal load into two triangles (see dashed line).

$$M_{\max} = \frac{1}{2}(q_1)(h)\left(\frac{2h}{3}\right) + \frac{1}{2}(q_2)(h)\left(\frac{h}{3}\right) = \frac{sh^2}{6}(2p_1 + p_2)$$

$$S = \frac{\pi d^3}{32} \qquad M_{\max} = \sigma_{\text{allow}} S \quad \text{or}$$

$$\frac{sh^2}{6}(2p_1 + p_2) = \sigma_{\text{allow}}\left(\frac{\pi d^3}{32}\right)$$

Solve for s:

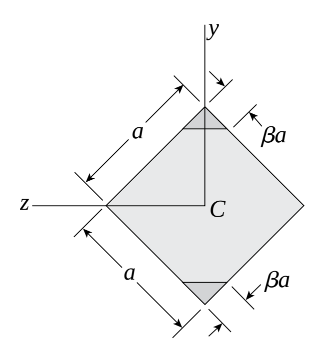
$$s = \frac{3\pi\sigma_{\text{allow}}\, d^3}{16h^2(2p_1 + p_2)} = 81.4 \text{ in.}$$

Plank governs $\quad s_{\max} = 72.0$ in. $\leftarrow$

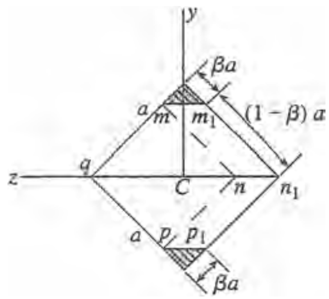
Problem 5.6-22 A beam of square cross section (a = length of each side) is bent in the plane of a diagonal (see figure). By removing a small amount of material at the top and bottom corners, as shown by the shaded triangles in the figure, we can increase the section modulus and obtain a stronger beam, even though the area of the cross section is reduced.

(a) Determine the ratio β defining the areas that should be removed in order to obtain the strongest cross section in bending.

(b) By what percent is the section modulus increased when the areas are removed?



Solution 5.6-22 Beam of square cross section with corners removed



a = length of each side
βa = amount removed
Beam is bent about the z axis.

ENTIRE CROSS SECTION (AREA 0)

$$I_0 = \frac{a^4}{12} \quad c_0 = \frac{a}{\sqrt{2}} \quad S_0 = \frac{I_0}{c_0} = \frac{a^3\sqrt{2}}{12}$$

SQUARE *mnpq* (AREA 1)

$$I_1 = \frac{(1-\beta)^4 a^4}{12}$$

PARALLELOGRAM *mm, n, n* (AREA 2)

$$I_2 = \frac{1}{3}(\text{base})(\text{height})^3$$

$$I_2 = \frac{1}{3}(\beta a\sqrt{2})\left[\frac{(1-\beta)a}{\sqrt{2}}\right]^3 = \frac{\beta a^4}{6}(1-\beta)^3$$

REDUCED CROSS SECTION (AREA *qmm, n, p, pq*)

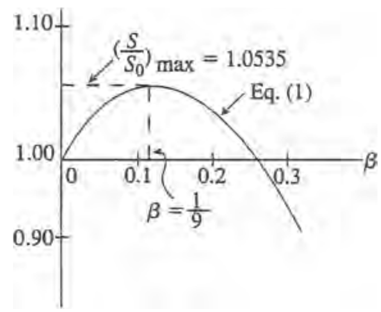
$$I = I_1 + 2I_2 = \frac{a^4}{12}(1+3\beta)(1-\beta)^3$$

$$c = \frac{(1-\beta)\,a}{\sqrt{2}} \quad S = \frac{I}{c} = \frac{\sqrt{2}\,a^3}{12}(1+3\beta)(1-\beta)^2$$

RATIO OF SECTION MODULI

$$\frac{S}{S_0} = (1+3\beta)(1-\beta)^2 \qquad \text{Eq. (1)}$$

GRAPH OF EQ. (1)



(a) VALUE OF β FOR A MAXIMUM VALUE OF S/S_0

$$\frac{d}{d\beta}\left(\frac{S}{S_0}\right) = 0$$

Take the derivative and solve this equation for β.

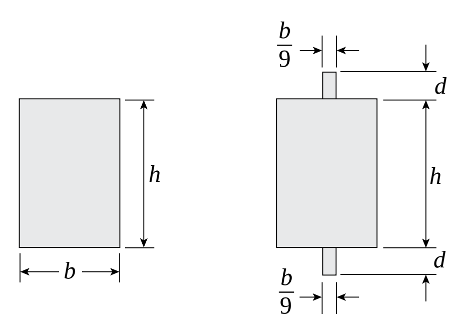
$$\beta = \frac{1}{9} \quad \longleftarrow$$

(b) MAXIMUM VALUE OF S/S_0

Substitute $\beta = 1/9$ into Eq. (1). $(S/S_0)_{max} = 1.0535$
The section modulus is increased by 5.35% when the triangular areas are removed. $\longleftarrow$

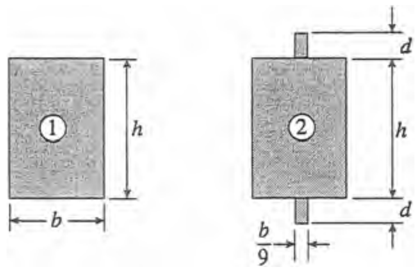
Problem 5.6-23 The cross section of a rectangular beam having width b and height h is shown in part (a) of the figure. For reasons unknown to the beam designer, it is planned to add structural projections of width $b/9$ and height d to the top and bottom of the beam [see part (b) of the figure].

For what values of d is the bending-moment capacity of the beam increased? For what values is it decreased?



(a) (b)

Solution 5.6-23 Beam with projections



(1) ORIGINAL BEAM

$$I_1 = \frac{bh^3}{12} \qquad c_1 = \frac{h}{2} \qquad S_1 = \frac{I_1}{c_1} = \frac{bh^2}{6}$$

(2) BEAM WITH PROJECTIONS

$$I_2 = \frac{1}{12}\left(\frac{8b}{9}\right)h^3 + \frac{1}{12}\left(\frac{b}{9}\right)(h + 2d)^3$$

$$= \frac{b}{108}[8h^3 + (h + 2d)^3]$$

$$c_2 = \frac{h}{2} + d = \frac{1}{2}(h + 2d)$$

$$S_2 = \frac{I_2}{c_2} = \frac{b[8h^3 + (h + 2d)^3]}{54(h + 2d)}$$

RATIO OF SECTION MODULI

$$\frac{S_2}{S_1} = \frac{b\,[8h^3 + (h + 2d)^3]}{9(h + 2d)(bh^2)} = \frac{8 + \left(1 + \frac{2d}{h}\right)^3}{9\left(1 + \frac{2d}{h}\right)}$$

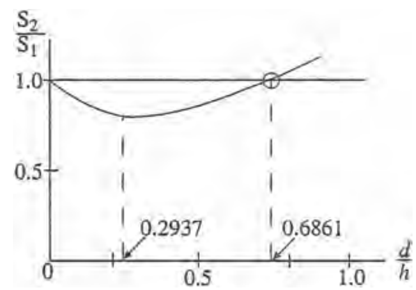
EQUAL SECTION MODULI

Set $\frac{S_2}{S_1} = 1$ and solve numerically for $\frac{d}{h}$.

$$\frac{d}{h} = 0.6861 \quad \text{and} \quad \frac{d}{h} = 0$$

Graph of $\frac{S_2}{S_1}$ versus $\frac{d}{h}$

$\frac{d}{h}$	$\frac{S_2}{S_1}$
0	1.000
0.25	0.8426
0.50	0.8889
0.75	1.0500
1.00	1.2963



Moment capacity is increased when

$\frac{d}{h} > 0.6861$ ←

Moment capacity is decreased when

$\frac{d}{h} < 0.6861$ ←

NOTES:

$$\frac{S_2}{S_1} = 1 \text{ when } \left(1 + \frac{2d}{h}\right)^3 - 9\left(1 + \frac{2d}{h}\right) + 8 = 0$$

or $\frac{d}{h} = 0.6861$ and 0

$\frac{S_2}{S_1}$ is minimum when $\frac{d}{h} = \frac{\sqrt[3]{4} - 1}{2} = 0.2937$

$$\left(\frac{S_2}{S_1}\right)_{\min} = 0.8399$$

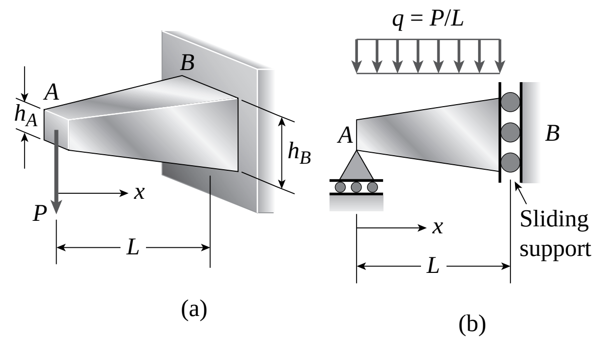
Nonprismatic Beams

Problem 5.7-1 A tapered cantilever beam AB of length L has square cross sections and supports a concentrated load P at the free end [see figure part (a)]. The width and height of the beam vary linearly from h_A at the free end to h_B at the fixed end.

Determine the distance x from the free end A to the cross section of maximum bending stress if $h_B = 3h_A$.

(a) What is the magnitude σ_{max} of the maximum bending stress? What is the ratio of the maximum stress to the largest stress B at the support?

(b) Repeat (a) if load P is now applied as a uniform load of intensity $q = P/L$ over the entire beam, A is restrained by a roller support and B is a sliding support [see figure, part (b)].



Solution 5.7-1

(a) Find max. bending stress for tapered cantilever

$$h(x) = h_A\left(1 + \frac{2x}{L}\right) \qquad S(x) = \frac{h(x)^3}{6}$$

$$\sigma(x) = \frac{M(x)}{S(x)} \qquad \sigma(x) = \frac{6(P)(x)}{\left[h_A\left(1 + \frac{2x}{L}\right)\right]^3}$$

$$\sigma(x) = \frac{6PxL^3}{h_A^3(L + 2x)^3}$$

$$\frac{d}{dx}\sigma(x) = 0 \quad \text{then solve for } x_{max}$$

$$\frac{d}{dx}\left[\frac{6PxL^3}{h_A^3(L + 2x)^3}\right] = 0$$

$$\left[6P\frac{L^3}{h_A^3(L+2x)^3} - 36Px\frac{L^3}{h_A^3(L+2x)^4}\right] = 0$$

$$\frac{-L + 4x}{h_A^3(L + 2x)^4} = 0 \quad \text{so} \quad x = \frac{L}{4}$$

$$\sigma_{max} = \sigma\left(\frac{L}{4}\right) \qquad \sigma_{max} = \frac{6P\frac{L}{4}L^3}{h_A^3\left(L + 2\frac{L}{4}\right)^3}$$

$$\sigma_{max} = \frac{4PL}{9h_A^3} \quad \leftarrow$$

$$\sigma_B = \sigma(L) \qquad \sigma_B = \frac{2PL}{9h_A^3}$$

$$\frac{\sigma_{max}}{\sigma_B} = \frac{\frac{4PL}{9h_A^3}}{\frac{2PL}{9h_A^3}} \qquad \frac{\sigma_{max}}{\sigma_B} = 2 \quad \leftarrow$$

(b) Repeat (a) but now for distributed uniform load of P/L over entire beam

$$\sum F_v = 0 \qquad R_A = P$$

$$M(x) = \left[\left[R_A x - \frac{P}{L}x\left(\frac{x}{2}\right)\right]\right]$$

$$M(x) = Px - \frac{1}{2}x^2\frac{P}{L}$$

$$\sigma(x) = \frac{M(x)}{S(x)} \qquad \sigma(x) = \frac{Px - \frac{1}{2}x^2\frac{P}{L}}{\frac{\left[h_A\left(1 + \frac{2x}{L}\right)\right]^3}{6}}$$

$$\sigma(x) = -3xP(-2L + x)\frac{L^2}{h_A^3(L + 2x)^3}$$

$$\frac{d}{dx}\sigma(x) = 0 \quad \text{then solve for } x_{max}$$

$$\frac{d}{dx}\left[-3xP(-2L+x)\frac{L^2}{h_A^3(L+2x)^3}\right] = 0$$

$$\left[-3P(-2L+x)\frac{L^2}{h_A^3(L+2x)^3}\right.$$

$$-3xP\frac{L^2}{h_A^3(L+2x)^3}$$

$$\left.+18xP(-2L+x)\frac{L^2}{h_A^3(L+2x)^4}\right] = 0$$

Simplifying

$$L^2 - 5xL + x^2 = 0 \quad \text{so}$$

$$\frac{x_{max}}{L} = \frac{5-\sqrt{5^2-4}}{2}$$

$$x_{max} = 0.20871\,L$$

$$\sigma_{max} = \sigma(0.20871\,L)$$

$$\sigma_{max} = 0.394\frac{PL}{h_A^3} \quad \leftarrow$$

$$\sigma_B = \sigma(L) \quad \text{So}$$

$$\sigma_B = \frac{PL}{9h_A^3} \qquad \frac{\sigma_{max}}{\sigma_B} = \frac{\left(0.39385\dfrac{PL}{h_A^3}\right)}{\dfrac{PL}{9h_A^3}}$$

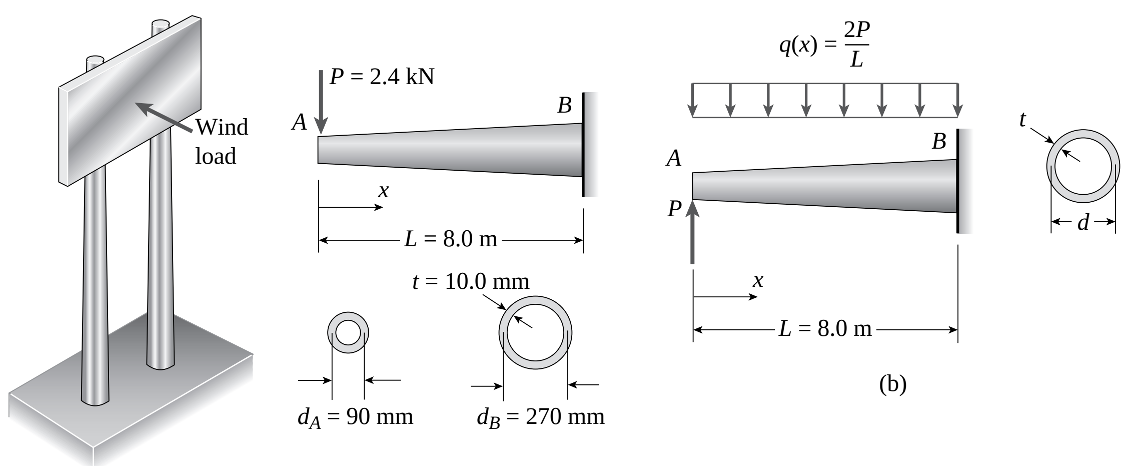
$$\frac{\sigma_{max}}{\sigma_B} = 3.54 \quad \leftarrow$$

Problem 5.7-2 A tall signboard is supported by two vertical beams consisting of thin-walled, tapered circular tubes [see figure]. For purposes of this analysis, each beam may be represented as a cantilever AB of length $L = 8.0$ m subjected to a lateral load $P = 2.4$ kN at the free end. The tubes have constant thickness $t = 10.0$ mm and average diameters $d_A = 90$ mm and $d_B = 270$ mm at ends A and B, respectively.

Because the thickness is small compared to the diameters, the moment of inertia at any cross section may be obtained from the formula $I = \pi d^3 t/8$ (see Case 22, Appendix D), and therefore, the section modulus may be obtained from the formula $S = \pi d^2 t/4$.

(a) At what distance x from the free end does the maximum bending stress occur? What is the magnitude σ_{max} of the maximum bending stress? What is the ratio of the maximum stress to the largest stress σ_B at the support?

(b) Repeat (a) if concentrated load P is applied upward at A and downward uniform load $q(x) = 2P/L$ is applied over the entire beam as shown. What is the ratio of the maximum stress to the stress at the location of maximum moment?



(a)

(b)

Solution 5.7-2

(a) FIND MAX. BENDING STRESS FOR TAPERED CANTILEVER

$$d(x) = d_A\left(1 + \frac{2x}{L}\right) \qquad S(x) = \frac{\pi d(x)^2\, t}{4}$$

$P = 2.4$ kN

$L = 8$ m $\quad t = 10$ mm

$d_A = 90$ mm

$d_B = 270$ mm

$$\sigma(x) = \frac{M(x)}{S(x)} \qquad \sigma(x) = \frac{4P}{\pi t}\left[\frac{x}{\left[d_A\left(1 + \frac{2x}{L}\right)\right]^2}\right]$$

$$\sigma(x) = \frac{4P}{\pi t}\left[\frac{xL^2}{d_A^2\,(L + 2x)^2}\right]$$

$\frac{d}{dx}\sigma(x) = 0$ then solve for x_{max}

$$\frac{d}{dx}\left[\frac{4P}{\pi t}\left[\frac{xL^2}{d_A^2\,(L + 2x)^2}\right]\right] = 0$$

$$\left[4\frac{P}{\pi t}\frac{L^2}{d_A^2\,(L + 2x)^2} - 16\frac{P}{\pi t}\frac{xL^2}{d_A^2\,(L + 2x)^3}\right] = 0$$

or $\quad \left[-4PL^2\frac{-L + 2x}{\pi t d_A^2\,(L + 2x)^3}\right] = 0$

so $\quad x_{max} = \frac{L}{2} = 4$ m $\quad\leftarrow$

$$\sigma_{max} = \sigma\left(\frac{L}{2}\right)$$

$$\sigma_{max} = \frac{4P}{\pi t}\left[\frac{\frac{L}{2}L^2}{d_A^2\left(L + 2\frac{L}{2}\right)^2}\right]$$

$$\sigma_{max} = \frac{PL}{2\pi t d_A^2}$$

Stress at support $\sigma_B = \sigma(L)$

$$\sigma_B = \frac{4}{9}\frac{P}{\pi t}\frac{L}{d_A^2}$$

$$\frac{\sigma_{max}}{\sigma_B} = \frac{\frac{PL}{2\pi t d_A^2}}{\left(\frac{4}{9}\frac{P}{\pi t}\frac{L}{d_A^2}\right)}$$

$\frac{\sigma_{max}}{\sigma_B} = \frac{9}{8} \quad\leftarrow$

Evaluate using numerical data

$$\sigma_{max} = \frac{(2400)\,(8)}{2\pi\,(0.010)\,(0.090)^2}$$

$\sigma_{max} = 37.7$ MPa $\quad\leftarrow$

(b) REPEAT (A) BUT NOW ADD DISTRIBUTED LOAD

$$M(x) = \left(Px - 2\frac{P}{L}x\frac{x}{2}\right)$$

$$M(x) = -Px\left(\frac{-L + x}{L}\right)$$

$$\sigma(x) = \frac{M(x)}{S(x)} \qquad \sigma(x) = \frac{-Px\left(\frac{-L + x}{L}\right)}{\frac{\pi t}{4}\left[d_A\left(1 + \frac{2x}{L}\right)\right]^2}$$

$$\sigma(x) = -4Px\,(-L + x)\frac{L}{\pi t d_A^2\,(L + 2x)^2}$$

tension on top, compression on bottom of beam

$\frac{d}{dx}\sigma(x) = 0$ then solve for x_{max}

$$\frac{d}{dx}\left[-4Px\,(-L + x)\frac{L}{\pi t d_A^2\,(L + 2x)^2}\right] = 0$$

$$\left[-4P\,(-L + x)\frac{L}{\pi t d_A^2\,(L + 2x)^2} - 4Px\frac{L}{\pi t d_A^2\,(L + 2x)^2} + 16Px\,(-L + x)\frac{L}{\pi t d_A^2\,(L + 2x)^3}\right] = 0$$

OR simplifying $\quad \left[-4PL^2\frac{-L + 4x}{\pi t d_A^2\,(L + 2x)^3}\right] = 0$

so $x_{max} = \frac{L}{4}$

$x_{max} = 2$ m $\quad\leftarrow$

$$\sigma_{max} = \sigma\left(\frac{L}{4}\right)$$

$$\sigma_{max} = \left[-4P\frac{L}{4}\left(-L+\frac{L}{4}\right)\frac{L}{\pi t d_A^2\left(L+2\frac{L}{4}\right)^2}\right]$$

$$\sigma_{max} = \frac{PL}{3\pi t d_A^2}$$

evaluate using numerical data

$P = 2.4$ kN $L = 8$ m

$t = 10$ mm $d_A = 90$ mm

$d_B = 270$ mm

$$\sigma_{max} = \frac{(2400)\,(8)}{3\pi\,(0.010)\,(0.090)^2}$$

$\sigma_{max} = 25.2$ MPa ←

stress at support

$$\sigma_B = \sigma(L)$$

$$\sigma_B = -4PL\,(-L+L)\frac{L}{\pi t d_A^2\,(L+2L^2)}$$

$\sigma_B = 0$ so no ratio of σ_{max}/σ_B is possible

MAX. MOMENT AT L/2 SO COMPARE

Stress at location of max. moment

$$\sigma\left(\frac{L}{2}\right) = -4P\frac{L}{2}\left(-L+\frac{L}{2}\right)\frac{L}{\pi t d_A^2\left(L+2\frac{L}{2}\right)^2}$$

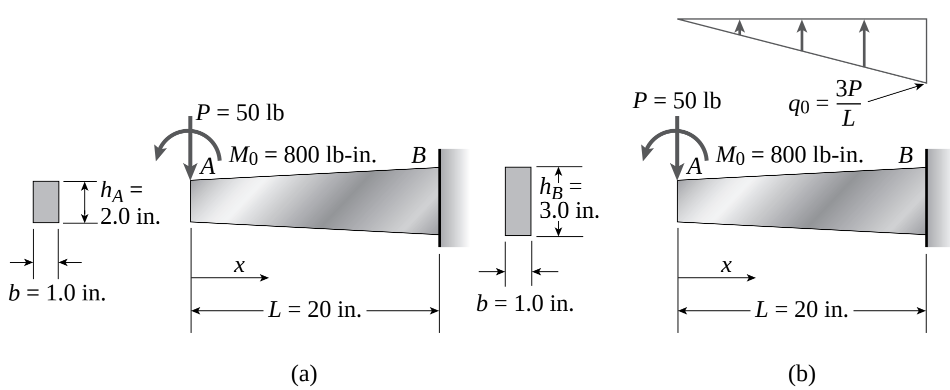
$$\sigma\left(\frac{L}{2}\right) = \frac{1}{4}P\frac{L}{\pi t d_A^2}$$

$$\sigma_{max}/\sigma(L/2) = \frac{\dfrac{PL}{3\pi t d_A^2}}{\left(\dfrac{1}{4}P\dfrac{L}{\pi t d_A^2}\right)} = \frac{4}{3} \quad \leftarrow$$

Problem 5.7-3 A tapered cantilever beam AB having rectangular cross sections is subjected to a concentrated load $P = 50$ lb and a couple $M_0 = 800$ lb-in. acting at the free end [see figure part (a)]. The width b of the beam is constant and equal to 1.0 in., but the height varies linearly from $h_A = 2.0$ in. at the loaded end to $h_B = 3.0$ in. at the support.

(a) At what distance x from the free end does the maximum bending stress σ_{max} occur? What is the magnitude σ_{max} of the maximum bending stress? What is the ratio of the maximum stress to the largest stress σ_B at the support?

(b) Repeat (a) if, in addition to P and M_0, a triangular distributed load with peak intensity $q_0 = 3P/L$ acts upward over the entire beam as shown. What is the ratio of the maximum stress to the stress at the location of maximum moment?



Solution 5.7-3

(a) Find max. bending stress for tapered cantilever fig. (a)

$$h(x) = h_A\left(1 + \frac{x}{2L}\right)$$

numerical data

$P = 50\text{ lb}$ $L = 20\text{ in.}$

$h_A = 2\text{ in.}$ $h_B = 3\text{ in.}$ $b = 1\text{ in.}$

$$M_0 = \frac{4}{5}PL \quad M_0 = 800\text{ in.-lb}$$

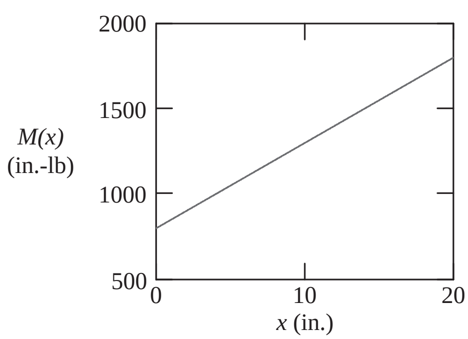
$$I(x) = \frac{bh(x)^3}{12} \qquad S(x) = \frac{I(x)}{\frac{h(x)}{2}}$$

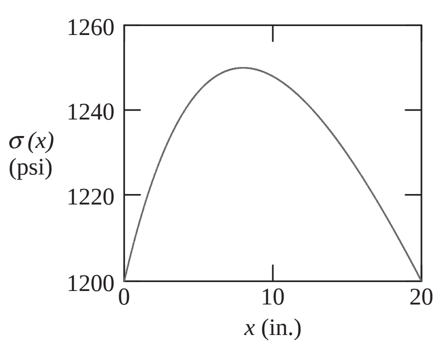
$$S(x) = \frac{bh(x)^2}{6}$$

$$S(x) = \frac{b\left[h_A\left(1 + \frac{x}{2L}\right)\right]^2}{6}$$

$$M(x) = Px + M_0$$

$$\sigma(x) = \frac{M(x)}{S(x)}$$





$$\sigma(x) = \frac{Px + M_0}{\frac{b\left[h_A\left(1 + \frac{x}{2L}\right)\right]^2}{6}}$$

$$\sigma(x) = 24\,(Px + M_0)\frac{L^2}{bh_A^2\,(2L + x)^2}$$

$$\frac{d}{dx}\sigma(x) = 0 \quad \text{then solve for } x_{max}$$

$$\frac{d}{dx}\left[24\,(Px + M_0)\frac{L^2}{bh_A^2\,(2L + x)^2}\right] = 0$$

$$24P\frac{L^2}{bh_A^2\,(2L + x)^2}$$

$$-\,2\,(24Px + 24M_0)\frac{L^2}{bh_A^2\,(2L + x)^3} = 0$$

$$\text{OR simplifying}\left[-24L^2\frac{-2PL + Px + 2M_0}{bh_A^2\,(2L + x)^3}\right] = 0$$

$$\text{so} \quad x = \frac{2\,(PL - M_0)}{P}$$

$x_{max} = 8\text{ in.}$ ←

agrees with plot at left

Evaluate max. stress & stress at B using numerical data

$\sigma_{max} = \sigma(8)$ $\sigma_{max} = 1250\text{ psi}$ ←

$\sigma_B = \sigma(20)$ $\sigma_B = 1200\text{ psi}$

$$\frac{\sigma_{max}}{\sigma_B} = 1.042 \quad \leftarrow$$

(b) Find max. bending stress for tapered cantilever, fig. (b)

$$h(x) = h_A\left(1 + \frac{x}{2L}\right)$$

$$M_0 = \frac{4}{5}PL \qquad M_0 = 800\text{ in.-lb}$$

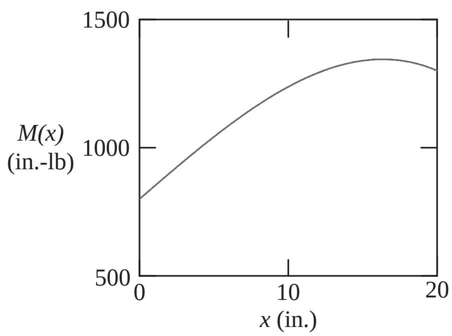
$$q_0 = 3\frac{P}{L}$$

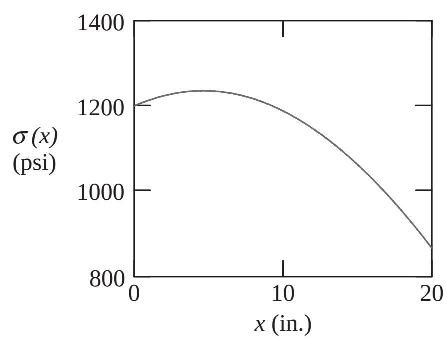
$$I(x) = \frac{bh(x)^3}{12} \qquad S(x) = \frac{I(x)}{\frac{h(x)}{2}} \qquad S(x) = \frac{bh(x)^2}{6}$$

$$S(x) = \frac{b\left[h_A\left(1 + \frac{x}{2L}\right)\right]^2}{6}$$

$$M(x) = Px + M_0 + \frac{-1}{2}\left(\frac{x}{L}q_0\right)x\frac{x}{3}$$

$$\sigma(x) = \frac{M(x)}{S(x)}$$





$$\sigma(x) = \frac{Px + M_0 - \frac{q_0x^3}{6L}}{\frac{b\left[h_A\left(1 + \frac{x}{2L}\right)\right]^2}{6}}$$

$$\sigma(x) = -4(-6PxL - 6M_0L + x^3q_0) \times \frac{L}{bh_A^2(2L + x)^2}$$

$$\frac{d}{dx}\sigma(x) = 0 \quad \text{then solve for } x_{\text{max}}$$

$$\frac{d}{dx}\sigma(x) = \left[(24PL - 12x^2q_0)\frac{L}{bh_A^2(2L + x)^2} - 2(24PxL + 24M_0L - 4x^3q_0) \times \frac{L}{bh_A^2(2L + x)^3}\right] = 0$$

Simplifying

$$-12PL^2 + 6PxL + 6x^2q_0L + x^3q_0 + 12M_0L = 0$$

Solve for x_{max}

$x_{\text{max}} = 4.642$ in. $\longleftarrow$

Max. stress & stress at B

$\sigma_{\text{max}} = \sigma(x_{\text{max}})$

$\sigma_{\text{max}} = 1235$ psi $\longleftarrow$

$\sigma_B = \sigma(20) \qquad \sigma_B = 867$ psi

Find max. moment and stress at location of max. moment

$$\frac{d}{dx}M(x) = 0 \qquad \frac{d}{dx}\left(Px + M_0 - \frac{q_0x^3}{6L}\right) = 0$$

$$x_{\text{m}} = \sqrt{\frac{P(2L)}{q_0}} \qquad x_{\text{m}} = 16.33 \text{ in.}$$

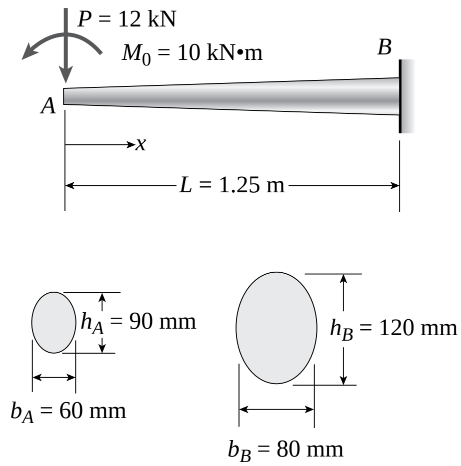
$\sigma_{\text{m}} = \sigma(x_{\text{m}}) \qquad \sigma_{\text{m}} = 1017$ psi

$\dfrac{\sigma_{\text{max}}}{\sigma_{\text{m}}} = 1.215$ $\longleftarrow$

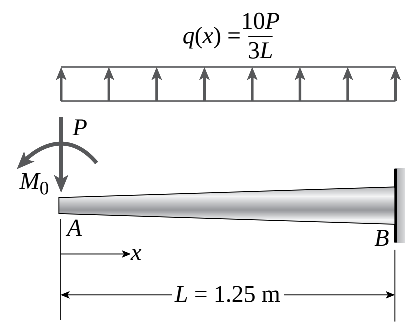
Problem 5.7-4 The spokes in a large flywheel are modeled as beams fixed at one end and loaded by a force P and a couple M_0 at the other (see figure). The cross sections of the spokes are elliptical with major and minor axes (height and width, respectively) having the lengths shown in the figure part (a). The cross-sectional dimensions vary linearly from end A to end B.

Considering only the effects of bending due to the loads P and M_0, determine the following quantities.

(a) The largest bending stress σ_A at end A
(b) The largest bending stress σ_B at end B
(c) The distance x to the cross section of maximum bending stress
(d) The magnitude σ_{max} of the maximum bending stress
(e) Repeat (d) if uniform load $q(x) = 10P/3L$ is added to loadings P and M_0, as shown in the figure part (b).



(a)



(b)

Solution 5.7-4

(a-d) FIND MAX. BENDING STRESS FOR TAPERED CANTILEVER

numerical data

$L = 1.25\ \text{m} \quad b_A = 60\ \text{mm} \quad h_A = 90\ \text{mm}$

$b_B = 80\ \text{mm} \quad h_B = 120\ \text{mm}$

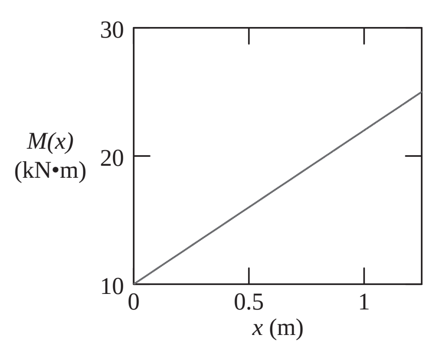
$P = 12\ \text{kn} \quad M_0 = 10\ \text{kN}\cdot\text{m}$

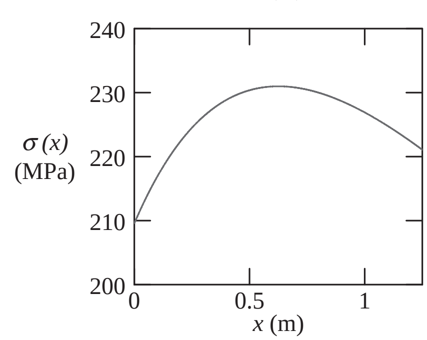
$$h(x) = h_A\left(1 + \frac{x}{3L}\right) \qquad b(x) = b_A\left(1 + \frac{x}{3L}\right)$$

$$I(x) = \frac{\pi\, b(x)\, h(x)^3}{64} \qquad S(x) = \frac{I(x)}{\frac{h(x)}{2}}$$

$$S(x) = \frac{\pi\, b(x)\, h(x)^2}{32}$$

$$S(x) = \frac{\pi\, b_A\, h_A^2\left(1 + \frac{x}{3L}\right)^3}{32}$$





$$M(x) = Px + M_0 \qquad \sigma(x) = \frac{M(x)}{S(x)}$$

$$\sigma(x) = \frac{Px + M_0}{\dfrac{\pi b_A h_A^2 \left(1 + \dfrac{x}{3L}\right)^3}{32}}$$

$$\sigma(x) = 864\left(\frac{Px + M_0}{\pi b_A h_A^2}\right)\left(\frac{L^3}{(3L + x)^3}\right)$$

$$\frac{d}{dx}\sigma(x) = 0 \quad \text{then solve for } x_{\max}$$

$$\frac{d}{dx}\left[864\frac{Px + M_0}{\pi b_A h_A^2}\frac{L^3}{(3L + x)^3}\right] = 0$$

$$864\frac{P}{\pi b_A h_A^2}\frac{L^3}{(3L + x)^3} - 2592\frac{Px + M_0}{\pi b_A h_A^2}\frac{L^3}{(3L + x)^4} = 0$$

OR simplfying

$$\left[-864L^3\frac{-3PL + 2Px + 3M_0}{\pi b_A h_A^2(3L + x)^4}\right] = 0$$

$$\text{so} \quad x_{\max} = \frac{3(PL - M_0)}{2P}$$

$$x_{\max} = 0.625 \text{ m} \quad \leftarrow$$

agrees with plot above

Evaluate using numerical data

$$\sigma_{\max} = \sigma(x_{\max}) \qquad \sigma_{\max} = 231 \text{ MPa} \quad \leftarrow$$

$$\sigma_A = \sigma(0) \qquad \sigma_A = 210 \text{ MPa} \quad \leftarrow$$

$$\sigma_B = \sigma(L) \qquad \sigma_B = 221 \text{ MPa} \quad \leftarrow$$

$$\frac{\sigma_{\max}}{\sigma_B} = 1.045$$

(e) FIND MAX. BENDING STRESS INCLUDING UNIFORM LOAD

$$L = 1.25 \text{ m} \qquad b_A = 60 \text{ mm} \qquad h_A = 90 \text{ mm}$$

$$b_B = 80 \text{ mm} \qquad h_B = 120 \text{ mm}$$

$$P = 12 \text{ kN} \qquad M_0 = 10 \text{ kN}\cdot\text{m}$$

$$h(x) = h_A\left(1 + \frac{x}{3L}\right)$$

$$b(x) = b_A\left(1 + \frac{x}{3L}\right)$$

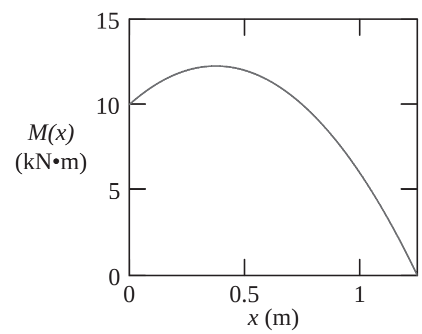
$$I(x) = \frac{\pi\, b(x)\, h(x)^3}{64} \qquad S(x) = \frac{I(x)}{\dfrac{h(x)}{2}}$$

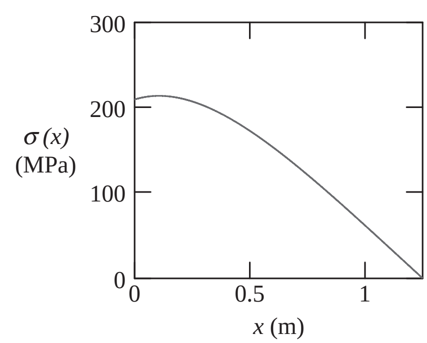
$$S(x) = \frac{\pi\, b(x)\, h(x)^2}{32}$$

$$S(x) = \frac{\pi b_A h_A^2\left(1 + \dfrac{x}{3L}\right)^3}{32}$$

$$M(x) = Px + M_0 - \frac{10}{3}\frac{P}{L}\frac{x^2}{2}$$

$$\sigma(x) = \frac{M(x)}{S(x)}$$





$$\sigma(x) = \frac{Px + M_0 - \dfrac{10}{3}\dfrac{P}{L}\dfrac{x^2}{2}}{\left[\dfrac{\pi b_A h_A^2\left(1 + \dfrac{x}{3L}\right)^3}{32}\right]}$$

$$\sigma(x) = -288\,(-3PxL - 3M_0L + 5Px^2)\frac{L^2}{\pi b_A h_A^2(3L + x)^3}$$

$$\frac{d}{dx}\sigma(x) = 0 \quad \text{then solve for } x_{\max}$$

$$\frac{d}{dx}\left[-288\,(-3PxL - 3M_0L + 5Px^2) \times \frac{L^2}{\pi b_A h_A^2\,(3L + x)^3}\right] = 0$$

$$\frac{d}{dx}\sigma(x) = \left[(864PL - 2880Px)\frac{L^2}{\pi\, b_A\, h_A^2\,(3L + x)^3} - 3\,(864PxL + 864M_0L - 1440Px^2) \times \frac{L^2}{\pi b_A h_A^2\,(3L + x)^4}\right] = 0$$

OR simplifying

$$(288L^2)\frac{\left[9PL^2 - 36PxL + 5Px^2 - 9M_0L\right]}{\left[\pi b_A h_A^2\,(3L + x)^4\right]} = 0$$

OR

$$9PL^2 - 36PxL + 5Px^2 - 9M_0L = 0$$

Solving for $x_{\max}$: $x_{\max} = 0.105\text{m}$

solution agrees with plot above, evaluate using numerical data

$\sigma_{\max} = \sigma(x_{\max})$ $\quad\sigma_{\max} = 214$ MPa $\leftarrow$

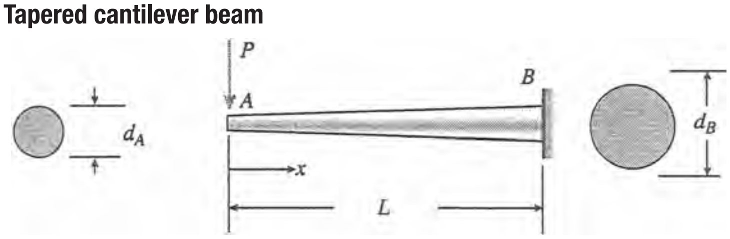
$\sigma_A = \sigma(0)$ $\quad\sigma_A = 210$ MPa $\leftarrow$

$\sigma_B = \sigma(L)$ $\quad\sigma_B = 0$ MPa $\leftarrow$

Problem 5.7-5 Refer to the tapered cantilever beam of solid circular cross section shown in Fig. 5-24 of Example 5-9.

(a) Considering only the bending stresses due to the load P, determine the range of values of the ratio d_B/d_A for which the maximum normal stress occurs at the support.
(b) What is the maximum stress for this range of values?

Solution 5.7-5 Tapered cantilever beam



FROM EQ. (5-32), EXAMPLE 5-9

$$\sigma_1 = \frac{32Px}{\pi\left[d_A + (d_B - d_A)\left(\frac{x}{L}\right)\right]^3} \qquad \text{Eq. (1)}$$

FIND THE VALUE OF x THAT MAKES σ_1 A MAXIMUM

$$\text{Let } \sigma_1 = \frac{u}{v} \quad \frac{d\sigma_1}{dx} = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2} = \frac{N}{D}$$

$$N = \pi\left[d_A + (d_B - d_A)\left(\frac{x}{L}\right)\right]^3[32P]$$

$$- [32Px][\pi]\,[3]\left[d_A + (d_B - d_A)\left(\frac{x}{L}\right)\right]^2\left[\frac{1}{L}(d_B - d_A)\right]$$

After simplification:

$$N = 32\pi P\left[d_A + (d_B - d_A)\left(\frac{x}{L}\right)\right]^2\left[d_A - 2(d_B - d_A)\frac{x}{L}\right]$$

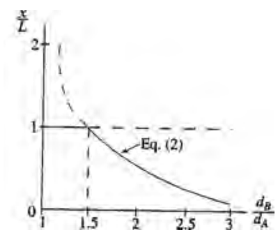
$$D = \pi^2\left[d_A + (d_B - d_A)\frac{x}{L}\right]^6$$

$$\frac{d\sigma_1}{dx} = \frac{N}{D} = \frac{32P\left[d_A - 2(d_B - d_A)\frac{x}{L}\right]}{\pi\left[d_A + (d_B - d_A)\left(\frac{x}{L}\right)\right]^4}$$

$$\frac{d\sigma_1}{dx} = 0 \quad d_A - 2(d_B - d_A)\left(\frac{x}{L}\right) = 0$$

$$\therefore \frac{x}{L} = \frac{d_A}{2(d_B - d_A)} = \frac{1}{2\left(\frac{d_B}{d_A} - 1\right)} \quad \text{Eq. (2)}$$

(a) Graph of x/L versus d_B/d_A (Eq. 2)



Maximum bending stress occurs at the support when

$$1 \leq \frac{d_B}{d_A} \leq 1.5 \qquad \leftarrow$$

(b) Maximum stress (at support B)
Substitute $x/L = 1$ into Eq. (1):

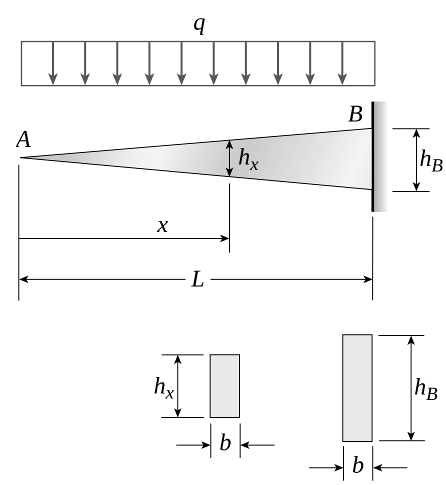
$$\sigma_{max} = \frac{32PL}{\pi d_B^3} \qquad \leftarrow$$

Fully Stressed Beams

Problems 5.7-6 to 5.7-8 pertain to fully stressed beams of rectangular cross section. Consider only the bending stresses obtained from the flexure formula and disregard the weights of the beams.

Problem 5.7-6 A cantilever beam AB having rectangular cross sections with constant width b and varying height h_x is subjected to a uniform load of intensity q (see figure).

How should the height h_x vary as a function of x (measured from the free end of the beam) in order to have a fully stressed beam? (Express h_x in terms of the height h_B at the fixed end of the beam.)



Solution 5.7-6 Fully stressed beam with constant width and varying height

h_x = height at distance x
h_B = height at end B
b = width (constant)

At distance x: $M = \dfrac{qx^2}{2} \qquad S = \dfrac{bh_x^2}{6}$

$$\sigma_{allow} = \frac{M}{S} = \frac{3qx^2}{bh_x^2}$$

$$h_x = x\sqrt{\frac{3q}{b\sigma_{allow}}}$$

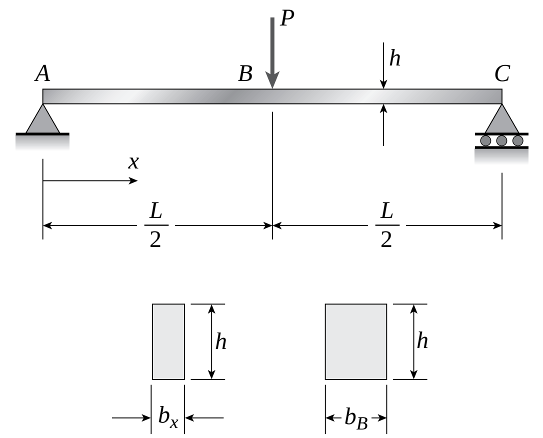
At the fixed end ($x = L$):

$$h_B = L\sqrt{\frac{3q}{b\sigma_{allow}}}$$

Therefore, $\dfrac{h_x}{h_B} = \dfrac{x}{L} \qquad h_x = \dfrac{h_B x}{L} \qquad \leftarrow$

Problem 5.7-7 A simple beam ABC having rectangular cross sections with constant height h and varying width b_x supports a concentrated load P acting at the midpoint (see figure).

How should the width b_x vary as a function of x in order to have a fully stressed beam? (Express b_x in terms of the width b_B at the midpoint of the beam.)



Solution 5.7-7 Fully stressed beam with constant height and varying width

h = height of beam (constant)

b_x = width at distance x from end $A\left(0 \le x \le \dfrac{L}{2}\right)$

b_B = width at midpoint B $\quad (x = L/2)$

AT DISTANCE x $\quad M = \dfrac{Px}{2} \quad S = \dfrac{1}{6} b_x h^2$

$$\sigma_{\text{allow}} = \frac{M}{S} = \frac{3Px}{b_x h^2} \qquad b_x = \frac{3Px}{\sigma_{\text{allow}} h^2}$$

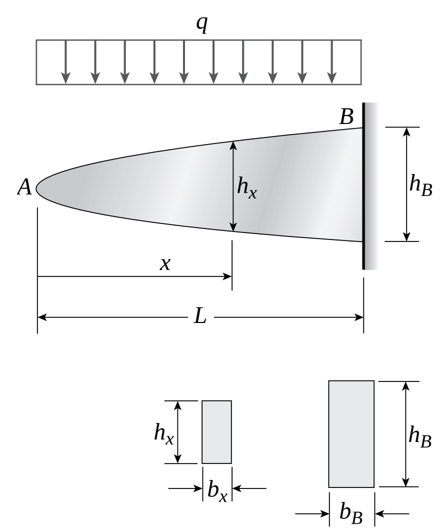
AT MIDPOINT B $(x = L/2)$

$$b_B = \frac{3PL}{2\sigma_{\text{allow}} h^2}$$

Therefore, $\dfrac{b_x}{b_b} = \dfrac{2x}{L}$ and $b_x = \dfrac{2b_B\,x}{L}$ $\quad \leftarrow$

NOTE: The equation is valid for $0 \le x \le \dfrac{L}{2}$ and the beam is symmetrical about the midpoint.

Problem 5.7-8 A cantilever beam AB having rectangular cross sections with varying width b_x and varying height h_x is subjected to a uniform load of intensity q (see figure). If the width varies linearly with x according to the equation $b_x = b_B\,x/L$, how should the height h_x vary as a function of x in order to have a fully stressed beam? (Express h_x in terms of the height h_B at the fixed end of the beam.)



Solution 5.7-8 Fully stressed beam with varying width and varying height

h_x = height at distance x
h_B = height at end B
b_x = width at distance x
b_B = width at end B

$$b_x = b_B\left(\frac{x}{L}\right)$$

At distance x

$$M = \frac{qx^2}{2} \quad S = \frac{b_x h_x^2}{6} = \frac{b_B\, x}{6L}(h_x)^2$$

$$\sigma_{\text{allow}} = \frac{M}{S} = \frac{3qLx}{b_B\, h_x^2}$$

$$h_x = \sqrt{\frac{3qLx}{b_B\sigma_{\text{allow}}}}$$

At the fixed end ($x = L$)

$$h_B = \sqrt{\frac{3qL^2}{b_B\sigma_{\text{allow}}}}$$

Therefore, $\dfrac{h_x}{h_B} = \sqrt{\dfrac{x}{L}} \quad h_x = h_B\sqrt{\dfrac{x}{L}} \quad \leftarrow$

Shear Stresses in Rectangular Beams

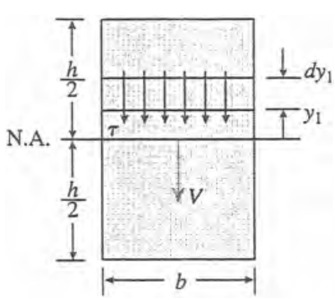
Problem 5.8-1 The shear stresses τ in a rectangular beam are given by Eq. (5-39):

$$\tau = \frac{V}{2I}\left(\frac{h^2}{4} - y_1^2\right)$$

in which V is the shear force, I is the moment of inertia of the cross-sectional area, h is the height of the beam, and y_1 is the distance from the neutral axis to the point where the shear stress is being determined (Fig. 5-30).

By integrating over the cross-sectional area, show that the resultant of the shear stresses is equal to the shear force V.

Solution 5.8-1 Resultant of the shear stresses



$$I = \frac{bh^3}{12}$$

$$\tau = \frac{V}{2I}\left(\frac{h^2}{4} - y_1^2\right)$$

V = shear force acting on the cross section

R = resultant of shear stresses τ

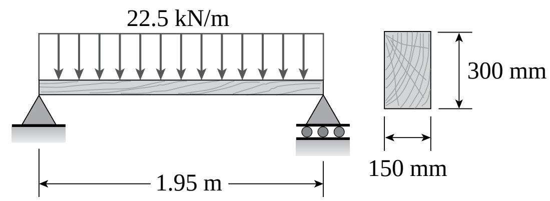
$$R = \int_{-h/2}^{h/2} \tau b dy_1 = 2\int_0^{h/2} \frac{V}{2I}\left(\frac{h^2}{4} - y_1^2\right) b dy_1$$

$$= \frac{12V}{bh^3}(b)\int_0^{h/2}\left(\frac{h^2}{4} - y_1^2\right) dy_1$$

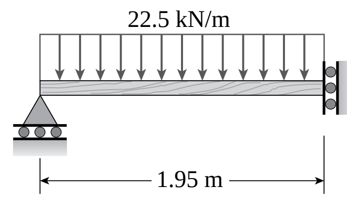
$$= \frac{12V}{h^3}\left(\frac{2h^3}{24}\right) = V$$

$\therefore R = V$ Q.E.D. $\leftarrow$

Problem 5.8-2 Calculate the maximum shear stress τ_{max} and the maximum bending stress σ_{max} in a wood beam (see figure) carrying a uniform load of 22.5 kN/m (which includes the weight of the beam) if the length is 1.95 m and the cross section is rectangular with width 150 mm and height 300 mm, and the beam is (a) simply supported as in the figure part (a) and (b) has a sliding support at right as in the figure part (b).



(a)



(b)

Solution 5.8-2

$$q = 22\frac{\text{kN}}{\text{m}} \qquad b = 150 \text{ mm}$$

$$h = 300 \text{ mm} \qquad L = 1.95 \text{ m}$$

(a) Maximum shear stress

$$V = \frac{qL}{2} \qquad A = bh$$

$$\tau_{max} = \frac{3}{2}\frac{V}{A} \qquad \tau_{max} = 715 \text{ kPa} \quad \longleftarrow$$

Maximum bending stress

$$M = \frac{qL^2}{8} \qquad S = \frac{bh^2}{6}$$

$$\sigma_{max} = \frac{M}{S} \qquad \sigma_{max} = 4.65 \text{ MPa} \quad \longleftarrow$$

(b) Maximum shear stress

$$V = qL$$

$$\tau_{max} = \frac{3}{2}\frac{V}{A} \qquad \tau_{max} = 1430 \text{ kPa} \quad \longleftarrow$$

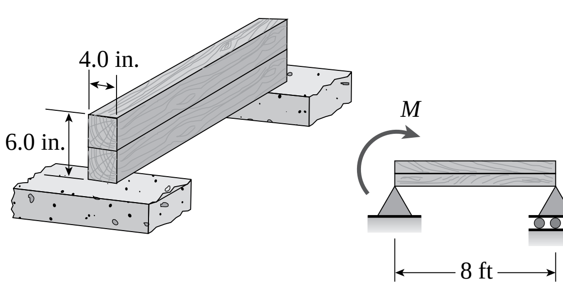
Maximum bending stress

$$M = \frac{qL^2}{2}$$

$$\sigma_{max} = \frac{M}{S} \qquad \sigma_{max} = 18.59 \text{ MPa} \quad \longleftarrow$$

Problem 5.8-3 Two wood beams, each of rectangular cross section (3.0 in. × 4.0 in., actual dimensions) are glued together to form a solid beam of dimensions 6.0 in. × 4.0 in. (see figure). The beam is simply supported with a span of 8 ft.

What is the maximum moment M_{max} that may be applied at the left support if the allowable shear stress in the glued joint is 200 psi? (Include the effects of the beam's own weight, assuming that the wood weighs 35 lb/ft^3.)



Solution 5.8-3

$L = 8\text{ ft} \qquad b = 4\text{ in.}$

$h = 6\text{ in.} \qquad \tau_{\text{allow}} = 200\text{ psi} \qquad A = b \cdot h$

$\gamma = 35\dfrac{\text{lb}}{\text{ft}^3}$

$q = \gamma A$ weight of beam per unit distance

$q = 5.833\,\dfrac{\text{1b}}{\text{ft}}$

Maximum load $M_{\max}$

$$V = \frac{M}{L} + \frac{qL}{2}$$

$$\tau_{\max} = \frac{3\,V}{2\,A} = \frac{3}{2\,A}\left(\frac{M}{L} + \frac{qL}{2}\right)$$

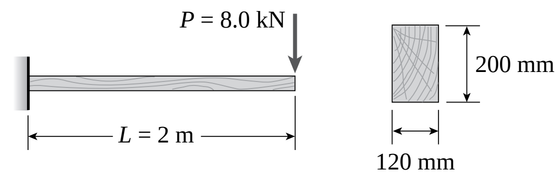
$$M = \frac{2\,AL}{3}\tau_{\max} - \frac{qL^2}{2}$$

$$M_{\max} = \frac{2\,AL}{3}\tau_{\text{allow}} - \frac{qL^2}{2}$$

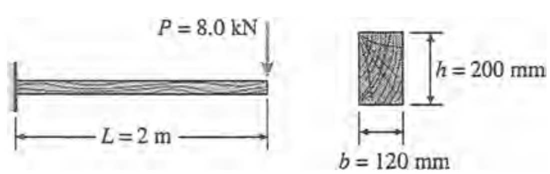
$$M_{\max} = 25.4\text{ k-ft} \qquad \longleftarrow$$

Problem 5.8-4 A cantilever beam of length $L = 2$ m supports a load $P = 8.0$ kN (see figure). The beam is made of wood with cross-sectional dimensions 120 mm × 200 mm.

Calculate the shear stresses due to the load P at points located 25 mm, 75 mm, and 100 mm from the top surface of the beam. from these results, plot a graph showing the distribution of shear stresses from top to bottom of the beam.



Solution 5.8-4 Shear stresses in a cantilever beam



Eq. (5-39): $\tau = \dfrac{V}{2I}\left(\dfrac{h^2}{4} - y_1^2\right)$

$V = P = 8.0\text{ kN} = 8{,}000\text{ N}$

$I = \dfrac{bh^3}{12} = 80 \times 10^6\text{ mm}^4$

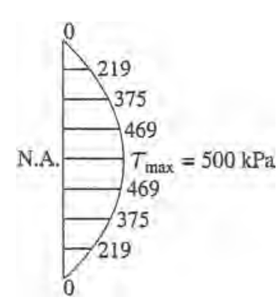
$h = 200\text{ mm} \quad (y_1 = \text{mm})$

$\tau = \dfrac{8{,}000}{2(80 \times 10^6)}\left[\dfrac{(200)^2}{4} - y_1^2\right] \quad (\tau = \text{N/mm}^2 = \text{MPa})$

$\tau = 50 \times 10^{-6}(10{,}000 - y_1^2) \quad (y_1 = \text{mm};\ \tau = \text{MPa})$

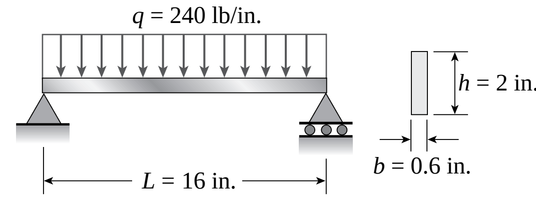
Distance from the top surface (mm)	y_1 (mm)	τ (MPa)	τ (kPa)
0	100	0	0
25	75	0.219	219
50	50	0.375	375
75	25	0.469	469
100 (N.A.)	0	0.500	500

Graph of shear stress τ

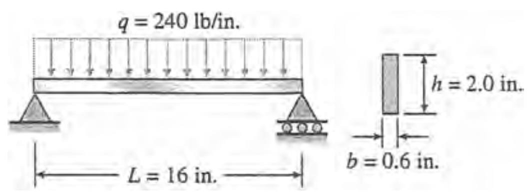


Problem 5.8-5 A steel beam of length $L = 16$ in. and cross-sectional dimensions $b = 0.6$ in. and $h = 2$ in. (see figure) supports a uniform load of intensity $q = 240$ lb/in., which includes the weight of the beam.

Calculate the shear stresses in the beam (at the cross section of maximum shear force) at points located 1/4 in., 1/2 in., 3/4 in., and 1 in. from the top surface of the beam. From these calculations, plot a graph showing the distribution of shear stresses from top to bottom of the beam.



Solution 5.8-5 Shear stresses in a simple beam



Eq. (5-39): $\tau = \dfrac{V}{2I}\left(\dfrac{h^2}{4} - y_1^2\right)$

$V = \dfrac{qL}{2} = 1920 \text{ lb} \quad I = \dfrac{bh^3}{12} = 0.4 \text{ in.}^4$

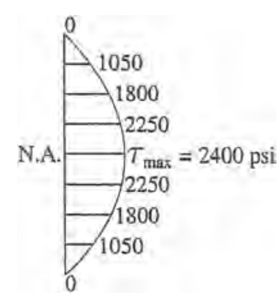
UNITS: POUNDS AND INCHES

$$\tau = \frac{1920}{2(0.4)}\left[\frac{(2)^2}{4} - y_1^2 = (2400)(1 - y_1^2)\right]$$

(τ = psi; y_1 = in.)

Distance from the top surface (in.)	y_1 (in.)	τ (psi)
0	1.00	0
0.25	0.75	1050
0.50	0.50	1800
0.75	0.25	2250
1.00 (N.A.)	0	2400

GRAPH OF SHEAR STRESS τ



Problem 5.8-6 A beam of rectangular cross section (width b and height h) supports a uniformly distributed load along its entire length L. The allowable stresses in bending and shear are σ_{allow} and τ_{allow}, respectively.

(a) If the beam is simply supported, what is the span length L_0 below which the shear stress governs the allowable load and above which the bending stress governs?
(b) If the beam is supported as a cantilever, what is the length L_0 below which the shear stress governs the allowable load and above which the bending stress governs?

Solution 5.8-6 Beam of rectangular cross section

$b =$ width $\quad h =$ height $\quad L =$ length

Uniform load $\quad q =$ intensity of load

ALLOWABLE STRESSES $\quad \sigma_{\text{allow}}$ and τ_{allow}

(a) SIMPLE BEAM

BENDING

$$M_{\max} = \frac{qL^2}{8} \quad S = \frac{bh^2}{6}$$

$$\sigma_{\max} = \frac{M_{\max}}{S} = \frac{3qL^3}{4bh^2}$$

$$q_{\text{allow}} = \frac{4\sigma_{\text{allow}}\, bh^2}{3L^2} \tag{1}$$

SHEAR

$$V_{\max} = \frac{qL}{2} \quad A = bh$$

$$\tau_{\max} = \frac{3V}{2A} = \frac{3qL}{4bh}$$

$$q_{\text{allow}} = \frac{4\tau_{\text{allow}} bh}{3L} \tag{2}$$

Equate (1) and (2) and solve for L_0:

$$L_0 = h\left(\frac{\sigma_{\text{allow}}}{\tau_{\text{allow}}}\right) \quad \leftarrow$$

(b) CANTILEVER BEAM

BENDING

$$M_{\max} = \frac{qL^2}{2} \quad S = \frac{bh^2}{6}$$

$$\sigma_{\max} = \frac{M_{\max}}{S} = \frac{3qL^2}{bh^2}$$

$$q_{\text{allow}} = \frac{\sigma_{\text{allow}} bh^2}{3L^2} \tag{3}$$

SHEAR

$$V_{\max} = qL \quad A = bh$$

$$\tau_{\max} = \frac{3V}{2A} = \frac{3qL}{2bh}$$

$$q_{\text{allow}} = \frac{2\tau_{\text{allow}}\, bh}{3L} \tag{4}$$

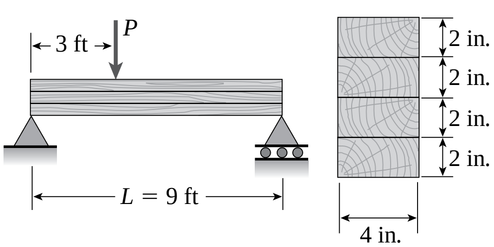
Equate (3) and (4) and solve for L_0:

$$L_0 = \frac{h}{2}\left(\frac{\sigma_{\text{allow}}}{\tau_{\text{allow}}}\right) \quad \leftarrow$$

NOTE: If the actual length is less than L_0, the shear stress governs the design. If the length is greater than L_0, the bending stress governs.

Problem 5.8-7 A laminated wood beam on simple supports is built up by gluing together four 2 in. × 4 in. boards (actual dimensions) to form a solid beam 4 in. × 8 in. in cross section, as shown in the figure. The allowable shear stress in the glued joints is 65 psi, and the allowable bending stress in the wood is 1800 psi.

If the beam is 9 ft long, what is the allowable load P acting at the one-third point along the beam as shown? (Include the effects of the beam's own weight, assuming that the wood weighs 35 lb/ft^3.)



Solution 5.8-7

$L = 9$ ft $\quad b = 4$ in.

$h = 8$ in. $\quad A = bh$

$\tau_{\text{allow}} = 65$ psi $\quad \sigma_{\text{allow}} = 1800$ psi

WEIGHT OF BEAM PER UNIT DISTANCE

$$\gamma = 35\,\frac{\text{lb}}{\text{ft}^3} \qquad q = \gamma A$$

$$q = 7.778\,\frac{\text{1b}}{\text{ft}}$$

ALLOWABLE LOAD BASED UPON SHEAR STRESS IN THE GLUED JOINTS; MAX. SHEAR STRESS AT NEUTRAL AXIS

$$\tau = \frac{VQ}{\text{Ib}} \qquad \tau_{\max} = \frac{3\,V}{2\,A}$$

$$V = P\frac{2}{3} + \frac{qL}{2}$$

$$\tau_{max} = \frac{3}{2}\frac{V}{A} = \frac{3}{2A}\left(P\frac{2}{3} + \frac{qL}{2}\right)$$

$$P = \left(A\tau_{max} - \frac{3\,qL}{4}\right)$$

$$P_{max} = A\tau_{allow} - \frac{3\,qL}{4}$$

$$P_{max} = 2.03\ k \text{ (governs)}$$

ALLOWABLE LOAD BASED UPON BENDING STRESS

$$M = P\frac{2}{3}3\text{ ft} + \frac{qL}{2}3\text{ ft} - \frac{q}{2}(3\text{ ft})^2$$

$$S = \frac{bh^2}{6}$$

$$\sigma_{max} = \frac{M}{S} = \frac{P\frac{2}{3}3\text{ ft} + \frac{qL}{2}3\text{ ft} - \frac{q}{2}(3\text{ ft})^2}{S}$$

$$P_{max} = \frac{\sigma_{allow}S3}{(3\text{ ft})\,2} - \frac{3}{2}\left(\frac{qL}{2} - \frac{q}{2}(3\text{ft})\right)$$

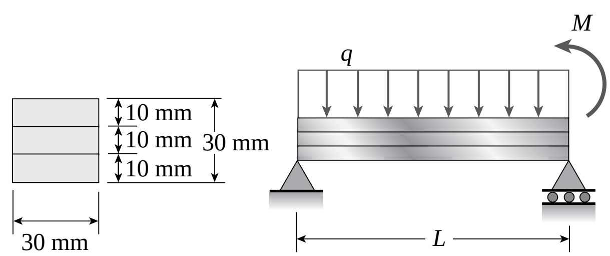
$$P_{max} = 3.165\text{ k}$$

$$P_{allow} = 2.03\text{ k} \quad \longleftarrow$$

Problem 5.8-8 A laminated plastic beam of square cross section is built up by gluing together three strips, each 10 mm × 30 mm in cross section (see figure). The beam has a total weight of 3.6 N and is simply supported with span length L = 360 mm.

Considering the weight of the beam (q) calculate the maximum permissible *CCW* moment M that may be placed at the right support.

(a) If the allowable shear stress in the glued joints is 0.3 MPa.
(b) If the allowable bending stress in the plastic is 8 MPa.



Solution 5.8-8

(a) FIND M BASED ON ALLOWABLE SHEAR STRESS IN GLUED JOINT

b = 30 mm h = 30 mm τ_a = 0.3 MPa

W = 3.6 N L = 360 mm

$$q = \frac{W}{L}$$

$$q = 10\ \frac{\text{N}}{\text{m}} \qquad \text{beam distributed weight}$$

MAX. SHEAR ST LEFT SUPPORT

$$V_m = \frac{qL}{2} + \frac{M}{L} \quad \text{and} \quad V_m = \tau_a\left(\frac{Ib}{Q}\right)$$

$$\tau_a = \frac{V_mQ}{Ib} \quad I = \frac{bh^3}{12} \quad Ib = \frac{b^2h^3}{12}$$

$$Q = \frac{bh}{3}\frac{h}{3} \qquad Q = \frac{bh^2}{9} \qquad \frac{Q}{Ib} = \frac{\frac{bh^2}{9}}{\frac{b^2h^3}{12}}$$

$$\frac{Q}{Ib} = \frac{4}{3bh}$$

$$M = L\left[\tau_a\left(\frac{Ib}{Q}\right) - \frac{qL}{2}\right]$$

$$M = L\left[\tau_a\left(\frac{3bh}{4}\right) - \frac{qL}{2}\right]$$

$$M_{max} = 72.2\ N \cdot M \quad \longleftarrow$$

(b) FIND M BASED ON ALLOWABLE BENDING STRESS AT h/2 FROM NA AT LOCATION (x_m) OF MAX. BENDING MOMENT, M_m

$$M(x) = \left(\frac{qL}{2} + \frac{M}{L}\right)x - \frac{qx^2}{2} \qquad \frac{d}{dx}M(x) = 0$$

use to find location of zero shear where max. moment occurs

$$\frac{d}{dx}\left[\left(\frac{qL}{2} + \frac{M}{L}\right)x - \frac{qx^2}{2}\right]$$

$$= \frac{1}{2}qL + \frac{M}{L} - qx = 0$$

$$x_m = \frac{L}{2} + \frac{M}{qL}$$

MAX. MOMENT M_{m}

$$M_m = \left(\frac{qL}{2} + \frac{M}{L}\right)x_m - \frac{q{x_m}^2}{2}$$

$$M_m = \left(\frac{qL}{2} + \frac{M}{L}\right)\left(\frac{L}{2} + \frac{M}{qL}\right) - \frac{q\left(\frac{L}{2} + \frac{M}{qL}\right)^2}{2}$$

simplifying

$$M_m = \frac{1}{8q}\,\frac{(qL^2 + 2M)^2}{L^2}$$

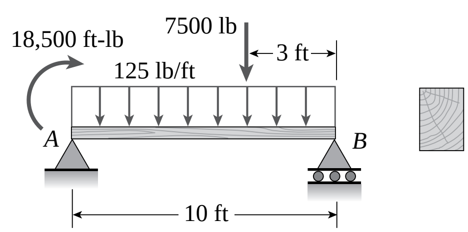
$$also \quad M_m = \sigma_a S \qquad M_m = \sigma_a\left(\frac{bh^2}{6}\right)$$

Equating both M_m expressions & solving for M where $\sigma_a = 8$ MPa

$$M = \frac{\sqrt{\sigma_a\left(\frac{bh^2}{6}\right)\left(8qL^2\right)} - qL^2}{2}$$

$$M_{\max} = 9.01\ \mathrm{N \cdot m} \quad \longleftarrow$$

Problem 5.8-9 A wood beam AB on simple supports with span length equal to 10 ft is subjected to a uniform load of intensity 125 lb/ft acting along the entire length of the beam, a concentrated load of magnitude 7500 lb acting at a point 3 ft from the right-hand support, and a moment at A of 18,500 ft-lb (see figure). The allowable stresses in bending and shear, respectively, are 2250 psi and 160 psi.



(a) From the table in Appendix F, select the lightest beam that will support the loads (disregard the weight of the beam).
(b) Taking into account the weight of the beam (weight density 5 35 lb/ft3), verify that the selected beam is satisfactory, or if it is not, select a new beam.

Solution 5.8-9

(a) $q = 125\,\frac{1\mathrm{b}}{\mathrm{ft}} \quad P = 75001\mathrm{b} \quad M = 18500$ ft-b

$L = 10$ ft $\qquad d = 3$ ft

$\sigma_{\mathrm{Allow}} = 2250$ psi $\qquad \tau_{\mathrm{allow}} = 160$ psi

$$R_A = \frac{qL}{2} + P\frac{d}{L} - \frac{M}{L}$$

$$R_A = 1.025 \times 10^3\ 1\mathrm{b}$$

$$R_B = \frac{qL}{2} + P\frac{L-d}{L} + \frac{M}{L}$$

$$R_B = 7.725 \times 10^3\ 1\mathrm{b}$$

$$V_{\max} = R_B \qquad V_{\max} = 7.725 \times 10^3\ 1\mathrm{b}$$

$$M_{\max} = R_B d - \frac{qd^2}{2}$$

$$M_{\max} = 2.261 \times 10^4\ 1\mathrm{b\text{-}ft}$$

$$\tau_{\max} = \frac{3}{2}\frac{V}{A} \qquad A_{\mathrm{req}} = \frac{3V_{\max}}{2\tau_{\mathrm{allow}}}$$

$$A_{\mathrm{req}} = 72.422\ \mathrm{in.}^2$$

$\sigma_{max} = \dfrac{M}{S} \qquad S_{req} = \dfrac{M_{max}}{\sigma_{allow}} \qquad S_{req} = 120.6 \text{ in.}^3$

From Appendix F: Select 8 × 12 in. beam (nominal dimensions) ←

$A = 86.25 \text{ in.}^2 \qquad S = 165.3 \text{ in.}^3$

(b) REPEAT (A) CONSIDERING THE WEIGHT OF THE BEAM

$\gamma = 35 \dfrac{1\text{b}}{\text{ft}^3} \qquad q_{beam} = \gamma A$

$q_{\ beam} = 20.964 \dfrac{1\text{b}}{\text{ft}}$

$R_B = 7.725 \times 10^3 \text{ 1b} + \dfrac{q_{beam} L}{2}$

$R_B = 7.83 \times 10^3 \text{ 1b}$

$V_{max} = R_B \qquad A_{req} = \dfrac{3V_{max}}{2\,\tau_{allow}}$

$A_{req} = 73.405 \text{ in.}^2 < \text{A}$

8 × 12 beam is still satisfactory for shear.

$q_{total} = q + q_{beam} \quad q_{\ total} = 145.964 \dfrac{1\text{b}}{\text{ft}}$

$M_{max} = R_B d - \dfrac{qd^2}{2}$

$M_{max} = 2.293 \times 10^4 \text{ 1b-ft}$

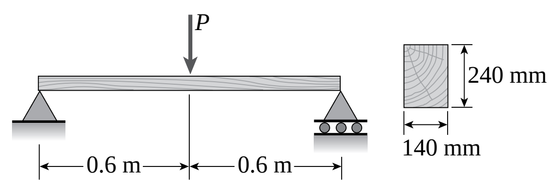
$S_{req} = \dfrac{M_{max}}{\sigma_{allow}} \qquad S_{req} = 122.3 \text{ in.}^3 < S$

8 × 12 beam is still satisfactory for moment.

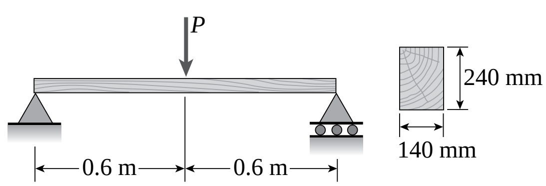
Use 8 × 12 in. beam ←

Problem 5.8-10 A simply supported wood beam of rectangular cross section and span length 1.2 m carries a concentrated load P at midspan in addition to its own weight (see figure). The cross section has width 140 mm and height 240 mm. The weight density of the wood is 5.4 kN/m^3.

Calculate the maximum permissible value of the load P if (a) the allowable bending stress is 8.5 MPa, and (b) the allowable shear stress is 0.8 MPa.



Solution 5.8-10 Simply supported wood beam



$b = 140 \text{ mm} \qquad h = 240 \text{ mm}$

$A = bh = 33{,}600 \text{ mm}^2$

$S = \dfrac{bh^2}{6} = 1344 \times 10^3 \text{ mm}^3$

$\gamma = 5.4 \text{ kN/m}^3$

$L = 1.2 \text{ m} \quad q = \gamma bh = 181.44 \text{ N/m}$

(a) ALLOWABLE P BASED UPON BENDING STRESS

$\sigma_{allow} = 8.5 \text{ MPa} \quad \sigma = \dfrac{M_{max}}{S}$

$$M_{max} = \frac{PL}{4} + \frac{qL^2}{8} = \frac{P(1.2 \text{ m})}{4} + \frac{(181.44 \text{ N/m})(1.2 \text{ m})^2}{8}$$

$= 0.3\,P + 32.66 \text{ N}\cdot\text{m}$

$(P = \text{newtons};\ M = \text{N}\cdot\text{m})$

$M_{max} = S\sigma_{allow} = (1344 \times 10^3 \text{ mm}^3)(8.5 \text{ MPa})$

$= 11{,}424 \text{ N}\cdot\text{m}$

Equate values of M_{max} and solve for P:

$0.3P + 32.66 = 11{,}424 \quad P = 37{,}970 \text{ N}$

or $\quad P = 38.0 \text{ kN} \quad$ ←

(b) Allowable load P based upon shear stress

$$\tau_{allow} = 0.8 \text{ MPa} \quad \tau = \frac{3V}{2A}$$

$$V = \frac{P}{2} + \frac{qL}{2} = \frac{P}{2} + \frac{(181.44 \text{ N/m})(1.2 \text{ m})}{2}$$

$$= \frac{P}{2} + 108.86 \text{ (N)}$$

$$V = \frac{2A\tau}{3} = \frac{2}{3}(33{,}600 \text{ mm}^2)(0.8 \text{ MPa}) = 17{,}920 \text{ N}$$

Equate values of V and solve for P:

$$\frac{P}{2} + 108.86 = 17{,}920 \quad P = 35{,}622\ N$$

or $P = 35.6$ kN $\leftarrow$

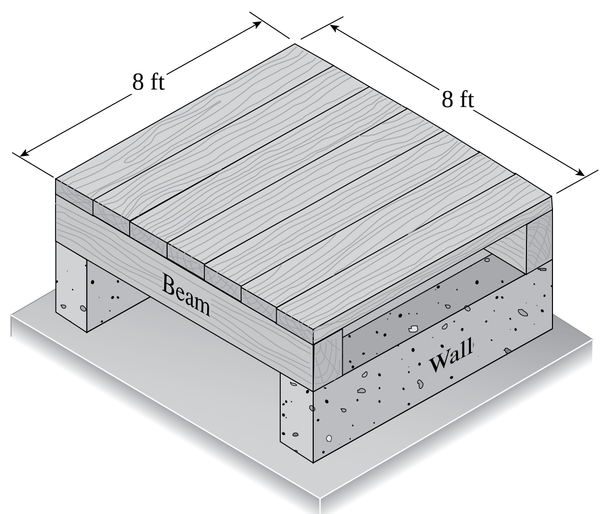
NOTE: The shear stress governs and $P_{allow} = 35.6$ kN

Problem 5.8-11 A square wood platform, 8 ft × 8 ft in area, rests on masonry walls (see figure). The deck of the platform is constructed of 2 in. nominal thickness tongue-and-groove planks (actual thickness 1.5 in.; see Appendix F) supported on two 8-ft long beams. The beams have 4 in. × 6 in. nominal dimensions (actual dimensions 3.5 in. × 5.5 in.).

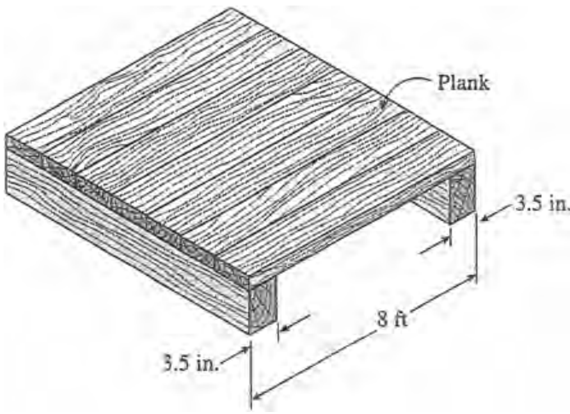
The planks are designed to support a uniformly distributed load w (lb/ft^2) acting over the entire top surface of the platform. The allowable bending stress for the planks is 2400 psi and the allowable shear stress is 100 psi. When analyzing the planks, disregard their weights and assume that their reactions are uniformly distributed over the top surfaces of the supporting beams.

(a) Determine the allowable platform load w_1 (lb/ft^2) based upon the bending stress in the planks.
(b) Determine the allowable platform load w_2 (lb/ft^2) based upon the shear stress in the planks.
(c) Which of the preceding values becomes the allowable load w_{allow} on the platform?

(*Hints*: Use care in constructing the loading diagram for the planks, noting especially that the reactions are distributed loads instead of concentrated loads. Also, note that the maximum shear forces occur at the inside faces of the supporting beams.)



Solution 5.8-11 Wood platform with a plank deck



Platform: 8 ft × 8 ft

t = thickness of planks

= 1.5 in.

w = uniform load on the deck (lb/ft^2)

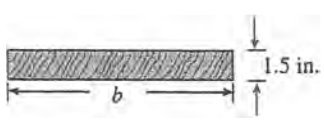
$\sigma_{allow} = 2400$ psi

$\tau_{allow} = 100$ psi

Find w_{allow} (lb/ft^2)

(a) ALLOWABLE LOAD BASED UPON BENDING STRESS IN THE PLANKS

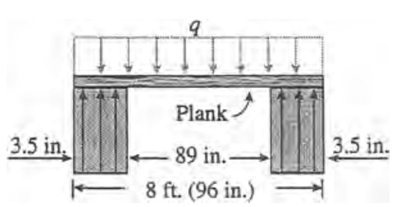
Let b = width of one plank (in.)



$$A = 1.5b \text{ (in.}^2)$$

$$S = \frac{b}{6}(1.5 \text{ in.})^2 = 0.375b \text{ (in.}^3)$$

Free-body diagram of one plank supported on the beams:



Load on one plank:

$$q = \left[\frac{w(\text{lb/ft}^2)}{144 \text{ in.}^2/\text{ft}^2}\right](b \text{ in.}) = \frac{wb}{144} \text{ (lb/in.)}$$

Reaction $R = q\left(\frac{96 \text{ in.}}{2}\right) = \left(\frac{wb}{144}\right)(48) = \frac{wb}{3}$

$$(R = \text{lb};\ w = \text{lb/ft}^2;\ b = \text{in.})$$

M_{max} occurs at midspan.

$$M_{max} = R\left(\frac{3.5 \text{ in.}}{2} + \frac{89 \text{ in.}}{2}\right) - \frac{q(48 \text{ in.})^2}{3}$$

$$= \frac{wb}{3}(46.25) - \frac{wb}{144}(1152) = \frac{89}{12}wb$$

$$(M = \text{lb-in.};\ w = \text{lb/ft}^2;\ b = \text{in.})$$

Allowable bending moment:

$$M_{allow} = \sigma_{allow} S = (2400 \text{ psi})(0.375\, b)$$

$$= 900\, b \text{ (lb-in.)}$$

Equate M_{max} and M_{allow} and solve for w:

$$\frac{89}{12}wb = 900\, b \quad w_1 = 121 \text{ lb/ft}^2 \quad \leftarrow$$

(b) ALLOWABLE LOAD BASED UPON SHEAR STRESS IN THE PLANKS

See the free-body diagram in part (a).

V_{max} occurs at the inside face of the support.

$$V_{max} = q\left(\frac{89 \text{ in.}}{2}\right) = 44.5q$$

$$= (44.5)\left(\frac{wb}{144}\right) = \frac{89\, wb}{288}$$

$$(V = \text{lb};\ w = \text{lb/ft}^2;\ b = \text{in.})$$

Allowable shear force:

$$\tau = \frac{3V}{2A} \quad V_{allow} = \frac{2A\tau_{allow}}{3}$$

$$= \frac{2(1.5\, b)(100 \text{ psi})}{3} = 100\, b \text{ (lb)}$$

Equate V_{max} and V_{allow} and solve for w:

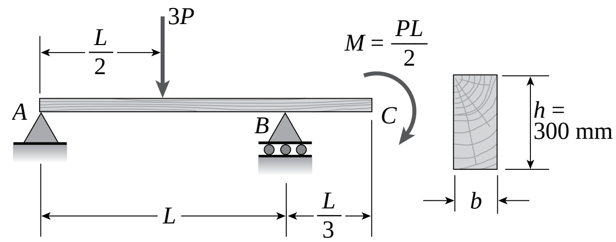
$$\frac{89wb}{288} = 100b \quad w_2 = 324 \text{ lb/ft}^2 \quad \leftarrow$$

(c) ALLOWABLE LOAD

Bending stress governs. $w_{allow} = 121$ lb/ft^2 $\leftarrow$

Problem 5.8-12 A wood beam *ABC* with simple supports at *A* and *B* and an overhang *BC* has height $h = 300$ mm (see figure). The length of the main span of the beam is $L = 3.6$ m and the length of the overhang is $L/3 = 1.2$ m. The beam supports a concentrated load $3P = 18$ kN at the midpoint of the main span and a moment $PL/2 = 10.8$ kN · m at the free end of the overhang. The wood has weight density $\gamma = 5.5$ kN/m^3.

(a) Determine the required width b of the beam based upon an allowable bending stress of 8.2 MPa.
(b) Determine the required width based upon an allowable shear stress of 0.7 MPa.



Solution 5.8-12

Numerical data:

$$L = 3.6 \text{ m} \qquad h = 300 \text{ mm}$$

$$A = bh \qquad P = 6 \text{ kN} \qquad M = \frac{PL}{2}$$

$$\gamma = 5.5\,\frac{\text{kN}}{\text{m}^3} \qquad q_{\text{beam}} = \gamma A$$

Reactions, max. shear and moment equations

$$R_A = \frac{3\,P}{2} - \frac{M}{L} + \frac{4}{9}\,q_{\text{beam}}L = P - \frac{4}{9}\,q_{\text{beam}}L$$

$$R_B = \frac{3\,P}{2} + \frac{M}{L} + \frac{8}{9}\,q_{\text{beam}}L = 2\,P + \frac{8}{9}\,q_{\text{beam}}L$$

$$V_{\max} = R_B = 2\,P + \frac{8}{9}\,q_{\text{beam}}L$$

$$M_D = R_A\frac{L}{2} - q_{\text{beam}}\frac{L^2}{2} = \frac{PL}{2} - \frac{17}{18}\,q_{\text{beam}}L^2$$

$$M_B = \frac{PL}{2}$$

(a) Required width b based upon bending stress

$$\sigma_{\text{allow}} = 8.2 \text{ MPa}$$

$$M_{\max} = M_B = \frac{PL}{2}$$

$$\sigma = \frac{M_{\max}}{S} = \frac{6\,M_{\max}}{bh^2}$$

$$b = \frac{3PL}{\sigma_{\text{allow}}h^2} \qquad b = 87.8 \text{ mm} \quad \leftarrow$$

(b) Required width b based upon shear stress

$$\tau_{\text{allow}} = 0.7 \text{ MPa}$$

$$V_{\max} = 2\,P + \frac{8}{9}\,q_{\text{beam}}L$$

$$\tau = \frac{3\,V_{\max}}{2\,A} = \frac{3\,V_{\max}}{2\,bh}$$

$$= \frac{3}{2\,bh}\left(2\,P + \frac{8}{9}\,q_{\text{beam}}\,L\right) = \frac{3\,P}{bh} + \frac{4}{3}\,\gamma L$$

$$b = \frac{3P}{h\left(\tau_{\text{allow}} - \dfrac{4}{3}\,\gamma L\right)} \qquad b = 89.074 \text{ mm}$$

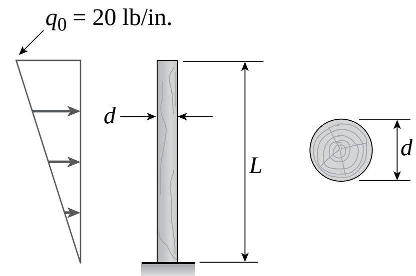
Shear stress governs

$$b = 89.1 \text{ mm} \qquad \leftarrow \text{(governs)}$$

Shear Stresses in Circular Beams

Problem 5.9-1 A wood pole of solid circular cross section (d = diameter) is subjected to a horizontal force $P = 450$ lb (see figure). The length of the pole is $L = 6$ ft, and the allowable stresses in the wood are 1900 psi in bending and 120 psi in shear.

Determine the minimum required diameter of the pole based upon (a) the allowable bending stress, and (b) the allowable shear stress.



Solution 5.9-1

$$q = 20\frac{\text{1b}}{\text{in}} \qquad L = 6\,\text{ft}$$

$$\sigma_{\text{allow}} = 1900\,\text{psi}$$

$$\tau_{\text{allow}} = 120\,\text{psi}$$

$$V_{\max} = \frac{qL}{2} \qquad V_{\max} = 720\ \text{1b}$$

$$M_{\max} = \frac{qL}{2}\,\frac{2\,L}{3} \qquad M_{\max} = 2.88 \times 10^3\ \text{1b-ft}$$

(a) Based upon bending stress

$$\sigma = \frac{M}{S} = \frac{32M}{\pi d^3}$$

$$d_{\min} = \sqrt[3]{\frac{32\,M_{\max}}{\pi\,\sigma_{\text{allow}}}}$$

$$d_{\min} = 5.701\ \text{in.}$$

(b) Based upon shear stress

$$\tau = \frac{4V}{3A} = \frac{16V}{3\pi d^2}$$

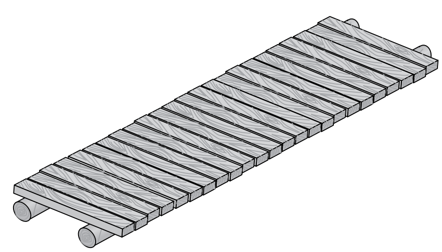
$$d_{\min} = \sqrt{\frac{16\ V_{\max}}{3\pi\ \tau_{\text{allow}}}} \qquad d_{\min} = 3.192\ \text{in.}$$

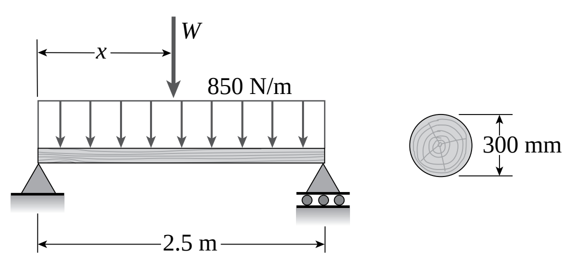
Bending stress governs $\quad d_{\min} = 5.70$ in. $\leftarrow$

Problem 5.9-2 A simple log bridge in a remote area consists of two parallel logs with planks across them (see figure). The logs are Douglas fir with average diameter 300 mm. A truck moves slowly across the bridge, which spans 2.5 m. Assume that the weight of the truck is equally distributed between the two logs.

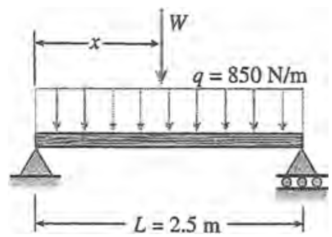
Because the wheelbase of the truck is greater than 2.5 m, only one set of wheels is on the bridge at a time. Thus, the wheel load on one log is equivalent to a concentrated load W acting at any position along the span. In addition, the weight of one log and the planks it supports is equivalent to a uniform load of 850 N/m acting on the log.

Determine the maximum permissible wheel load W based upon (a) an allowable bending stress of 7.0 MPa, and (b) an allowable shear stress of 0.75 MPa.





Solution 5.9-2 Log bridge



Diameter $d = 300$ mm
$\sigma_{\text{allow}} = 7.0$ MPa
$\tau_{\text{allow}} = 0.75$ MPa
Find allowable load W

(a) BASED UPON BENDING STRESS

Maximum moment occurs when wheel is at midspan ($x = L/2$).

$$M_{\max} = \frac{WL}{4} + \frac{qL^2}{8}$$

$$= \frac{W}{4}(2.5 \text{ m}) + \frac{1}{8}(850 \text{ N/m})(2.5 \text{ m})^2$$

$$= 0.625W + 664.1 \text{ (N}\cdot\text{m)} \quad (W = \text{newtons})$$

$$S = \frac{\pi d^3}{32} = 2.651 \times 10^{-3} \text{m}^3$$

$$M_{\max} = S\sigma_{\text{allow}} = (2.651 \times 10^{-3} \text{ m}^3)(7.0 \text{ MPa})$$

$$= 18{,}560 \text{ N}\cdot\text{m}$$

$$\therefore\ 0.625W + 664.1 = 18{,}560$$

$$W = 28{,}600 \text{ N} = 28.6 \text{ kN} \quad \leftarrow$$

(b) BASED UPON SHEAR STRESS

Maximum shear force occurs when wheel is adjacent to support ($x = 0$).

$$V_{\max} = W + \frac{qL}{2} = W + \frac{1}{2}(850 \text{ N/m})(2.5 \text{ m})$$

$$= W + 1062.5 \text{ N} \quad (W = \text{newtons})$$

$$A = \frac{\pi d^2}{4} = 0.070686 \text{ m}^2$$

$$\tau_{\max} = \frac{4V_{\max}}{3A}$$

$$V_{\max} = \frac{3A\tau_{\text{allow}}}{4} = \frac{3}{4}(0.070686 \text{ m}^2)(0.75 \text{ MPa})$$

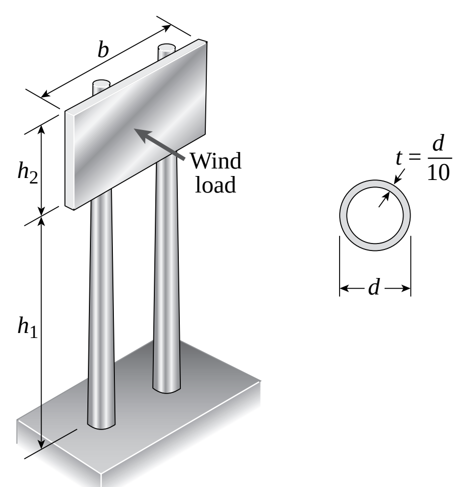
$$= 39{,}760 \text{ N}$$

$$\therefore\ W + 1062.5 \text{ N} = 39{,}760 \text{ N}$$

$$W = 38{,}700 \text{ N} = 38.7 \text{ kN} \quad \leftarrow$$

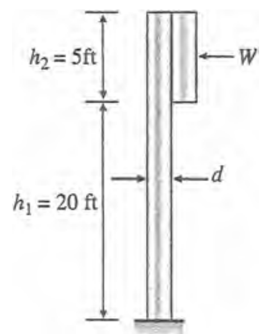
Problem 5.9-3 A sign for an automobile service station is supported by two aluminum poles of hollow circular cross section, as shown in the figure. The poles are being designed to resist a wind pressure of 75 lb/ft^2 against the full area of the sign. The dimensions of the poles and sign are $h_1 = 20$ ft, $h_2 = 5$ ft, and $b = 10$ ft. To prevent buckling of the walls of the poles, the thickness t is specified as one-tenth the outside diameter d.

(a) Determine the minimum required diameter of the poles based upon an allowable bending stress of 7500 psi in the aluminum.
(b) Determine the minimum required diameter based upon an allowable shear stress of 2000 psi.



Probs. 5.9.3 and 5.9.4

Solution 5.9-3 Wind load on a sign



$b =$ width of sign

$b = 10$ ft

$p = 75$ lb/ft^2

$\sigma_{allow} = 7500$ psi

$\tau_{allow} = 2000$ psi

$d =$ diameter $\quad W =$ wind force on one pole

$$t = \frac{d}{10} \qquad W = ph_2\left(\frac{b}{2}\right) = 1875 \text{ lb}$$

(a) Required diameter based upon bending stress

$$M_{max} = W\left(h_1 + \frac{h_2}{2}\right) = 506{,}250 \text{ lb-in.}$$

$$I = \frac{\pi}{64}(d_2^4 - d_2^4) \quad d_2 = d \quad d_1 = d - 2t = \frac{4}{5}d$$

$$I = \frac{\pi}{64}\left[d^4 - \left(\frac{4d}{5}\right)^4\right] = \frac{\pi d^4}{64}\left(\frac{369}{625}\right)$$

$$= \frac{369\pi d^4}{40{,}000} \text{ (in.}^4\text{)}$$

$$c = \frac{d}{2} \quad (d = \text{inches})$$

$$\sigma = \frac{Mc}{I} = \frac{M(d/2)}{369\pi d^4/40{,}000} = \frac{17.253\,M}{d^3}$$

$$d^3 = \frac{17.253\,M_{max}}{\sigma_{allow}} = \frac{(17.253)(506{,}250 \text{ lb-in.})}{7500 \text{ psi}}$$

$$= 1164.6 \text{ in.}^3 \quad d = 10.52 \text{ in.} \quad \leftarrow$$

(b) Required diameter based upon shear stress

$$V_{max} = W = 1875 \text{ lb}$$

$$\tau = \frac{4V}{3A}\left(\frac{r_2^2 + r_2 r_1 + r_1^2}{r_2^2 + r_1^2}\right) \quad r_2 = \frac{d}{2}$$

$$r_1 = \frac{d}{2} - t = \frac{d}{2} - \frac{d}{10} = \frac{2d}{5}$$

$$\frac{r_2^2 + r_2 r_1 + r_1^2}{r_2^2 + r_1^2}$$

$$= \frac{\left(\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)\left(\frac{2d}{5}\right) + \left(\frac{2d}{5}\right)^2}{\left(\frac{d}{2}\right)^2 + \left(\frac{2d}{5}\right)^2} = \frac{61}{41}$$

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = \frac{\pi}{4}\left[d^2 - \left(\frac{4d}{5}\right)^2\right] = \frac{9\pi d^2}{100}$$

$$\tau = \frac{4V}{3}\left(\frac{61}{41}\right)\left(\frac{100}{9\pi d^2}\right) = 7.0160\,\frac{V}{d^2}$$

$$d^2 = \frac{7.0160\,V_{max}}{\tau_{allow}}$$

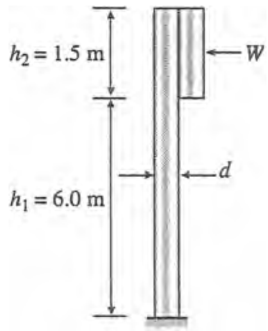
$$= \frac{(7.0160)(1875 \text{ lb})}{2000 \text{ psi}} = 6.5775 \text{ in.}^2$$

$$d = 2.56 \text{ in.} \quad \leftarrow$$

(Bending stress governs.)

Problem 5.9-4 Solve the preceding problem for a sign and poles having the following dimensions: $h_1 = 6.0$ m, $h_2 = 1.5$ m, $b = 3.0$ m, and $t = d/10$. The design wind pressure is 3.6 kPa, and the allowable stresses in the aluminum are 50 MPa in bending and 14 MPa in shear.

Solution 5.9-4 Wind load on a sign



$b =$ width of sign

$b = 3.0$ m

$p = 3.6$ kPa

$\sigma_{allow} = 50$ MPa

$\tau_{allow} = 16$ MPa

$d =$ diameter $\quad W =$ wind force on one pole

$$t = \frac{d}{10} \quad W = ph_2\left(\frac{b}{2}\right) = 8.1 \text{ kN}$$

(a) Required diameter based upon bending stress

$$M_{max} = W\left(h_1 + \frac{h_2}{2}\right) = 54.675 \text{ kN} \cdot \text{m}$$

$$\sigma = \frac{Mc}{I} \quad I = \frac{\pi}{64}(d_2^4 - d_1^4)$$

$$d_2 = d \quad d_1 = d - 2t = \frac{4}{5}d$$

$$I = \frac{\pi}{64}\left[d^4 - \left(\frac{4d}{5}\right)^4\right]$$

$$= \frac{\pi d^4}{64}\left(\frac{369}{625}\right) = \frac{369\pi d^4}{40{,}000} \text{ (m}^4)$$

$$c = \frac{d}{2} \quad (d = \text{meters})$$

$$\sigma = \frac{Mc}{I} = \frac{M(d/2)}{369\pi d^4/40{,}000} = \frac{17.253\,M}{d^3}$$

$$d^3 = \frac{17.253 M_{max}}{\sigma_{allow}} = \frac{(17.253)(54.675 \text{ kN} \cdot \text{m})}{50 \text{ MPa}}$$

$$= 0.018866 \text{ m}^3$$

$$d = 0.266 \text{ m} = 266 \text{ m} \quad \leftarrow$$

(b) Required diameter based upon shear stress

$$V_{max} = W = 8.1 \text{ kN}$$

$$\tau = \frac{4V}{3A}\left(\frac{r_2^2 + r_1 r_2 + r_1^2}{r_2^2 + r_1^2}\right) \quad r_2 = \frac{d}{2}$$

$$r_1 = \frac{d}{2} - t = \frac{d}{2} - \frac{d}{10} = \frac{2d}{5}$$

$$\frac{r_2^2 + r_1 r_2 + r_1^2}{r_2^2 + r_1^2} = \frac{\left(\frac{d}{2}\right)^2 + \left(\frac{d}{5}\right)\left(\frac{2d}{5}\right) + \left(\frac{2d}{5}\right)^2}{\left(\frac{d}{2}\right)^2 + \left(\frac{2d}{5}\right)^2} = \frac{61}{41}$$

$$A = \frac{\pi}{4}(d_2^2 - d_1^2)$$

$$= \frac{\pi}{4}\left[d^2 - \left(\frac{4d}{5}\right)^2\right] = \frac{9\pi d^2}{100}$$

$$\tau = \frac{4V}{3}\left(\frac{61}{41}\right)\left(\frac{100}{9\pi d^2}\right) = 7.0160\,\frac{V}{d^2}$$

$$d^2 = \frac{7.0160\,V_{max}}{\tau_{allow}} = \frac{(7.0160)(8.1 \text{ kN})}{14 \text{ MPa}}$$

$$= 0.004059 \text{ m}^2$$

$$d = 0.06371 \text{ m} = 63.7 \text{ mm} \quad \leftarrow$$

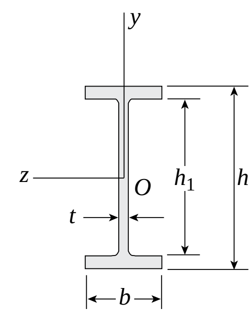
Bending stress governs

Shear Stresses in Beams with Flanges

Problem 5.10-1 through 5.10-6 A wide-flange beam (see figure) having the cross section described below is subjected to a shear force V. Using the dimensions of the cross section, calculate the moment of inertia and then determine the following quantites:

(a) The maximum shear stress τ_{max} in the web.
(b) The minimum shear stress τ_{min} in the web.
(c) The average shear stress τ_{aver} (obtained by dividing the shear force by the area of the web) and the ratio τ_{max}/τ_{aver}.
(d) The shear force V_{web} carried in the web and the ratio V_{web}/V.

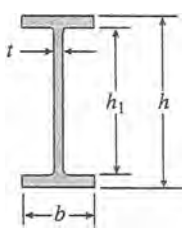
NOTE: Disregard the fillets at the junctions of the web and flanges and determine all quantities, including the moment of inertia, by considering the cross section to consist of three rectangles.



Probs 5.10.1through 5.-10.6

Problem 5.10-1 Dimensions of cross section: $b = 6$ in., $t = 0.5$ in., $h = 12$ in., $h_1 = 10.5$ in., and $V = 30$ k.

Solution 5.10-1 Wide-flange beam



$b = 6.0$ in.

$t = 0.5$ in.

$h = 12.0$ in.

$h_1 = 10.5$ in.

$V = 30$ k

Moment of inertia (Eq.5-47)

$$I = \frac{1}{12}(bh^3 - bh_1^3 + th_1^3) = 333.4 \text{ in.}^4$$

(a) Maximum shear stress in the web (Eq. 5-48a)

$$\tau_{max} = \frac{V}{8It}(bh^2 - bh_1^2 + th_1^2) = 5795 \text{ psi} \quad \leftarrow$$

(b) Minimum shear stress in the web (Eq. 5-48b)

$$\tau_{min} = \frac{Vb}{8It}(h^2 - h_1^2) = 4555 \text{ psi} \quad \leftarrow$$

(c) Average shear strear in the web (Eq. 5-50)

$$\tau_{aver} = \frac{V}{th_1} = 5714 \text{ psi} \quad \leftarrow$$

$$\frac{\tau_{max}}{\tau_{aver}} = 1.014 \quad \leftarrow$$

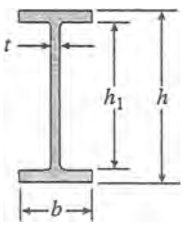
(d) Shear force in the web (Eq. 5-49)

$$V_{web} = \frac{th_1}{3}(2\tau_{max} + \tau_{min}) = 28.25 \text{ k} \quad \leftarrow$$

$$\frac{V_{web}}{V} = 0.942 \quad \leftarrow$$

Problem 5.10-2 Dimensions of cross section: $b = 180$ mm, $t = 12$ mm, $h = 420$ mm, $h_1 = 380$ mm, and $V = 125$ kN.

Solution 5.10-2 Wide-flange beam



$b = 180$ mm

$t = 12$ mm

$h = 420$ mm

$h_1 = 380$ mm

$V = 125$ kN

MOMENT OF INERTIA (Eq. 5-47)

$$I = \frac{1}{12}(bh^3 - bh_1^3 + th_1^3) = 343.1 \times 10^6 \text{ mm}^4$$

(a) MAXIMUM SHEAR STRESS IN THE WEB (Eq. 5-48a)

$$\tau_{max} = \frac{V}{8It}(bh^2 - bh_1^2 + th_1^2) = 28.43 \text{ MPa} \quad \leftarrow$$

(b) MINIMUM SHEAR STRESS IN THE WEB (Eq. 5-48b)

$$\tau_{min} = \frac{Vb}{8It}(h^2 - h_1^2) = 21.86 \text{ MPa} \quad \leftarrow$$

(c) AVERAGE SHEAR STRESS IN THE WEB (Eq. 5-50)

$$\tau_{aver} = \frac{V}{th_1} = 27.41 \text{ MPa} \quad \leftarrow$$

$$\frac{\tau_{max}}{\tau_{aver}} = 1.037 \quad \leftarrow$$

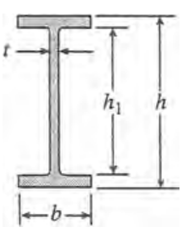
(d) SHEAR FORCE IN THE WEB (Eq. 5-49)

$$V_{web} = \frac{th_1}{3}(2\tau_{max} + \tau_{min}) = 119.7 \text{ kN} \quad \leftarrow$$

$$\frac{V_{web}}{V} = 0.957 \quad \leftarrow$$

Problem 5.10-3 Wide-flange shape, W 8 × 28 (see Table E-1(a), Appendix E); $V = 10$ k.

Solution 5.10-3 Wide-flange beam



$W\,8 \times 28$

$b = 6.535$ in.

$t = 0.285$ in.

$h = 8.06$ in.

$h_1 = 7.13$ in.

$V = 10$ k

MOMENT OF INERTIA (Eq. 5-47)

$$I = \frac{1}{12}(bh^3 - bh_1^3 + th_1^3) = 96.36 \text{ in.}^4$$

(a) MAXIMUM SHEAR STRESS IN THE WEB (Eq. 5-48a)

$$\tau_{max} = \frac{V}{8It}(bh^2 - bh_1^2 + th_1^2) = 4861 \text{ psi} \quad \leftarrow$$

(b) MINIMUM SHEAR STRESS IN THE WEB (Eq. 5-48b)

$$\tau_{min} = \frac{Vb}{8It}(h^2 - h_1^2) = 4202 \text{ psi} \quad \leftarrow$$

(c) AVERAGE SHEAR STRESS IN THE WEB (Eq. 5-50)

$$\tau_{aver} = \frac{V}{th_1} = 4921 \text{ psi} \quad \leftarrow$$

$$\frac{\tau_{max}}{\tau_{aver}} = 0.988 \quad \leftarrow$$

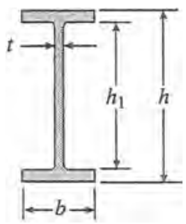
(d) SHEAR FORCE IN THE WEB (EQ. 5-49)

$$V_{web} = \frac{th_1}{3}(2\tau_{max} + \tau_{min}) = 9.432 \text{ k} \quad \leftarrow$$

$$\frac{V_{web}}{V} = 0.943 \quad \leftarrow$$

Problem 5.10-4 Dimensions of cross section: $b = 220$ mm, $t = 12$ mm, $h = 600$ mm, $h_1 = 570$ mm, and $V = 200$ kN.

Solution 5.10-4 Wide-flange beam



$b = 220$ mm

$t = 12$ mm

$h = 600$ mm

$h_1 = 570$ mm

$V = 200$ kN

Moment of inertia (Eq. 5-47)

$$I = \frac{1}{12}(bh^3 - bh_1^3 + th_1^3) = 750.0 \times 10^6 \text{ mm}^4$$

(a) Maximum shear stress in the web (Eq. 5-48a)

$$\tau_{max} = \frac{V}{8It}(bh^2 - bh_1^2 + th_1^2) = 32.28 \text{ MPa} \quad \leftarrow$$

(b) Minimum shear stress in the web (Eq. 5-48b)

$$\tau_{min} = \frac{Vb}{8It}(h^2 - h_1^2) = 21.45 \text{ MPa} \quad \leftarrow$$

(c) Average shear stress in the web (Eq. 5-50)

$$\tau_{aver} = \frac{V}{th_1} = 29.24 \text{ MPa} \quad \leftarrow$$

$$\frac{\tau_{max}}{\tau_{aver}} = 1.104$$

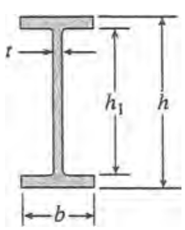
(d) Shear force in the web (Eq. 5-49)

$$V_{web} = \frac{th_1}{3}(2\tau_{max} + \tau_{min}) = 196.1 \text{ kN} \quad \leftarrow$$

$$\frac{V_{web}}{V} = 0.981 \quad \leftarrow$$

Problem 5.10-5 Wide-flange shape, W 18×71 (see Table E-1(a), Appendix E); $V = 21$ k.

Solution 5.10-5 Wide-flange beam



$W\ 18 \times 71$

$b = 7.635$ in.

$t = 0.495$ in.

$h = 18.47$ in.

$h_1 = 16.85$ in.

$V = 21$ k

Moment of inertia (Eq. 5-47)

$$I = \frac{1}{12}(bh^3 - bh_1^3 + th_1^3) = 1162 \text{ in.}^4$$

(a) Maximum shear stress in the web (Eq. 5-48a)

$$\tau_{max} = \frac{V}{8It}(bh^2 - bh_1^2 + th_1^2) = 2634 \text{ psi} \quad \leftarrow$$

(b) Minimum shear stress in the web (Eq. 5-48b)

$$\tau_{min} = \frac{Vb}{8It}(h^2 - h_1^2) = 1993 \text{ psi} \quad \leftarrow$$

(c) Average shear stress in the web (Eq. 5-50)

$$\tau_{aver} = \frac{V}{th_1} = 2518 \text{ psi} \quad \leftarrow$$

$$\frac{\tau_{max}}{\tau_{aver}} = 1.046 \quad \leftarrow$$

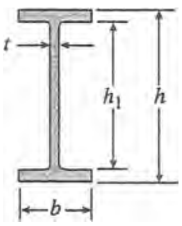
(d) Shear force in the web (Eq. 5-49)

$$V_{web} = \frac{th_1}{3}(2\tau_{max} + \tau_{min}) = 20.19 \text{ k} \quad \leftarrow$$

$$\frac{V_{web}}{V} = 0.961 \quad \leftarrow$$

Problem 5.10-6 Dimensions of cross section: $b = 120$ mm, $t = 7$ mm, $h = 350$ mm, $h_1 = 330$ mm, and $V = 60$ kN

Solution 5.10-6 Wide-flange beam



$b = 120$ mm

$t = 7$ mm

$h = 350$ mm

$h_1 = 330$ mm

$V = 60$ kN

MOMENT OF INERTIA (Eq. 5-47)

$$I = \frac{1}{12}(bh^3 - bh_1^3 + th_1^3) = 90.34 \times 10^6 \text{ mm}^4$$

(a) MAXIMUM SHEAR STRESS IN THE WEB (Eq. 5-48a)

$$\tau_{max} = \frac{V}{8It}(bh^2 - bh_1^2 + th_1^2) = 28.40 \text{ MPa} \quad \leftarrow$$

(b) MINIMUM SHEAR STRESS IN THE WEB (Eq. 5-48)

$$\tau_{min} = \frac{Vb}{8It}(h^2 - h_1^2) = 19.35 \text{ MPa} \quad \leftarrow$$

(c) AVERAGE SHEAR STRESS IN THE WEB (Eq. 5-50)

$$\tau_{aver} = \frac{V}{th_1} = 25.97 \text{ MPa} \quad \leftarrow$$

$$\frac{\tau_{max}}{\tau_{aver}} = 1.093 \quad \leftarrow$$

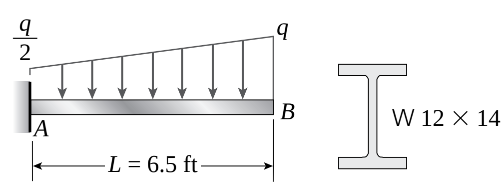
(d) SHEAR FORCE IN THE WEB (Eq. 5-49)

$$V_{web} = \frac{th_1}{3}(2\tau_{max} + \tau_{min}) = 58.63 \text{ kN} \quad \leftarrow$$

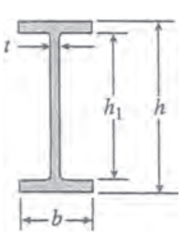
$$\frac{V_{web}}{V} = 0.977 \quad \leftarrow$$

Problem 5.10-7 A cantilever beam AB of length $L = 6.5$ ft supports a trapezoidal distributed load of peak intensity q, and minimum intensity $q/2$, that includes the weight of the beam (see figure). The beam is a steel W 12 × 14 wide-flange shape (see Table E-1(a), Appendix E).

Calculate the maximum permissible load q based upon (a) an allowable bending stress $\sigma_{allow} = 18$ ksi and (b) an allowable shear stress $\tau_{allow} = 7.5$ ksi. (*Note:* Obtain the moment of inertia and section modulus of the beam from Table E-1(a))



Solution 5.10-7



$b = 3.97$ in. $\quad I = 88.6 \cdot \text{in.}^4$

$t = 0.2$ in.

$t_f = 0.225$ in.

$S = 14.9 \text{ in.}^3$

$h = 11.9$ in.

$h_1 = h - 2t_f$

$h_1 = 11.45$ in.

$L = 6.5$ ft $\quad \sigma_{allow} = 18$ ksi $\quad \tau_{allow} = 7.5$ ksi

$$V_{max} = \frac{\left(\frac{q}{2} + q\right)L}{2} \qquad V_{max} = \frac{3}{4}qL$$

$$M_{max} = \frac{1}{2}\frac{q}{2}L^2 + \frac{1}{2}\frac{q}{2}L\frac{2L}{3}$$

$$M_{max} = \frac{5}{12}qL^2$$

(a) Maximum load based upon bending stress

$$\sigma = \frac{M}{S} = \frac{\frac{5}{12}qL^2}{S} \qquad q = \frac{12S\,\sigma_{allow}}{5L^2}$$

$q = 1270$ lb/ft

(b) Maximum load upon shear stress

$$\tau_{max} = \frac{V_{max}}{8It}(bh^2 - bh_1^2 + th_1^2)$$

$$= \frac{3\,qL}{32It}(bh^2 - bh_1^2 + th_1^2)$$

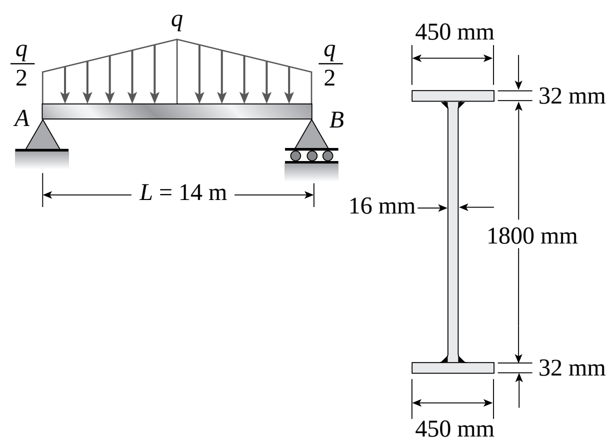
$$q = \frac{\tau_{allow}32It}{3\,L(bh^2 - bh_1^2 + th_1^2)}$$

$$q = 3210\ \frac{\text{lb}}{\text{ft}}$$

Shear stress governs $\quad q = 1270$ lb/ft $\quad\leftarrow$

Problem 5.10-8 A bridge girder *AB* on a simple span of length $L = 14$ m supports a distributed load of maximum intensity q at midspan and minimum intensity $q/2$ at supports *A* and *B* that includes the weight of the girder (see figure). The girder is constructed of three plates welded to form the cross section shown.

Determine the maximum permissible load q based upon (a) an allowable bending stress $\sigma_{allow} = 110$ MPa and (b) an allowable shear stress $\tau_{allow} = 50$ MPa.



Solution 5.10-8

$L = 14$ m

$h = 1864$ mm $\quad h_1 = 1800$ mm

$b = 450$ mm $\quad t_f = 32$ mm $\quad t_w = 16$ mm

$$I = \frac{1}{12}(bh^3 - bh_1^3 + t_w h_1^3)$$

$$I = 3.194 \times 10^{10}\ \text{mm}^4$$

$$S = \frac{2\,I}{h} \qquad S = 3.427 \times 10^7\ \text{mm}^3$$

$$R_A = R_B = \frac{q}{2}\frac{L}{2} + \frac{q}{4}\frac{L}{2} = \frac{3}{8}qL$$

(a) Maximum load based upon bending stress

$\sigma_{allow} = 110$ MPa

$$M_{max} = \frac{3}{8}qL\frac{L}{2} - \frac{q}{2}\frac{L}{2}\frac{L}{4} - \frac{q}{2}\frac{L}{4}\frac{L}{6}$$

$$= \frac{5}{48}qL^2$$

$$\sigma = \frac{M_{max}}{S} = \frac{\frac{5}{48}qL^2}{S}$$

$$q_{max} = \frac{\sigma_{allow}S}{\frac{5}{48}L^2} \quad \leftarrow$$

$$q_{max} = 184.7\ \frac{\text{kN}}{\text{m}} \quad \leftarrow$$

(b) Maximum load based upon shear stress

$$\tau_{allow} = 50 \text{ MPa}$$

$$V_{max} = R_A = \frac{3}{8} qL$$

$$\tau_{max} = \frac{V_{max}}{8It}\left(bh^2 - bh_1^2 + th_1^2\right)$$

$$= \frac{3\,qL}{64It}\left(bh^2 - bh_1^2 + th_1^2\right)$$

$$q_{max} = \frac{64\,\tau_{allow} It_w}{3\,L\,(bh^2 - bh_1{}^2 + t_w h_1{}^2)}$$

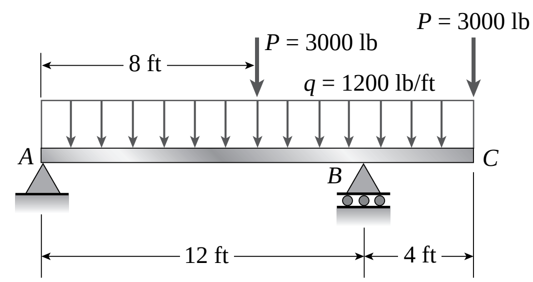
$$q_{max} = 247 \text{ kN/m} \quad \leftarrow$$

$\therefore$ Bending stress governs: $q_{max} = 184.7$ kN/m $\leftarrow$

Problem 5.10-9 A simple beam with an overhang supports a uniform load of intensity $q = 1200$ lb/ft and a concentrated load $P = 3000$ lb (see figure). The uniform load includes an allowance for the wight of the beam. The allowable stresses in bending and shear are 18 ksi and 11 ksi, respectively.

Select from Table E-2 (a), Appendix E, the lightest I-beam (S shape) that will support the given loads.

(*Hint:* Select a beam based upon the bending stress and then calculate the maximum shear stress. If the beam is overstressed in shear, select a heavier beam and repeat.)



Solution 5.10-9 Beam with an overhand

$\sigma_{allow} = 18$ ksi $\qquad \tau_{allow} = 11$ ksi $\qquad L = 12$ ft

$$q = 1200 \frac{\text{lb}}{\text{ft}} \qquad P = 3000 \text{ lb}$$

Sum moments about A & Solve for R_B

$$R_B = \frac{q\left(\frac{4}{3}L\right)^2 \frac{1}{2} + P(8 \text{ ft} + 16 \text{ ft})}{12 \text{ ft}}$$

$$R_B = 1.88 \times 10^4 \text{ lb}$$

Sum forces in vertical direction

$$R_A = q\,(16 \text{ ft}) + 2P - R_B$$

$$R_A = 6.4 \times 10^3 \text{ lb}$$

$$V_{max} = R_B - (P + q4 \text{ ft})$$

$$V_{max} = 1.1 \times 10^4 \text{ lb} \quad \text{at } B$$

$$M_B = -P(4 \text{ ft}) - q\frac{(4 \text{ ft})^2}{2}$$

$$M_B = -2.16 \times 10^4 \text{ lb-ft}$$

Find moment at D (at Load P between A and B)

$$M_D = R_A 8 \text{ ft} - q\frac{(8 \text{ ft})^2}{2}$$

$$M_D = 1.28 \times 10^4 \text{ lb-ft}$$

$$M_{max} = |M_B| \qquad M_{max} = 2.16 \times 10^4 \text{ lb-ft}$$

Required section modulus:

$$S = \frac{M_{max}}{\sigma_{allow}} \qquad S = 14.4 \text{ in.}^3$$

Lightest beam is $S\,8 \times 23$ (from Table E-2(a))

$I = 64.7$ in.^{4} $\qquad S = 16.2$ in.^{3}

$b = 4.17$ in. $\qquad t = 0.441$ in.

$t_f = 0.425$ in. $\qquad h = 8$ in.

$h_1 = h - 2t_f \qquad h_1 = 7.15$ in.

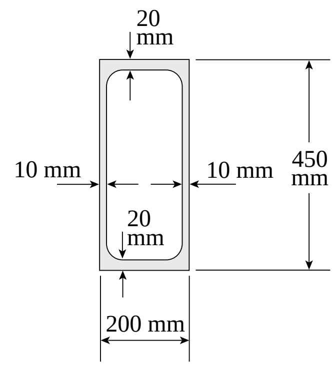
Check max. shear stress

$$\tau_{max} = \frac{V_{max}}{8\,It}\left(bh^2 - bh_1^2 + th_1^2\right)$$

$\tau_{max} = 3674 < 11{,}000$ psi so ok for shear

Select S 8×23 beam $\leftarrow$

Problem 5.10-10 A hollow steel box beam has the rectangular cross section shown in the figure. Determine the maximum allowable shear force V that may act on the beam if the allowable shear stress in 36 Mpa.



Solution 5.10-10 Rectangular box beam

$\tau_{allow} = 36$ MPa

Find V_{allow}

$$\tau = \frac{VQ}{It}$$

$$V_{allow} = \frac{\tau_{allow} It}{Q}$$

$$I = \frac{1}{12}(200)(450)^3 - \frac{1}{12}(180)(410)^3$$

$$= 484.9 \times 10^6 \text{mm}^4$$

$$t = 2(10 \text{ mm}) = 20 \text{ mm}$$

$$Q = (200)\left(\frac{450}{2}\right)\left(\frac{450}{4}\right) - (180)\left(\frac{410}{2}\right)\left(\frac{410}{4}\right)$$

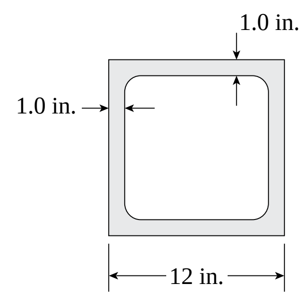
$$= 1.280 \times 10^6 \text{ mm}^3$$

$$V_{allow} = \frac{\tau_{allow} It}{Q}$$

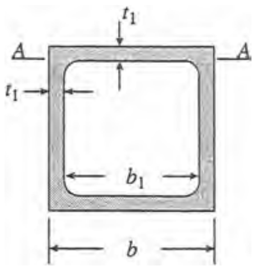
$$= \frac{(36 \text{ MPa})(484.9 \times 10^6 \text{ mm}^4)(20 \text{ mm})}{1.280 \times 10^6 \text{ mm}^3}$$

$$= 273 \text{ kN} \quad \leftarrow$$

Problem 5.10-11 A hollow aluminum box beam has the square cross section shown in the figure. Calculate the maximum and minimum shear stresses τ_{max} and τ_{min} in the webs of the beam due to a shear force $V = 28$ k.



Solution 5.10-11 Square box beam



$V = 28 \text{ k} = 28{,}000 \text{ lb}$

$t_1 = 1.0 \text{ in.}$

$b = 12 \text{ in.}$

$b_1 = 10 \text{ in.}$

$$\tau = \frac{VQ}{It} \qquad t = 2t_1 = 2.0 \text{ in.}$$

Moment of inertia

$$I = \frac{1}{12}(b^4 - b_1^4) = 894.67 \text{ in.}^4$$

Maximum shear stress in the web (at neutral axis)

$$Q = A_1\bar{y}_1 - A_2\bar{y}_2 \quad A_1 = b\left(\frac{b}{2}\right) = \frac{b^2}{2}$$

$$A_2 = b_1\left(\frac{b_1}{2}\right) = \frac{b_1^2}{2}$$

$$\bar{y}_1 = \frac{1}{2}\left(\frac{b}{2}\right) = \frac{b}{4} \quad \bar{y}_2 = \frac{1}{2}\left(\frac{b_1}{2}\right) = \frac{b_1}{4}$$

$$Q = \left(\frac{b^2}{2}\right)\left(\frac{b}{4}\right) - \left(\frac{b_1^2}{2}\right)\left(\frac{b_1}{4}\right)$$

$$= \frac{1}{8}(b^3 - b_1^3) = 91.0 \text{ in.}^3$$

$$\tau_{\max} = \frac{VQ}{It} = \frac{(28{,}000 \text{ lb})(91.0 \text{ in.}^3)}{(894.67 \text{ in.}^4)(2.0 \text{ in.})} = 1424 \text{ psi}$$

$$= 1.42 \text{ ksi} \quad \leftarrow$$

Minimum shear stress in the web (at level $A.A$)

$$Q = A\bar{y} = (bt_1)\left(\frac{b}{2} - \frac{t_1}{2}\right) = \left(\frac{bt_1}{2}\right)(b - t_1)$$

$$t_1 = \frac{b - b_1}{2} \qquad Q = \frac{b}{8}(b^2 - b_1^2)$$

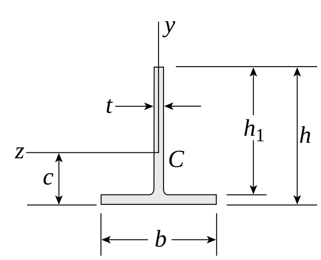
$$Q = \frac{(12 \text{ in.})}{8}[(12 \text{ in.})^2 - (10 \text{ in.})^2] = 66.0 \text{ in.}^3$$

$$\tau_{\min} = \frac{VQ}{It} = \frac{(28{,}000 \text{ lb})(66.0 \text{ in.}^3)}{(894.67 \text{ in}^4)(2.0 \text{ in.})} = 1033 \text{ psi}$$

$$= 1.03 \text{ ksi} \quad \leftarrow$$

Problem 5.10-12 The T-beam shown in the figure has cross-sectional dimensions as follows: $b = 220$ mm, $t = 15$ mm, $h = 300$ mm, and $h_1 = 275$ mm. The beam is subjected to a shear force $V = 60$ kN.

Determine the maximum shear stress $\tau_{\max}$ in the web of the beam.



Probs 5.10.12 and 5.-10.13

Solution 5.10-12

$h = 300 \text{ mm} \qquad h_1 = 280 \text{ mm}$

$b = 210 \text{ mm} \qquad t = 16 \text{ mm}$

$t_f = h - h_1 \qquad V = 68 \text{ kN}$

$t_f = 20 \text{ mm}$

Location of neutral axis

$$c = \frac{b(h - h_1)\left(\frac{h - h_1}{2}\right) + th_1\left(h - \frac{h_1}{2}\right)}{b(h - h_1) + th_1}$$

$$c = 87.419 \text{ mm}$$

$$c_1 = c \qquad c_1 = 87.419 \text{ mm}$$

$$c_2 = h - c \quad c_2 = 212.581 \text{ mm}$$

MOMENT OF INERTIA ABOUT THE Z-AXIS

$$I_{web} = \frac{1}{3}tc_2^3 + \frac{1}{3}t(c_1 - t_f)^3$$

$$I_{web} = 5.287 \times 10^7 \text{ mm}^4$$

$$I_{flange} = \frac{1}{12}bt_f^3 + bt_f\left(c_1 - \frac{t_f}{2}\right)^2$$

$$I_{flange} = 2.531 \times 10^7 \text{ mm}^4$$

$$I = I_{web} + I_{flange} \qquad I = 7.818 \times 10^7 \text{ mm}^4$$

FIRST MOMENT OF AREA ABOVE THE Z AXIS

$$Q = tc_2\frac{c_2}{2}$$

$$\tau_{max} = \frac{VQ}{It} \qquad \tau_{max} = 19.7 \text{ MPa} \quad \leftarrow$$

Problem 5.10-13 Calculate the maximum shear stress τ_{max} in the web of the T-beam shown in the figure if $b = 10$ in., $t = 0.5$ in., $h = 7$ in., $h_1 = 6.2$ in., and the shear force $V = 5300$ lb.

Solution 5.10-13 T-beam

$h = 7$ in. $\qquad h_1 = 6.2$ in.

$b = 10$ in. $\qquad t = 0.5$ in.

$t_f = h - h_1 \qquad t_f = 0.8$ in.

$V = 5300$ lb

LOCATION OF NEUTRAL AXIS

$$c = \frac{b(h - h_1)\left(\frac{h - h_1}{2}\right) + th_1\left(h - \frac{h_1}{2}\right)}{b(h - h_1) + th_1}$$

$c = 1.377$ in.

$c_1 = c \qquad c_1 = 1.377$ in.

$c_2 = h - c \qquad c_2 = 5.623$ in.

MOMENT OF INERTIA ABOUT THE Z-AXIS

$$I_{web} = \frac{1}{3}tc_2^3 + \frac{1}{3}t(c_1 - t_f)^3$$

$$I_{web} = 29.656 \text{ in.}^4$$

$$I_{flange} = \frac{1}{12}bt_f^3 + bt_f\left(c_1 - \frac{t_f}{2}\right)^2$$

$$I_{flange} = 8.07 \text{ in.}^4$$

$$I = I_{web} + I_{flange} \qquad I = 37.726 \text{ in.}^4$$

FIRST MOMENT OF AREA ABOVE THE Z AXIS

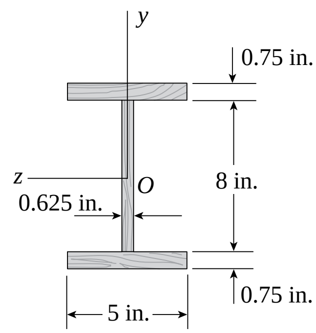
$$Q = tc_2\frac{c_2}{2}$$

$$\tau_{max} = \frac{VQ}{It} \qquad \tau_{max} = 2221 \text{ psi} \quad \leftarrow$$

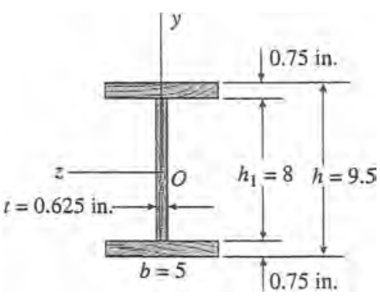
Built-Up Beams

Problem 5.11-1 A prefabricated wood I-beam serving as a floor joist has the cross section shown in the figure. The allowable load in shear for the glued joints between the web and the flanges is 65 lb/in. in the longitudinal direction.

Determine the maximum allowable shear force V_{max} for the beam.



Solution 5.11-1 Wood I-beam



All dimensions in inches.

Find V_{max} based upon shear in the glued joints.

Allowable load in shear for the glued joints is 65 lb/in.

$$\therefore f_{allow} = 65 \text{ lb/in.}$$

$$f = \frac{VQ}{I} \qquad V_{max} = \frac{f_{allow}\,I}{Q} \qquad \leftarrow$$

$$I = \frac{bh^3}{12} - \frac{(b - t)h_1^3}{12}$$

$$= \frac{1}{12}(5)(9.5)^3 - \frac{1}{12}(4.375)(8)^3 = 170.57 \text{ in.}^4$$

$$Q = Q_{flange} = A_f\,d_f$$

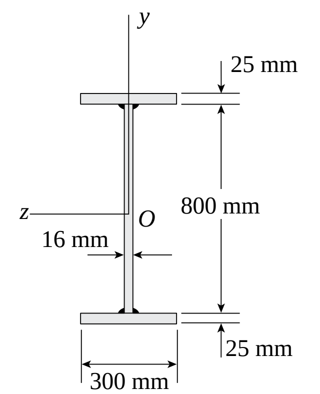
$$= (5)(0.75)(4.375) = 16.406 \text{ in.}^3$$

$$V_{max} = \frac{f_{allow} I}{Q}$$

$$= \frac{(65 \text{ lb/in.})(170.57 \text{ in.}^4)}{16.406 \text{ in.}^3} = 676 \text{ lb} \qquad \leftarrow$$

Problem 5.11-2 A welded steel girder having the cross section shown in the figure is fabricated of two 300 mm × 25 mm flange plates and a 800 mm × 16 mm web plate. The plates are joined by four fillet welds that run continuously for the length of the girder. Each weld has an allowable load in shear of 920 kN/m.

Calculate the maximum allowable shear force V_{max} for the girder.



Solution 5.11-2

$h = 850 \text{ mm} \qquad h_1 = 800 \text{ mm}$

$b = 300 \text{ mm} \qquad t = 16 \text{ mm}$

$t_f = 25 \text{ mm}$

$$I = \frac{bh^3}{12} - \frac{(b - t)h_1^{\,3}}{12}$$

$I = 3.236 \times 10^9 \text{ mm}^4$

$$Q_{\text{flange}} = A_f d_f \quad Q_{\text{flange}} = b\,t_f\left(\frac{h - t_f}{2}\right)$$

$Q_{\text{flange}} = 3.094 \times 10^6 \text{ mm}^3$

$$f_{\text{allow}} = 920\,\frac{\text{kN}}{\text{m}} \qquad f = 2\,f_{\text{allow}}$$

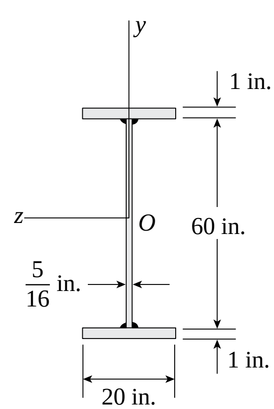
(2 welds, one either side of web)

$$f = \frac{VQ}{I} \qquad V_{\max} = \frac{fI}{Q_{\text{flange}}}$$

$V_{\max} = 1.924 \text{ MN} \quad \leftarrow$

Problem 5.11-3 A welded steel girder having the cross section shown in the figure is fabricated of two 20 in. × 1 in. flange plates and a 60 in. × 5/16 in. web plate. The plates are joined by four longitudinal fillet welds that run continuously throughout the length of the girder.

If the girder is subjected to a shear force of 280 kips, what force F (per inch of length of weld) must be resisted by each weld?



Solution 5.11-3

$h = 62$ in. $\quad h_1 = 60$ in.

$b = 20$ in. $\quad t = \dfrac{5}{16}$ in.

$t_f = 1$ in.

$$I = \frac{bh^3}{12} - \frac{(b - t)h_1^3}{12}$$

$I = 4.284 \times 10^4$ in.^{4}

$Q_{\text{flange}} = A_f\, d_f$

$$Q_{\text{flange}} = bt_f\left(\frac{h - t_f}{2}\right)$$

$Q_{\text{flange}} = 610$ in^3

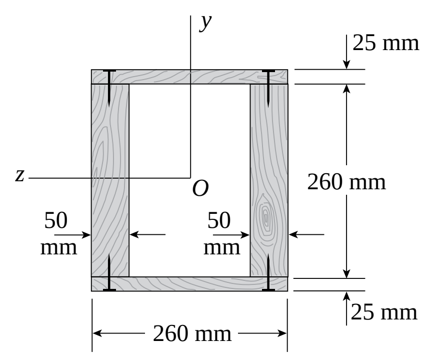
$V = 280\ k \qquad f = 2F = \dfrac{VQ}{I}$

$F = \dfrac{VQ_{\text{flange}}}{2I} \qquad F = 1994 \times 10^3$ lb.in.

$F = 1994$ lb/in. $\leftarrow$

Problem 5.11-4 A box beam of wood is constructed of two 260 mm $\times$ 50 mm boards and two 260 mm $\times$ 25 mm boards (see figure). The boards are nailed at a longitudinal spacing $s = 100$ mm.

If each nail has a allowable shear force $F = 1200$ N, what is the maximum allowable shear force V_{max}?



Solution 5.11-4 Wood box beam

All dimensions in millimeters.

$b = 260 \quad b_1 = 260 - 2(50) = 160$

$h = 310 \quad h_1 = 260$

$s =$ nail spacing $= 100$ mm

$F =$ allowable shear force for one nail $= 1200$ N

$f =$ shear flow between one flange and both webs

$$f_{\text{allow}} = \frac{2F}{s} = \frac{2(1200\text{ N})}{100\text{ mm}} = 24\text{ kN/ m}$$

$$f = \frac{VQ}{I} \qquad V_{\text{max}} = \frac{f_{\text{allow}}\,\mathrm{I}}{Q}$$

$$I = \frac{1}{12}(bh^3 - b_1h_1^3) = 411.125 \times 10^6\text{ mm}^4$$

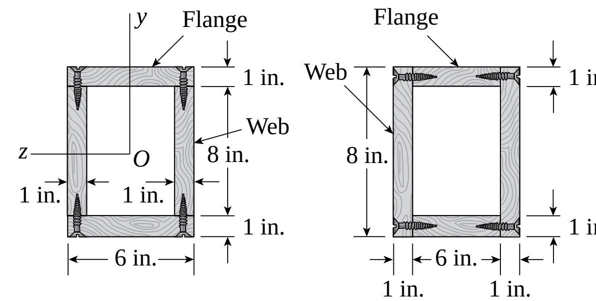
$$Q = Q_{\text{flange}} = A_f d_f = (260)(25)(142.5)$$
$$= 926.25 \times 10^3\text{ mm}^4$$

$$V_{\text{max}} = \frac{f_{\text{allow}}I}{Q} = \frac{(24\text{ kN/ m})(411.25 \times 10^6\text{ mm}^4)}{926.25 \times 10^3\text{ mm}^3}.$$
$$= 10.7\text{ kN} \quad \leftarrow$$

Problem 5.11-5 A box beam is constructed of four wood boards as shown in the figure part (a). The webs are 8 in. × 1 in. and the flanges are 6 in. × 1 in. boards (actual dimensions), joined by screws for which the allowable load in shear is $F = 250$ lb per screw.

(a) Calculate the maximum permissible longitudinal spacing s_{max} of the screws if the shear force V is 1200 lb.

(b) Repeat (a) if the flanges are attached to the webs using a *horizontal* arrangement of screws as shown in the figure part (b).



(a) (b)

Solution 5.11-5 Wood box beam

$V = 1200$ lb $\quad F = 250$ lb

(a) Vertical screws

$h = 10$ in. $\quad h_1 = 8$ in.

$b = 6$ in. $\quad t = 1$ in.

$$I = \frac{bh^3}{12} - \frac{(b - 2t)h_1^3}{12} \qquad I = 329.333\text{ in.}^4$$

$$Q_a = bt(4.5\text{ in.}) \qquad Q_a = 27\text{ in.}^3$$

$$f = \frac{VQ}{I} = \frac{2F}{S}$$

$$s_{max} = \frac{2FI}{VQ_a}$$

$$s_{max} = 5.08\text{ in.} \quad \leftarrow$$

(b) Horizontal screws

$h = 8$ in. $\quad h_1 = 6$ in.

$b = 8$ in. $\quad t = 1$ in.

$$I = \frac{bh^3}{12} - \frac{(b - 2t)h_1^3}{12} \qquad I = 233.333\text{ in.}^4$$

$$Q_b = (b - 2\,t)t(3.5\text{ in.}) \qquad Q_b = 21\text{ in.}^3$$

$$f = \frac{VQ}{I} = \frac{2F}{s}$$

$$s_{max} = \frac{2FI}{VQ_b}$$

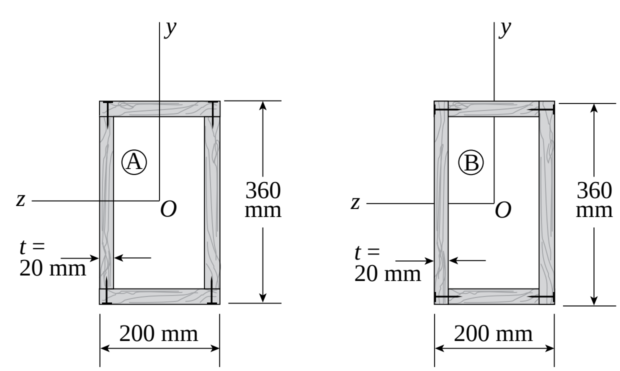
$$s_{max} = 4.63\text{ in.} \quad \leftarrow$$

Problem 5.11-6 Two wood box beams (beams A and B) have the same outside dimensions (200 mm × 360 mm) and the same thickness ($t = 20$ mm) throughout, as shown in the figure on the next page. Both beams are formed by nailing, with each nail having an allowable shear load of 250 N. The beams are designed for a shear force $V = 3.2$ kN.

(a) What is the maximum longitudinal spacing S_A for the nails in beam A?

(b) What is the maximum longitudinal spacing s_B for the nails in beam B?

(c) Which beam is more efficient in resisting the shear force?



Solution 5.11-6 Two wood box beams

Cross-sectional dimensions are the same.

All dimensions in millimeters.

$b = 200 \quad b_1 = 200 - 2(20) = 160$

$h = 360 \quad h_1 = 360 - 2(20) = 320$

$t = 20$

$F =$ allowable load per nail $= 250\ N$
$V =$ shear force $= 3.2$ kN

$$I = \frac{1}{12}(bh^3 - b_1 h_1^3) = 340.69 \times 10^6 \text{ mm}^4$$

$s =$ longitudinal spacing of the nails

$f =$ shear flow between one flange and both webs

$$f = \frac{2F}{s} = \frac{VQ}{I} \quad \therefore s_{\max} = \frac{2FI}{VQ}$$

(a) Beam A

$$Q = A_f d_f = (bt)\left(\frac{h - t}{2}\right) = (200)(20)\left(\frac{1}{2}\right)(340)$$

$$= 680 \times 10^3 \text{ mm}^3$$

$$s_A = \frac{2FI}{VQ} = \frac{(2)(250 \text{ N})(340.7 \times 10^6 \text{ mm}^4)}{(3.2 \text{ kN})(680 \times 10^3 \text{ mm}^3)}$$

$$= 78.3 \text{ mm} \quad \leftarrow$$

(b) Beam B

$$Q = A_f d_f = (b - 2t)(t)\left(\frac{h - t}{2}\right)$$

$$= (160)(20)\frac{1}{2}(340)$$

$$= 544 \times 10^3 \text{ mm}^3$$

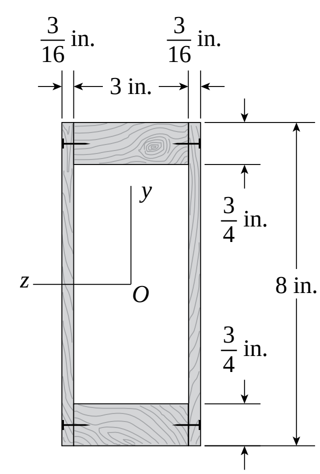
$$s_B = \frac{2FI}{VQ} = \frac{(2)(250\ N)(340.7 \times 10^6 \text{ mm}^4)}{(3.2 \text{ kN})(544 \times 10^3 \text{ mm}^3)}$$

$$= 97.9 \text{ mm} \quad \leftarrow$$

(c) Beam B is more efficient because the shear flow on the contact surfaces is smaller and therefore fewer nails are needed. $\leftarrow$

Problem 5.11-7 A hollow wood beam with plywood webs has the cross-sectional dimensions shown in the figure. The plywood is attached to the flanges by means of small nails. Each nail has an allowable load in shear of 30 lb.

Find the maximum allowable spacing s of the nails at cross sections where the shear force V is equal to (a) 200 lb and (b) 300 lb.



Solution 5.11-7 Wood beam with plywood webs

All dimensions in inches.

$b = 3.375 \quad b_1 = 3.0$

$h = 8.0 \quad h_1 = 6.5$

F = allowable shear force for one nail = 30 lb

s = longitudinal spacing of the nails

f = shear flow between one flange and both webs

$$f = \frac{VQ}{I} = \frac{2F}{s} \qquad \therefore\ s_{\max} = \frac{2FI}{VQ}$$

$$I = \frac{1}{12}(bh^3 - b_1h_1^3) = 75.3438 \text{ in.}^4$$

$$Q = Q_{\text{flange}} = A_f d_f = (3.0)(0.75)(3.625) = 8.1563 \text{ in.}^3$$

(a) $V = 200$ lb

$$s_{\max} = \frac{2FI}{VQ} = \frac{2(30\text{ lb})(75.344\text{ in.}^4)}{(200\text{ lb})(8.1563\text{ in.}^3)}$$

$$= 2.77 \text{ in.} \quad \leftarrow$$

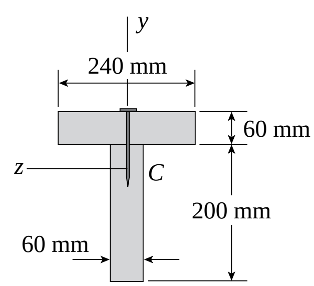
(b) $V = 300$ lb

By proportion,

$$s_{\max} = (2.77\text{ in.})\left(\frac{200}{300}\right) = 1.85 \text{ in.} \quad \leftarrow$$

Problem 5.11-8 A beam of T cross section is formed by nailing together two boards having the dimensions shown in the figure.

If the total shear force V acting on the cross section is 1500 N and each nail may carry 760 N in shear, what is the maximum allowable nail spacing s?



Solution 5.11-8

$V = 1500$ N $\qquad F_{\text{allow}} = 760$ N

$h_1 = 200$ mm $\qquad b = 240$ mm

$t = 60$ mm $\qquad h = 260$ mm

$A = bt + h_1t \quad A = 2.64 \times 10^4 \text{ mm}^2$

Location of neutral axis (z axis)

$$c_2 = \frac{bt\left(h_1 + \dfrac{t}{2}\right) + th_1\dfrac{h_1}{2}}{A}$$

$c_2 = 170.909$ mm

$c_1 = h - c_2$

$c_1 = 89.091$ mm

Moment of inertia about the neutral axis

$$I = \frac{1}{3}tc_2^3 + \frac{1}{3}t(h_1 - c_2)^3$$

$$+ \frac{1}{12}bt^3 + bt\left(c_1 - \frac{t}{2}\right)^2$$

$I = 1.549 \times 10^8 \text{ mm}^4$

First moment of area of flange

$$Q = bt\left(c_1 - \frac{t}{2}\right)$$

$Q = 8.509 \times 10^5 \text{ mm}^3$

Maximum allowable spacing of nails

$$f = \frac{VQ}{I} = \frac{F}{s}$$

$$s_{\max} = \frac{F_{\text{allow}}I}{VQ} \qquad s_{\max} = 92.3 \text{ mm} \quad \leftarrow$$

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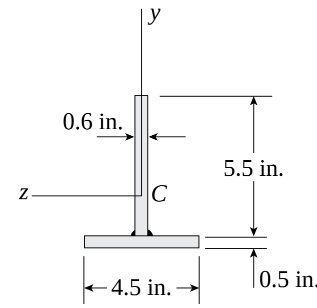
Problem 5.11-9 The T-beam shown in the figure is fabricated by welding together two steel plates. If the allowable load for each weld is 1.8 k/in. in the longitudinal direction, what is the maximum allowable shear force *V*?



Solution 5.11-9 T-beam (welded)

$$F_{allow} = 1.8\,\frac{\text{k}}{\text{in.}}$$

$h_1 = 5.5$ in. $\quad b = 4.5$ in.

$t_1 = 0.6$ in. $\quad t_2 = 0.5$ in.

$h = 6$ in.

$A = bt_2 + h_1t_1 \quad A = 5.55\ \text{in.}^2$

LOCATION OF NEUTRAL AXIS (Z AXIS)

$$c_2 = \frac{bt_2\dfrac{t_2}{2} + t_1h_1\left(\dfrac{h_1}{2} + t_2\right)}{A}$$

$c_2 = 2.034$ in. $\quad c_1 = h - c_2$

$c_1 = 3.966$ in.

MOMENT OF INERTIA ABOUT THE NEUTRAL AXIS

$$I = \frac{1}{3}t_1c_1^3 + \frac{1}{3}t_1(c_2 - t_2)^3 + \frac{1}{12}bt_2^3 + bt_2\left(c_2 - \frac{t_2}{2}\right)^2$$

$I = 20.406\ \text{in.}^4$

FIRST MOMENT OF AREA OF FLANGE

$$Q = bt_2\left(c_2 - \frac{t_2}{2}\right) \qquad Q = 4.014\ \text{in.}^3$$

MAXIMUM ALLOWABLE SHEAR FORCE

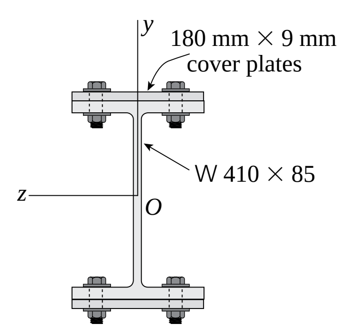
$$f = \frac{VQ}{I} = 2F$$

$$V_{max} = \frac{2F_{allow}I}{Q}$$

$$V_{max} = 18.30\ \text{k} \quad \leftarrow$$

Problem 5.11-10 A steel beam is built up from a W 410 × 85 wide-flange beam and two 180 mm × 9 mm cover plates (see figure). The allowable load in shear on each bolt is 9.8 kN.

What is the required bolt spacing *s* in the longitudinal direction if the shear force *V* = 110kN (*Note:* Obtain the dimensions and moment of inertia of the W shape from Table E-1(b).)



Solution 5.11-10

$V = 110 \text{ kN} \qquad F_{\text{allow}} = 9.8 \text{ kN}$

$W\, 410 \times 85$

$A_w = 10800 \text{ mm}^2 \quad h_w = 417 \text{ mm}$

$I_w = 310 \times 10^6 \text{ mm}^4$

$A_{cp} = (180)\,(9)\,(2) \text{ mm}^2$ for two plates

$h = h_w + (9 \text{ mm})\,(2)$

$A = A_w + A_{cp} \quad A = 1.404 \times 10^4 \text{ mm}^2$

LOCATION OF NEUTRAL AXIS (Z AXIS)

$$c = \frac{h}{2} \qquad c = 217.5 \text{ mm}$$

Moment of inertia about the neutral axis

$$I = I_w + \frac{180 \text{ mm } (9 \text{ mm})^3}{12}(2) + A_{cp}\left(c - \frac{9 \text{ mm}}{2}\right)^2$$

$I = 4.57 \times 10^8 \text{ mm}^4$

First moment of area of one flange

$$Q = 180 \text{ mm } (9 \text{ mm})\left(c - \frac{9 \text{ mm}}{2}\right)$$

$Q = 3.451 \times 10^5 \text{ mm}^3$

Maximum allowable spacing of nails

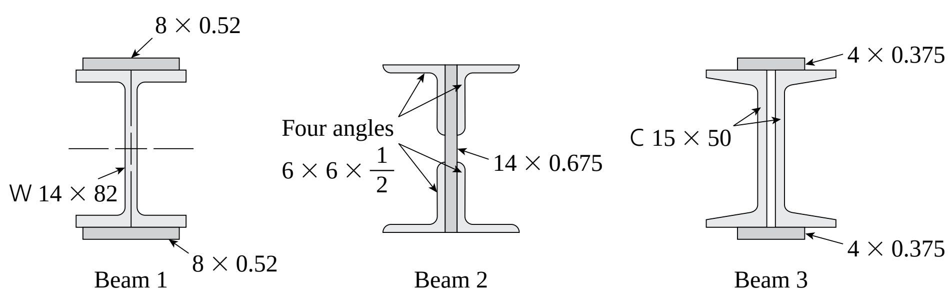
$$f = \frac{VQ}{I} = \frac{2\,F}{s}$$

$$s_{\max} = \frac{2\,F_{\text{allow}}\,I}{VQ} \qquad s_{\max} = 236 \text{ mm} \quad \leftarrow$$

Problem 5.11-11 The three beams shown have approximately the same cross-sectional area. Beam 1 is a W 14 × 82 with flange plates; Beam 2 consists of a web plate with four angles; and Beam 3 is constructed of 2 C shapes with flange plates.

(a) Which design has the largest moment capacity?
(b) Which has the largest shear capacity?
(c) Which is the most economical in bending?
(d) Which is the most economical in shear?

Assume allowable stress values are: $\sigma_a = 18$ ksi and $\tau_a = 11$ ksi. The most economical beam is that having the largest capacity-to-weight ratio. Neglect fabrication costs in answering (c) and (d) above. (*Note*: Obtain the dimensions and properties of all rolled shapes from tables in Appendix E.)



Solution 5.11-11 Built-up steel beam

Beam 1: properties and dimensions for W14 × 82 with flange plates

$A_W = 24 \text{ in.}^2 \qquad h_w = 14.3 \text{ in.} \qquad I_w = 881 \text{ in.}^4$

$b_1 = 8 \text{ in.} \qquad t_1 = 0.52 \text{ in.}$

$h_1 = h_w + 2t_1 \qquad bf_1 = 10.1 \text{ in.}$

$tf_1 = 0.855 \text{ in.} \qquad tw_1 = 0.51 \text{ in.}$

$A_I = A_W + 2b_1t_1 \qquad A_I = 32.32 \text{ in.}^2$

$$I_1 = I_w + \frac{b_1 + t_1^3}{12} 2 + b_1 t_1 \left(\frac{h_w}{2} + \frac{t_1}{2}\right)^2 2$$

$$I_1 = 1.338 \times 10^3 \text{ in}^4$$

Beam 2: properties and dimensions for L6 × 6 × 1/2 angles with web plate

$A_a = 5.77 \text{ in.}^2$ $c_a = 1.67 \text{ in.}$ $h_a = 6 \text{ in.}$

$I_a = 19.9 \text{ in.}^4$ $b_2 = 14 \text{ in.}$

$t_2 = 0.675 \text{ in.}$ $h_2 = b_2$

$A_2 = 4A_a + b_2 t_2$ $A_2 = 32.53 \text{ in.}^2$

$$I_2 = 4I_a + A_a\left(\frac{b_2}{2} - c_a\right)^2 4 + \frac{t_2 b_2^3}{12}$$

$$I_2 = 889.627 \text{ in.}^4$$

Beam 3: properties and dimensions for C15 × 50 with flange plates

$A_c = 14.7 \text{ in.}^2$ $h_c = 15 \text{ in.}$ $I_c = 404 \text{ in.}^4$

$b_3 = 4 \text{ in.}$ $t_3 = 0.375 \text{ in.}$ $h_3 = h_c + 2t_3$

$bf_3 = 3.72 \text{ in.}$ $tf_3 = 0.65 \text{ in.}$ $tw_3 = 0.716 \text{ in.}$

$A_3 = 2A_c + 2b_3 t_3$ $A_3 = 32.4 \text{ in.}^2$

$$I_3 = I_c 2 + \frac{b_3 t_3^3}{12} 2 + b_3 t_3 \left(\frac{h_c}{2} + \frac{t_3}{2}\right)^2 2$$

$$I_3 = 985.328 \text{ in.}^4$$

(a) Beam with largest moment capacity; largest section modulus controls

$$M_{\max} = \sigma_{\text{allow}} S$$

$S_1 = \frac{2I_1}{h_1}$ $S_1 = 174.449 \text{ in.}^3$ largest value

$S_2 = \frac{2I_2}{h_2}$ $S_2 = 127.09 \text{ in.}^3$

$S_3 = \frac{2I_3}{h_3}$ $S_3 = 125.121 \text{ in.}^3$

case (1) with maximum S has the largest moment capacity ←

(b) BEAM WITH LARGEST SHEAR CAPACITY: LARGEST It_w/Q RATIO CONTROLS

$$V_{\max} = \frac{\tau_{\text{allow}} I t_w}{Q}$$

$$Q_1 = b_1 t_1 \left(\frac{h_1}{2} - \frac{t_1}{2}\right) + bf_1 tf_1 \left(\frac{h_w}{2} - \frac{tf_1}{2}\right) + tw_1 \frac{\left(\frac{h_w}{2} - tf_1\right)^2}{2} \qquad Q_1 = 98.983 \text{ in.}^3$$

$$Q_2 = 2A_a\left(\frac{h_2}{2} - c_a\right) + t_2 \frac{\left(\frac{b_2}{2}\right)^2}{2}$$

$$Q_2 = 78.046 \text{ in.}^3$$

$$Q_3 = b_3 t_3 \left(\frac{h_3}{2} - \frac{t_3}{2}\right) + 2bf_3 tf_3 \left(\frac{h_c}{2} - \frac{tf_3}{2}\right) + 2tw_3 \frac{\left(\frac{h_c}{2} - tf_3\right)^2}{2} \qquad Q_3 = 79.826 \text{ in.}^3$$

$$\frac{I_1 tw_1}{Q_1} = 4.448 \times 10^3 \text{ mm}^2$$

$$\frac{I_2 t_2}{Q_2} = 4.964 \times 10^3 \text{ mm}^2$$

$$\frac{I_2 2tw_3}{Q_3} = 1.14 \times 10^4 \text{ mm}^2 \qquad \text{largest value}$$

Case (3) with maximum $\frac{It_w}{Q}$ has the largest shear capacity ←

(c) MOST ECONOMICAL BEAM IN BENDING HAS LARGEST BENDING CAPACITY-TO-WEIGHT RATIO

$$\frac{S_3}{A_3} = 3.862 \text{ in.} < \frac{S_2}{A_2} = 3.907 \text{ in.} < \frac{S_1}{A_1} = 5.398 \text{ in.}$$

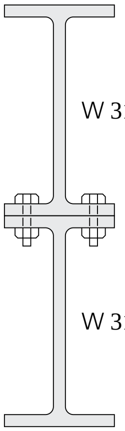
Case (1) is the most economical in bending. ←

(d) MOST ECONOMICAL BEAM IN SHEAR HAS LARGEST SHEAR CAPACITY-TO-WEIGHT RATIO

$$\frac{I_1 tw_1}{Q_1 A_1} = 0.213 < \frac{I_2 t_2}{Q_2 A_2} = 0.237 < \frac{I_3 tw_3}{Q_3 A_3} = 0.273$$

Case (3) is the most economical in shear. ←

Problem 5.11-12 Two W 310 × 74 steel wide-flange beams are bolted together to form a built-up beam as shown in the figure. What is the maximum permissible bolt spacing s if the shear force $V = 80$ kN and the allowable load in shear on each bolt is $F = 13.5$ kN (*Note:* Obtain the dimensions and properties of the W shapes from Table E-1(b).)



Solution 5.11-12

$V = 80$ kN $\quad W\,310 \times 74$

$F_{allow} = 13.5$ kN $\quad A_w = 9420\ \text{mm}^2$

$h_w = 310$ mm $\quad I_w = 163 \times 10^6\ \text{mm}^4$

Location of neutral axis (z axis)

$c = h_w \quad c = 310$ mm

Moment of inertia about the neutral axis

$$I = \left[I_w + A_w \left(\frac{h_w}{2} \right)^2 \right] (2)$$

$$I = 7.786 \times 10^8\ \text{mm}^4$$

First moment of area of flange

$$Q = A_w \frac{h_w}{2} \qquad Q = 1.46 \times 10^6\ \text{mm}^3$$

Maximum allowable spacing of nails

$$f = \frac{VQ}{I} = \frac{2F}{s}$$

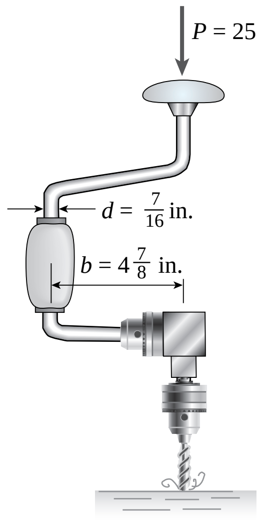
$$s_{max} = \frac{2F_{allow} I}{VQ} \qquad s_{max} = 180\ \text{mm} \quad \leftarrow$$

Beams with Axial Loads

When solving the problems for Section 5.12, assume that the bending moments are not affected by the presence of lateral deflections.

Problem 5.12-1 While drilling a hole with a brace and bit, you exert a downward force $P = 25$ lb on the handle of the brace (see figure). The diameter of the crank arm is $d = 7/16$ in. and its lateral offset is $b = 4\text{-}7/8$ in.

Determine the maximum tensile and compressive stresses σ_t and σ_c, respectively, in the crank.



Solution 5.12-1 Brace and bit

$P = 25 \text{ lb (compression)}$

$M = Pb = (25 \text{ lb})(4\ 7/8 \text{ in.})$

$= 121.9 \text{ lb-in.}$

d = diameter

$d = 7/16$ in.

$$A = \frac{\pi d^2}{4} = 0.1503 \text{ in.}^2$$

$$S = \frac{\pi d^3}{32} = 0.008221 \text{ in.}^3$$

MAXIMUM STRESSES

$$\sigma_t = -\frac{P}{A} + \frac{M}{S} = -\frac{25 \text{ lb}}{0.1503 \text{ in.}^2} + \frac{121.9 \text{ lb-in.}}{0.008221 \text{ in.}^3}$$

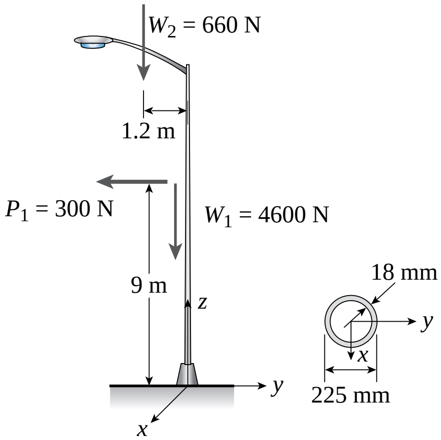
$$= -166 \text{ psi} + 14{,}828 \text{ psi} = 14{,}660 \text{ psi} \quad \leftarrow$$

$$\sigma_c = -\frac{P}{A} - \frac{M}{S} = -166 \text{ psi} - 14{,}828 \text{ psi}$$

$$= -14{,}990 \text{ psi} \quad \leftarrow$$

Problem 5.12-2 An aluminum pole for a street light weights 4600 N and supports an arm that weights 660 N (see figure). The center of gravity of the arm is 1.2 m from the axis of the pole. A wind force of 300 N also acts in the ($-y$) direction at 9 m above the base. The outside diameter of the pole (at its base) is 225 mm, and its thickness is 18 mm.

Determine the maximum tensile and compressive stresses σ_t and σ_c, respectively, in the pole (at its base) due to the weights and the wind force.



Solution 5.12-2

$W_1 = 4600$ N $\quad b = 1.2$ m

$W_2 = 660$ N $\quad h = 9$ m

$P_1 = 300$ N $\quad d_1 = 225$ mm $\quad t = 18$ mm

$d_2 = d_1 - 2t$

$$A = \frac{\pi}{4}(d_1^2 - d_2^2) \qquad I = \frac{\pi}{64}(d_1{}^4 - d_2{}^4)$$

$$A = 1.171 \times 10^4 \text{ mm}^2 \qquad I = 6.317 \times 10^7 \text{ mm}^4$$

AT BASE OF POLE

$P_z = W_1 + W_2$

$P_z = 5.26 \times 10^3$ N (Axial force)

$V_y = P_1 \quad V_y = 300\ N$ (Shear force)

$M_x = W_2 b + P_1 h$

$M_x = 3.492 \times 10^3 \text{ N} \cdot \text{m}$ (Moment)

MAXIMUM STRESS

$$\sigma_t = \left(-\frac{P_z}{A} + \frac{M_x}{I}\frac{d_1}{2}\right)$$

$\sigma_t = 5.77 \times 10^3$ kPa

$= 5770 \text{ kPa} \quad \leftarrow$

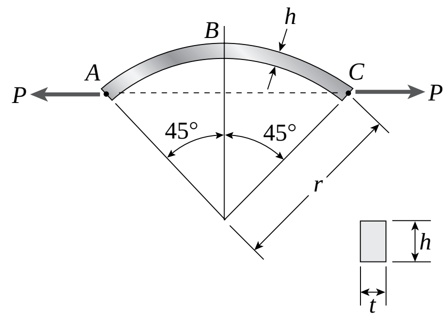
$$\sigma_c = \left(-\frac{P_z}{A} - \frac{M_x}{I}\frac{d_1}{2}\right) \quad \leftarrow$$

$\sigma_c = -6.668 \times 10^3$

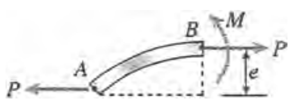
$= -6668 \text{ kPa} \quad \leftarrow$

Problem 5.12-3 A curved bar *ABC* having a circular axis (radius $r = 12$ in.) is loaded by forces $P = 400$ lb (see figure). The cross section of the bar is rectangular with height h and thickness t.

If the allowable tensile stress in the bar is 12,000 psi and the height $h = 1.25$ in., what is the minimum required thickness t_{min}?



Solution 5.12-3 Curved bar



r = radius of curved bar

$$e = r - r\cos 45^\circ$$

$$= r\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$M = Pe = \frac{Pr}{2}(2 - \sqrt{2})$$

CROSS SECTION

h = height $\quad t$ = thickness $\quad A = ht \quad S = \frac{1}{6}th^2$

TENSILE STRESS

$$\sigma_t = \frac{P}{A} + \frac{M}{S} = \frac{P}{ht} + \frac{3Pr(2 - \sqrt{2})}{th^2}$$

$$= \frac{P}{ht}\left[1 + 3(2 - \sqrt{2})\frac{r}{h}\right]$$

MINIMUM THICKNESS

$$t_{min} = \frac{P}{h\sigma_{allow}}\left[1 + 3(2 - \sqrt{2})\frac{r}{h}\right]$$

SUBSTITUE NUMERICAL VALUES:

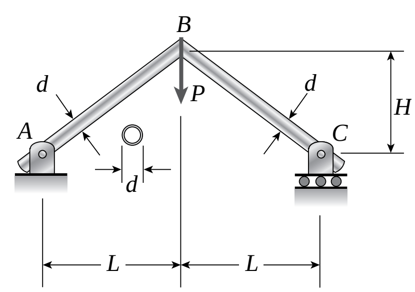
$P = 400$ lb $\quad \sigma_{allow} = 12{,}000$ psi

$r = 12$ in. $\quad h = 1.25$ in.

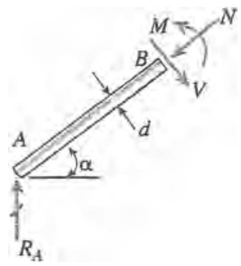
$t_{min} = 0.477$ in. $\quad \leftarrow$

Problem 5.12-4 A rigid frame *ABC* is formed by welding two steel pipes at *B* (see figure). Each pipe has cross-sectional area $A = 11.31 \times 10^3$ mm^2, moment of inertia $I = 46.37 \times 10^6$ mm^4, and outside diameter $d = 200$ mm.

Find the maximum tensile and compressive stresses σ_t and σ_c, respectively, in the frame due to the load $P = 8.0$ kN if $L = H = 1.4$ m.



Solution 5.12-4 Rigid frame



Load P at midpoint B

REACTIONS: $R_A = R_C = \dfrac{P}{2}$

BAR AB:

$$\tan\alpha = \frac{H}{L}$$

$$\sin\alpha = \frac{H}{\sqrt{H^2 + L^2}}$$

$d =$ diameter

$c = d/2$

AXIAL FORCE: $N = R_A \sin\alpha = \dfrac{P}{2}\sin\alpha$

BENDING MOMENT: $M = R_A L = \dfrac{PL}{2}$

TENSILE STRESS

$$\sigma_t = -\frac{N}{A} + \frac{Mc}{I} = -\frac{P\sin\alpha}{2A} + \frac{PLd}{4I}$$

SUBSTITUTE NUMERICAL VALUES

$P = 8.0\ \text{kN} \quad L = H = 1.4\ \text{m} \quad \alpha = 45°$

$\sin\alpha = 1/\sqrt{2} \quad d = 200\ \text{mm}$

$A = 11.31 \times 10^3\ \text{mm}^2 \quad I = 46.37 \times 10^6\ \text{mm}^4$

$$\sigma_t = -\frac{(8.0\ \text{kN})(1/\sqrt{2})}{2(11.31 \times 10^3\ \text{mm}^2)} + \frac{(8.0\ \text{kN})(1.4\ \text{m})(200\ \text{mm})}{4(46.37 \times 10^6\ \text{mm}^4)}$$

$$= -0.250\ \text{MPa} + 12.08\ \text{MPa}$$

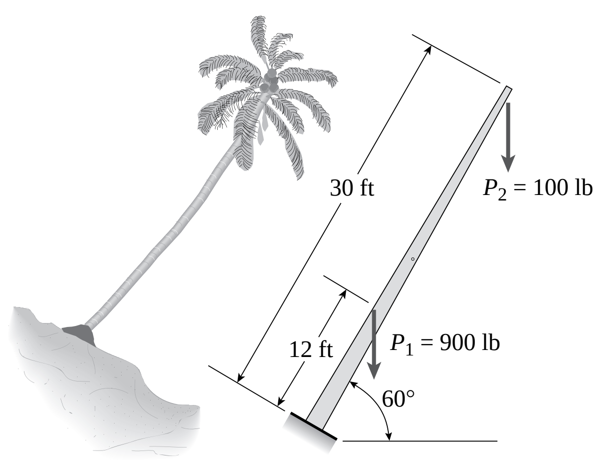
$$= 11.83\ \text{MPa (tension)} \quad \leftarrow$$

$$\sigma_c = -\frac{N}{A} - \frac{Mc}{I} = -0.250\ \text{MPa} - 12.08\ \text{MPa}$$

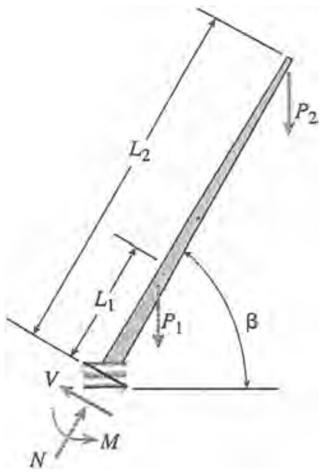
$$= -12.33\ \text{MPa (compression)} \quad \leftarrow$$

Problem 5.12-5 A palm tree weighing 1000 lb is inclined at an angle of 60° (see figure). The weight of the tree may be resolved into two resultant forces, a force $P_1 = 900$ lb acting at a point 12 ft from the base and a force $P_2 = 100$ lb acting at the top of the tree, which is 30 ft long. The diameter at the base of the tree is 14 in.

Calculate the maximum tensile and compressive stresses σ_t and σ_c, respectively, at the base of the tree due to its weight.



Solution 5.12-5 Palm tree



FREE-BODY DIAGRAM

$P_1 = 900$ lb

$P_2 = 100$ lb

$L_1 = 12$ ft $= 144$ in.

$L_2 = 30$ ft $= 360$ in.

$d = 14$ in.

$$A = \frac{\pi d^2}{4} = 153.94 \text{ in.}^2$$

$$S = \frac{\pi d^3}{32} = 269.39 \text{ in.}^3$$

$$M = P_1L_1 \cos 60° + P_2L_2 \cos 60°$$
$$= [(900 \text{ lb})(144 \text{ in.}) + (100 \text{ lb})(360 \text{ in.})] \cos 60°$$
$$= 82{,}800 \text{ lb-in.}$$
$$N = (P_1 + P_2) \sin 60° = (1000 \text{ lb}) \sin 60° = 866 \text{ lb}$$

MAXIMUM TENSILE STRESS

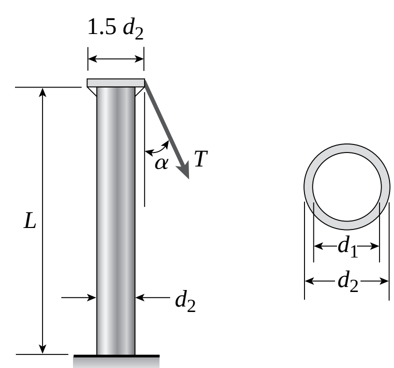
$$\sigma_t = -\frac{N}{A} + \frac{M}{S} = -\frac{866 \text{ lb}}{153.94 \text{ in.}^2} + \frac{82{,}800 \text{ lb-in.}}{269.39 \text{ in.}^3}$$
$$= -5.6 \text{ psi} + 307.4 \text{ psi} = 302 \text{ psi} \quad \leftarrow$$

MAXIMUM COMPRESSIVE STRESS

$$\sigma_c = -5.6 \text{ psi} - 307.4 \text{ psi} = -313 \text{ psi} \quad \leftarrow$$

Problem 5.12-6 A vertical pole of aluminum is fixed at the base and pulled at the top by a cable having a tensile force T (see figure). The cable is attached at the outer edge of a stiffened cover plate on top of the pole and makes an angle $\alpha = 20°$ at the point of attachment. The pole has length $L = 2.5$ m and a hollow circular cross section with outer diameter $d_2 = 280$ mm and inner diameter $d_1 = 220$ mm. The circular cover plate has diameter $1.5d_2$.

Determine the allowable tensile force T_{allow} in the cable if the allowable compressive stress in the aluminum pole is 90 MPa.



Solution 5.12-6

$\sigma_{allow} = 90$ MPa $\qquad d_1 = 220$ mm

$d_2 = 280$ mm

$$t = \frac{d_2 - d_1}{2} \qquad \alpha = 20° \qquad L = 2.5 \text{ m}$$

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) \qquad I = \frac{\pi}{64}(d_2^4 - d_1^4)$$

$$A = 2.356 \times 10^4 \text{ mm}^2 \qquad I = 1.867 \times 10^8 \text{ mm}^4$$

$P_N = T\cos(\alpha)$ (Axial force)

$V = T\sin(\alpha)$ (Shear force)

$$M = VL + P_N\left(\frac{1.5\,d_2}{2}\right) \qquad \text{(Moment)}$$

Allowable Tensile Force

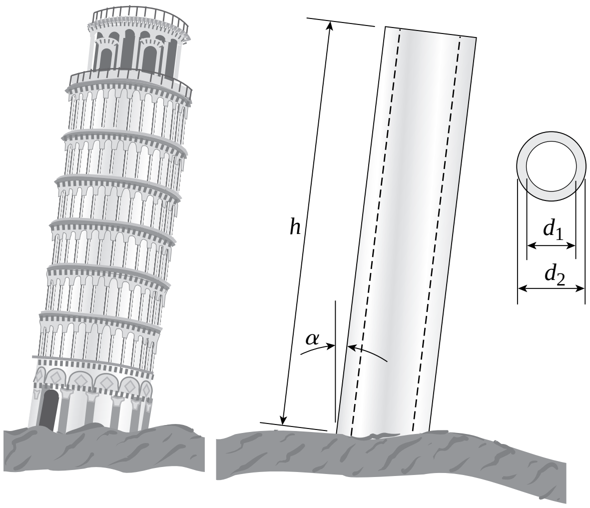
$$\sigma_c = -\frac{P_N}{A} - \frac{M}{I}\frac{d_2}{2} = -\frac{T\cos(\alpha)}{A} - \frac{T\sin(\alpha)\,L + T\cos(\alpha)\left(\frac{1.5\,d_2}{2}\right)}{I}\frac{d_2}{2}$$

$$T_{allow} = \frac{\sigma_{allow}}{\dfrac{\cos(\alpha)}{A} + \dfrac{\sin(\alpha)\,L + \cos(\alpha)\left(\frac{1.5\,d_2}{2}\right)}{I}\dfrac{d_2}{2}}$$

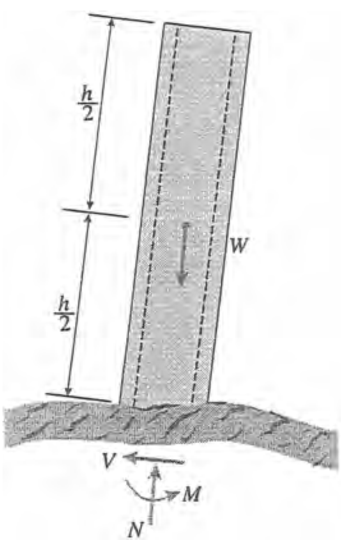
$$T_{allow} = 108.6\text{ kN} \quad \leftarrow$$

Problem 5.12-7 Because of foundation settlement, a circular tower is leaning at an angle α to the vertical (see figure). The structural core of the tower is a circular cylinder of height h, outer diameter d_2, and inner diameter d_1. For simplicity in the analysis, assume that the weight of the tower is uniformly distributed along the height.

Obtain a formula for the maximum permissible angle α if there is to be no tensile stress in the tower.



Solution 5.12-7 Leaning tower



W = weight of tower
α = angle of tilt

CROSS SECTION

$$A = \frac{\pi}{4}(d_2^2 - d_1^2)$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4)$$

$$= \frac{\pi}{64}(d_2^2 - d_1^2)(d_2^2 + d_1^2)$$

$$\frac{I}{A} = \frac{d_2^2 + d_1^2}{16}$$

$$c = \frac{d_2}{2}$$

AT THE BASE OF THE TOWER

$$N = W\cos\alpha \quad M = W\left(\frac{h}{2}\right)\sin\alpha$$

TENSILE STRESS (EQUAL TO ZERO)

$$\sigma_t = -\frac{N}{A} + \frac{Mc}{I} = -\frac{W\cos\alpha}{A} + \frac{W}{I}\left(\frac{h}{2}\sin\alpha\right)\left(\frac{d_2}{2}\right) = 0$$

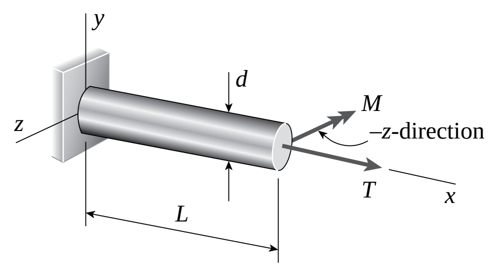
$$\therefore \frac{\cos\alpha}{A} = \frac{hd_2\sin\alpha}{4I} \qquad \tan\alpha = \frac{4I}{hd_2A} = \frac{d_2^2 + d_1^2}{4hd_2}$$

MAXIMUM ANGLE α

$$\alpha = \arctan\frac{d_2^2 + d_1^2}{4hd_2} \quad \leftarrow$$

Problem 5.12-8 A steel bar of solid circular cross section and length $L = 2.5$ m is subjected to an axial tensile force $T = 24$ kN and a bending moment $M = 3.5$ kN m (see figure).

(a) Based upon an allowable stress in tension of 110 MPa, determine the required diameter d of the bar; disregard the weight of the bar itself.
(b) Repeat (a) including the weight of the bar.



Solution 5.12-8

$M = 3.5\ \text{kN}\cdot\text{m} \qquad T = 24\ \text{kN}$

$\gamma_{steel} = 77\ \frac{\text{kN}}{\text{m}^3} \qquad L = 2.5\ \text{m}$

$\sigma_{allow} = 110\ \text{MPa}$

$A = \frac{\pi}{4}d^2 \qquad c = \frac{d}{2} \qquad I = \frac{\pi}{64}d^4$

(a) DISREGARD WEIGHT OF BAR

MAX. TENSILE STRESS AT TOP OF BEAM AT SUPPORT

$$\sigma_{max} = \frac{T}{A} + \frac{M}{I}\frac{d}{2} = \frac{T}{\frac{\pi}{4}d^2} + \frac{M}{\frac{\pi}{64}d^4}\frac{d}{2}$$

$$\sigma_{allow} = \frac{4\,T}{\pi\,d^2} + \frac{32\ \text{M}}{\pi\,d^3}$$

SOLVE NUMERICALLY FOR d (SUBSTITUTE σ_{allow})

$d = 70$ mm $\leftarrow$

(b) INCLUDE WEIGHT OF BAR

$$M_{max} = M + \frac{A\gamma_{steel}L^2}{2}$$

AT TOP OF BEAM AT SUPPORT

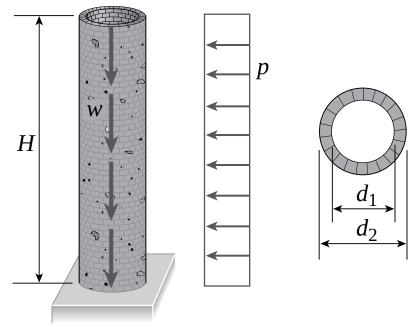
$$\sigma_t = \sigma_{allow} = \frac{T}{A} + \frac{M_{max}}{I}\frac{d}{2}$$

SUBSTITUTE M_{MAX} FROM ABOVE, SOLVE FOR d NUMERICALLY

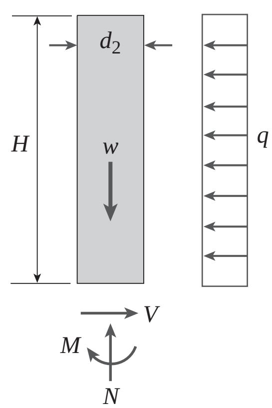
$d = 76.5$ mm $\leftarrow$

Problem 5.12-9 A cylindrical brick chimney of height H weighs $w = 825$ lb/ft of height (see figure). The inner and outer diameters are $d_1 = 3$ ft and $d_2 = 4$ ft, respectively. The wind pressure against the side of the chimney is $p = 10$ lb/ft^2 of projected area.

Determine the maximum height H if there is to be no tension in the brickwork



Solution 5.12-9 Brick Chimney



p = wind pressure

q = intensity of load $= pd_2$

d_2 = outer diameter

d_1 = inner diameter

W = total weight of chimney $= wH$

CROSS SECTION

$$A = \frac{\pi}{4}(d_2^2 - d_1^2)$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = \frac{\pi}{64}(d_2^2 - d_1^2)(d_2^2 - d_1^2)$$

$$\frac{I}{A} = \frac{1}{16}(d_2^2 + d_1^2) \qquad c = \frac{d_2}{2}$$

AT BASE OF CHIMNEY

$$N = W = wH \qquad M = qH\left(\frac{H}{2}\right) = \frac{1}{2}pd_2H^2$$

TENSILE STRESS (EQUAL TO ZERO)

$$\sigma_1 = -\frac{N}{A} + \frac{Md_2}{2I} = 0 \qquad \text{or} \qquad \frac{M}{N} = \frac{2I}{Ad_2}$$

$$\frac{pd_2H^2}{2wH} = \frac{d_2^2 + d_1^2}{8d_2}$$

SOLVE FOR H $\qquad H = \dfrac{w(d_2^2 + d_1^2)}{4pd_2^2} \quad \leftarrow$

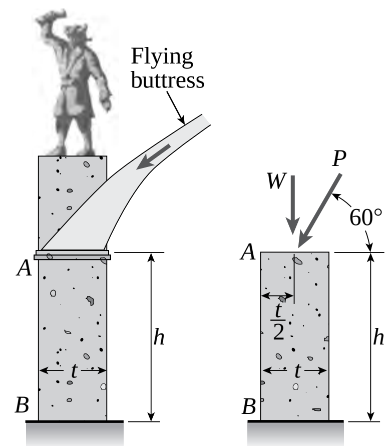
SUBSTITUTE NUMERICAL VALUES

$w = 825$ lb/ft $\qquad d_2 = 4$ ft $\qquad d_1 = 3$ ft

$q = 10$ lb/ft^2 $\qquad H_{max} = 32.2$ ft $\quad \leftarrow$

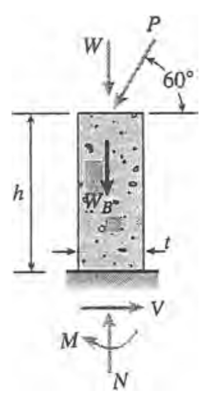
Problem 5.12-10 A flying buttress transmits a load $P = 25$ kN, acting at an angle of 60° to the horizontal, to the top of a vertical buttress AB (see figure). The vertical buttress has height $h = 5.0$ m and rectangular cross section of thickness $t = 1.5$ m and width $b = 1.0$ m (perpendicular to the plane of the figure). The stone used in the construction weighs $y = 26$ kN/m^3.

What is the required weight W of the pedestal and statue above the vertical buttress (that is, above section A) to avoid any tensile stresses in the vertical buttress?



Solution 5.12-10 Flying buttress

Free-body diagram of vertical buttress



$P = 25$ kN

$h = 5.0$ m

$t = 1.5$ m

$b =$ width of buttress perpendicular to the figure

$b = 1.0$ m

$\gamma = 26$ kN/m^3

$W_B =$ weight of vertical buttress

$= bth\gamma$

$= 195$ kN

Cross section

$$A = bt = (1.0 \text{ m})(1.5 \text{ m}) = 1.5 \text{ m}^2$$

$$S = \frac{1}{6}bt^2 = \frac{1}{6}(1.0 \text{ m})(1.5 \text{ m})^2 = 0.375 \text{ m}^3$$

At the base

$$N = W + W_B + P \sin 60°$$

$$= W + 195 \text{ kN} + (25 \text{ kN}) \sin 60°$$

$$= W + 216.651 \text{ kN}$$

$$M = (P\cos 60°)\,h = (25 \text{ kN})(\cos 60°)(5.0 \text{ m})$$

$$= 62.5 \text{ kN} \cdot \text{m}$$

Tensile stress (equal to zero)

$$\sigma_t = -\frac{N}{A} + \frac{M}{S}$$

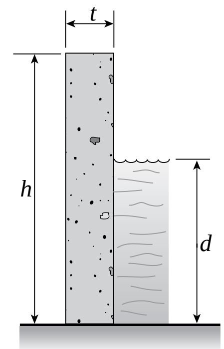
$$= -\frac{W + 216.651 \text{ kN}}{1.5 \text{ m}^2} + \frac{62.5 \text{ kN} \cdot \text{m}}{0.375 \text{ m}^3} = 0$$

or $-W - 216.651 \text{ kN} + 250 \text{ kN} = 0$

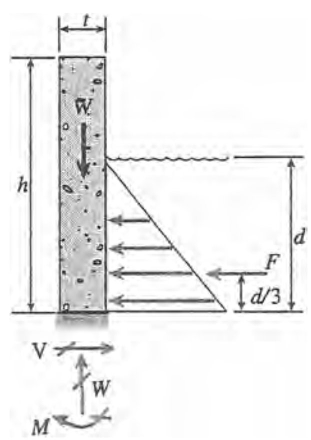
$W = 33.3$ kN ←

Problem 5.12-11 A plain concrete wall (i.e., a wall with no steel reinforcement) rests on a secure foundation and serves as a small dam on a creek (see figure). The height of the wall is $h = 6.0$ ft and the thickness of the wall is $t = 1.0$ ft.

(a) Determine the maximum tensile and compressive stresses σ_t and σ_c, respectively, at the base of the wall when the water level reaches the top ($d = h$). Assume plain concrete has weight density $\gamma_c = 145$ Ib/ft^3.
(b) Determine the maximum permissible depth d_{max} of the water if there is to be no tension in the concrete.



Solution 5.12-11 Concrete wall



h = height of wall

t = thickness of wall

b = width of wall (perpendicular to the figure)

γ_c = width density of concrete

γ_w = weight density of water

d = depth of water

W = weight of wall

$W = bht\gamma_c$

F = resultant force for the water pressure

MAXIMUM WATER PRESSURE $= \gamma_w d$

$$F = \frac{1}{2}(d)(\gamma_w d)(b) = \frac{1}{2}bd^2\gamma_w$$

$$M = F\left(\frac{d}{3}\right) = \frac{1}{6}bd^3\gamma_w$$

$$A = bt \quad S = \frac{1}{6}bt^2$$

STRESSES AT THE BASE OF THE WALL (d = DEPTH OF WATER)

$$\sigma_t = -\frac{W}{A} + \frac{M}{S} = -h\gamma_c + \frac{d^3\gamma_w}{t^2} \qquad \text{Eq. (1)}$$

$$\sigma_c = -\frac{W}{A} - \frac{M}{S} = -h\gamma_c - \frac{d^3\gamma_w}{t^2} \qquad \text{Eq.(2)}$$

(a) STRESSES AT THE BASE WHEN $d = h$

$h = 6.0$ ft $= 72$ in. $\quad d = 72$ in.

$t = 1.0$ ft $= 12$ in.

$$\gamma_c = 145 \text{ lb/ft}^3 = \frac{145}{1728} \text{ lb/in.}^3$$

$$\gamma_w = 62.4 \text{ Ib/ft}^3 = \frac{62.4}{1728} \text{ lb/in.}^3$$

Substitute numerical values into Eqs. (1) and (2):

$\sigma_t = -6.042 \text{ psi} + 93.600 \text{ psi} = 87.6 \text{ psi} \quad \leftarrow$

$\sigma_c = -6.042 \text{ psi} - 93.600 \text{ psi} = -99.6 \text{ psi} \quad \leftarrow$

(b) MAXIMUM DEPTH FOR NO TENSION

Set $\sigma_t = 0$ in Eq. (1):

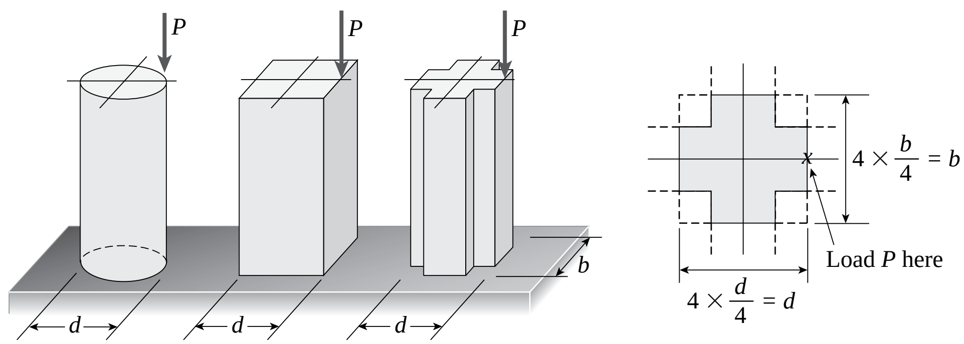
$$-h\gamma_c + \frac{d^3\gamma_w}{t^2} = 0 \quad d^3 = ht^2\left(\frac{\gamma_c}{\gamma_w}\right)$$

$$d^3 = (72 \text{ in.})(12 \text{ in.})^2\left(\frac{145}{62.4}\right) = 24{,}092 \text{ in.}^3$$

$d_{max} = 28.9 \text{ in.} \quad \leftarrow$

Problem 5.12-12 A circular post, a rectangular post, and a post of cruciform cross section are each compressed by loads that produce a resultant force P acting at the edge of the cross section (see figure). The diameter of the circular post and the depths of the rectangular and cruciform posts are the same.

(a) For what width b of the rectangular post will the maximum tensile stresses be the same in the circular and rectangular posts?
(b) Repeat (a) for the post with cruciform cross section.
(c) Under the conditions described in parts (a) and (b), which post has the largest compressive stress?



Solution 5.12-12

(a) EQUAL MAXIMUM TENSILE STRESSES

CIRCULAR POST

$$A = \frac{\pi}{4}d^2 \qquad S = \frac{\pi}{32}d^3 \qquad M = \frac{Pd}{2}$$

Tension

$$\sigma_t = -\frac{P}{A} + \frac{M}{S} = -\frac{4P}{\pi d^2} + \frac{16P}{\pi d^2} = \frac{12P}{\pi d^2}$$

COMPRESSION $\sigma_c = -\dfrac{P}{A} - \dfrac{M}{S}$

$$= -\frac{4P}{\pi d^2} - \frac{16P}{\pi d^2} = -\frac{20P}{\pi d^2}$$

RECTANGULAR POST

$$A = bd \qquad S = \frac{bd^2}{6} \qquad M = \frac{Pd}{2}$$

TENSION $\quad \sigma_t = -\dfrac{P}{A} + \dfrac{M}{S} = -\dfrac{P}{bd} + \dfrac{3P}{bd} = \dfrac{2P}{bd}$

COMPRESSION $\sigma_c = -\frac{P}{A} - \frac{M}{S} = -\frac{P}{bd} - \frac{3P}{bd} = -\frac{4P}{bd}$

Equate tensile stress expressions, solve for b

$\frac{12\,P}{\pi d^2} = \frac{2\,P}{bd} \quad \frac{6}{\pi d} = \frac{1}{b} \quad b = \frac{\pi d}{6} \quad \leftarrow$

(b) CRUCIFORM CROSS SECTION

$$A = \left[bd - \left(\frac{b}{2}\frac{d}{2}\right)\right]$$

$$S = \left[\frac{b}{2}\frac{d^3}{12} + \frac{b}{2}\left(\frac{d}{2}\right)^3 \frac{1}{12}\right]\frac{2}{d} = \frac{3}{32}bd^2$$

$$M = \frac{Pd}{2\left(\frac{3}{32}bd^2\right)} = \frac{16P}{3bd}$$

TENSION $\sigma_t = -\frac{P}{A} + \frac{M}{S}$

$$= -\frac{4P}{3bd} + \frac{16P}{3bd} = \frac{12P}{3bd}$$

COMPRESSION $\sigma_c = -\frac{P}{A} - \frac{M}{S}$

$$= -\frac{4P}{3bd} - \frac{16P}{3bd} = -\frac{20P}{3bd}$$

Equate compressive stresses & solve for b

$\frac{12\,P}{\pi d^2} = \frac{2\,P}{3bd} \quad \frac{3}{\pi d} = \frac{1}{b} \quad b = \frac{\pi d}{3} \quad \leftarrow$

(c) THE LARGEST COMPRESSIVE STRESS

substitute expressions for b above & compare compressive stresses

CIRCULAR POST

$$\sigma_c = -\frac{20\,P}{\pi d^2}$$

RECTANGULAR POST

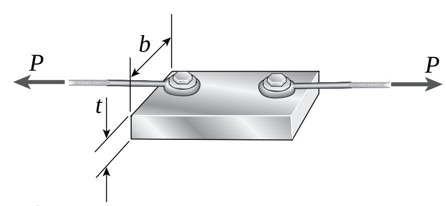
$$\sigma_c = -\frac{4\,P}{\left(\frac{\pi d}{6}\right)d} = -\frac{24\,P}{\pi d^2}$$

CRUCIFORM POST

$$\sigma_c = -\frac{20\,P}{3\,\frac{\pi d}{3}\,d} = -\frac{20\,P}{\pi d^2}$$

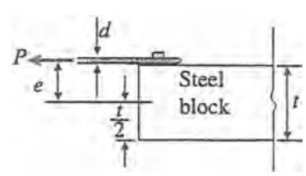
Rectangular post has the largest compressive stress $\leftarrow$

Problem 5.12-13 Two cables, each carrying a tensile force $P = 1200$ lb, are bolted to a block of steel (see figure). The block has thickness $t = 1$ in. and width $b = 3$ in.



(a) If the diameter d of the cable is 0.25 in., what are the maximum tensile and compressive stresses σ_t and σ_c, respectively, in the block?
(b) If the diameter of the cable is increased (without changing the force P), what happens to the maximum tensile and compressive stresses?

Solution 5.12-13 Steel block loaded by cables



$P = 1200$ lb $\quad d = 0.25$ in.

$t = 1.0$ in. $\quad e = \frac{t}{2} + \frac{d}{2} = 0.625$ in.

$b =$ width of block

$= 3.0$ in.

CROSS SECTION OF BLOCK

$$A = bt = 30 \text{ in.}^2 \qquad I = \frac{1}{12}bt^3 = 0.25 \text{ in.}^4$$

(a) MAMIMUM TENSILE STRESS (AT TOP OF BLOCK)

$$y = \frac{t}{2} = 0.5 \text{ in.}$$

$$\sigma_t = \frac{P}{A} + \frac{Pey}{I}$$

$$= \frac{1200 \text{ lb}}{3 \text{ in.}^2} + \frac{(1200 \text{ lb})(0.625 \text{ in.})(0.5 \text{ in.})}{0.25 \text{ in.}^4}$$

$$= 400 \text{ psi} + 1500 \text{ psi} = 1900 \text{ psi} \quad \leftarrow$$

MAXIMUM COMPRESSIVE STRESS (AT BOTTOM OF BLOCK)

$$y = -\frac{t}{2} = -0.5 \text{ in.}$$

$$\sigma_c = \frac{P}{A} + \frac{Pey}{I}$$

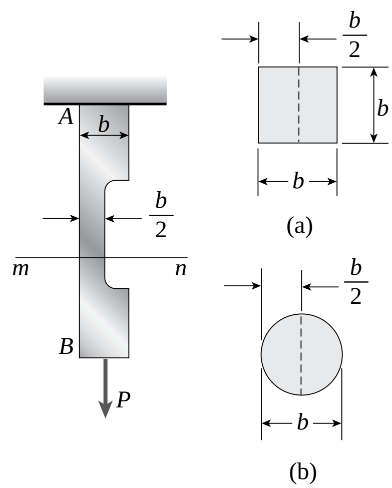
$$= \frac{1200 \text{ lb}}{3 \text{ in.}^2} + \frac{(1200 \text{ lb})(0.625 \text{ in.})(-0.5 \text{ in.})}{0.25 \text{ in.}^4}$$

$$= 400 \text{ psi} - 1500 \text{ psi} = -1100 \text{ psi} \quad \leftarrow$$

(b) IF d IS INCREASED, increase the eccentricity e increases and both stresses in magnitude.

Problem 5.12-14 A bar AB supports a load P acting at the centroid of the end cross section (see figure). In the middle region of the bar the cross-sectional area is reduced by removing one-half of the bar.

(a) If the end cross sections of the bar are square with sides of length b, what are the maximum tensile and compressive stresses σ_t and σ_c, respectively, at cross section mn within the reduced region?
(b) If the end cross sections are circular with diameter b, what are the maximum stresses σ_t and σ_c?



Solution 5.12-14 Bar with reduced cross section

(a) SQUARE BAR

Cross section mn is a rectangle.

$$A = (b)\left(\frac{b}{2}\right) = \frac{b^2}{2} \quad I = \frac{1}{12}(b)\left(\frac{b}{2}\right)^3 = \frac{b^4}{96}$$

$$M = P\left(\frac{b}{4}\right) \quad c = \frac{b}{4}$$

STRESSES

$$\sigma_t = \frac{P}{A} + \frac{Mc}{I} = \frac{2P}{b^2} + \frac{6P}{b^2} = \frac{8P}{b^2} \quad \leftarrow$$

$$\sigma_c = \frac{P}{A} - \frac{Mc}{I} = \frac{2P}{b^2} - \frac{6P}{b^2} = -\frac{4P}{b^2} \quad \leftarrow$$

(b) CIRCULAR BAR

Cross section mn is a semicircle

$$A = \frac{1}{2}\left(\frac{\pi b^2}{4}\right) = \frac{\pi b^2}{8} = 0.3927\, b^2$$

From Appendix D, Case 10:

$$I = 0.1098\left(\frac{b}{2}\right)^4 = 0.006860\, b^4$$

$$M = P\left(\frac{2b}{3\pi}\right) = 0.2122\, Pb$$

For tension

$$c_t = \frac{4r}{3\pi} = \frac{2b}{3\pi} = 0.2122\ b$$

For compression:

$$c_c = r - c_t = \frac{b}{2} - \frac{2b}{3\pi} = 0.2878\ b$$

Stresses

$$\sigma_t = \frac{P}{A} + \frac{Mc_t}{I} = \frac{P}{0.3927\ b^2} + \frac{(0.2122\ Pb)(0.2122\ b)}{0.006860\ b^4}$$

$$= 2.546\frac{P}{b^2} + 6.564\frac{P}{b^2} = 9.11\frac{P}{b^2} \quad \leftarrow$$

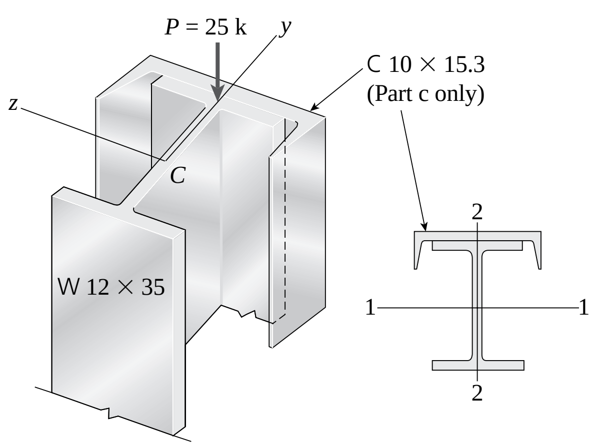
$$\sigma_c = \frac{P}{A} - \frac{Mc_c}{I}$$

$$= \frac{P}{0.3927\ b^2} - \frac{(0.2122\ Pb)(0.2878\ b)}{0.006860\ b^4}$$

$$= 2.546\frac{P}{b^2} - 8.903\frac{P}{b^2} = -6.36\frac{P}{b^2} \quad \leftarrow$$

Problem 5.12-15 A short column constructed of a W 12 × 35 wide-flange shape is subjected to a resultant compressive load $P = 12$ k having its line of action at the midpoint of one flange (see figure).

(a) Determine the maximum tensile and compressive stresses σ_t and σ_c, respectively, in the column.
(b) Locate the neutral axis under this loading condition.
(c) Recompute maximum tensile and compressive stresses if a C 10 × 15.3 is attached to one flange, as shown.



Solution 5.12-15 Column of wide-flange shape

Properties of each shape:

W 12 × 35	C 10 × 15.3
$A_w = 10.3$ in.^{3}	$A_c = 4.48$ in.^{2}
$h_w = 12.5$ in.	$t_{wc} = 0.24$ in.
$t_f = 0.52$ in.	$x_p = 0.634$ in.
$I_w = 285$ in.^{4}	$I_c = 2.27$ in.^{4} (2-2 axis)

(a) The maximum tensile and compressive stresses
Location of centroid for W 12 × 35 alone

$$c_w = \frac{h_w}{2} \qquad c_w = 6.25\text{ in.}$$

$$P = 25\text{ k} \qquad e_w = \frac{h_w}{2} - \frac{t_f}{2} \qquad e_w = 5.99\text{ in.}$$

$$\sigma_t = -\frac{P}{A_w} + \frac{Pe_w}{I_w}c_w \qquad \sigma_t = 857\text{ psi} \quad \leftarrow$$

$$\sigma_c = -\frac{P}{A_w} - \frac{Pe_w}{I_w}c_w \qquad \sigma_c = -5711\text{ psi} \quad \leftarrow$$

(b) Neutral axis (W shape alone)

$$y_0 = -\frac{I_w}{A_w e_w} \qquad y_0 = -4.62\text{ in.} \quad \leftarrow$$

(c) Combined Column, W 12 × 35 with C 10 × 15.3

$$h = h_w + t_{wc}$$

$$h = 12.74\text{ in.}$$

$$A = A_w + A_c \qquad A = 14.78\text{ in.}^2$$

LOCATION OF CENTROID OF COMBINED SHAPE

$$c = \frac{A_w\left(\frac{h_w}{2}\right) + A_c(h - x_p)}{A} \qquad c = 8.025 \text{ in.}$$

$$I = I_w + A_w\left(c - \frac{h_w}{2}\right)^2 + I_c + A_c(h - x_p - c)^2$$

$$I = 394.334 \text{ in.}^4$$

$$P = 25 \text{ k} \qquad e = h_w - \frac{t_f}{2} - c \qquad e = 4.215 \text{ in.}$$

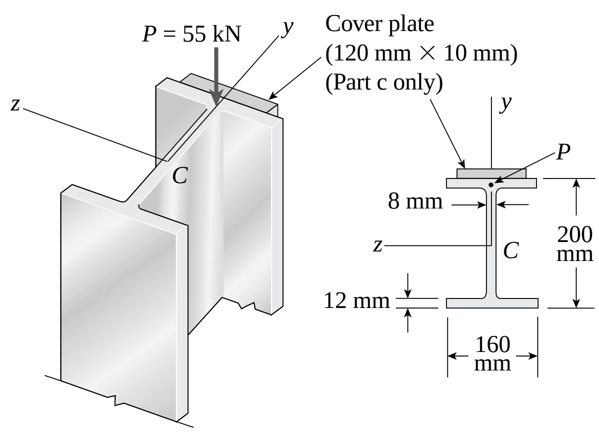
$$\sigma_t = -\frac{P}{A} + \frac{Pe}{I}c \qquad \sigma_t = 453 \text{ psi} \quad \longleftarrow$$

$$\sigma_c = -\frac{P}{A} - \frac{Pe}{I}(h - c) \qquad \sigma_c = -2951 \text{ psi} \quad \longleftarrow$$

$$y_0 = -\frac{I}{Ae} \qquad y_0 = -6.33 \text{ in. (from centroid)} \quad \longleftarrow$$

Problem 5.12-16 A short column of wide-flange shape is subjected to a compressive load that produces a resultant force $P = 55$ kN acting at the midpoint of one flange (see figure).

(a) Determine the maximum tensile and compressive stresses σ_t and σ_c, respectively, in the column.
(b) Locate the neutral axis under this loading condition.
(c) Recompute maximum tensile and compressive stresses if a 120 mm × 10 mm cover plate is added to one flange as shown.



Solution 5.12-16

$P = 55$ kN

(a) MAXIMUM TENSILE AND COMPRESSIVE STRESSES FOR W SHAPE ALONE

PROPERTIES AND DIMENSIONS FOR W SHAPE

$b = 160$ mm $\quad d = 200$ mm

$t_f = 12$ mm $\quad t_w = 8$ mm

$$A_w = bd - (b - t_w)(d - 2t_f)$$

$$A_w = 5.248 \times 10^3 \text{ mm}^2$$

$$I_w = \frac{bd^3}{12} - \frac{(b - t_w)(d - 2t_f)^3}{12}$$

$$I_w = 3.761 \times 10^7 \text{ mm}^4$$

$$e = \frac{d}{2} - \frac{t_f}{2} \qquad e = 94 \text{ mm}$$

$$\sigma_t = -\frac{P}{A_w} + \frac{Pe}{I_w}\frac{d}{2} \qquad \sigma_t = 3.27 \text{ MPa} \quad \longleftarrow$$

$$\sigma_c = -\frac{P}{A_w} - \frac{Pe}{I_w}\frac{d}{2} \qquad \sigma_c = -24.2 \text{ MPa} \quad \longleftarrow$$

(b) NEUTRAL AXIS (W SHAPE ALONE)

$$y_0 = -\frac{I_w}{A_w e} \qquad y_0 = -76.2 \text{ mm} \quad \longleftarrow$$

(c) COMBINED COLUMN-W SHAPE & COVER PLATE

$b_p = 120$ mm $\quad t_p = 10$ mm

$$h = d + t_p$$

$$h = 210 \text{ mm}$$

$$A = A_w + b_p t_p \qquad A = 6.448 \times 10^3 \text{ mm}^2$$

CENTROID OF COMPOSITE SECTION

$$c = \frac{A_w \dfrac{d}{2} + b_p t_p \left(d + \dfrac{t_p}{2} \right)}{A}$$

$$c = 119.541 \text{ mm}$$

$$I = I_w + A_w \left(c - \frac{d}{2} \right)^2 + \frac{b_p t_p^3}{12} + b_p t_p \left(d + \frac{t_p}{2} - c \right)^2$$

$$I = 4.839 \times 10^7 \text{ mm}^4$$

$$e = d - \frac{t_f}{2} - c \qquad e = 74.459 \text{ mm}$$

$$\sigma_t = -\frac{P}{A} + \frac{Pe}{I} c \qquad \sigma_t = 1.587 \text{ MPa} \quad \longleftarrow$$

$$\sigma_c = -\frac{P}{A_w} - \frac{Pe}{I_w}(h - c) \qquad \sigma_c = -20.3 \text{ MPa} \quad \longleftarrow$$

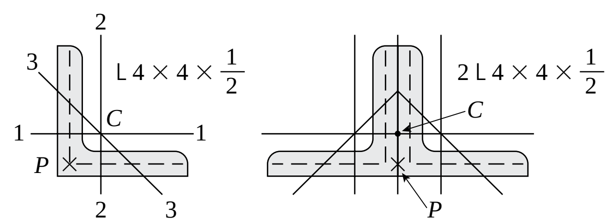
NEUTRAL AXIS

$$y_0 = -\frac{I}{Ae} \qquad y_0 = -100.8 \text{ mm (from centrioid)}$$

Problem 5.12-17 A tension member constructed of an L $4 \times 4 \times \frac{1}{2}$ inch angle section (see Table E-4(a) in Appendix E) is subjected to a tensile load $P = 12.5$ kips that acts through the point where the midlines of the legs intersect [see figure part (a)].

(a) Determine the maximum tensile stress σ_t in the angle section.

(b) Recompute the maximum tensile stress if two angels are used and P is applied as shown in the figure part (b).



(a) (b)

Solution 5.12-17 Angle section in tension

(a) ONE ANGLE: $L\ 4 \times 4 \times 1/2$

$$A_L = 3.75 \text{ in.}^2 \qquad r_{min} = 0.776 \text{ in.}$$

$$t = 0.5 \text{ in.}$$

$$c = 1.18 \text{ in.}$$

$$e = \left(c - \frac{t}{2} \right) \sqrt{2} \qquad e = 1.315 \text{ in}$$

$$P = 12.5 \text{ k}$$

$$c_1 = c\sqrt{2} \qquad c_1 = 1.699 \text{ in.}$$

$$I_3 = A_L r_{min}^2 \qquad I_3 = 2.258 \text{ in.}^4$$

$$M = Pe \qquad M = 16.44 \text{ k-in.}$$

MAXIMUM TENSILE STRESS OCCURS AT CORNER

$$\sigma_t = \frac{P}{A_L} + \frac{Mc_1}{I_3} \qquad \sigma_t = 15.48 \text{ ksi} \quad \longleftarrow$$

(b) TWO ANGLES: $L\ 4 \times 4 \times 1/2$

$$A = 2A_L$$

$$t = 0.5 \text{ in}$$

$$c = 1.18 \text{ in}$$

$$I_L = 5.52 \text{ in.}^4 \text{ (2-2 axis)}$$

$$e = \left(c - \frac{t}{2} \right) \qquad e = 0.93 \text{ in.}$$

$$P = 12.5 \text{ k}$$

$$I = 2I_L \qquad I = 11.04 \text{ in.}^4$$

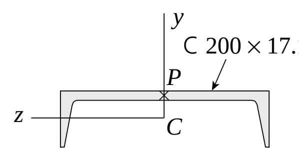
$$M = Pe \qquad M = 11.625 \text{ k-in.}$$

MAXIMUM TENSILE STRESS OCCURS AT THE LOWER EDGE

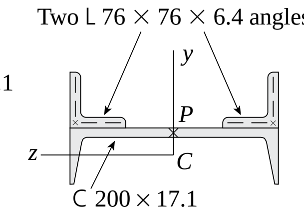
$$\sigma_t = \frac{P}{A} + \frac{Mc}{I} \qquad \sigma_t = 2.91 \text{ ksi} \quad \longleftarrow$$

Problem 5.12-18 A short length of a 200 × 17.1 channel is subjected to an axial compressive force P that has its line of action through the midpoint of the web of the channel [(see figure(a)].

(a) Determine the equation of the neutral axis under this loading condition.
(b) If the allowable stresses in tension and compression are 76 MPa and 52 MPa respectively, find the maximum permissible load P_{max}.
(c) Repeat (a) and (b) if two L 76 × 76 × 6.4 angles are added to the channel as shown in the figure part (b).
See Table E-3(b) in Appendix E for channel properties and Table E-4(b) for angle properties.

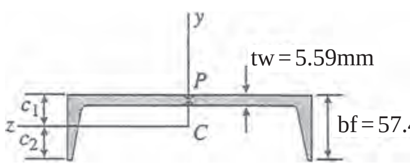


(a)



(b)

Solution 5.12-18



C 200 × 17.1

$A_c = 2170\ \text{mm}^2 \quad d_c = 203\ \text{mm} \quad c_1 = 14.5\ \text{mm}$

$I_c = 0.545 \times 10^6\ \text{mm}^4$ (z-axis)

$c_2 = b_f - c_1 \quad c_2 = 42.9\ \text{mm}$

Allowable stresses

$\sigma_t = 76\ \text{MPa} \quad \sigma_c = -52\ \text{MPa}$

Eccentricity of the load

$$e = c_1 - \frac{t_w}{2} \qquad e = 11.705\ \text{mm}$$

(a) Location of the neutral axis (channel alone)

$$y_0 = \frac{-I_c}{A_c \cdot e} \qquad y_0 = -21.5\ \text{mm} \quad \leftarrow$$

(b) Find P_{max}

$$\sigma_t = -\frac{P}{A} + \frac{Pe}{I}c_2 \qquad P = \frac{\sigma_t}{-\dfrac{1}{A_c} + \dfrac{e}{I_c}c^2}$$

$P = 165.025$ kN

$$\sigma_c = -\frac{P}{A} - \frac{Pe}{I}c_1$$

$$P = \frac{\sigma_c}{-\dfrac{1}{A_c} - \dfrac{e}{I_c}c_1}$$

$$P_{max} = 67.3\ \text{kN} \quad \leftarrow$$

(c) Combined column with 2-angles

L 76 × 76 × 6.4

$A_L = 929\ \text{mm}^2 \qquad I_L = 0.512 \times 10^6\ \text{mm}^4$

$c_L = 21.2$ mm

Composite section

$A = A_c + 2A_L \qquad A = 4.028 \times 10^3\ \text{mm}^2$

$h = b_f + 76\ \text{mm} \qquad h = 133.4\ \text{mm}$

Centroid of composite section

$$c = \frac{A_c(b_f - c_1) + 2A_L(b_f + c_L)}{A}$$

$c = 59.367$ mm

$$I = I_c + A_c(b_f - c_1 - c)^2 + 2I_L + 2A_L(bf + c_L - c)^2$$

$I = 2.845 \times 10^6\ \text{mm}^4$

$$e = b_f - \frac{t_w}{2} - c \qquad e = -4.762\ \text{mm}$$

$b_f = 57.4$ mm

Location of the neutral axis

$$y_0 = -\frac{I}{Ae} \qquad y_0 = 148.3\ \text{mm} \quad \leftarrow$$

$y_0 = 148.3 \text{ mm} > h = 133.4 \text{ mm}$ ←

Thus, this composite section has no tensile stress

$$\sigma_c = -\frac{P}{A} + \frac{Pe}{I}c$$

$$P = \frac{\sigma_c}{-\frac{1}{A} + \frac{e}{I}c} \qquad P_{max} = 149.6 \text{ kN} \quad \leftarrow$$

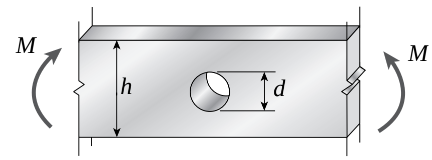
Stress Concentrations

The problems for Section 5.13 are to be solved considering the stress-concentration factors.

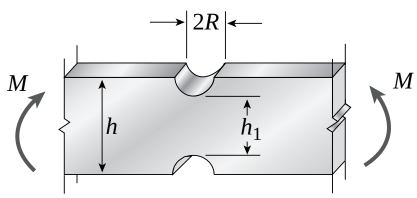
Problem 5.13-1 The beams shown in the figure are subjected to bending moments $M = 2100$ lb-in. Each beam has a rectangular cross section with height $h = 1.5$ in. and width $b = 0.375$ in. (perpendicular to the plane of the figure).

(a) For the beam with a hole at midheight, determine the maximum stresses for hole diameters $d = 0.25$, 0.50, 0.75, and 1.00 in.

(b) For the beam with two identical notches (inside height $h_1 = 1.25$ in.), determine the maximum stresses for notch radii $R = 0.05$, 0.10, 0.15, and 0.20 in.



(a)



Probs. 5.13.1 through 5.13-4 (b)

Solution 5.13-1

$M = 2100$ lb-in. $h = 1.5$ in. $b = 0.375$ in.

(a) Beam with a hole

$$\frac{d}{h} \leq \frac{1}{2} \quad \text{Eq.(5-57):} \quad \sigma_c = \frac{6Mh}{b(h^3 - d^3)}$$

$$= \frac{50{,}400}{3.375 - d^3} \qquad (1)$$

$$\frac{d}{h} \geq \frac{1}{2} \quad \text{Eq.(5-56):} \quad \sigma_B = \frac{12Md}{b(h^3 - d^3)}$$

$$= \frac{67{,}200\, d}{3.375 - d^3} \qquad (2)$$

d (in.)	$\frac{d}{h}$	σ_c Eq. (1) (psi)	σ_B Eq. (2) (psi)	σ_{max} (psi)
0.25	0.1667	15,000	—	15,000
0.50	0.3333	15,500	—	15,500
0.75	0.5000	17,100	17,100	17,100
1.00	0.6667	—	28,300	28,300

NOTE: The larger the hole, the larger the stress.

(b) Beam with notches

$$h_1 = 1.25 \text{ in.} \qquad \frac{h}{h_1} = \frac{1.5 \text{ in.}}{1.25 \text{ in.}} = 1.2$$

Eq. (5-58)

$$\sigma_{nom} = \frac{6M}{bh_1^2} = 21{,}500 \text{ psi}$$

R (in)	$\frac{R}{h_1}$	K (Fig. 5-50)	$\sigma_{max} = K\sigma_{nom}$ σ_{max} (psi)
0.05	0.04	3.0	65,000
0.10	0.08	2.3	49,000
0.15	0.12	2.1	45,000
0.20	0.16	1.9	41,000

NOTE: The larger the notch radius, the smaller the stress.

Problem 5.13-2 The beams shown in the figure are subjected to bending moments $M = 250\ \text{N} \cdot \text{m}$. Each beam has a rectangular cross section with height $h = 44$ mm and width $b = 10$ mm (perpendicular to the plane of the figure).

(a) For the beam with a hole at midheight, determine the maximum stresses for hole diameters $d = 10$, 16, 22 and 28 mm.
(b) For the beam with two identical notches (inside height $h_1 = 40$ mm), determine the maximum stresses for notch radii $R = 2$, 4, 6, and 8 mm.

Solution 5.13-2

$M = 250\ \text{N} \cdot \text{m} \quad h = 44\ \text{mm} \quad b = 10\ \text{mm}$

(a) Beam with a hole

$$\frac{d}{h} \leq \frac{1}{2} \quad \text{Eq. (5-57):}$$

$$\sigma_c = \frac{6Mh}{b(h^3 - d^3)} = \frac{6.6 \times 10^6}{85{,}180 - d^3}\ \text{MPa} \qquad (1)$$

$$\frac{d}{h} \geq \frac{1}{2} \quad \text{Eq. (5-56):}$$

$$\sigma_B = \frac{12Md}{b(h^3 - d^3)} = \frac{300 \times 10^3 d}{85{,}180 - d^3}\ \text{MPa} \qquad (2)$$

d (mm)	$\frac{d}{h}$	σ_c Eq. (1) (MPa)	σ_B Eq. (2) (MPa)	σ_{max} (MPa)
10	0.227	78	—	78
16	0.364	81	—	81
22	0.500	89	89	89
28	0.636	—	133	133

NOTE: The larger the hole, the larger the stress.

(b) Beam with notches

$$h_1 = 40\ \text{mm} \qquad \frac{h}{h_1} = \frac{44\ \text{mm}}{40\ \text{mm}} = 1.1$$

$$\text{Eq. (5-58): } \sigma_{nom} = \frac{6M}{bh_1^2} = 93.8\ \text{MPa}$$

R (mm)	$\frac{R}{h_1}$	K (Fig. 5-50)	$\sigma_{max} = K\sigma_{nom}$ σ_{max} (MPa)
2	0.05	2.6	240
4	0.10	2.1	200
6	0.15	1.8	170
8	0.20	1.7	160

NOTE: The larger the notch radius, the smaller the stress.

Problem 5.13-3 A rectangular beam with semicircular notches, as shown in part (b) of the figure, has dimensions $h = 0.88$ in. and $h_1 = 0.80$ in. The maximum allowable bending stress in the metal beam is $\sigma_{max} = 60$ ksi, and the bending moment is $M = 600$ lb-in.

Determine the minimum permissible width b_{min} of the beam.

Solution 5.13-3 Beam with semicircular notches

$h = 0.88$ in. $\quad h_1 = 0.80$ in.

$\sigma_{max} = 60$ ksi $\quad M = 600$ lb-in.

$h = h_1 + 2R \quad R = \frac{1}{2}(h - h_1) = 0.04$ in.

$$\frac{R}{h_1} = \frac{0.04 \text{ in.}}{0.80 \text{ in.}} = 0.05$$

From Fig. 5-50: $K \approx 2.57$

$$\sigma_{max} = K\sigma_{nom} = K\left(\frac{6M}{bh_1^2}\right)$$

$$60 \text{ ksi} = 2.57\left[\frac{6(600 \text{ lb-in.})}{b(0.80 \text{ in.})^2}\right]$$

Solve for b:

$b_{min} \approx 0.24$ in. $\quad \leftarrow$

Problem 5.13-4 A rectangular beam with semicircular notches, as shown in part (b) of the figure, has dimension $h = 120$ mm and $h_1 = 100$ mm. The maximum allowable bending stress in the plastic beam is $\sigma_{max} = 6$ MPa, and the bending moment is $M = 150$ N·m.

Determine the minimum permissible width b_{min} of the beam.

Solution 5.13-4 Beam with semicircular notches

$h = 120$ mm $\quad h_1 = 100$ mm

$\sigma_{max} = 6$ MPa $\quad M = 150$ N·m

$h = h_1 + 2R \quad R = \frac{1}{2}(h - h_1) = 10$ mm

$$\frac{R}{h_1} = \frac{10 \text{ mm}}{100 \text{ mm}} = 0.10$$

From Fig.5-50: $K \approx 2.20$

$$\sigma_{max} = K\sigma_{nom} = K\left(\frac{6M}{bh_1^2}\right)$$

$$6 \text{ MPa} = 2.20\left[\frac{6(150 \text{ N} \cdot \text{m})}{b(100 \text{ mm})^2}\right]$$

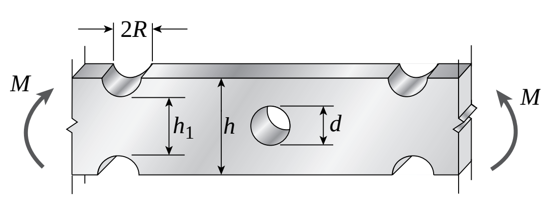
Solve for b:

$b_{min} \approx 33$ mm $\quad \leftarrow$

Problem 5.13-5 A rectangular beam with notches and a hole (see figure) has dimensions $h = 5.5$ in., $h_1 = 5$ in., and width $b = 1.6$ in. The beam is subjected to a bending moment $M = 130$ k-in., and the maximum allowable bending stress in the material (steel) is $\sigma_{max} = 42{,}000$ psi.

(a) What is the smallest radius R_{min} that should be used in the notches?

(b) What is the diameter d_{max} of the largest hole that should be drilled at the midheight of the beam?



Solution 5.13-5 Beam with notches and a hole

$h = 5.5 \text{ in.} \quad h_1 = 5 \text{ in.} \quad b = 1.6 \text{ in.}$

$M = 130 \text{ k-in.} \quad \sigma_{max} = 42{,}000 \text{ psi}$

(a) MINIMUM NOTCH RADIUS

$$\frac{h}{h_1} = \frac{5.5 \text{ in.}}{5 \text{ in.}} = 1.1$$

$$\sigma_{nom} = \frac{6M}{bh_1^2} = 19{,}500 \text{ psi}$$

$$K = \frac{\sigma_{max}}{\sigma_{nom}} = \frac{42{,}000 \text{ psi}}{19{,}500 \text{ psi}} = 2.15$$

From Fig. 5-50, with $K = 2.15$ and $\dfrac{h}{h_1} = 1.1$, we get

$$\frac{R}{h_1} \approx 0.090$$

$$\therefore R_{min} \approx 0.090h_1 = 0.45 \text{ in.} \quad \leftarrow$$

(b) LARGEST HOLE DIAMETER

Assume $\dfrac{d}{h} > \dfrac{1}{2}$ and use Eq. (5-56).

$$\sigma_B = \frac{12Md}{b(h^3 - d^3)}$$

$$42{,}000 \text{ psi} = \frac{12(130 \text{ k-in.})d}{(1.6 \text{ in.})[(5.5 \text{ in.})^3 - d^3]} \quad \text{or}$$

$$d^3 + 23.21d - 166.4 = 0$$

Solve numerically:

$$d_{max} = 4.13 \text{ in.} \quad \leftarrow$$

6

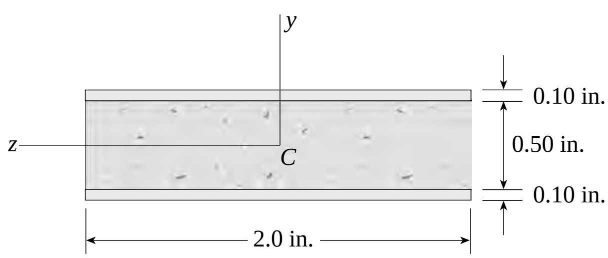
Stresses in Beams (Advanced Topics)

Composite Beams

When solving the problems for Section 6.2, assume that the component parts of the beams are securely bonded by adhesives or connected by fasteners. Also, be sure to use the general theory for composite beams described in Sect. 6.2.

Problem 6.2-1 A composite beam consisting of fiberglass faces and a core of particle board has the cross section shown in the figure. The width of the beam is 2.0 in., the thickness of the faces is 0.10 in., and the thickness of the core is 0.50 in. The beam is subjected to a bending moment of 250 lb-in. acting about the z axis.

Find the maximum bending stresses σ_{face} and σ_{core} in the faces and the core, respectively, if their respective moduli of elasticity are 4×10^6 psi and 1.5×10^6 psi.



Solution 6.2-1 Composite beam

$b = 2$ in. $\quad h = 0.7$ in.

$h_c = 0.5$ in. $\quad M = 250$ lb-in.

$E_1 = 4 \times 10^6$ psi $\quad E_2 = 1.5 \times 10^6$ psi

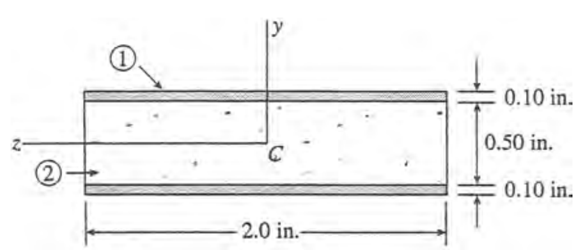
$$I_1 = \frac{b}{12}(h^3 - h_c^3) = 0.03633 \text{ in.}^4$$

$$I_2 = \frac{bh_c^3}{12} = 0.02083 \text{ in.}^4$$

$$E_1I_1 + E_2I_2 = 176{,}600 \text{ lb-in.}^2$$

From Eq. (6-6a): $$\sigma_{face} = \pm\frac{M(h/2)E_1}{E_1I_1 + E_2I_2}$$

$$= \pm 1980 \text{ psi} \quad \longleftarrow$$

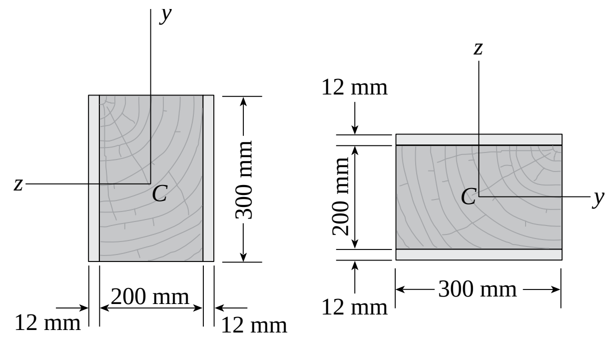


From Eq. (6-6b): $$\sigma_{core} = \pm\frac{M(h_c/2)E_2}{E_1I_1 + E_2I_2}$$

$$= \pm 531 \text{psi} \quad \longleftarrow$$

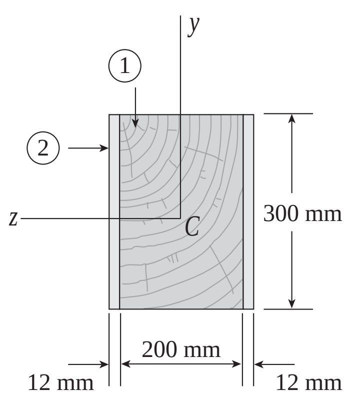
Problem 6.2-2 A wood beam with cross-sectional dimensions 200 mm $\times$ 300 mm is reinforced on its sides by steel plates 12 mm thick (see figure). The moduli of elasticity for the steel and wood are $E_s = 190$ GPa and $E_w = 11$ GPa, respectively. Also, the corresponding allowable stresses are $\sigma_s = 110$ MPa and $\sigma_w = 7.5$ MPa.

(a) Calculate the maximum permissible bending moment M_{max} when the beam is bent about the z axis.
(b) Repeat part a if the beam is now bent about its y axis.



(a) (b)

Solution 6.2-2



(a) Bent about the z axis

$$b = 200\,\text{mm} \qquad t = 12\,\text{mm} \qquad h = 300\text{ mm}$$

$$E_w = 11\ GPa \qquad E_s = 190\ GPa$$

$$\sigma_{allow\_w} = 7.5\text{ MPa} \qquad \sigma_{allow\_s} = 110\text{ MPa}$$

$$I_w = \frac{bh^3}{12} \qquad I_w = 4.50 \times 10^8\text{ mm}^4$$

$$I_s = \frac{2th^3}{12} \qquad I_s = 5.40 \times 10^7\text{ mm}^4$$

$$E_wI_w + E_sI_s = 1.52 \times 10^7\text{ N}\cdot\text{m}^2$$

Maximum moment based upon the wood

$$M_{max\_w} = \sigma_{allow\_w}\left[\frac{E_wI_w + E_sI_s}{\left(\frac{h}{2}\right)E_w}\right]$$

$$M_{max\_w} = 69.1\text{ kN}\cdot\text{m}$$

Maximum moment based upon the steel

$$M_{max\_s} = \sigma_{allow\_s}\left[\frac{E_wI_w + E_sI_s}{\left(\frac{h}{2}\right)E_s}\right]$$

$$M_{max\_s} = 58.7\text{ kN}\cdot\text{m}$$

$$M_{max} = \min\left(M_{max\_w}, M_{max\_s}\right)$$

Steel Governs. $\quad M_{max} = 58.7\text{ kN}\cdot\text{m} \quad \leftarrow$

(b) Bent about the y axis

$$I_w = \frac{b^3h}{12} \qquad I_w = 2.00 \times 10^8\text{ mm}^4$$

$$I_s = 2\left[\frac{t^3h}{12} + th\left(\frac{b+t}{2}\right)^2\right]$$

$$I_s = 8.10 \times 10^7\text{ mm}^4$$

$$E_wI_w + E_sI_s = 1.76 \times 10^7\text{ N}\cdot\text{m}^2$$

Maximum moment based upon the wood

$$M_{\text{max\_w}} = \sigma_{\text{allow\_w}}\left[\frac{E_w I_w + E_s I_s}{\left(\frac{b}{2}\right)E_w}\right]$$

$$M_{\text{max\_w}} = 119.9 \text{ kN}\cdot\text{m}$$

Maximum moment based upon the steel

$$M_{\text{max\_s}} = \sigma_{\text{allow\_s}}\left[\frac{E_w I_w + E_s I_s}{\left(\frac{b}{2}+t\right)E_s}\right]$$

$$M_{\text{max\_s}} = 90.9 \text{ kN}\cdot\text{m} \quad \leftarrow$$

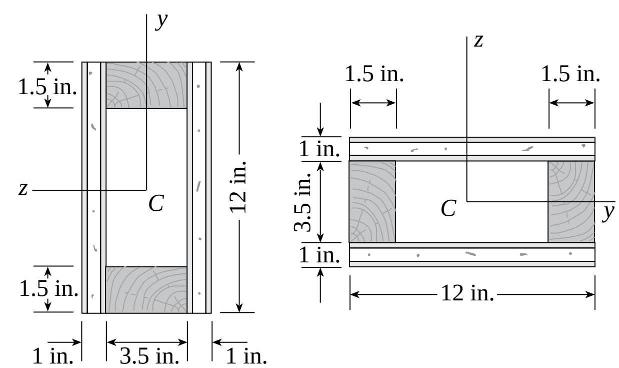
$$M_{\text{max}} = \min\left(M_{\text{max\_w}}, M_{\text{max\_s}}\right)$$

Steel Governs. $M_{\text{max}} = 90.9 \text{ kN}\cdot\text{m} \quad \leftarrow$

Problem 6.2-3 A hollow box beam is constructed with webs of Douglas-fir plywood and flanges of pine, as shown in the figure in a cross-sectional view. The plywood is 1 in. thick and 12 in. wide; the flanges are 2 in. × 4 in. (nominal size). The modulus of elasticity for the plywood is 1,800,000 psi and for the pine is 1,400,000 psi.

(a) If the allowable stresses are 2000 psi for the plywood and 1750 psi for the pine, find the allowable bending moment M_{max} when the beam is bent about the z axis.

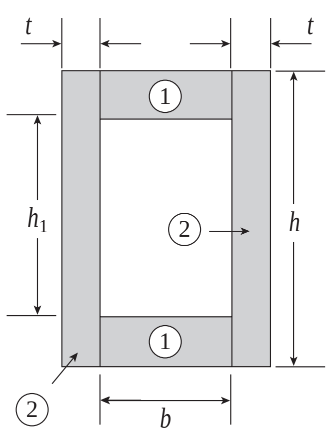
(b) Repeat part a if the b.eam is now bent about its y axis.



(a) (b)

Solution 6.2-3

(a) Bent about the z axis



$b = 3.5$ in. $\quad t = 1$ in. $\quad h = 12$ in. $\quad h_1 = 9$ in.

$E_1 = 1.4 \times 10^6$ psi $\quad E_2 = 1.8 \times 10^6$ psi

$\sigma_{\text{allow\_1}} = 1750$ psi $\quad \sigma_{\text{allow\_2}} = 2000$ psi

$$I_1 = \frac{b(h^3 - h_1^3)}{12} \qquad I_1 = 291 \text{ in.}^4$$

$$I_2 = \frac{2th^3}{12} \qquad I_2 = 288 \text{ in.}^4$$

$$E_1 I_1 + E_2 I_2 = 9.26 \times 10^8 \text{ lb}\cdot\text{in.}^2$$

Maximum moment based upon the wood

$$M_{\text{max\_1}} = \sigma_{\text{allow\_1}}\left[\frac{E_1 I_1 + E_2 I_2}{\left(\frac{h}{2}\right)E_1}\right]$$

$$M_{\text{max\_1}} = 193 \text{ k}\cdot\text{in.} \quad \leftarrow$$

Maximum moment based upon the plywood

$$M_{\text{max\_2}} = \sigma_{\text{allow\_2}}\left[\frac{E_1 I_1 + E_2 I_2}{\left(\frac{h}{2}\right)E_2}\right]$$

$$M_{\text{max\_2}} = 172 \text{ k}\cdot\text{in.} \quad \leftarrow$$

$$M_{max} = \min(M_{max\_1}, M_{max\_2})$$

PLYWOOD GOVERNS. $\quad M_{max} = 172 \text{ k} \cdot \text{in.} \quad \leftarrow$

(b) BENT ABOUT THE Y AXIS

$$I_1 = \frac{b^3(h - h_1)}{12} \qquad I_1 = 11 \text{ in.}^4$$

$$I_2 = 2\left[\frac{t^3 h}{12} + th\left(\frac{b + t}{2}\right)^2\right] \qquad I_2 = 123 \text{ in.}^4$$

$$E_1 I_1 + E_2 I_2 = 2.37 \times 10^8 \text{ lb} \cdot \text{in.}^2$$

MAXIMUM MOMENT BASED UPON THE WOOD

$$M_{max\_1} = \sigma_{allow\_1}\left[\frac{E_1 I_1 + E_2 I_2}{\left(\frac{b}{2}\right) E_1}\right]$$

$$M_{max\_1} = 170 \text{ k} \cdot \text{in.}$$

MAXIMUM MOMENT BASED UPON THE PLYWOOD

$$M_{max\_2} = \sigma_{allow\_2}\left[\frac{E_1 I_1 + E_2 I_2}{\left(\frac{b}{2} + t\right) E_2}\right]$$

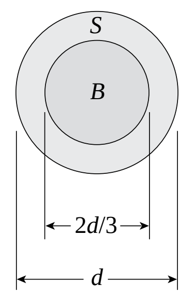
$$M_{max\_2} = 96 \text{ k} \cdot \text{in.}$$

$$M_{max} = \min(M_{max\_1}, M_{max\_1})$$

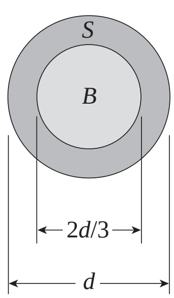
PLYWOOD GOVERNS. $\quad M_{max} = 96 \text{ k} \cdot \text{in.} \quad \leftarrow$

Problem 6.2-4 A round steel tube of outside diameter d and an brass core of diameter $2d/3$ are bonded to form a composite beam, as shown in the figure.

Derive a formula for the allowable bending moment M that can be carried by the beam based upon an allowable stress σ_s in the steel. (Assume that the moduli of elasticity for the steel and brass are E_s and E_b, respectively.)



Solution 6.2-4



Tube (1): $\quad I_1 = \frac{\pi}{64}\left[d^4 - \left(\frac{2d}{3}\right)^4\right] = \frac{65}{5184}\pi d^4$

Core (2): $\quad I_2 = \frac{\pi}{64}\left(\frac{2d}{3}\right)^4 = \frac{\pi d^4}{324}$

$$E_1 I_1 + E_2 I_2 = E_s I_1 + E_b I_2$$

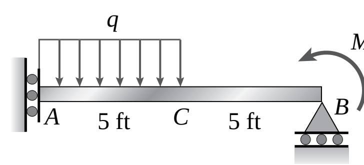
$$= \frac{\pi d^4}{5184}(65 E_s + 16 E_b)$$

$$M_{allow} = \sigma_s\left[\frac{E_1 I_1 + E_2 I_2}{\left(\frac{d}{2}\right) E_s}\right]$$

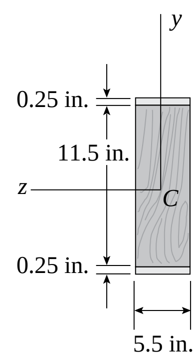
$$M_{allow} = \frac{\sigma_s \pi d^3}{2592}\left(65 + 16\frac{E_b}{E_s}\right) \quad \leftarrow$$

Problem 6.2-5 A beam with a guided support and 10 ft span supports a distributed load of intensity q = 660 lb/ft over its first half (see figure part a) and a moment M_0 = 300 ft-lb at joint B. The beam consists of a wood member (nominal dimensions 6 in. $\times$ 12 in., actual dimensions 5.5 in. $\times$ 11.5 in. in cross section, as shown in the figure part b) that is reinforced by 0.25-in.-thick steel plates on top and bottom. The moduli of elasticity for the steel and wood are $E_s = 30 \times 10^6$ psi and $E_w = 1.5 \times 10^6$ psi, respectively.

(a) Calculate the maximum bending stresses σ_s in the steel plates and σ_w in the wood member due to the applied loads.
(b) If the allowable bending stress in the steel plates is σ_{as} = 14,000 psi and that in the wood is σ_{aw} = 900 psi, find q_{max}. (Assume that the moment at B, M_0, remains at 300 ft-lb.)
(c) If q = 660 lb/ft and allowable stress values in (b) apply, what is $M_{0,max}$ at B?



(a)



(b)

Solution 6.2-5

$q = 660$ lb/it $\quad M_0 = 300$ lb · ft $\quad L = 10$ ft

(a) Maximum bending stresses

$$M_{max} = q\left(\frac{L}{2}\right)\left(\frac{3L}{4}\right) + M_0$$

$$M_{max} = 25050 \text{ lb} \cdot \text{ft}$$

Wood (1): $b = 5.5$ in. $\quad h_1 = 11.5$ in.

$E_w = 1.5 \times 10^6$ psi

$$I_1 = \frac{bh_1^3}{12} \qquad I_1 = 697.07 \text{ in.}^4$$

Plate (2): $b = 5.5$ in. $\quad t = 0.25$ in.
$h = 12$ in. $\quad E_s = 30 \times 10^6$ psi

$$I_2 = \frac{b}{12}(h^3 - h_1^3) \qquad I_2 = 94.93 \text{ in.}^4$$

$$E_wI_1 + E_sI_2 = 3.894 \times 10^9 \text{ lb} \cdot \text{in.}^2$$

$$\sigma_w = \frac{M_{max}\left(\frac{h_1}{2}\right)E_w}{E_wI_1 + E_sI_2} \qquad \sigma_w = 666 \text{ psi} \quad \leftarrow$$

$$\sigma_s = \frac{M_{max}\left(\frac{h}{2}\right)E_s}{E_wI_1 + E_sI_2} \qquad \sigma_s = 13897 \text{ psi} \quad \leftarrow$$

(b) Maximum uniform distributed load

Maximum moment based upon wood

$$\sigma_{allow\_w} = 900 \text{ psi}$$

From $\quad \sigma_{allow\_w} = \dfrac{M_{allow\_w}\left(\frac{h_1}{2}\right)E_w}{E_wI_1 + E_sI_2}$

$$M_{allow\_w} = 33857 \text{ lb-ft}$$

Maximum moment based upon steel plate

$$\sigma_{allow\_s} = 14000 \text{ psi}$$

From $\quad \sigma_{allow\_s} = \dfrac{M_{allow\_s}\left(\frac{h}{2}\right)E_s}{E_wI_1 + E_sI_2}$

$$M_{allow\_s} = 25236 \text{ lb-ft}$$

Maximum allowable moment

$$M_{allow} = \min(M_{allow\_s}, M_{allow\_w})$$

Steel plates govern

$$M_{allow} = 25236 \text{ lb-ft} \quad \leftarrow$$

MAXIMUM UNIFORM DISTRIBUTED LOAD

From $M_{allow} = q_{max}\left(\frac{L}{2}\right)\left(\frac{3L}{4}\right) + M_0$

$q_{max} = 665\text{lb/ft} \quad \leftarrow$

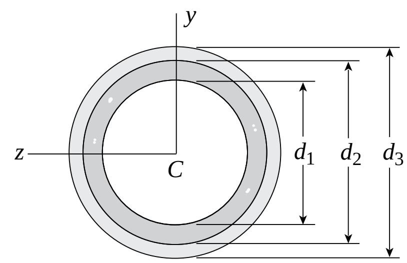
(c) MAXIMUM APPLIED MOMENT

From $M_{allow} = q\left(\frac{L}{2}\right)\left(\frac{3L}{4}\right) + M_{o\_}\text{max}$

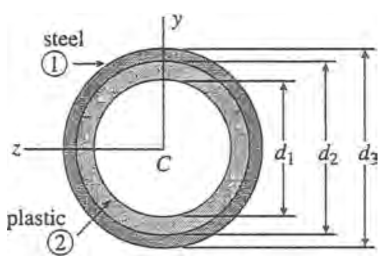
$M_{0\_max} = 486 \text{ lb-ft} \quad \leftarrow$

Problem 6.2-6 A plastic-lined steel pipe has the cross-sectional shape shown in the figure. The steel pipe has outer diameter $d_3 = 100$ mm and inner diameter $d_2 = 94$ mm. The plastic liner has inner diameter $d_1 = 82$ mm. The modulus of elasticity of the steel is 75 times the modulus of the plastic.

Determine the allowable bending moment M_{allow} if the allowable stress in the steel is 35 MPa and in the plastic is 600 kPa.



Solution 6.2-6 Steel pipe with plastic liner



(1) Pipe: $d_s = 100$ mm $\quad d_2 = 94$ mm

$E_s = E_1$ = modulus of elasticity

$(\sigma_1)_{allow} = 35$ MPa

(2) Liner: $d_2 = 94$ mm $\quad d_1 = 32$ mm

$E_p = E_2$ = modulus of elasticity

$(\sigma_2)_{allow} = 600$ kPa

$E_1 = 75E_2 \quad E_1/E_2 = 75$

$$I_1 = \frac{\pi}{64}(d_3^4 - d_2^4) = 1.076 \times 10^{-6}\text{m}^4$$

$$I_2 = \frac{\pi}{64}(d_2^4 - d_1^4) = 1.613 \times 10^{-6}\text{m}^4$$

MAXIMUM MOMENT BASED UPON THE STEEL (1)

From Eq. (6-6a):

$$M_{max} = (\sigma_1)_{allow}\left[\frac{E_1I_1 + E_2I_2}{(d_3/2)E_1}\right]$$

$$= (\sigma_1)_{allow}\frac{(E_1/E_2)I_1 + I_2}{(d_3/2)(E_1/E_2)} = 768\,\text{N}\cdot\text{m}$$

MAXIMUM MOMENT BASED UPON THE PLASTIC (2)

From Eq. (6-6b):

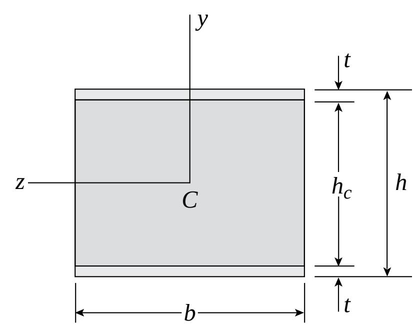
$$M_{max} = (\sigma_2)_{allow}\left[\frac{E_1I_1 + E_2I_2}{(d_2/2)E_2}\right]$$

$$= (\sigma_2)_{allow}\left[\frac{(E_1/E_2)I_1 + I_2}{(d_2/2)}\right] = 1051\,\text{N}\cdot\text{m}$$

STEEL GOVERNS. $\quad M_{allow} = 768 \text{ N}\cdot\text{m} \quad \leftarrow$

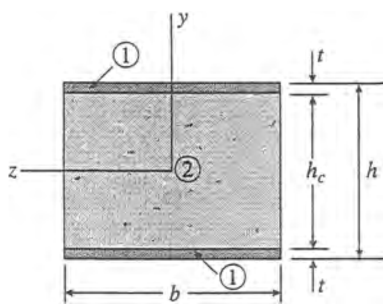
Problem 6.2-7 The cross section of a sandwich beam consisting of aluminum alloy faces and a foam core is shown in the figure. The width b of the beam is 8.0 in., the thickness t of the faces is 0.25 in., and the height h_c of the core is 5.5 in. (total height $h = 6.0$ in.). The moduli of elasticity are 10.5×10^6 psi for the aluminum faces and 12,000 psi for the foam core. A bending moment $M = 40$ k-in. acts about the z axis.

Determine the maximum stresses in the faces and the core using (a) the general theory for composite beams, and (b) the approximate theory for sand wich beams.



Probs. 6.2-7 and 6.2-8

Solution 6.2-7 Sandwich beam



(1) ALUMINUM FACES:

$b = 8.0$ in. $\quad t = 0.25$ in. $\quad h = 6.0$ in.

$E_1 = 10.5 \times 10^6$ psi

$$I_1 = \frac{b}{12}(h^3 - h_c^3) = 33.08 \text{ in.}^4$$

(2) Foam core:

$b = 8.0$ in. $\quad h_c = 5.5$ in. $\quad E_2 = 12{,}000$ psi

$$I_2 = \frac{bh_c^3}{12} = 110.92 \text{ in.}^4$$

$M = 40$ k.in. $\quad E_1I_1 + E_2I_2 = 348.7 \times 10^6$ lb-in.^{2}

(a) GENERAL THEORY (EQS. 6-6a AND b)

$$\sigma_{\text{face}} = \sigma_1 = \frac{M(h/2)E_1}{E_1I_1 + E_2I_2} = 3610 \text{ psi} \quad \leftarrow$$

$$\sigma_{\text{core}} = \sigma_2 = \frac{M(h_c/2)E_2}{E_1I_1 + E_2I_2} = 4 \text{ psi} \quad \leftarrow$$

(b) APPROXIMATE THEORY (EQS. 6-8 AND 6-9)

$$I_1 = \frac{b}{12}(h^3 - h_c^3) = 33.08 \text{ in.}^4$$

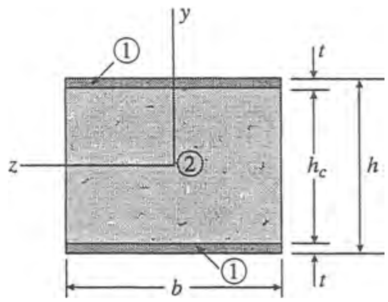
$$\sigma_{\text{face}} = \frac{Mh}{2I_1} = 3630 \text{ psi} \quad \leftarrow$$

$$\sigma_{\text{core}} = 0 \quad \leftarrow$$

Problem 6.2-8 The cross section of a sandwich beam consisting of fiberglass faces and a lightweight plastic core is shown in the figure. The width b of the beam is 50 mm, the thickness t of the faces is 4 mm, and the height h_c of the core is 92 mm (total height $h = 100$ mm). The moduli of elasticity are 75 GPa for the fiberglass and 1.2 GPa for the plastic. A bending moment $M = 275$ N·m acts about the z axis.

Determine the maximum stresses in the faces and the core using (a) the general theory for composite beams, and (b) the approximate theory for sandwich beams.

Solution 6.2-8 Sandwich beam



(1) Fiber glass faces:

$b = 50$ mm $\quad t = 4$ mm $\quad h = 100$ mm

$E_1 = 75$ GPa

$$I_1 = \frac{b}{12}(h^3 - h_c^3) = 0.9221 \times 10^{-6}\,\text{m}^4$$

(2) Plastic core:

$b = 50$ mm $\quad h_c = 92$ mm $\quad E_2 = 1.2$ GPa

$$I_2 = \frac{bh_c^3}{12} = 3.245 \times 10^{-6}\,\text{m}^4$$

$M = 275\ \text{N}\cdot\text{m} \qquad E_1I_1 + E_2I_2 = 73{,}050\ \text{N}\cdot\text{m}^2$

(a) General theory (Eqs. 6-6a and b)

$$\sigma_{\text{face}} = \sigma_1 = \frac{M(h/2)E_1}{E_1I_1 + E_2I_2} = 14.1\ \text{MPa} \quad \leftarrow$$

$$\sigma_{\text{core}} = \sigma_2 = \frac{M(h_c/2)E_2}{E_1I_1 + E_2I_2} = 0.21\ \text{MPa} \quad \leftarrow$$

(b) Approximate theory (Eqs. 6-8 and 6-9)

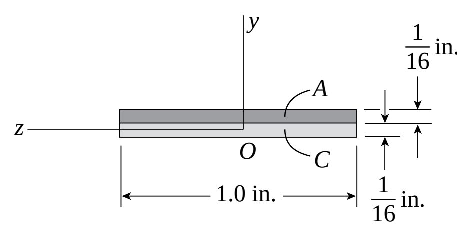
$$I_1 = \frac{b}{12}(h^3 - h_c^3) = 0.9221 \times 10^6\,\text{m}^4$$

$$\sigma_{\text{face}} = \frac{Mh}{2I_1} = 14.9\ \text{MPa} \quad \leftarrow$$

$$\sigma_{\text{core}} = 0 \quad \leftarrow$$

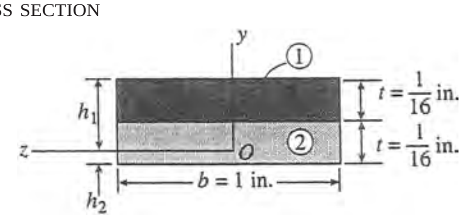
Problem 6.2-9 A bimetallic beam used in a temperature-control switch consists of strips of aluminum and copper bonded together as shown in the figure, which is a cross-sectional view. The width of the beam is 1.0 in., and each strip has a thickness of 1/16 in.

Under the action of a bending moment $M = 12$ lb-in. acting about the z axis, what are the maximum stresses σ_a and σ_c in the aluminum and copper, respectively? (Assume $E_a = 10.5 \times 10^6$ psi and $E_c = 16.8 \times 10^6$ psi.)



Solution 6.2-9 Bimetallic beam

Cross section



(1) Aluminum $E_1 = E_a = 10.5 \times 10^6$ psi

(2) Copper $\quad E_2 = E_c = 16.8 \times 10^6$ psi

$M = 12$ lb-in.

Neutral axis (Eq. 6-3)

$$\int_1 ydA = \bar{y}_1A_1 = (h_1 - t/2)(bt)$$

$$= (h_1 - 1/32)(1)(1/16)\ \text{in.}^3$$

$$\int_2 ydA = \bar{y}_2A_2 = (h_1 - t - t/2)(bt)$$

$$= (h_1 - 3/32)(1)(1/16)\ \text{in.}^3$$

Eq. (6-3): $E_1\int_1 ydA + E_2\int_2 ydA = 0$

$$(10.5 \times 10^6)(h_1 - 1/32)(1/16)$$
$$+ (16.8 \times 10^6)(h_1 - 3/32)(1/16) = 0$$

Solve for h_1: $h_1 = 0.06971$ in.

$$h_2 = 2(1/16 \text{ in.}) - h_1 = 0.05529 \text{ in.}$$

MOMENTS OF INERTIA (FROM PARALLEL-AXIS THEOREM)

$$I_1 = \frac{bt^3}{12} + bt(h_1 - t/2)^2 = 0.0001128 \text{ in.}^4$$

$$I_2 = \frac{bt^3}{12} + bt(h_2 - t/2)^2 = 0.00005647 \text{ in.}^4$$

$$E_1I_1 + E_2I_2 = 2133 \text{ lb-in.}^2$$

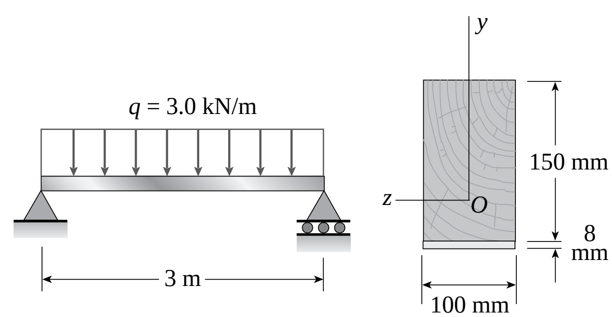
MAXIMUM STRESSES (EQS. 6-6a AND b)

$$\sigma_a = \sigma_1 = \frac{Mh_1E_1}{E_1I_1 + E_2I_2} = 4120 \text{ psi} \quad \leftarrow$$

$$\sigma_c = \sigma_2 = \frac{Mh_2E_2}{E_1I_1 + E_2I_2} = 5230 \text{ psi} \quad \leftarrow$$

Problem 6.2-10 A simply supported composite beam 3 m long carries a uniformly distributed load of intensity $q = 3.0$ kN/m (see figure). The beam is constructed of a wood member, 100 mm wide by 150 mm deep, reinforced on its lower side by a steel plate 8 mm thick and 100 mm wide.

Find the maximum bending stresses σ_w and σ_s in the wood and steel, respectively, due to the uniform load if the moduli of elasticity are $E_w = 10$ GPa for the wood and $E_s = 210$ GPa for the steel.

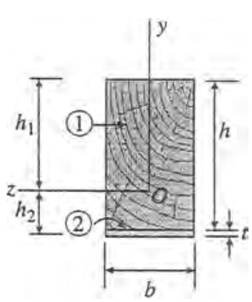


Solution 6.2-10 Simply supported composite beam

BEAM: $L = 3$ m $\quad q = 3.0$ kN/m

$$M_{max} = \frac{qL^2}{8} = 3375 \text{ N} \cdot \text{m}$$

CROSS SECTION



$b = 100$ mm $\quad h = 150$ mm $\quad t = 8$ mm

(1) Wood: $E_1 = E_w = 10$ GPa

(2) Steel: $E_2 = E_s = 210$ GPa

NEUTRAL AXIS

$$\int_1 ydA = \bar{y}_1A_1 = (h_1 - h/2)(bh)$$
$$= (h_1 - 75)(100)(150) \text{ mm}^3$$

$$\int_2 ydA = \bar{y}_2A_2 = -(h + t/2 - h_1)(bt)$$
$$= -(154 - h_1)(100)(18) \text{ mm}^3$$

Eq. (6-3): $E_1\int_1 ydA + E_2\int_2 ydA = 0$

$$(10\text{ GPa})(h_1 - 75)(100)(150)(10^{-9})$$
$$+ (210\text{ GPa})(h_1 - 154)(100)(8)(10^{-9}) = 0$$

Solve for h_1: $h_1 = 116.74$ mm

$h_2 = h + t - h_1 = 41.26$ mm

MOMENTS OF INERTIA (FROM PARALLEL-AXIS THEOREM)

$$I_1 = \frac{bh^3}{12} + bh(h_1 - h/2)^2 = 54.26 \times 10^6\text{ mm}^4$$

$$I_2 = \frac{bt^2}{12} + bt(h_2 - t/2)^2 = 1.115 \times 10^6\text{ mm}^4$$

$$E_1I_1 + E_2I_2 = 776{,}750\text{ N}\cdot\text{m}^2$$

MAXIMUM STRESSES (EQS. 6-6a AND b)

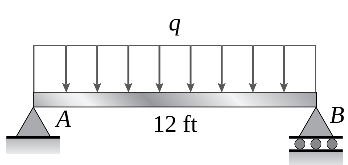
$$\sigma_w = \sigma_1 = \frac{Mh_1E_1}{E_1I_1 + E_2I_2}$$
$$= 5.1\text{MPa (Compression)} \quad \leftarrow$$

$$\sigma_s = \sigma_2 = \frac{Mh_2E_2}{E_1I_1 + E_2I_2}$$
$$= 37.6\text{MPa (Tension)} \quad \leftarrow$$

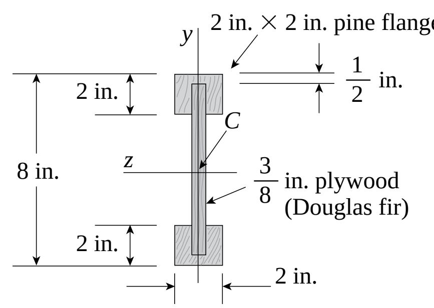
Problem 6.2-11 A simply supported wooden I-beam with a 12 ft span supports a distributed load of intensity $q = 90$ lb/ft over its length (see figure part a). The beam is constructed with a web of Douglas-fir plywood and flanges of pine glued to the web as shown in the figure part b. The plywood is 3/8 in. thick; the flanges are 2 in. × 2 in. (actual size). The modulus of elasticity for the plywood is 1,600,000 psi and for the pine is 1,200,000 psi.

(a) Calculate the maximum bending stresses in the pine flanges and in the plywood web.

(b) What is q_{max} if allowable stresses are 1600 psi in the flanges and 1200 psi in the web?



(a)



(b)

Solution 6.2-11

$q = 90$ lb/f $\qquad L = 12$ft

(a) MAXIMUM BENDING STRESSES

$$M_{max} = \frac{qL^2}{8} \qquad M_{max} = 1620\text{ lb}\cdot\text{ft}$$

Plywood (1): $\quad t = \frac{3}{8}$ in. $\quad h_1 = 7$ in.

$E_{plywood} = 1.6 \times 10^6$ psi

$$I_1 = \frac{th_1^3}{12} \qquad I_1 = 10.72\text{ in.}^4$$

Pine (2): $b = 2$ in. $\quad h_2 = 2$ in. $\quad a = \frac{1}{2}$ in.

$E_{pine} = 1.2 \times 10^6$psi

$$I_2 = 2\left[\frac{ba^3}{12} + ba\left(\frac{h_1 + a}{2}\right)^2 + \frac{(b - t)(h_2 - a)^3}{12} + (b - t)(h_2 - a)\left(\frac{h_1}{2} - \frac{h_2 - a}{2}\right)^2\right]$$

$I_2 = 65.95$ in.$^4 \quad \leftarrow$

$E_{plywood}I_1 + E_{pine}I_2 = 96.287 \times 10^6$ lbin.^{2}

$$\sigma_{plywood} = \frac{M_{max}\left(\frac{h_1}{2}\right)E_{plywood}}{E_{plywood}I_1 + E_{pine}I_2}$$

$\sigma_{plywood} = 1131$ psi $\quad \leftarrow$

$$\sigma_{pine} = \frac{M_{max}\left(\dfrac{h_1}{2} + a\right)E_{pine}}{E_{plywood}I_1 + E_{pine}I_2}$$

$$\sigma_{pine} = 969 \text{ psi} \quad \leftarrow$$

(b) MAXIMUM UNIFORM DISTRIBUTED LOAD

MAXIMUM MOMENT BASED UPON PLYWOOD

$\sigma_{allow\_plywood} = 1200$ psi

From $\quad \sigma_{allow\_plywood} = \dfrac{M_{allow\_plywood}\left(\dfrac{h_1}{2}\right)E_{plywood}}{E_{plywood}I_1 + E_{pine}I_2}$

$$M_{allow\_plywood} = 1719 \text{ lb} \cdot \text{ft}$$

MAXIMUM MOMENT BASED UPON PINE

$\sigma_{allow\_pine} = 1600$ psi

From $\quad \sigma_{allow\_pine} = \dfrac{M_{allow\_pine}\left(\dfrac{h_1}{2} + a\right)E_{pine}}{E_{plywood}I_1 + E_{pine}I_2}$

$M_{allow\_pine} = 2675 \text{ lb} \cdot \text{ft}$

MAXIMUM ALLOWABLE MOMENT

$M_{allow} = \min(M_{allow\_plywood}, M_{allow\_pine})$

PLYWOOD GOVERNS. $\quad M_{allow} = 1719 \text{ lb}\cdot\text{ft} \quad \leftarrow$

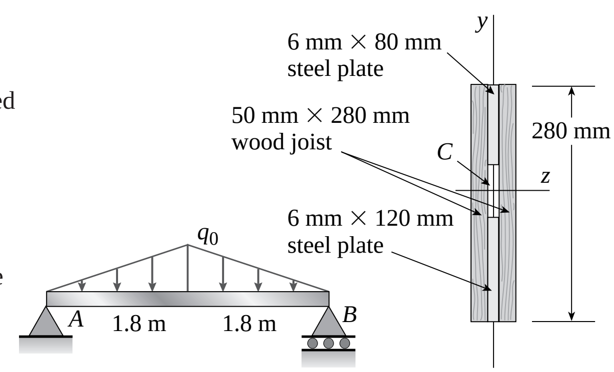
MAXIMUM UNIFORM DISTRIBUTED LOAD

From $\quad M_{allow} = \dfrac{q_{max}L^2}{8}$

$$q_{max} = 95.5 \text{ lb/ft} \quad \leftarrow$$

Problem 6.2-12 A simply supported composite beam with a 3.6 m span supports a triangularly distributed load of peak intensity q_0 at midspan (see figure part a). The beam is constructed of two wood joists, each 50 mm $\times$ 280 mm, fastened to two steel plates, one of dimensions 6 mm $\times$ 80 mm and the lower plate of dimensions 6 mm $\times$ 120 mm (see figure part b). The modulus of elasticity for the wood is 11 GPa and for the steel is 210 GPa.

If the allowable stresses are 7 MPa for the wood and 120 MPa for the steel, find the allowable peak load intensity $q_{0,max}$ when the beam is bent about the z axis. Neglect the weight of the beam.



(a) (b)

Solution 6.2-12

$L = 3.6$ m

DETERMINE NEUTRAL AXIS

WOOD (1): $\quad t_1 = 50$ mm $\quad h = 280$ mm

$E_w = 11$ GPa

$$\int y_1\, dA = \bar{y}_1 A_1 = \left(\frac{h}{2} - h_1\right)(2t_1 h)$$

Steel (2): $\quad t_2 = 6$ mm $\quad b_1 = 80$ mm

$b_2 = 120$ mm $\quad E_s = 210$ GPa

$$\int y_2\, dA = \bar{y}_2 A_2 = \left(h - h_1 - \frac{b_1}{2}\right)(t_2 b_1) - \left(h_1 - \frac{b_2}{2}\right)(t_2 b_2)$$

From $\quad E_1 \int y_1\, dA + E_2 \int y_2\, dA = 0$

$$E_w\left(\frac{h}{2} - h_1\right)(2t_1 h) + E_s\left[\left(h - h_1 - \frac{b_1}{2}\right)(t_2 b_1) - \left(h_1 - \frac{b_2}{2}\right)(t_2 b_2)\right] = 0$$

$h_1 = 136.4\,\text{mm}$

MOMENT OF INERTIA

Wood (1): $\quad I_1 = 2\left[\dfrac{t_1 h^3}{12} + (t_1 h)\left(\dfrac{h}{2} - h_1\right)^2\right]$

$I_1 = 183.30 \times 10^6\ \text{mm}^4$

Steel (2): $\quad I_2 = \dfrac{t_2 b_1^3}{12} + t_2 b_1\left(h - h_1 - \dfrac{b_1}{2}\right)^2 + \dfrac{t_2 b_2^3}{12} + t_2 b_2\left(h_1 - \dfrac{b_2}{2}\right)^2$

$I_2 = 10.47 \times 10^6\ \text{mm}^4$

$E_w I_1 + E_s I_2 = 4.22 \times 10^{12}\ \text{N} \cdot \text{mm}^2$

MAXIMUM MOMENT BASED UPON WOOD

$\sigma_{\text{allow\_w}} = 7\ \text{MPa}$

From $\quad \sigma_{\text{allow\_w}} = \dfrac{M_{\text{allow\_w}}(h - h_1)E_w}{E_w I_1 + E_s I_2}$

$M_{\text{allow\_w}} = 18.68\,\text{kN} \cdot \text{m}$

MAXIMUM MOMENT BASED UPON STEEL

$\sigma_{\text{allow\_s}} = 120\ \text{MPa}$

From $\quad \sigma_{\text{allow\_s}} = \dfrac{M_{\text{allow\_s}}(h - h_1)E_s}{E_w I_1 + E_s I_2}$

$M_{\text{allow\_s}} = 16.78\,\text{kN} \cdot \text{m}$

MAXIMUM ALLOWABLE MOMENT

$M_{\text{allow}} = \min(M_{\text{allow\_w}}, M_{\text{allow\_s}})$

STEEL GOVERNS. $\quad M_{\text{allow}} = 16.78\,\text{kN} \cdot \text{m} \quad \leftarrow$

MAXIMUM UNIFORM DISTRIBUTED LOAD

From $\quad M_{\text{allow}} = \dfrac{q_{\text{omax}} L^2}{12}$

$q_{\text{omax}} = 15.53\ \text{kN/m} \quad \leftarrow$

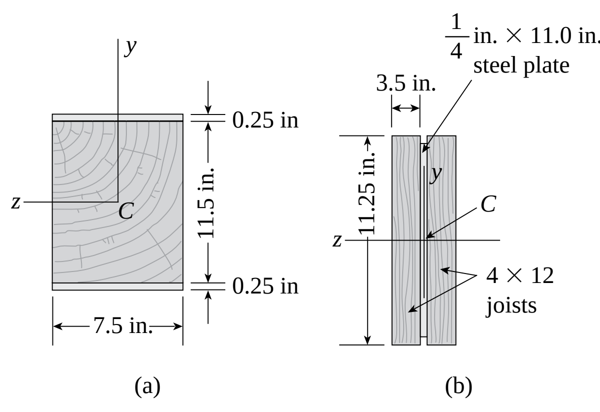
Transformed-Section Method

When solving the problems for Section 6.3, assume that the component parts of the beams are securely bonded by adhesives or connected by fasteners. Also, be sure to use the transformed-section method in the solutions.

Problem 6.3-1 A wood beam 8 in. wide and 12 in. deep (nominal dimensions) is reinforced on top and bottom by 0.25-in.-thick steel plates (see figure part a).

(a) Find the allowable bending moment M_{max} about the z axis if the allowable stress in the wood is 1,100 psi and in the steel is 15,000 psi. (Assume that the ratio of the moduli of elasticity of steel and wood is 20.)

(b) Compare the moment capacity of the beam in part a with that shown in the figure part b which has two 4 in. × 12 in. joists (nominal dimensions) attached to a 1/4 in. × 11.0 in. steel plate.



Solution 6.3-1

(a) FIND M_{max}

(1) Wood beam $b = 7.5$ in. $h_1 = 11.5$ in.

$\sigma_{allow\_w} = 1100$ psi

(2) Steel plates $b = 7.5$ in. $h_2 = 12$ in.

$t = 0.25$ in.

$\sigma_{allow\_s} = 15000$ psi

TRANSFORMED SECTION (WOOD)

$n = 20$

WIDTH OF STEEL PLATES

$b_T = nb \qquad b_T = 150\,\text{in.}$

$$I_T = \frac{bh_1^3}{12} + 2\left[\frac{t^3 b_T}{12} + t b_T\left(\frac{h_2 - t}{2}\right)^2\right]$$

$I_T = 3540\ \text{in.}^4$

MAXIMUM MOMENT BASED UPON THE WOOD (1)

$$M_1 = \frac{\sigma_{allow\_w} I_T}{\frac{h_1}{2}} \qquad M_1 = 677\,\text{k}\cdot\text{in.}$$

MAXIMUM MOMENT BASED UPON THE STEEL (2)

$$M_2 = \frac{\sigma_{allow\_s} I_T}{\frac{h_2 n}{2}} \qquad M_2 = 442\ \text{k}\cdot\text{in.}$$

$M_{max} = \min(M_1, M_2)$

STELL GOVERNS $\qquad M_{max} = 422\,\text{k-in.}$ $\leftarrow$

(b) COMPARE MOMENT CAPACITIES

(1) Wood beam $b = 3.5$ in. $h_1 = 11.25$ in.

(2) Steel plates $h_2 = 11$ in. $t = 0.25$ in.

WIDTH OF STEEL PLATES

$b_T = nt \quad b_T = 5$ in.

$$I_T = 2\frac{bh_1^3}{12} + \frac{b_T h_2^3}{12} \qquad I_T = 1385\ \text{in.}^4$$

MAXIMUM MOMENT BASED UPON THE WOOD (1)

$$M_1 = \frac{\sigma_{allow\_w} I_T}{\frac{h_1}{2}} \qquad M_1 = 271\ \text{k}\cdot\text{in.}$$

MAXIMUM MOMENT BASED UPON THE STEEL (2)

$$M_2 = \frac{\sigma_{allow\_s} I_T}{\frac{h_2 n}{2}} \qquad M_2 = 189\ \text{k}\cdot\text{in.}$$

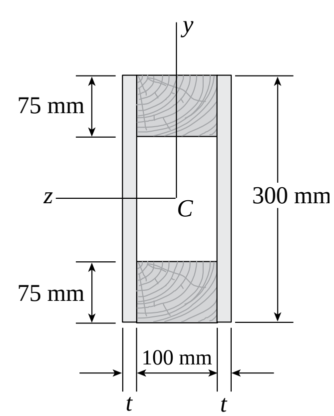
$M_{max} = \min(M_1, M_2)$

STELL GOVERNS. $\qquad M_{max} = 189\,\text{k-in.}$ $\leftarrow$

THE MOMENT CAPACITY OF THE BEAM IN (a) IS 2.3 TIMES MORE THAN THE BEAM IN (b)

Problem 6.3-2 A simple beam of span length 3.2 m carries a uniform load of intensity 48 kN/m. The cross section of the beam is a hollow box with wood flanges and steel side plates, as shown in the figure. The wood flanges are 75 mm by 100 mm in cross section, and the steel plates are 300 mm deep.

What is the required thickness t of the steel plates if the allowable stresses are 120 MPa for the steel and 6.5 MPa for the wood? (Assume that the moduli of elasticity for the steel and wood are 210 GPa and 10 GPa, respectively, and disregard the weight of the beam.)



Solution 6.3-2 Box beam

$$M_{max} = \frac{qL^2}{8} = 61.44 \text{ kN} \cdot \text{m}$$

SIMPLE BEAM: $L = 3.2$ m $\quad q = 48$ kN/m

(1) Wood flanges: $b = 100$ mm $\quad h = 300$ mm

$h_1 = 150$ mm

$(\sigma_1)_{allow} = 6.5$ MPa

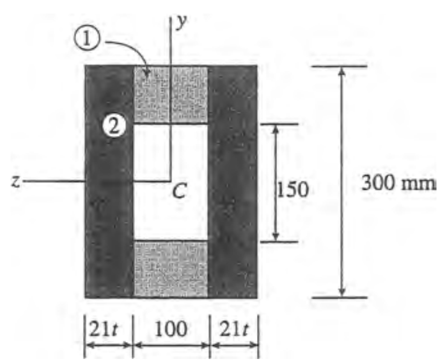
$E_w = 10$ GPa

(2) Steel plates: $t =$ thickness $\quad h = 300$ mm

$(\sigma_2)_{allow} = 120$ MPa

$E_s = 210$ GPa

TRANSFORMED SECTION (WOOD)



Wood flanges are not changed

$$n = \frac{E_s}{E_w} = 21$$

Width of steel plates

$= nt = 21t$

All dimensions in millimeters.

$$I_T = \frac{1}{12}(100 + 42t)(300)^3 - \frac{1}{12}(100)(150)^3$$

$$= 196.9 \times 10^6 \text{ mm}^4 + 94.5t \times 10^6 \text{ mm}^4$$

REQUIRED THICKNESS BASED UPON THE WOOD (1) (EQ. 6-15)

$$\sigma_1 = \frac{M(h/2)}{I_T} \qquad (I_T)_1 = \frac{M_{max}(h/2)}{(\sigma_1)_{allow}}$$

$$= 1.418 \times 10^9 \text{ mm}^4$$

Equate I_T and $(I_T)_1$ and solve for t : $t_1 = 12.92$ mm

REQUIRED THICKNESS BASED UPON THE STEEL (2) (EQ. 6-17)

$$\sigma_2 = \frac{M(h/2)n}{I_T} \qquad (I_T)_2 = \frac{M_{max}(h/2)n}{(\sigma_2)_{allow}}$$

$$= 1.612 \times 10^9 \text{ mm}^4$$

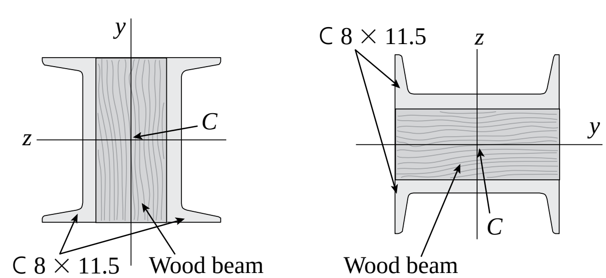
Equate I_T and $(I_T)_2$ and solve for t : $t_2 = 14.97$ mm

STEEL GOVERNS. $\quad t_{min} = 15.0$ mm $\quad \leftarrow$

Problem 6.3-3 A simple beam that is 18 ft long supports a uniform load of intensity q. The beam is constructed of two C 8 × 11.5 sections (channel sections or C shapes) on either side of a 4 × 8 (actual dimensions) wood beam (see the cross section shown in the figure part a). The modulus of elasticity of the steel ($E_s = 30{,}000$ ksi) is 20 times that of the wood (E_w).

(a) If the allowable stresses in the steel and wood are 12,000 psi and 900 psi, respectively, what is the allowable load q_{allow}? (*Note*: Disregard the weight of the beam, and see Table E-3a of Appendix E for the dimensions and properties of the C-shape beam.)

(b) If the beam is rotated 90° to bend about its y axis (see figure part b), and uniform load $q = 250$ lb/ft is applied, find the maximum stresses σ_s and σ_w in the steel and wood, respectively. Include the weight of the beam. (Assume weight densities of 35 lb/ft^3 and 490 lb/ft^3 for the wood and steel, respectively.)



(a) (b)

Solution 6.3-3

$L = 18$ ft

(a) BENT ABOUT THE Z AXIS

(1) Wood beam $b = 4$ in. $h = 8$ in.

$$\sigma_{allow\_w} = 900 \text{ psi}$$

(2) Steel Channel $h = 8.0$ in. $I_z = 32.5 \text{ in.}^4$

$$I_y = 1.31 \text{ in.}^4 \quad c = 0.572 \text{ in.}$$

$$\sigma_{allow\_s} = 12000 \text{ psi}$$

TRANSFORMED SECTION (WOOD)

$$n = 20$$

$$I_T = \frac{bh^3}{12} + 2I_z n \qquad I_T = 1471 \text{ in.}^4$$

MAXIMUM MOMENT BASED UPON THE WOOD (1)

$$M_1 = \frac{\sigma_{allow\_w} I_T}{h/2} \qquad M_1 = 331 \text{ k} \cdot \text{in.}$$

MAXIMUM MOMENT BASED UPON THE STEEL (2)

$$M_2 = \frac{\sigma_{allow\_s} I_T}{hn/2} \qquad M_2 = 221 \text{ k} \cdot \text{in.}$$

$$M_{max} = \min(M_1, M_2)$$

STEEL GOVERNS. $M_{max} = 221 \text{ k} \cdot \text{in}$

ALLOWABLE LOAD ON a 18-FT-LONG SIMPLE BEAM

From $M_{max} = \dfrac{q_{allow} L^2}{8}$ $\quad q_{allow} = 454 \text{ lb/ft.}$ $\leftarrow$

(b) BENT ABOUT THE Y AXIS (INCLUDING THE WEIGHT OF THE BEAM) $q = 250$ lb/ft.

(1) Wood beam $\rho_w = 35$ lb/ft $\quad q_w = bh\rho_w$

$$q_w = 7.778 \text{ lb/ft.}$$

(2) Steel Channels $q_s = 11.5$ lb/ft.

$$A_s = 3.37 \text{ in.}^2 \qquad b_s = 2.26 \text{ in.}$$

$$q_{total} = q + q_w + 2q_s \qquad q_{total} = 281 \text{ lb/ft.}$$

$$M_{max} = \frac{q_{total} L^2}{8} \qquad M_{max} = 11.4 \text{ k} \cdot \text{ft.}$$

TRANSFORMED SECTION (WOOD)

$$I_T = \frac{b^3 h}{12} + 2n\left[I_y + A_s\left(c + \frac{b}{2}\right)^2\right]$$

$$I_T = 987 \text{ in.}^4$$

MAXIMUM STRESS IN THE WOOD (1)

$$\sigma_{w\_max} = \frac{M_{max}\left(\frac{b}{2}\right)}{I_T} \qquad \sigma_{w\_max} = 277 \text{ psi} \quad \leftarrow$$

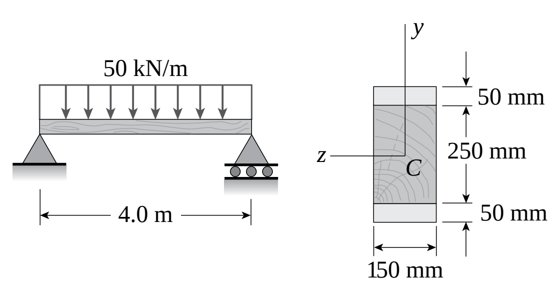
MAXIMUM MOMENT BASED UPON THE STEEL (2)

$$\sigma_{s\_max} = \frac{nM_{max}\left(\frac{b}{2} + b_s\right)}{I_T}$$

$$\sigma_{s\_max} = 11782 \text{ psi} \quad \leftarrow$$

Problem 6.3-4 The composite beam shown in the figure is simply supported and carries a total uniform load of 50 kN/m on a span length of 4.0 m. The beam is built of a wood member having cross-sectional dimensions 150 mm $\times$ 250 mm and two steel plates of cross-sectional dimensions 50 mm $\times$ 150 mm.

Determine the maximum stresses σ_s and σ_w in the steel and wood, respectively, if the moduli of elasticity are $E_s = 209$ GPa and $E_w = 11$ GPa. (Disregard the weight of the beam.)



Solution 6.3-4 Composite beam

SIMPLE BEAM: $L = 4.0$ m $\quad q = 50$ kN/m

$$M_{max} = \frac{qL^2}{8} = 100\,\text{kN}\cdot\text{m}$$

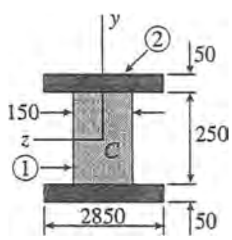
(1) Wood beam: $b = 150$ mm $\quad h_1 = 250$ mm

$E_w = 11$ GPa

(2) Steel plates: $b = 150$ mm $\quad t = 50$ mm

$h_2 = 350$ mm $\quad E_s = 209$ GPa

TRANSFORMED SECTION (WOOD)



Wood beam is not changed.

$$n = \frac{E_s}{E_w} = \frac{209}{11} = 19$$

Width of steel plates

$= nb = (19)\,(150\text{ mm}) = 2850$ mm

All dimensions in millimeters.

$$I_T = \frac{1}{12}(2850)(350)^3 - \frac{1}{12}(2850 - 150)(250)^3$$

$$= 6.667 \times 10^9\,\text{mm}^4$$

MAXIMUM STRESS IN THE WOOD (1) (EQ. 6-15)

$$\sigma_w = \sigma_1 = \frac{M_{max}(h_1/2)}{I_T} = 1.9\ \text{MPa} \quad \longleftarrow$$

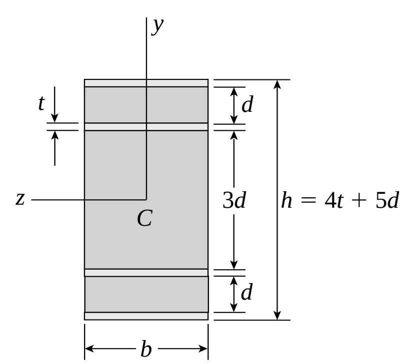
MAXIMUM STRESS IN THE STEEL (2) (EQ. 6-17)

$$\sigma_s = \sigma_2 = \frac{M_{max}(h_2/2)n}{I_T} = 49.9\ \text{MPa} \quad \longleftarrow$$

Problem 6.3-5 The cross section of a beam made of thin strips of aluminum separated by a lightweight plastic is shown in the figure. The beam has width $b = 3.0$ in., the aluminum strips have thickness $t = 0.1$ in., and the plastic segments have heights $d = 1.2$ in. and $3d = 3.6$ in. The total height of the beam is $h = 6.4$ in.

The moduli of elasticity for the aluminum and plastic are $E_a = 11 \times 10^6$ psi and $E_p = 440 \times 10^3$ psi, respectively.

Determine the maximum stresses σ_a and σ_p in the aluminum and plastic, respectively, due to a bending moment of 6.0 k-in.



Probs. 6.3-5 and 6.3-6

Solution 6.3-5 Plastic beam with aluminum strips

(1) Plastic segments: $b = 3.0$ in. $d = 1.2$ in.

$3d = 3.6$ in.

$E_p = 440 \times 10^3$ psi

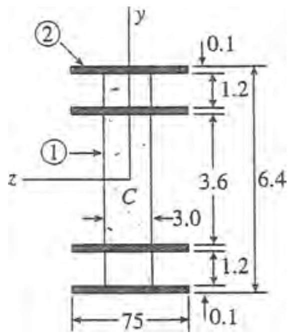
(2) Aluminum strips: $b = 3.0$ in. $t = 0.1$ in.

$E_a = 11 \times 10^6$ psi

$h = 4t + 5d = 6.4$ in.

$M = 6.0$ k-in.

TRANSFORMED SECTION (PLASTIC)



Plastic segments are not changed.

$$n = \frac{E_a}{E_p} = 25$$

Width of aluminum strips

$= nb = (25)(3.0 \text{ in.}) = 75$ in.

All dimensions in inches.

$$\text{Plastic: } I_1 = 2\left[\frac{1}{12}(3.0)(1.2)^3 + (3.0)(1.2)(2.50)^2\right] + \frac{1}{12}(3.0)(3.6)^3 = 57.528 \text{ in.}^4$$

Aluminum:

$$I_2 = 2\left[\frac{1}{12}(75)(0.1)^3 + (75)(0.1)(3.15)^2 + \frac{1}{12}(75)(0.1)^3 + (75)(0.1)(1.85)^2\right]$$

$$= 200.2 \text{ in.}^4$$

$$I_T = I_1 + I_2 = 257.73 \text{ in.}^4$$

MAXIMUM STRESS IN THE PLASTIC (1) (EQ. 6-15)

$$\sigma_p = \sigma_1 = \frac{M(h/2 - t)}{I_T} = 72 \text{ psi} \quad \leftarrow$$

MAXIMUM STRESS IN THE ALUMINUM (2) (EQ. 6-17)

$$\sigma_a = \sigma_2 = \frac{M(h/2)n}{I_T} = 1860 \text{ psi} \quad \leftarrow$$

Problem 6.3-6 Consider the preceding problem if the beam has width $b = 75$ mm, the aluminum strips have thickness $t = 3$ mm, the plastic segments have heights $d = 40$ mm and $3d = 120$ mm, and the total height of the beam is $h = 212$ mm. Also, the moduli of elasticity are $E_a = 75$ GPa and $E_p = 3$ GPa, respectively.

Determine the maximum stresses σ_a and σ_p in the aluminum and plastic, respectively, due to a bending moment of 1.0 kN · m.

Solution 6.3-6 Plastic beam with aluminum strips

(1) Plastic segments: $b = 75$ mm $\quad d = 40$ mm

$3d = 120$ mm $\quad E_p = 3$ GPa

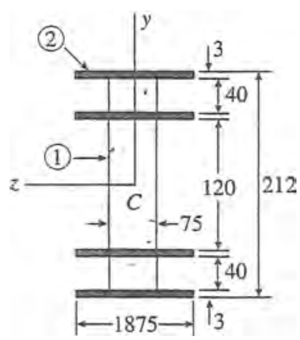
(2) Aluminum strips: $b = 75$ mm $\quad t = 3$ mm

$E_a = 75$ GPa

$h = 4t + 5d = 212$ mm

$M = 1.0$ kN·m

TRANSFORMED SECTION (PLASTIC)



Plastic segments are not changed.

$$n = \frac{E_a}{E_p} = 25$$

Width of aluminum strips

$= nb = (25)(75\text{ mm}) = 1875$ mm

All dimensions in millimeters.

$$\text{Plastic: } I_1 = 2\left[\frac{1}{12}(75)(40)^3 + (75)(40)(83)^2\right] + \frac{1}{12}(75)(120)^3$$

$$= 52.934 \times 10^6 \text{ mm}^4$$

ALUMINUM:

$$I_2 = 2\left[\frac{1}{12}(1875)(3)^3 + (1875)(3)(104.5)^2 + \frac{1}{12}(1875)(3)^3 + (1875)(3)(61.5)^2\right]$$

$$= 165.420 \times 10^6 \text{ mm}^4$$

$$I_T = I_1 + I_2 = 218.35 \times 10^6 \text{ mm}^4$$

MAXIMUM STRESS IN THE PLASTIC (1) (EQ. 6-15)

$$\sigma_p = \sigma_1 = \frac{M(h/2 - t)}{I_T} = 0.47\text{ MPa} \quad \leftarrow$$

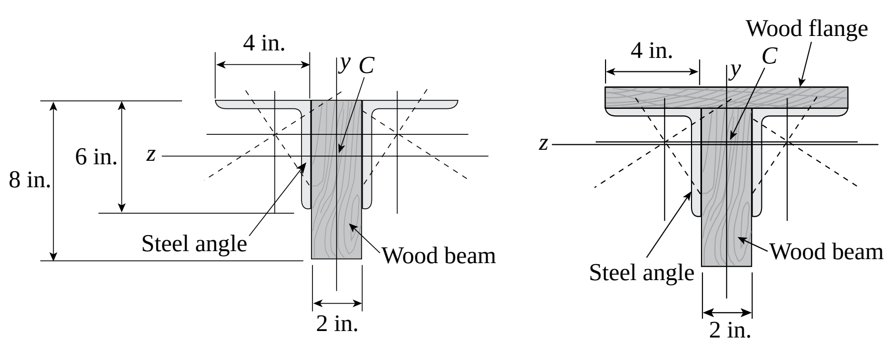
MAXIMUM STRESS IN THE ALUMINUM (2) (EQ. 6-17)

$$\sigma_a = \sigma_2 = \frac{M(h/2)n}{I_T} = 12.14\text{ MPa} \quad \leftarrow$$

Problem 6.3-7 A simple beam that is 18 ft long supports a uniform load of intensity q. The beam is constructed of two angle sections, each L $6 \times 4 \times 1/2$, on either side of a 2 in. $\times$ 8 in. (actual dimensions) wood beam (see the cross section shown in the figure part a). The modulus of elasticity of the steel is 20 times that of the wood.

(a) If the allowable stresses in the steel and wood are 12,000 psi and 900 psi, respectively, what is the allowable load q_{allow}? (*Note*: Disregard the weight of the beam, and see Table E-5a of Appendix E for the dimensions and properties of the angles.)

(b) Repeat part a if a 1 in. $\times$ 10 in. wood flange (actual dimensions) is added (see figure part b).



(a) (b)

Solution 6.3-7

$L = 18$ ft

(a) WOOD BEAM AND STEEL ANGLES

(1) Wood beam $b = 2$ in. $h = 8$ in.

$\sigma_{allow\_w} = 900$ psi

(2) Steel Channel $I_z = 17.3 \text{ in.}^4$ $d = 1.98$ in.

$A_s = 4.75 \text{ in.}^2$ $h_s = 6$ in.

$\sigma_{allow\_s} = 12000$ psi

TRANSFORMED SECTION (WOOD)

$n = 20$

NEUTRAL AXIS

From $\int y_1\, dA + \int n y_2\, dA = 0$

$$bh\left(\frac{h}{2} - h_1\right) - 2nA_s(h_1 - d) = 0$$

$h_1 = 2.137$ in.

$$I_T = \left[\frac{bh^3}{12} + bh\left(\frac{h}{2} - h_1\right)^2\right] + 2n[I_z + A_s(h_1 - d)^2] \quad I_T = 838 \text{ in.}^4$$

MAXIMUM MOMENT BASED UPON THE WOOD (1)

$$M_1 = \frac{\sigma_{allow\_w} I_T}{h - h_1} \qquad M_1 = 128.6 \text{ k} \cdot \text{in.}$$

MAXIMUM MOMENT BASED UPON THE STEEL (2)

$$M_2 = \frac{\sigma_{allow\_s} I_T}{(h_s - h_1)n} \qquad M_2 = 130.1 \text{ k} \cdot \text{in.}$$

$M_{max} = \min(M_1, M_2)$

WOOD GOVERNS $M_{max} = 128.6 \text{ k} \cdot \text{in.}$ ←

ALLOWABLE LOAD ON A 18-FT-LONG SIMPLE BEAM

From $M_{max} = \dfrac{q_{allow} L^2}{8}$

$q_{allow} = 264$ lb/ft. ←

(b) ADDITIONAL WOOD FLANGE $b_f = 1$ in.

$h_f = 10$ in.

TRANSFORMED SECTION (WOOD)

NEUTRAL AXIS

From $\int y_1\, dA + \int n y_2\, dA = 0$

$$(bh + 2nA_s)(h_1 + b_f - h_{1\_b}) - b_f h_f\left(h_{1\_}b - \frac{b_f}{2}\right) = 0$$

$h_{1\_}b = 3.015$ in.

$$I_{T\_}b = [I_T + (bh + 2nA_s)(h_1 + b_f - h_{1\_}b)^2] + \left[\frac{b_f^3 h_f}{12} + b_f h_f\left(h_{1\_b} - \frac{b_f}{2}\right)^2\right]$$

$I_{T\_}b = 905 \text{ in.}^4$

MAXIMUM MOMENT BASED UPON THE WOOD (1)

$$M_1 = \frac{\sigma_{allow\_w} I_{T\_b}}{h + b_f - h_{1\_b}} \qquad M_1 = 136.0 \text{ k} \cdot \text{in.}$$

MAXIMUM MOMENT BASED UPON THE STEEL (2)

$$M_2 = \frac{\sigma_{allow\_s} I_{T\_b}}{(h_s + b_f - h_{1\_b})n} \qquad M_2 = 136.2 \text{ k} \cdot \text{in.}$$

$M_{max} = \min(M_1, M_2)$

WOOD GOVERNS $M_{max} = 136.0 \text{ k} \cdot \text{in}$ ←

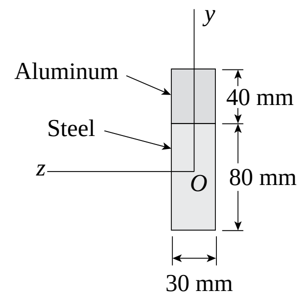
ALLOWABLE LOAD ON A 18-FT-LONG SIMPLE BEAM

From $M_{max} = \dfrac{q_{allow} L^2}{8}$

$q_{allow} = 280$ lb/ft ←

Problem 6.3-8 The cross section of a composite beam made of aluminum and steel is shown in the figure. The moduli of elasticity are $E_a = 75$ GPa and $E_s = 200$ GPa.

Under the action of a bending moment that produces a maximum stress of 50 MPa in the aluminum, what is the maximum stress σ_s in the steel?



Solution 6.3-8 Composite beam of aluminum and steel

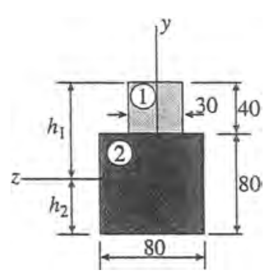
(1) Aluminum: $b = 30$ mm $\quad h_a = 40$ mm

$E_a = 75$ GPa $\quad \sigma_a = 50$ MPa

(2) Steel: $b = 30$ mm $\quad h_s = 80$ mm

$E_s = 200$ GPa $\quad \sigma_s = ?$

TRANSFORMED SECTION (ALUMINUM)



Aluminum part is not changed.

$$n = \frac{E_s}{E_a} = \frac{200}{75} = 2.667$$

Width of steel part

$= nb = (2.667)(30\text{ mm}) = 80$ mm

All dimensions in millimeters.

Use the base of the cross section as a reference line.

$$h_2 = \frac{\Sigma y_i A_i}{\Sigma A_i} = \frac{(40)(80)(80) + (100)(30)(40)}{(80)(80) + (30)(40)}$$

$$= 49.474\text{ mm}$$

$$h_1 = 120 - h_2 = 70.526\text{ mm}$$

MAXIMUM STRESS IN THE ALUMINUM (1) (EQ. 6-15)

$$\sigma_a = \sigma_1 = \frac{Mh_1}{I_T}$$

MAXIMUM STRESS IN THE STEEL (2) (EQ. 6-17)

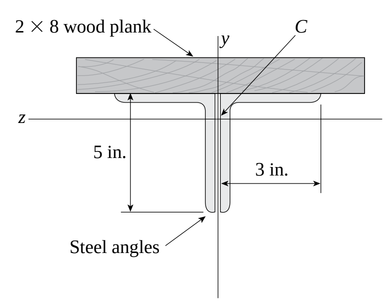
$$\sigma_s = \sigma_2 = \frac{Mh_2 n}{I_T}$$

$$\frac{\sigma_s}{\sigma_a} = \frac{h_2 n}{h_1} = \frac{(49.474)(2.667)}{70.526} = 1.8707$$

$$\sigma_s = 1.8707\,(50\text{ MPa}) = 93.5\text{ MPa} \quad \leftarrow$$

Problem 6.3-9 A beam is constructed of two angle sections, each L $5 \times 3 \times 1/2$, which reinforce a 2×8 (actual dimensions) wood plank (see the cross section shown in the figure). The modulus of elasticity for the wood is $E_w = 1.2 \times 10^6$ psi and for the steel is $E_s = 30 \times 10^6$ psi.

Find the allowable bending moment M_{allow} for the beam if the allowable stress in the wood is $\sigma_w = 1100$ psi and in the steel is $\sigma_s = 12{,}000$ psi. (*Note*: Disregard the weight of the beam, and see Table E-5a of Appendix E for the dimensions and properties of the angles.)



Solution 6.3-9

(1) Wood beam $b = 2$ in. $h = 8$ in.

$\sigma_{allow\_w} = 1100$ psi $E_w = 1.2 \cdot 10^6$ psi

(2) Steel Angle $I_z = 9.43$ in.^{4} $d = 1.74$ in.

$A_s = 3.75$ in.^{2} $h_s = 5$ in.

$\sigma_{allow\_s} = 12000$ psi $E_s = 30 \cdot 10^6$ psi

TRANSFORMED SECTION (WOOD)

$$n = \frac{E_s}{E_w} \qquad n = 25$$

NEUTRAL AXIS

From $\int y_1 \, dA + \int n y_2 \, dA = 0$

$$bh\left(h_1 - \frac{b}{2}\right) - 2nA_s(b + d - h_1) = 0$$

$h_1 = 3.525$ in.

$$I_T = \left[\frac{b^3h}{12} + bh\left(h_1 - \frac{b}{2}\right)^2\right] + 2n[I_z + A_s(b + d - h_1)^2]$$

$I_T = 588$ in.^{4}

MAXIMUM MOMENT BASED UPON THE WOOD (1)

$$M_1 = \frac{\sigma_{allow\_w} I_T}{h_1} \qquad M_1 = 183.4 \text{ k} \cdot \text{in.}$$

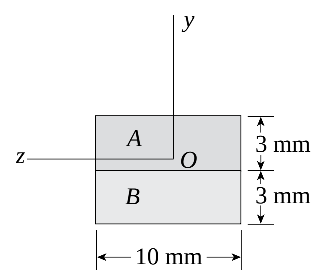
MAXIMUM MOMENT BASED UPON THE STEEL (2)

$$M_2 = \frac{\sigma_{allow\_s} I_T}{(h_s + b - h_1)n} \qquad M_2 = 81.1 \text{ k} \cdot \text{in.}$$

$M_{max} = \min(M_1, M_2)$

STEEL GOVERNS $M_{max} = 81.1$ k-in. $\leftarrow$

Problem 6.3-10 The cross section of a bimetallic strip is shown in the figure. Assuming that the moduli of elasticity for metals A and B are $E_A = 168$ GPa and $E_B = 90$ GPa, respectively, determine the smaller of the two section moduli for the beam. (Recall that section modulus is equal to bending moment divided by maximum bending stress.) In which material does the maximum stress occur?



Solution 6.3-10 Bimetallic strip

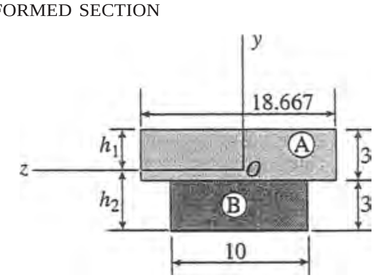
Metal A: $b = 10$ mm $h_A = 3$ mm

$E_A = 168$ GPa

Metal B: $b = 10$ mm

$h_B = 3$ mm $E_B = 90$ GPa

TRANSFORMED SECTION



TRANSFORMED SECTION (METAL B)

Metal B does not change.

$$n = \frac{E_A}{E_B} = \frac{168}{90} = 1.8667$$

Width of metal A

$= nb = (1.8667)(10 \text{ mm}) = 18.667$ mm

All dimensions in millimeters.

Use the base of the cross section as a reference line.

$$h_2 = \frac{\Sigma y_i A_i}{\Sigma A_i} = \frac{(1.5)(10)(13) + (4.5)(18.667)(3)}{(10)(3) + (18.667)(3)}$$

$= 3.4535$ mm

$h_1 = 6 - h_2 = 2.5465$ mm

$$I_T = \frac{1}{12}(10)(3)^3 + (10)(3)(h_2 - 1.5)^2$$

$$+ \frac{1}{12}(18.667)(3)^3 + (18.667)(3)(h_1 - 1.5)^2$$

$$= 240.31\text{mm}^4$$

MAXIMUM STRESS IN MATERIAL B (EQ. 6-15)

$$\sigma_B = \sigma_1 = \frac{Mh_2}{I_T} \qquad S_B = \frac{M}{\sigma_B} = \frac{I_T}{h_2} = 69.6 \text{ mm}^3$$

MAXIMUM STRESS IN MATERIAL A (EQ. 6-17)

$$\sigma_A = \sigma_2 = \frac{Mh_1 n}{I_T} \qquad S_A = \frac{M}{\sigma_A} = \frac{I_T}{h_1 n}$$

$$= 50.6 \text{ mm}^3$$

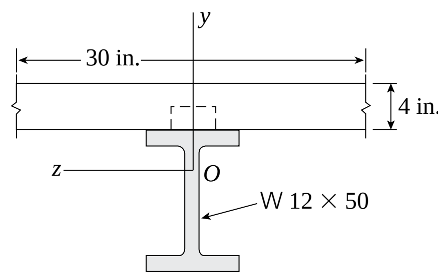
SMALLER SECTION MODULUS

$$S_A = 50.6\text{mm}^3 \quad \leftarrow$$

$\therefore$ Maximum stress occurs in metal A. $\leftarrow$

Problem 6.3-11 A W 12 × 50 steel wide-flange beam and a segment of a 4-inch thick concrete slab (see figure) jointly resist a positive bending moment of 95 k-ft. The beam and slab are joined by shear connectors that are welded to the steel beam. (These connectors resist the horizontal shear at the contact surface.) The moduli of elasticity of the steel and the concrete are in the ratio 12 to 1.

Determine the maximum stresses σ_s and σ_c in the steel and concrete, respectively. (*Note*: See Table E-1a of Appendix E for the dimensions and properties of the steel beam.)



Solution 6.3-11 Steel beam and concrete slab

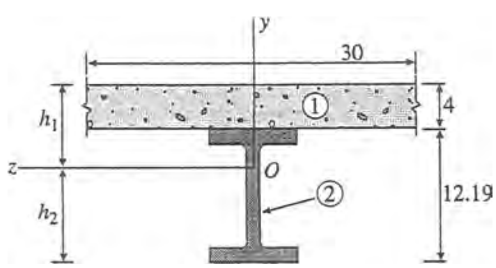
(1) Concrete: $b = 30$ in. $\quad t = 4$ in.

(2) Wide-flange beam: W 12 × 50

$d = 12.19$ in. $\quad I = 394$ in.^{4}

$A = 14.7$ in.^{2} $\quad M = 95$ k-ft $= 1140$ k-in.

TRANSFORMED SECTION (CONCRETE)



No change in dimensions of the concrete.

$$n = \frac{E_s}{E_c} = \frac{E_2}{E_1} = 12$$

Width of steel beam is increased by the factor n to transform to concrete.

All dimensions in inches.

Use the base of the cross section as a reference line.

$nI = 4728$ in.^{4} $\qquad nA = 176.4$ in.^{2}

$$h_2 = \frac{\Sigma y_i A_i}{\Sigma A_i} = \frac{(12.19/2)(176.4) + (14.19)(30)(4)}{176.4 + (30)(4)}$$

$$= 9.372\text{ in.}$$

$$h_1 = 16.19 - h_2 = 6.818 \text{ in.}$$

$$I_T = \frac{1}{12}(30)(4)^3 + (30)(4)(h_1 - 2)^2 + 4728$$

$$+ (176.4)(h_2 - 12.19/2)^2 = 9568\text{ in.}^4$$

MAXIMUM STRESS IN THE CONCRETE (1) (EQ. 6-15)

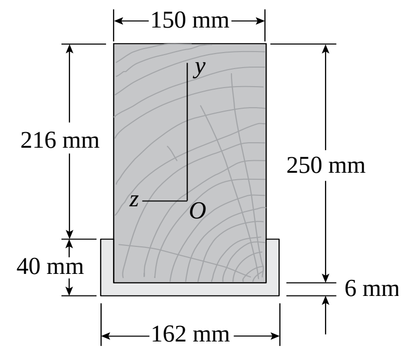
$$\sigma_c = \sigma_1 = \frac{Mh_1}{I_T} = 812 \text{ psi (Compression)} \quad \leftarrow$$

MAXIMUM STRESS IN THE STEEL (2) (EQ. 6-17)

$$\sigma_s = \sigma_2 = \frac{Mh_2 n}{I_T} = 13{,}400 \text{ psi (Tension)} \quad \leftarrow$$

Problem 6.3-12 A wood beam reinforced by an aluminum channel section is shown in the figure. The beam has a cross section of dimensions 150 mm by 250 mm, and the channel has a uniform thickness of 6 mm.

If the allowable stresses in the wood and aluminum are 8.0 MPa and 38 MPa, respectively, and if their moduli of elasticity are in the ratio 1 to 6, what is the maximum allowable bending moment for the beam?



Solution 6.3-12 Wood beam and aluminum channel

(1) Wood beam: $b_w = 150$ mm $\quad h_w = 250$ mm

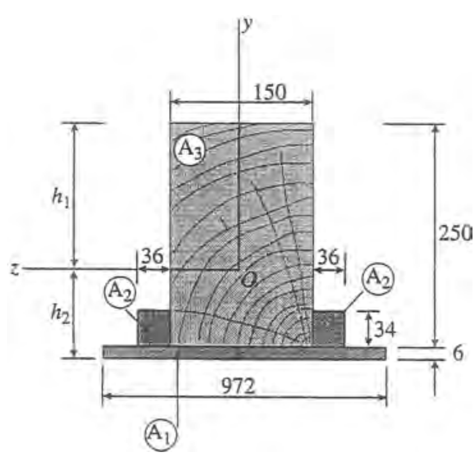
$(\sigma_w)_{\text{allow}} = 8.0$ MPa

(2) Aluminum channel: $t = 6$ mm $\quad b_a = 162$ mm

$h_a = 40$ mm

$(\sigma_a)_{\text{allow}} = 38$ MPa

Transformed section (wood)



Wood beam is not changed.

$$n = \frac{E_a}{E_w} = 6$$

Width of aluminum channel is increased.

$nb = (6)(162 \text{ mm}) = 972$ mm

$nt = (6)(6 \text{ mm}) = 36$ mm

All dimensions in millimeters.

Use the base of the cross section as a reference line.

$$h_2 = \frac{\Sigma y_i A_i}{\Sigma A_i}$$

Area A_1: $y_1 = 3 \quad A_1 = (972)(6) = 5832$

$y_1 A_1 = 17{,}496 \text{ mm}^3$

Area A_2: $y_2 = 23 \quad A_2 = (36)(34) = 1224$

$y_2 A_2 = 28{,}152 \text{ mm}^3$

Area A_3: $y_3 = 131 \quad A_3 = (150)(250) = 37{,}500$

$y_3 A_3 = 4{,}912{,}500 \text{ mm}^3$

$$h_2 = \frac{y_1 A_1 + 2y_2 A_2 + y_3 A_3}{A_1 + 2A_2 + A_3} = \frac{4{,}986{,}300 \text{ mm}^3}{45{,}780 \text{ mm}^2}$$

$$= 108.92 \text{ mm}$$

$$h_1 = 256 - h_2 = 147.08 \text{ mm}$$

Moment of inertia

Area A_1: $I_1 = \frac{1}{12}(972)(6)^3 + (972)(6)(h_2 - 3)^2$

$= 65{,}445{,}000 \text{ mm}^4$

Area A_2: $I_2 = \frac{1}{12}(36)(34)^3$

$+ (36)(34)(h_2 - 6 - 17)^2$

$= 9{,}153{,}500 \text{ mm}^4$

$$\text{Area } A_3\text{: } I_3 = \frac{1}{12}(150)(250)^3$$
$$+ (150)(250)(h_1 - 125)^2$$
$$= 213{,}597{,}000 \text{ mm}^4$$

$$I_T = I_1 + 2I_2 + I_3 = 297.35 \times 10^6 \text{ mm}^4$$

MAXIMUM MOMENT BASED UPON THE WOOD (1) (EQ. 6-15)

$$\sigma_w = \sigma_1 = \frac{Mh_1}{I_T} \qquad M_1 = \frac{(\sigma_w)_{\text{allow}} I_T}{h_1} = 16.2 \text{ kN} \cdot \text{m}$$

MAXIMUM MOMENT BASED UPON ALUMINUM (2) (EQ. 6-17)

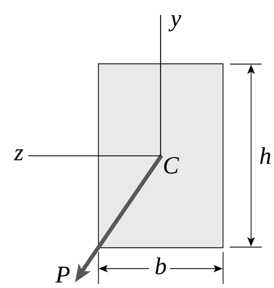
$$\sigma_a = \sigma_2 = \frac{Mh_2 n}{I_T} \qquad M_2 = \frac{(\sigma_a)_{\text{allow}} I_T}{h_2 n} = 17.3 \text{ kN} \cdot \text{m}$$

WOOD GOVERNS. $\quad M_{\text{allow}} = 16.2 \text{ kN} \cdot \text{m} \quad \leftarrow$

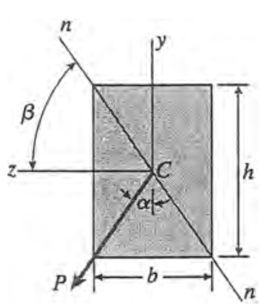
Beams with Inclined Loads

When solving the problems for Section 6.4, be sure to draw a sketch of he cross section showing the orientation of the neutral axis and the locations of the points where the stresses are being found.

Problem 6.4-1 A beam of rectangular cross section supports an inclined load P having its line of action along a diagonal of the cross section (see figure). Show that the neutral axis lies along the other diagonal.



Solution 6.4-1 Location of neutral axis



Load P acts along a diagonal.

$$\tan \alpha = \frac{b/2}{h/2} = \frac{b}{h}$$

$$I_z = \frac{bh^3}{12}$$

$$I_y = \frac{hb^3}{12}$$

$$\frac{I_z}{I_y} = \frac{h^2}{b^2}$$

See Figure 6-15b.

β = angle between the z axis and the neutral axis nn
θ = angle between the y axis and the load P
$\theta = \alpha + 180°$
$\tan \theta = \tan(\alpha + 180°) = \tan \alpha$

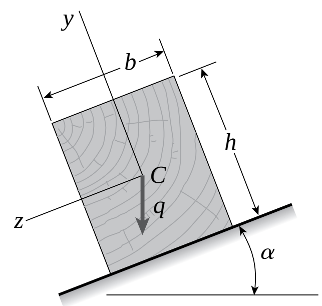
(Eq. 6-23): $\quad \tan \beta = \dfrac{I_z}{I_y} \tan \theta = \dfrac{h^2}{b^2} \tan \theta$

$$= \left(\frac{h^2}{b^2}\right)\left(\frac{b}{h}\right) = \frac{h}{b}$$

$\therefore$ The neutral axis lies along the other diagonal. QED $\quad \leftarrow$

Problem 6.4-2 A wood beam of rectangular cross section (see figure) is simply supported on a span of length L. The longitudinal axis of the beam is horizontal, and the cross section is tilted at an angle α. The load on the beam is a vertical uniform load of intensity q acting through the centroid C.

Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_{max} if $b = 80$ mm, $h = 140$ mm, $L = 1.75$ m, $\alpha = 22.5°$, and $q = 7.5$ kN/m.



Probs. 6.4-2 and 6.4-3

Solution 6.4-2

$L = 1.75\text{ m} \qquad q = 7.5\text{ kN/m} \qquad b = 80\text{ mm}$

$h = 140\text{ mm} \qquad \alpha = 22.5\text{ deg}$

Bending moments

$$M_y = \frac{q\sin(\alpha)L^2}{8} \qquad M_y = 1099\text{ N}\cdot\text{m}$$

$$M_z = \frac{q\cos(\alpha)L^2}{8} \qquad M_z = 2653\text{ N}\cdot\text{m}$$

Moment of inertia

$$I_y = \frac{hb^3}{12} \qquad I_y = 5.973 \times 10^6\text{ mm}^4$$

$$I_z = \frac{bh^3}{12} \qquad I_z = 18.293 \times 10^6\text{ mm}^4$$

Neutral axis nn

$$\beta = a\tan\left(\frac{I_z}{I_y}\tan(\alpha)\right) \qquad \beta = 51.8° \qquad \leftarrow$$

Maximum tensile stress (at point A)

$$\sigma_{max} = \frac{M_y\left(\frac{b}{2}\right)}{I_y} - \frac{M_z\left(\frac{-h}{2}\right)}{I_z}$$

$$\sigma_{max} = 17.5\text{ MPa} \qquad \leftarrow$$

Problem 6.4-3 Solve the preceding problem for the following data: $b = 6$ in., $h = 10$ in., $L = 12.0$ ft, $\tan\alpha = 1/3$, and $q = 325$ lb/ft.

Solution 6.4-3

$L = 12\text{ ft} \qquad q = 325\text{ lb/ft} \qquad b = 6\text{ in.}$

$h = 10\text{ in.} \qquad \alpha = a\tan\left(\frac{1}{3}\right)$

Bending moments

$$M_y = \frac{q\sin(\alpha)L^2}{8} \qquad M_y = 22199\text{ lb-in.}$$

$$M_z = \frac{q\cos(\alpha)L^2}{8} \qquad M_z = 66598\text{ lb-in.}$$

Moment of inertia

$$I_y = \frac{hb^3}{12} \qquad I_y = 180\text{ in.}^4$$

$$I_z = \frac{bh^3}{12} \qquad I_z = 500\text{ in.}^4$$

NEUTRAL AXIS *nn*

$$\beta = a\tan\left(\frac{I_z}{I_y}\tan(\alpha)\right) \qquad \beta = 42.8° \quad \leftarrow$$

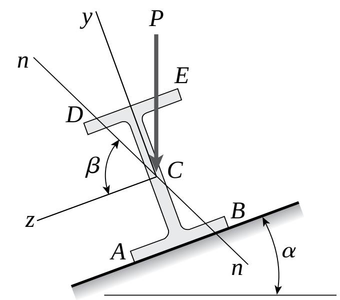
MAXIMUM TENSILE STRESS (AT POINT A)

$$\sigma_{max} = \frac{M_y\left(\frac{b}{2}\right)}{I_y} - \frac{M_z\left(\frac{-h}{2}\right)}{I_z}$$

$$\sigma_{max} = 1036 \text{ psi} \quad \leftarrow$$

Problem 6.4-4 A simply supported wide-flange beam of span length L carries a vertical concentrated load P acting through the centroid C at the midpoint of the span (see figure). The beam is attached to supports inclined at an angle α to the horizontal.

Determine the orientation of the neutral axis and calculate the maximum stresses at the outside corners of the cross section (points A, B, D, and E) due to the load P. Data for the beam are as follows: W 250 × 44.8 section, $L = 3.5$ m, $P = 18$ kN, and $\alpha = 26.57°$. (*Note*: See Table E-1b of Appendix E for the dimensions and properties of the beam.)



Probs. 6.4-4 and 6.4-5

Solution 6.4-4

$L = 3.5$ m $\qquad P = 18$ kN $\qquad \alpha = 26.57$ deg

Wide-flange beam: W 250 × 44.8

$I_y = 6.95 \times 10^6 \text{ mm}^4 \qquad I_z = 70.8 \times 10^6 \text{ mm}^4$

$d = 267$ mm $\qquad b = 148$ mm

BENDING MOMENTS

$$M_y = \frac{P\sin(\alpha)L}{4} \qquad M_y = 7045 \text{ N}\cdot\text{m}$$

$$M_z = \frac{P\cos(\alpha)L}{4} \qquad M_z = 14087 \text{ N}\cdot\text{m}$$

NEUTRAL AXIS *NN*

$$\beta = a\tan\left(\frac{I_z}{I_y}\tan(\alpha)\right) \qquad \beta = 78.9° \quad \leftarrow$$

BENDING STRESSES

$$\sigma_x(z,y) = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

POINT A: $\quad z_A = \frac{b}{2} \qquad y_A = \frac{-d}{2} \qquad \sigma_A = \sigma_x(z_A, y_A)$

$$\sigma_A = 102 \text{ MPa} \quad \leftarrow$$

POINT B: $\quad z_B = -\frac{b}{2} \qquad y_B = \frac{-d}{2}$

$$\sigma_B = \sigma_x(z_B, y_B) \qquad \sigma_B = -48 \text{ MPa} \quad \leftarrow$$

POINT D: $\quad \sigma_D = -\sigma_B \qquad \sigma_D = 48 \text{ MPa} \quad \leftarrow$

POINT E: $\quad \sigma_E = -\sigma_A \qquad \sigma_E = -102 \text{ MPa} \quad \leftarrow$

Problem 6.4-5 Solve the preceding problem using the following data: W 8 × 21 section, $L = 84$ in., $P = 4.5$ k, and $\alpha = 22.5°$.

Solution 6.4-5

$L = 84$ in. $\quad P = 4.5$ k $\quad \alpha = 22.5°$

WIDE-FLANGE BEAM:

W 8 × 21 $\quad I_y = 9.77\text{ in.}^4 \quad I_z = 75.3\text{ in.}^4$

$d = 8.28$ in. $\quad b = 5.270$ in.

BENDING MOMENTS

$$M_y = \frac{P\sin(\alpha)L}{4} \qquad M_y = 36164\text{ lb-in.}$$

$$M_z = \frac{P\cos(\alpha)L}{4} \qquad M_z = 87307\text{ lb-in.}$$

NEUTRAL AXIS *nn*

$$\beta = a\tan\left(\frac{I_z}{I_y}\tan(\alpha)\right) \qquad \beta = 72.6° \qquad \leftarrow$$

BENDING STRESSES

$$\sigma_x(z, y) = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

POINT A: $\quad z_A = \frac{b}{2} \qquad y_A = \frac{-d}{2}$

$$\sigma_A = \sigma_x(z_A, y_A)$$

$$\sigma_A = 14554\text{ psi} \qquad \leftarrow$$

POINT B: $\quad z_B = -\frac{b}{2} \qquad y_B = \frac{-d}{2}$

$$\sigma_B = \sigma_x(z_B, y_B)$$

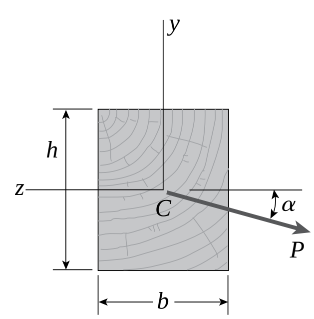
$$\sigma_B = -4953\text{ psi} \qquad \leftarrow$$

POINT D: $\quad \sigma_D = -\sigma_B \qquad \sigma_D = 4953\text{ psi} \qquad \leftarrow$

POINT E: $\quad \sigma_E = -\sigma_A \qquad \sigma_E = -14554\text{ psi} \qquad \leftarrow$

Problem 6.4-6 A wood cantilever beam of rectangular cross section and length L supports an inclined load P at its free end (see figure).

Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_{max} due to the load P_t Data for the beam are as follows: $b = 80$ mm, $h = 140$ mm, $L = 2.0$ m, $P = 575$ N, and $\alpha = 30°$.



Probs. 6.4-6 and 6.4-7

Solution 6.4-6

$L = 2.0$ m $\quad P = 575$ N $\quad b = 80$ mm

$h = 140$ mm $\quad \alpha = 30°$

BENDING MOMENTS

$$M_y = P\cos(\alpha)L \qquad M_y = 996\text{ N}\cdot\text{m}$$

$$M_z = -P\sin(\alpha)L \qquad M_z = -575\text{ N}\cdot\text{m}$$

MOMENT OF INERTIA

$$I_y = \frac{hb^3}{12} \qquad I_y = 5.973 \times 10^6\text{ mm}^4$$

$$I_z = \frac{bh^3}{12} \qquad I_z = 18.293 \times 10^6\text{ mm}^4$$

NEUTRAL AXIS *nn*

$$\beta = a\tan\left(\frac{I_z}{I_y}\tan(\alpha + 90°)\right)$$

$$\beta = -79.3° \qquad \leftarrow$$

MAXIMUM TENSILE STRESS (AT POINT A)

$$\sigma_{max} = \frac{M_y\left(\frac{b}{2}\right)}{I_y} - \frac{M_z\left(\frac{h}{2}\right)}{I_z}$$

$$\sigma_{max} = 8.87 \text{ MPa} \qquad \leftarrow$$

Problem 6.4-7 Solve the preceding problem for a cantilever beam with data as follows: $b = 4$ in., $h = 9$ in., $L = 10.0$ ft, $P = 325$ lb, and $\alpha = 45°$.

Solution 6.4-7

$L = 10.0$ ft $\quad P = 325$ lb $\quad b = 4$ in.

$h = 9$ in. $\quad \alpha = 45°$

BENDING MOMENTS

$M_y = P\cos(\alpha)L \qquad M_y = 27577 \text{ lb} \cdot \text{in.}$

$M_z = -P\sin(\alpha)L \qquad M_z = -27577 \text{ lb} \cdot \text{in.}$

MOMENT OF INERTIA

$$I_y = \frac{hb^3}{12} \qquad I_y = 48.000 \text{ in.}^4$$

$$I_z = \frac{bh^3}{12} \qquad I_z = 243.000 \text{ in.}^4$$

NEUTRAL AXIS *nn*

$$\beta = a\tan\left(\frac{I_z}{I_y}\tan(\alpha + 90°)\right)$$

$$\beta = -78.8 \deg \qquad \leftarrow$$

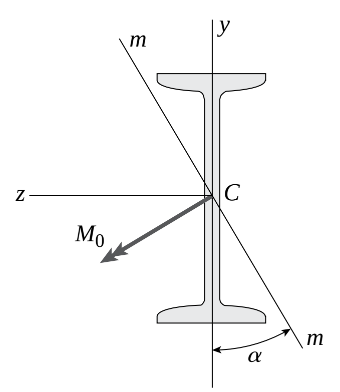
MAXIMUM TENSILE STRESS (AT POINT A)

$$\sigma_{max} = \frac{M_y\left(\frac{b}{2}\right)}{I_y} - \frac{M_z\left(\frac{h}{2}\right)}{I_z}$$

$$\sigma_{max} = 1660 \text{ psi} \qquad \leftarrow$$

Problem 6.4-8 A steel beam of I-section (see figure) is simply supported at the ends. Two equal and oppositely directed bending moments M_0 act at the ends of the beam, so that the beam is in pure bending. The moments act in plane *mm*, which is oriented at an angle α to the *xy* plane.

Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_{max} due to the moments M_0. Data for the beam are as follows: S 200 × 27.4 section, $M_0 = 4 \text{ kN} \cdot \text{m.}$, and $\alpha = 24°$. (*Note*: See Table E-2b of Appendix E for the dimensions and properties of the beam.)



Solution 6.4-8

$M_o = 4.0\ \text{kN} \cdot \text{m} \qquad \alpha = 24°$

$\text{S } 200 \times 27.4 \qquad I_y = 1.54 \times 10^6\ \text{mm}^4$

$I_z = 23.9 \times 10^6\ \text{mm}^4 \qquad d = 203\ \text{mm} \qquad b = 102\ \text{mm}$

BENDING MOMENTS

$M_y = -M_0 \sin(\alpha) \qquad M_y = -1627\ \text{N} \cdot \text{m}$

$M_z = M_0 \cos(\alpha) \qquad M_z = 3654\ \text{N} \cdot \text{m}$

NEUTRAL AXIS nn

$$\beta = a\tan\left(\frac{I_z}{I_y}\tan(-\alpha)\right) \qquad \beta = -81.8° \qquad \longleftarrow$$

MAXIMUM TENSILE STRESS (AT POINT A)

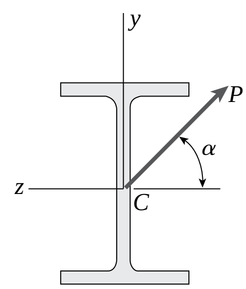
$$\sigma_{\max} = \frac{M_y\left(-\dfrac{b}{2}\right)}{I_y} - \frac{M_z\left(-\dfrac{d}{2}\right)}{I_z}$$

$$\sigma_{\max} = 69.4\ \text{MPa} \qquad \longleftarrow$$

Problem 6.4-9 A cantilever beam of wide-flange cross section and length L supports an inclined load P at its free end (see figure).

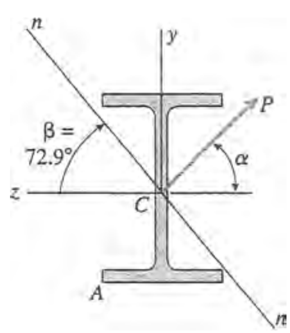
Determine the orientation of the neutral axis and calculate the maximum tensile stress $\sigma_{\max}$ due to the load P.

Data for the beam are as follows: W 10 × 45 section, $L = 8.0$ ft, $P = 1.5$ k, and $\alpha = 55°$. (*Note*: See Table E-1a of Appendix E for the dimensions and properties of the beam.)



Probs. 6.4-9 and 6.4-10

Solution 6.4-9 Cantilever beam with inclined load



$P = 1.5\ \text{k} = 1500\ \text{lb}$

$L = 8.0\ \text{ft} = 96\ \text{in.}$

$\alpha = 55°$

W 10 × 45

$I_y = 53.4\ \text{in.}^4 \qquad I_z = 248\ \text{in.}^4$

$d = 10.10\ \text{in.} \qquad b = 8.02\ \text{in.}$

BENDING MOMENTS

$M_y = (P \cos \alpha)L = 82{,}595\ \text{lb-in.}$

$M_z = (P \sin \alpha)L = 117{,}960\ \text{lb-in.}$

NEUTRAL AXIS nn (EQ. 6-23)

$\theta = 90° - \alpha = 35° \quad$ (see Fig. 6-15)

$$\tan\beta = \frac{I_z}{I_y}\tan\theta = \frac{248}{53.4}\tan 35° = 3.2519$$

$$\beta = 72.91° \qquad \longleftarrow$$

MAXIMUM TENSILE STRESS (POINT A) (EQ. 6-18)

$z_A = b/2 = 4.01\ \text{in.}$

$y_A = -d/2 = -5.05\ \text{in.}$

$$\sigma_{\max} = \sigma_A = \frac{M_y z_A}{I_y} - \frac{M_z y_A}{I_z} = 8600\ \text{psi} \qquad \longleftarrow$$

Problem 6.4-10 Solve the preceding problem using the following data: W 310 $\times$ 129 section, $L = 1.8$ m, $P = 9.5$ kN, and $\alpha = 60°$. (*Note*: See Table E-1b of Appendix E for the dimensions and properties of the beam.)

Solution 6.4-10

$P = 9.5$ kN $\quad L = 1.8$ m $\quad \alpha = 60°$

W 310 $\times$ 129 $\quad I_y = 100 \times 10^6$ mm^4

$I_z = 308 \times 10^6$ mm^4 $\quad d = 318$ mm $\quad b = 307$ mm

Bending moments

$M_y = P\cos(\alpha)L \qquad M_y = 8550$ N · m

$M_z = P\sin(\alpha)L \qquad M_z = 14809$ N · m

Neutral axis *nn*

$$\beta = a\tan\left(\frac{I_z}{I_y}\tan(90° - \alpha)\right) \qquad \beta = 60.6°$$

$\beta = 60.6°$ $\quad \leftarrow$

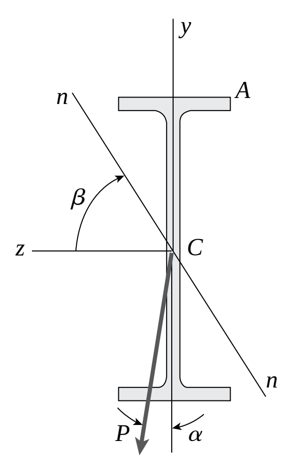
Maximum tensile stress (at point A)

$$\sigma_{max} = \frac{M_y\left(\frac{b}{2}\right)}{I_y} - \frac{M_z\left(-\frac{d}{2}\right)}{I_z}$$

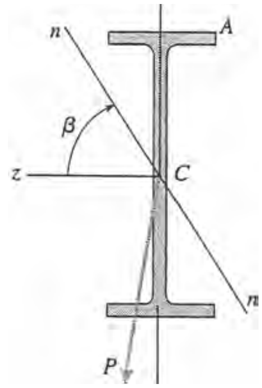
$\sigma_{max} = 20.8$ MPa $\quad \leftarrow$

Problem 6.4-11 A cantilever beam of W 12 $\times$ 14 section and length $L = 9$ ft supports a slightly inclined load $P = 500$ lb at the free end (see figure).

(a) Plot a graph of the stress σ_A at point A as a function of the angle of inclination α.
(b) Plot a graph of the angle β, which locates the neutral axis *nn*, as a function of the angle α. (When plotting the graphs, let α vary from 0 to 10°.) (*Note*: See Table E-1a of Appendix *E* for the dimensions and properties of the beam.)



Solution 6.4-11 Cantilever beam with inclined load



$P = 500$ lb $\quad L = 9$ ft $= 108$ in. $\quad$ W 12×14

$I_y = 2.36$ in.^{4} $\quad I_z = 88.6$ in.^{4}

$d = 11.91$ in. $\quad b = 3.970$ in.

Bending moments

$M_y = -(P \sin \alpha)L = -54{,}000 \sin \alpha$

$M_z = -(P \cos \alpha)L = -54{,}000 \cos \alpha$

(a) Stress at point A (Eq. 6-18)

$z_A = -b/2 = -1.985$ in.

$y_A = d/2 = 5.955$ in.

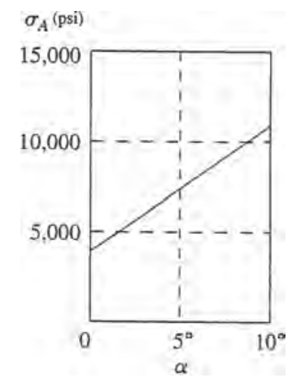
$$\sigma_A = \frac{M_y z_A}{I_y} - \frac{M_z y_A}{I_z} = 45{,}420 \sin \alpha$$
$$+ 3629 \cos \alpha \text{ (psi)} \quad \leftarrow$$

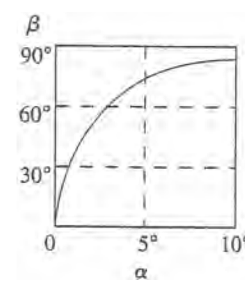
(b) Neutral axis nn (Eq. 6-23)

$\theta = 180° + \alpha \qquad$ (see Fig. 6-15)

$$\tan \beta = \frac{I_z}{I_y} \tan \theta = \frac{I_z}{I_y} \tan(180° + \alpha)$$
$$= \frac{88.6}{2.36} \tan(180° + \alpha) = 37.54 \tan \alpha$$

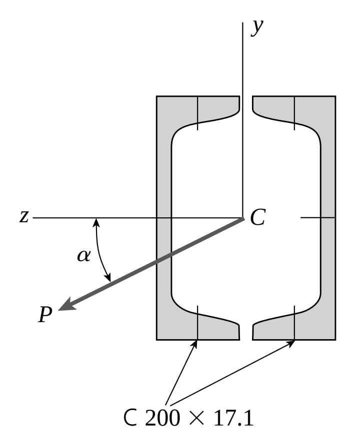
$$\beta = \arctan(37.54 \tan \alpha) \quad \leftarrow$$





Problem 6.4-12 A cantilever beam built up from two channel shapes, each C 200×17.1, and of length L supports an inclined load P at its free end (see figure).

Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_{max} due to the load P. Data for the beam are as follows: $L = 4.5$ m, $P = 500$ N, and $\alpha = 30°$.



Solution 6.4-12

$L = 4.5\text{ m} \qquad P = 500\text{ N} \qquad \alpha = 30°$

BUILT UP BEAM: Double C 200 × 17.1

$I_{cy} = 0.545 \times 10^6\text{ mm}^4 \qquad I_{cz} = 13.5 \times 10^6\text{ mm}^4$

$c = 14.5\text{ mm} \qquad b_c = 57.4\text{ mm} \qquad A_c = 2170\text{ mm}^2$

$I_y = 2\left[I_{cy} + A_c(b_c - c)^2\right] \qquad I_y = 9.08 \times 10^6\text{ mm}^4$

$I_z = 2I_{cz} \qquad I_z = 27.0 \times 10^6\text{ mm}^4$

$d = 203\text{ mm} \qquad b = 2b_c \qquad b = 114.8\text{ mm}$

BENDING MOMENTS

$M_y = -P\cos(\alpha)L \qquad M_y = -1949\text{ N}\cdot\text{m}$

$M_z = -P\sin(\alpha)L \qquad M_z = -1125\text{ N}\cdot\text{m}$

NEUTRAL AXIS *nn*

$$\beta = a\tan\left(\frac{I_z}{I_y}\tan(90° - \alpha)\right)$$

$$\beta = 79.0° \quad \leftarrow$$

MAXIMUM TEMSILE STRESS

$$\sigma_x(z, y) = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

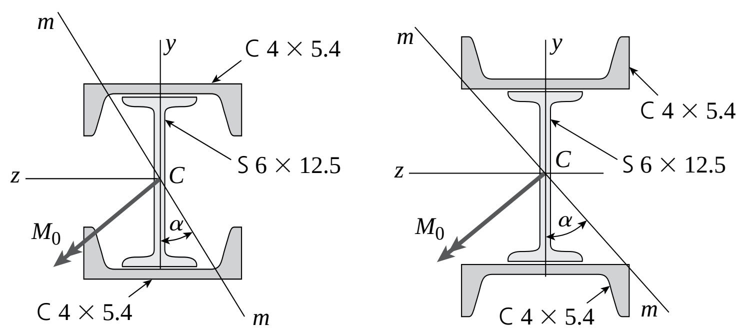
POINT A: $z_A = -\dfrac{b}{2} \quad y_A = \dfrac{d}{2}$

$$\sigma_A = \sigma_x(z_A, y_A) \qquad \sigma_A = 16.6\text{ MPa} \quad \leftarrow$$

Problem 6.4-13 A built-up steel beam of I-section with channels attached to the flanges (see figure part a) is simply supported at the ends. Two equal and oppositely directed bending moments M_0 act at the ends of the beam, so that the beam is in pure bending. The moments act in plane *mm*, which is oriented at an angle α to the *xy* plane.

(a) Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_{max} due to the moments M_0.

(b) Repeat part a if the channels are now with their flanges pointing away from the beam flange, as shown in figure part b. Data for the beam are as follows: S 6 × 12.5 section with C 4 × 5.4 sections attached to the flanges, $M_0 = 45$ k-in., and $\alpha = 40°$. (*Note*: See Tables E-2a and E-3a of Appendix E for the dimensions and properties of the S and C shapes.)



(a) (b)

Solution 6.4-13

$M_o = 45\text{ k}\cdot\text{in.} \qquad \alpha = 40°$

S 6 × 12.5:	$I_{sy} = 1.80\text{ in.}^4$	$I_{sz} = 22.0\text{ in.}^4$
$d_s = 6.0\text{ in.}$	$b_s = 3.33\text{ in.}$	$A_s = 3.66\text{ in.}^2$
C 4 × 5.4:	$I_{cy} = 0.312\text{ in.}^4$	$I_{cz} = 3.85\text{ in.}^4$
$d_c = 4.0\text{ in.}$	$b_c = 1.58\text{ in.}$	$A_c = 1.58\text{ in.}^2$
$t_{wc} = 0.184\text{ in.}$	$c = 0.457\text{ in.}$	

(a) BUILT UP SECTION: $I_y = I_{sy} + 2I_{cz} \quad I_y = 9.50\text{ in.}^4$

$$I_z = I_{sz} + 2\left[I_{cy} + A_c\left(\frac{d_s}{2} + t_{wc} - c\right)^2\right]$$

$$I_z = 46.1\text{ in.}^4$$

$d = d_s + 2t_{wc} \qquad d = 6.368\text{ in.}$

$b = d_c \qquad b = 4.0\text{ in.}$

BENDING MOMENTS

$M_y = -M_o \sin(\alpha) \qquad M_y = -28.9 \text{ k} \cdot \text{in.}$

$M_z = M_o \cos(\alpha) \qquad M_z = 34.5 \text{ k} \cdot \text{in.}$

NEUTRAL AXIS *nn*

$$\beta = a\tan\left(\frac{I_z}{I_y}\tan(-\alpha)\right)$$

$$\beta = -76.2° \qquad \leftarrow$$

MAXIMUM TENSILE STRESS

$$\sigma_x(z, y) = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

POINT A: $z_A = -\dfrac{b}{2} \qquad y_A = -\dfrac{d}{2}$

$$\sigma_A = \sigma_x(z_A, y_A) \qquad \sigma_A = 8469 \text{ psi} \qquad \leftarrow$$

(b) BUILT UP SECTION: $I_y = I_{sy} + 2I_{cz} \qquad I_y = 9.50 \text{ in.}^4$

$$I_z = I_{sz} + 2\left[I_{cy} + A_c\left(\frac{d_s}{2} + c\right)^2\right]$$

$$I_z = 60.4 \text{ in.}^4$$

$d = d_s + 2b_c \qquad d = 9.160 \text{ in.}$

$b = d_c \qquad b = 4.000 \text{ in.}$

NEUTRAL AXIS *nn*

$$\beta = a\tan\left(\frac{I_z}{I_y}\tan(-\alpha)\right)$$

$$\beta = -79.4° \qquad \leftarrow$$

MAXIMUM TENSILE STRESS

$$\sigma_x(z, y) = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

POINT A: $z_A = -\dfrac{b}{2} \qquad y_A = -\dfrac{d}{2}$

$$\sigma_A = \sigma_x(z_A, y_A) \qquad \sigma_A = 8704 \text{ psi} \qquad \leftarrow$$

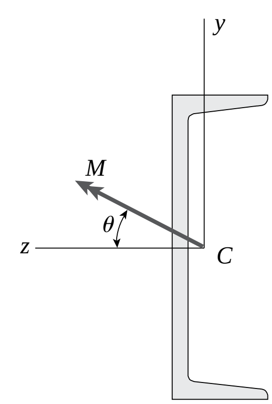
Bending of Unsymmetric Beams

When solving the problems for Section 6.5, be sure to draw a sketch of the cross section showing the orientation of the neutral axis and the locations of the points where the stresses are being found.

Problem 6.5-1 A beam of channel section is subjected to a bending moment M having its vector at an angle θ to the z axis (see figure).

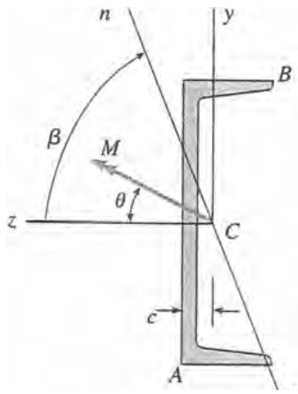
Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c in the beam.

Use the following data: C 8 × 11.5 section, M = 20 k-in., tan θ = 1/3. (*Note*: See Table E-3a of Appendix E for the dimensions and properties of the channel section.)



Probs. 6.5-1 and 6.5-2

Solution 6.5-1 Channel section



$M = 20$ k-in. $\tan\theta = 1/3$ $\theta = 18.435°$

C 8×11.5

$c = 0.571$ in. $I_y = 1.32$ in.^{4} $I_z = 32.6$ in.^{4}

$d = 8.00$ in. $b = 2.260$ in.

NEUTRAL AXIS *nn* (EQ. 6-40)

$$\tan\beta = \frac{I_z}{I_y}\tan\theta = \frac{32.6}{1.32}(1/3) = 8.2323$$

$$\beta = 83.07° \quad \leftarrow$$

MAXIMUM TENSILE STRESS (POINT *A*) (EQ. 6-38)

$z_A = c = 0.571$ in. $y_A = -d/2 = -4.00$ in.

$$\sigma_t = \sigma_A = \frac{(M\sin\theta)z_A}{I_y} - \frac{(M\cos\theta)y_A}{I_z}$$

$$= 5060 \text{ psi} \quad \leftarrow$$

MAXIMUM COMPRESSIVE STRESS (POINT *B*) (EQ. 6-38)

$z_B = -(b - c) = -(2.260 - 0.571) = -1.689$ in.

$y_z = d/2 = 4.00$ in.

$$\sigma_c = \sigma_B = \frac{(M\sin\theta)z_B}{I_y} - \frac{(M\cos\theta)y_B}{I_z}$$

$$= -10{,}420 \text{ psi} \quad \leftarrow$$

Problem 6.5-2 A beam of channel section is subjected to a bending moment M having its vector at an angle θ to the z axis (see figure).

Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c in the beam. Use a C 200×20.5 channel section with $M = 0.75$ kN·m and $\theta = 20°$.

Solution 6.5-2

$M = 0.75$ kN·m $\theta = 20°$

C 200×20.5 $I_y = 0.633 \cdot 10^6$ mm^4

$I_z = 15.0 \times 10^6$ mm^4 $d = 203$ mm $b = 59.4$ mm

$c = 14.1$ mm

BENDING MOMENTS

$M_y = M\sin(\theta)$ $M_y = 257$ N·m

$M_z = M\cos(\theta)$ $M_z = 705$ N·m

NEUTRAL AXIS *nn*

$$\beta = a\tan\left(\frac{I_z}{I_y}\tan(\theta)\right) \qquad \beta = 83.4° \quad \leftarrow$$

MAXIMUM TENSILE STRESS (AT POINT A)

$$\sigma_{max} = \frac{M_y(c)}{I_y} - \frac{M_z\left(-\frac{d}{2}\right)}{I_z}$$

$$\sigma_{max} = 10.5\,\text{MPa} \quad \leftarrow$$

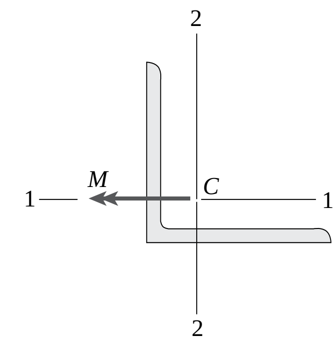
MAXIMUM TENSILE STRESS (AT POINT B)

$$\sigma_{max} = \frac{M_y(-b + c)}{I_y} - \frac{M_z\left(\frac{d}{2}\right)}{I_z}$$

$$\sigma_{max} = -23.1\,\text{MPa} \quad \leftarrow$$

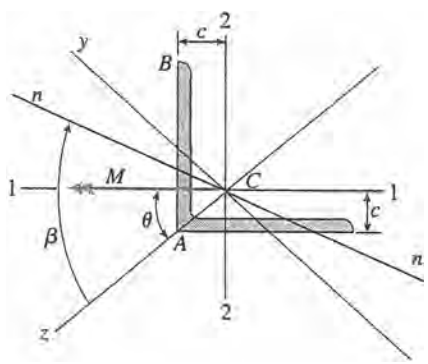
Problem 6.5-3 An angle section with equal legs is subjected to a bending moment M having its vector directed along the 1-1 axis, as shown in the figure.

Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c if the angle is an L 6 × 6 × 3/4 section and M = 20 k-in. (*Note*: See Table E-4a of Appendix E for the dimensions and properties of the angle section.)



Probs. 6.5-3 and 6.5-4

Solution 6.5-3 Angle section with equal legs



$M = 20$ k-in. L 6 × 6 × 3/4 in.

$h = b = 6$ in. $c = 1.78$ in.

$I_1 = I_2 = 28.2 \text{ in.}^4$

$\theta = 45°$ $A = 8.44 \text{ in.}^2$ $r_{\min} = 1.17$ in.

$I_y = Ar^2_{\min} = 11.55 \text{ in.}^4$

$I_z = I_1 + I_2 - I_y = 44.85 \text{ in.}^4$

NEUTRAL AXIS *nn* (EQ. 6-40)

$$\tan\beta = \frac{I_z}{I_y}\tan\theta = \frac{44.85}{11.55}\tan 45° = 3.8831$$

$$\beta = 75.56° \quad \leftarrow$$

MAXIMUM TENSILE STRESS (POINT *A*) (EQ. 6-38)

$z_A = c\sqrt{2} = 2.517$ in. $y_A = 0$

$$\sigma_t = \sigma_A = \frac{(M\sin\theta)z_A}{I_y} - \frac{(M\cos\theta)y_A}{I_z}$$

$$= 3080 \text{ psi} \quad \leftarrow$$

MAXIMUM COMPRESSIVE STRESS (POINT *B*) (EQ. 6-38)

$z_B = c\sqrt{2} - h/\sqrt{2} = -1.725$ in.

$y_B = h/\sqrt{2} = 4.243$ in.

$$\sigma_c = \sigma_B = \frac{(M\sin\theta)z_B}{I_y} - \frac{(M\cos\theta)y_B}{I_z}$$

$$= -3450 \text{ psi} \quad \leftarrow$$

Problem 6.5-4 An angle section with equal legs is subjected to a bending moment M having its vector directed along the 1–1 axis, as shown in the figure.

Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c if the section is an L 152 × 152 × 12.7 section and $M = 2.5$ kN · m. (*Note*: See Table E-4b of Appendix E for the dimensions and properties of the angle section.)

Solution 6.5-4

$M = 2.5\ \text{kN} \cdot \text{m} \qquad \theta = 45°$

$\text{L } 152 \times 152 \times 12.7 \qquad I_1 = 8.28 \times 10^6\ \text{mm}^4 \qquad I_2 = I_1$

$r_{\min} = 30.0\ \text{mm} \qquad A = 3720\ \text{mm}^2$

$I_y = A r_{\min}{}^2 \qquad I_y = 3.348 \times 10^6\ \text{mm}^4$

$I_z = I_1 + I_2 - I_y \qquad I_z = 13.212 \times 10^6\ \text{mm}^4$

$h = 152\ \text{mm} \qquad b = h \qquad c = 42.4\ \text{mm}^4$

Bending moments

$M_y = M\sin(\theta) \qquad M_y = 1768\ \text{N} \cdot \text{m}$

$M_z = M\cos(\theta) \qquad M_z = 1768\ \text{N} \cdot \text{m}$

Neutral axis *nn*

$$\beta = a\tan\left(\frac{I_z}{I_y}\tan(\theta)\right) \qquad \beta = 75.8° \qquad \leftarrow$$

Maximum tensile stress (at point A)

$$\sigma_{\max} = \frac{M_y(c\sqrt{2})}{I_y} - \frac{M_z(0)}{I_z}$$

$$\sigma_{\max} = 31.7\ \text{MPa} \qquad \leftarrow$$

Maximum tensile stress (at point B)

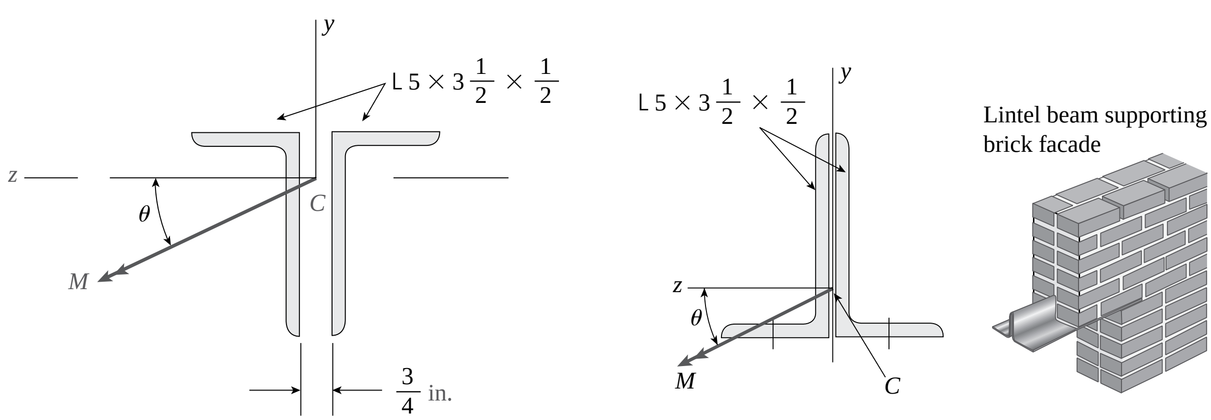
$$\sigma_{\max} = \frac{M_y\left(c\sqrt{2} - \dfrac{h}{\sqrt{2}}\right)}{I_y} - \frac{M_z\left(\dfrac{h}{\sqrt{2}}\right)}{I_z}$$

$$\sigma_{\max} = -39.5\ \text{MPa} \qquad \leftarrow$$

Problem 6.5-5 A beam made up of two unequal leg angles is subjected to a bending moment M having its vector at an angle θ to the z axis (see figure part a).

(a) For the position shown in the figure, determine the orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c in the beam. Assume that $\theta = 30°$ and $M = 30$ k-in.

(b) The two angles are now inverted and attached back-to-back to form a lintel beam which supports two courses of brick façade (see figure part b). Find the new orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c in the beam using $\theta = 30°$ and $M = 30$ k-in.



(a) (b)

Solution 6.5-5

$M = 30\,\text{k}\cdot\text{in.} \qquad \theta = 30°$

$\text{L5} \times 3\text{–}1/2 \times 1/2 \qquad I_{L1} = 10.0\,\text{in.}^4$

$I_{L2} = 4.02\ \text{in.}^4 \qquad d = 1.65\,\text{in.} \qquad c = 0.901\,\text{in.}$

$h_{L1} = 5\ \text{in.} \qquad h_{L2} = 3.5\ \text{in.}$

$A_L = 4\,\text{in.}^2 \qquad gap = \dfrac{3}{4}\ \text{in.} \qquad t = \dfrac{1}{2}\ \text{in.}$

(a) Built up section: $I_z = 2I_{L1} \qquad I_z = 20.000\ \text{in.}^4$

$$I_y = 2\left[I_{L2} + A_L\left(\frac{gap}{2} + c\right)^2\right]$$

$I_y = 21.065\ \text{in.}^4$

$h = h_{L1} \qquad h = 5.000\ \text{in.} \qquad b = gap + 2h_{L2}$

$b = 7.750\ \text{in.}$

$h_1 = d \qquad h_1 = 1.650\ \text{in.}$

Bending moments

$M_y = -M\sin(\theta) \qquad M_y = -1.250\ \text{k}\cdot\text{ft}$

$M_z = M\cos(\theta) \qquad M_z = 2.165\ \text{k}\cdot\text{ft}$

Neutral axis nn

$$\beta = a\tan\left(\frac{I_z}{I_y}\tan(-\theta)\right)$$

$\beta = -28.7° \qquad \leftarrow$

Maximum tensile stress

$$\sigma_x(z, y) = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

Point A: $z_A = -\dfrac{gap}{2} + t \qquad y_A = -h + h_1$

$\sigma_t = \sigma_x(z_A, y_A) \qquad \sigma_t = 4263\ \text{psi} \qquad \leftarrow$

Maximum compressive stress

Point B: $z_B = \dfrac{b}{2} \qquad y_B = h_1$

$\sigma_c = \sigma_x(z_B, y_B) \qquad \sigma_c = -4903\ \text{psi} \qquad \leftarrow$

(b) Built up section: $I_z = 2I_{L1} \qquad I_z = 20.000\ \text{in.}^4$

$I_y = 2(I_{L2} + A_L c^2) \qquad I_y = 14.534\ \text{in.}^4$

$h = h_{L1} \qquad h = 5.000\ \text{in.} \qquad b = 2h_{L2}$

$b = 7.000\ \text{in.} \qquad h_1 = h - d \qquad h_1 = 3.350\ \text{in.}$

Neutral axis nn

$$\beta = a\tan\left(\frac{I_z}{I_y}\tan(-\theta)\right)$$

$\beta = -38.5\,\text{deg} \qquad \leftarrow$

Maximum tensile stress

$$\sigma_x(z, y) = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

Point A: $z_A = -\dfrac{b}{2} \qquad y_A = -h + h_1$

$\sigma_t = \sigma_x(z_A, y_A) \qquad \sigma_t = 5756\ \text{psi} \qquad \leftarrow$

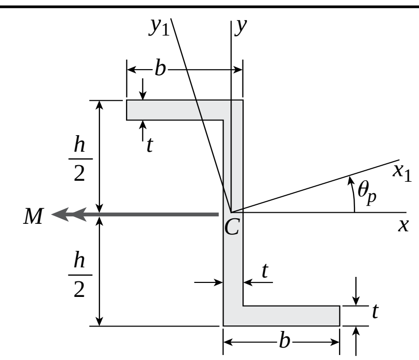
Maximum compressive stress

Point B: $z_B = t \qquad y_B = h_1$

$\sigma_c = \sigma_x(z_B, y_B) \qquad \sigma_c = -4868\ \text{psi} \qquad \leftarrow$

Problem 6.5-6 The Z-section of Example 12-7 is subjected to $M = 5\ \text{kN}\cdot\text{m}$, as shown.

Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c in the beam. Use the following numerical data: height $h = 200$ mm, width $b = 90$ mm, constant thickness $t = 15$ mm, and $\theta_p = 19.2°$. Use $I_1 = 32.6 \times 10^6\ \text{mm}^4$ and $I_2 = 2.4 \times 10^6\ \text{mm}^4$ from Example 12-7.



Solution 6.5-6

$M = 5\ \text{kN}\cdot\text{m} \qquad \theta = 19.2°$

Z-SECTION

$I_{zp} = I_1 \qquad I_{zp} = 32.6 \times 10^6\ \text{mm}^4$

$I_{yp} = I_2 \qquad I_{yp} = 2.4 \times 10^6\ \text{mm}^4$

$h = 200\ \text{mm} \qquad b = 90\ \text{mm} \qquad t = 15\ \text{mm}$

BENDING MOMENTS

$M_{yp} = M\sin(\theta) \qquad M_{yp} = 1644\ \text{N}\cdot\text{m}$

$M_{zp} = M\cos(\theta) \qquad M_{zp} = 4722\ \text{N}\cdot\text{m}$

NEUTRAL AXIS *nn*

$$\beta = a\tan\left(\frac{I_{zp}}{I_{yp}}\tan(\theta)\right) \qquad \beta = 78.1° \qquad \leftarrow$$

MAXIMUM TENSILE STRESS (AT POINT A)

$$zp_A = \left(\frac{t}{2}\right)\cos(\theta) - \left(\frac{-h}{2}\right)\sin(\theta) \qquad zp_A = 39.97\ \text{mm}$$

$$yp_A = \left(\frac{t}{2}\right)\sin(\theta) + \left(\frac{-h}{2}\right)\cos(\theta)$$

$$yp_A = -91.97\ \text{mm}$$

$$\sigma_t = \frac{M_{yp}(zp_A)}{I_{yp}} - \frac{M_{zp}(yp_A)}{I_{zp}}$$

$$\sigma_t = 40.7\ \text{MPa} \qquad \leftarrow$$

MAXIMUM COMPRESSIVE STRESS (AT POINT B)

$$zp_B = \left(-\frac{t}{2}\right)\cos(\theta) - \left(\frac{h}{2}\right)\sin(\theta)$$

$$zp_B = -39.97\ \text{mm}$$

$$yp_B = \left(-\frac{t}{2}\right)\sin(\theta) + \left(\frac{h}{2}\right)\cos(\theta)$$

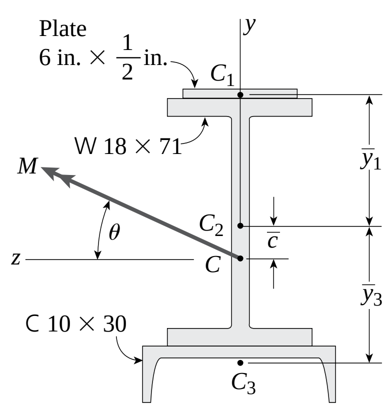
$$yp_B = 91.97\ \text{mm}$$

$$\sigma_c = \frac{M_{yp}(zp_B)}{I_{yp}} - \frac{M_{zp}(yp_B)}{I_{zp}}$$

$$\sigma_c = -40.7\ \text{MPa} \qquad \leftarrow$$

Problem 6.5-7 The cross section of a steel beam is constructed of a W 18 × 71 wide-flange section with a 6 in × 1/2 in. cover plate welded to the top flange and a C 10 × 30 channel section welded to the bottom flange. This beam is subjected to a bending moment M having its vector at an angle θ to the z axis (see figure).

Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c in the beam. Assume that $\theta = 30°$ and $M = 75$ k-in. (*Note*: The cross sectional properties of this beam were computed in Examples 12-2 and 12-5.)



Solution 6.5-7

$M = 75\ \text{k} \cdot \text{in.} \qquad \theta = 30°$

PLATE: $b_p = \dfrac{1}{2}\ \text{in.} \qquad h_p = 6\ \text{in.}$

$$I_{\text{plate}} = \frac{b_p h_p^3}{12} \qquad I_{\text{plate}} = 9.00\ \text{in.}^4$$

W SECTION: $h_w = 18.47\ \text{in.} \qquad b_w = 7.635\ \text{in.}$

$I_{wy} = 60.3\ \text{in.}^4$

C SECTION: $h_c = 10.0\ \text{in.} \qquad b_c = 3.033\ \text{in.}$

$I_{cz} = 103\ \text{in.}^4$

BUILT-UP SECTION:

$c_{\text{bar}} = 1.80\ \text{in.}$

$I_z = 2200\ \text{in.}^4 \qquad I_y = I_{wy} + I_{cz} + I_{\text{plate}}$

$I_y = 172.3\ \text{in.}^4$

BENDING MOMENTS

$M_y = M\sin(\theta) \qquad M_y = 3.125\ \text{k} \cdot \text{ft}$

$M_z = M\cos(\theta) \qquad M_z = 5.413\ \text{k} \cdot \text{ft}$

NEUTRAL AXIS *nn*

$$\beta = a\tan\left(\frac{I_z}{I_y}\tan(\theta)\right) \qquad \beta = 82.3° \qquad \longleftarrow$$

MAXIMUM TENSILE STRESS

$$\sigma_x(z, y) = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

POINT A: $z_A = \dfrac{h_c}{2} \qquad y_A = -\dfrac{h_w}{2} - b_c + c_{\text{bar}}$

$$\sigma_t = \sigma_x(z_A, y_A) \qquad \sigma_t = 1397\ \text{psi} \qquad \longleftarrow$$

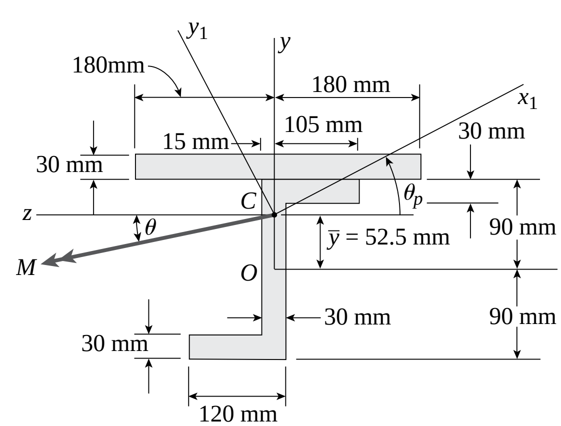
MAXIMUM COMPRESSIVE STRESS

POINT B: $z_B = -\dfrac{b_w}{2} \qquad y_B = \dfrac{h_w}{2} + c_{\text{bar}}$

$$\sigma_c = \sigma_x(z_B, y_B) \qquad \sigma_c = -1157\ \text{psi} \qquad \longleftarrow$$

Problem 6.5-8 The cross section of a steel beam is shown in the figure. This beam is subjected to a bending moment M having its vector at an angle θ to the z axis.

Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c in the beam. Assume that $\theta = 22.5°$ and $M = 4.5\ \text{kN} \cdot \text{m}$. Use cross sectional properties $I_{x1} = 93.14 \times 10^6\ \text{mm}^4$, $I_{y1} = 152.7 \times 10^6\ \text{mm}^4$, and $\theta_p = 27.3°$.



Solution 6.5-8

$M = 4.5\ \text{kN} \cdot \text{m} \qquad \theta = 22.5°$

BUILT-UP SECTION:

$b = 120\,\text{mm} \qquad t = 30\,\text{mm}$

$h = 180\,\text{mm} \qquad b_1 = 360\,\text{mm}$

$$y_{\text{bar}} = \frac{tb_1\left(\dfrac{t+h}{2}\right)}{tb_1 + [2tb + (h - 2t)t]}$$

$y_{\text{bar}} = 52.5\,\text{mm}$

$\theta_P = 27.3\,\text{deg}$

$I_{zp} = I_{x1} \qquad I_{yp} = I_{y1}$

$I_{zp} = 93.14 \times 10^6\ \text{mm}^4 \qquad I_{yp} = 152.7 \times 10^6\,\text{mm}^4$

BENDING MOMENTS

$M_{yp} = M\sin(\theta_P - \theta) \qquad M_{yp} = 377\ \text{N} \cdot \text{m}$

$M_{zp} = M\cos(\theta_P - \theta) \qquad M_{zp} = 4484\ \text{N} \cdot \text{m}$

NEUTRAL AXIS *nn*

$$\beta = a\tan\left(\frac{I_{zp}}{I_{yp}}\tan(\theta_P - \theta)\right)$$

$\beta = 2.93° \qquad \leftarrow$

MAXIMUM TENSILE STRESS (AT POINT A)

$$z_A = -\frac{t}{2} \qquad y_A = -\frac{h}{2} - y_{bar}$$

$zp_A = (z_A)\cos(\theta_P) - (y_A)\sin(\theta_P) \qquad zp_A = 52.03\ \text{mm}$

$yp_A = (z_A)\sin(\theta_P) - (y_A)\cos(\theta_P)$

$yp_A = -133.51\ \text{mm}$

$$\sigma_t = \frac{M_{yp}(zp_A)}{I_{yp}} - \frac{M_{zp}(yp_A)}{I_{zp}}$$

$\sigma_t = 6.56\ \text{MPa} \qquad \leftarrow$

MAXIMUM COMPRESSIVE STRESS (AT POINT B)

$$z_B = \frac{b_1}{2} \qquad y_B = \frac{h}{2} + t - y_{\text{bar}}$$

$zp_B = (z_B)\cos(\theta_P) - (y_B)\sin(\theta_P) \qquad zp_B = 128.99\ \text{mm}$

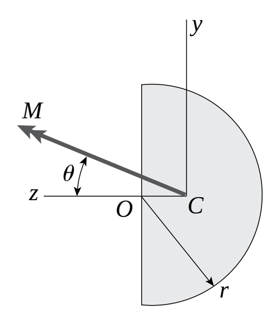
$yp_B = (z_B)\sin(\theta_P) + (y_B)\cos(\theta_P) \qquad yp_B = 142.5\ \text{mm}$

$$\sigma_c = \frac{M_{yp}(zp_B)}{I_{yp}} - \frac{M_{zp}(yp_B)}{I_{zp}}$$

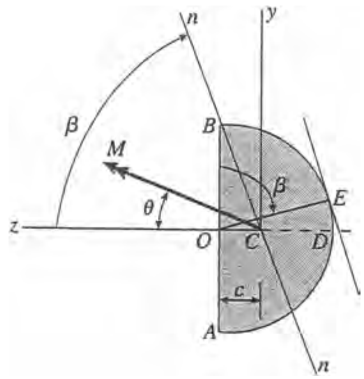
$\sigma_c = -6.54\ \text{MPa} \qquad \leftarrow$

Problem 6.5-9 A beam of semicircular cross section of radius r is subjected to a bending moment M having its vector at an angle θ to the z axis (see figure).

Derive formulas for the maximum tensile σ, and the maximum compressive stress σ_c in the beam for $\theta = 0$, 45°, and 90°. (*Note*: Express the results in the from $\alpha\ M/r^3$, where α is a numerical value.)



Soultion 6.5.-9 Semicircle



r = radius

$$c = \frac{4r}{3\pi} = 0.42441r$$

$$I_y = \frac{(9\pi^2 - 64)}{72\pi} r^4$$

$$= 0.109757r^4$$

$$I_z = \frac{\pi r^4}{8}$$

σ_t = maximum tensile stress

σ_c = maximum compressive stress

For $\theta = 0°$: $\sigma_t = \sigma_A = \frac{Mr}{I_z} = \frac{8M}{\pi r^3}$

$$= 2.546 \frac{M}{r^3} \quad \leftarrow$$

$$\sigma_c = \sigma_B = -\sigma_A = -\frac{8M}{\pi r^3}$$

$$= -2.546 \frac{M}{r^3} \quad \leftarrow$$

For $\theta = 90°$: $\sigma_t = \sigma_o = \frac{Mc}{I_y}$

$$= 3.867 \frac{M}{r^3} \quad \leftarrow$$

$$\sigma_c = \sigma_D = \frac{M(r - c)}{I_y}$$

$$= -5.244 \frac{M}{r^3} \quad \leftarrow$$

For $\theta = 45°$: Eq. (6–40): $\tan \beta = \frac{I_z}{I_y} \tan \theta$

$$\tan \beta = \frac{9\pi^2}{9\pi^2 - 64}(1) = 3.577897$$

$$\beta = 74.3847°$$

$$90° - \beta = 15.6153°$$

Maximum Tensile stress for $\theta = 45°$ occurs at point A.

$z_A = c = 0.42441r \qquad y_A = -r$

From (Eq. 6-38):

$$\sigma_t = \sigma_A = \frac{(M \sin \theta) z_A}{I_y} - \frac{(M \cos \theta) y_A}{I_z}$$

$$= 4.535 \frac{M}{r^3} \quad \leftarrow$$

Maximum compressive stress for $\theta = 45°$ occurs at point E. where the tangent to the circle is parallel to the **neutral** axis nn.

$z_E = c - r \cos(90° - \beta) = -0.53868r$

$y_E = r \sin(90° - \beta) = 0.26918r$

From (Eq. 6-38):

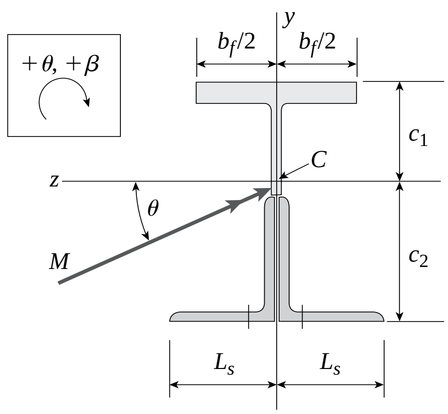
$$\sigma_C = \sigma_E = \frac{(M \sin \theta) z_E}{I_y} - \frac{(M \cos \theta) y_E}{I_z}$$

$$= -3.955 \frac{M}{r^3} \quad \leftarrow$$

Problem 6.5-10 A built-up beam supporting a condominium balcony is made up of a structural T (one half of a W 200 × 31.3) for the top flange and web and two angles (2L 102 × 76 × 6.4, long legs back-to-back) for the bottom flange and web, as shown. The beam is subjected to a bending moment M having its vector at an angle θ to the z axis (see figure).

Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c in the beam. Assume that $\theta = 30°$ and $M = 15$ kN·m.

Use the following numerical properties: $c_1 = 4.111$ mm, $c_2 = 4.169$ mm, $b_f = 134$ mm, $L_s = 76$ mm, $A = 4144$ mm^2, $I_y = 3.88 \times 10^6$ mm^4, and $I_z = 34.18 \times 10^6$ mm^4.



Solution 6.5-10

$M = 15$ kN·m $\quad \theta = 30°$

BUILT-UP SECTION:

$c_1 = 4.111$ mm $\quad c_2 = 4.169$ mm $\quad b_f = 134$ mm

$L_s = 76$ mm $\quad A = 4144$ mm^2

$I_y = 3.88 \times 10^6$ mm^4 $\quad I_z = 34.18 \times 10^6$ mm^4

BENDING MOMENTS

$M_y = M\sin(180° - \theta) \qquad M_y = 7500$ N·m

$M_z = M\cos(180° - \theta) \qquad M_z = -12990$ N·m

NEUTRAL AXIS *nn*

$$\beta = a\tan\left(\frac{I_z}{I_y}\tan(180° - \theta)\right)$$

$\beta = -78.9°$ ←

MAXIMUM TENSILE STRESS (AT POINT A)

$$z_A = \frac{b_f}{2} \qquad y_A = c_1$$

$$\sigma_t = \frac{M_y(z_A)}{I_y} - \frac{M_z(y_A)}{I_z}$$

$\sigma_t = 131.1$ MPa ←

MAXIMUM COMPRESSIVE STRESS (AT POINT B)

$$z_B = -L_s \qquad y_B = -c_2$$

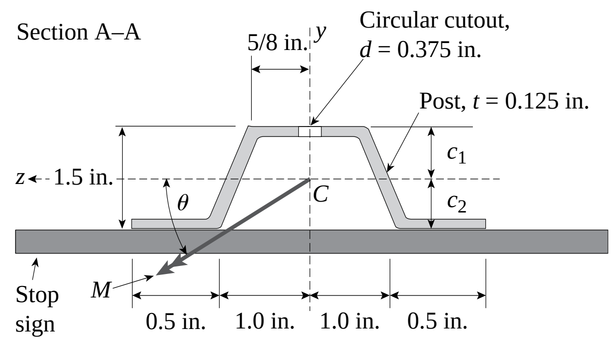
$$\sigma_c = \frac{M_y(z_B)}{I_y} - \frac{M_z(y_B)}{I_z}$$

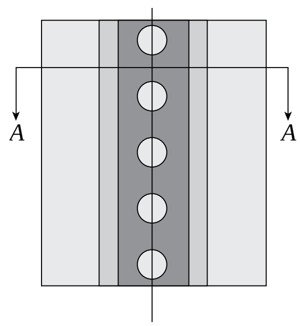
$\sigma_c = -148.5$ MPa ←

Problem 6.5-11 A steel post ($E = 30 \times 10^6$ psi) having thickness $t = 1/8$ in. and height $L = 72$ in. supports a stop sign (see figure). The stop sign post is subjected to a bending moment M having its vector at an angle θ to the z axis.

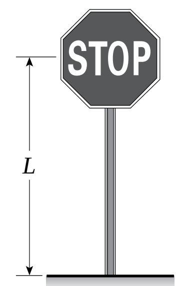
Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c in the beam. Assume that $\theta = 30°$ and $M = 5.0$ k-in.

Use the following numerical properties for the post: $A = 0.578$ in^2, $c_1 = 0.769$ in., $c_2 = 0.731$ in., $I_y = 0.44867$ in^4, and $I_z = 0.16101$ in^4.





Elevation
view of post



Solution 6.5-11

$M = 5\ \text{k} \cdot \text{in} \qquad \theta = -30°$

Post: $A = 0.578\ \text{in.}^2 \qquad c_1 = 0.769$ in.

$c_2 = 0.731$ in.

$I_y = 0.44867\ \text{in.}^4 \qquad I_z = 0.16101\ \text{in.}^4$

Bending moments

$M_y = M\sin(\theta) \qquad M_y = -0.208\ \text{k} \cdot \text{ft}$

$M_z = M\cos(\theta) \qquad M_z = 0.361\ \text{k} \cdot \text{ft}$

Neutral axis nn

$$\beta = atan\left(\frac{I_z}{I_y}\tan(\theta)\right) \qquad \beta = -11.7° \qquad \longleftarrow$$

Maximum tensile stress

$$\sigma_x(z, y) = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

Point A: $z_A = -1.5$ in. $\qquad y_A = -c_2$

$\sigma_t = \sigma_x(z_A, y_A) \qquad \sigma_t = 28.0\ \text{ksi} \qquad \longleftarrow$

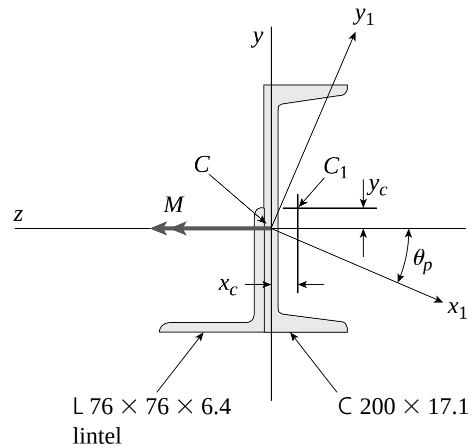
Maximum compressive stress

Point B: $z_B = \frac{5}{8}$ in. $\qquad y_B = c_1$

$\sigma_c = \sigma_x(z_B, y_B) \qquad \sigma_c = -24.2\ \text{ksi} \qquad \longleftarrow$

Problem 6.5-12 A C 200 × 17.1 channel section has an angle with equal legs attached as shown; the angle serves as a lintel beam. The combined steel section is subjected to a bending moment M having its vector directed along the z axis, as shown in the figure. The centroid C of the combined section is located at distances x_c and y_c from the centroid (C_1) of the channel alone. Principal axes x_1 and y_1 are also shown in the figure and properties I_{x1}, I_{y1} and θ_p are given below.

Find the orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c if the angle is an L 76 × 76 × 6.4 section and $M = 3.5$ kN·m. Use the following properties for principal axes for the combined section: $I_{x1} = 18.49 \times 10^6$ mm^4 $I_{y1} = 1.602 \times 10^6$ mm^4, $\theta_p = 7.448°$ (CW), $x_c = 10.70$ mm, $y_c = 24.07$ mm.



Solution 6.5-12

$M = 3.5\text{ kN}\cdot\text{m} \qquad \theta_p = 7.448°$

Angle: $c_a = 21.2$ mm $\qquad L_a = 76$ mm

Channel: $c_c = 14.5$ mm $\qquad d_c = 203$ mm

$b_c = 57.4$ mm

Built-up section:

$y_{bar} = 24.07\text{ mm} \qquad x_{bar} = 10.70\text{ mm}$

$I_{zp} = I_{x1} \qquad I_{yp} = I_{y1}$

$I_{zp} = 18.49 \times 10^6\text{ mm}^4 \qquad I_{yp} = 1.602 \times 10^6\text{ mm}^4$

Bending moments

$M_{yp} = M\sin(-\theta_p) \qquad M_{yp} = -454\text{ N}\cdot\text{m}$

$M_{zp} = M\cos(-\theta_p) \qquad M_{zp} = 3470\text{ N}\cdot\text{m}$

Neutral axis nn

$$\beta = a\tan\left(\frac{I_{zp}}{I_{yp}}\tan(-\theta_p)\right) \qquad \beta = -56.5° \quad \leftarrow$$

Maximum tensile stress (at point A)

$$z_A = -x_{bar} + c_c - b_c \qquad y_A = -\frac{d_c}{2} + y_{bar}$$

$zp_A = (z_A)\cos(-\theta_p) - (y_A)\sin(-\theta_p)$

$zp_A = -63.18\text{ mm}$

$yp_A = (z_A)\sin(-\theta_p) + (y_A)\cos(-\theta_p)$

$yp_A = -69.83\text{ mm}$

$$\sigma_t = \frac{M_{yp}(zp_A)}{I_{yp}} - \frac{M_{zp}(yp_A)}{I_{zp}}$$

$\sigma_t = 31.0\text{ MPa} \quad \leftarrow$

Maximum compressive stress (at point B)

$$z_B = -x_{bar} + c_c \qquad y_B = \frac{d_c}{2} + y_{bar}$$

$zp_B = (z_B)\cos(-\theta_p) - (y_B)\sin(-\theta_p)$

$zp_B = 20.05\text{ mm}$

$yp_B = (z_B)\sin(-\theta_p) + (y_B)\cos(-\theta_p)$

$yp_B = 124.0\text{ mm}$

$$\sigma_c = \frac{M_{yp}(zp_B)}{I_{yp}} - \frac{M_{zp}(yp_B)}{I_{zp}}$$

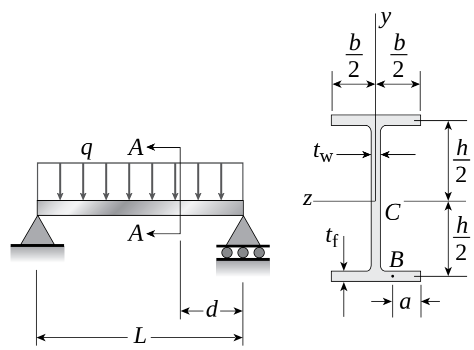
$\sigma_c = -29.0\text{ MPa} \quad \leftarrow$

Shear Stresses in Wide-Flange Beams

When solving the problems for Section 6.8, assume the cross sections are thin-walled. Use centerline dimensions for all calculations and derivations, unless otherwise specified

Problem 6.8-1 A simple beam of W 10 × 30 wide-flange cross section supports a uniform load of intensity $q = 3.0$ k/ft on a span of length $L = 12$ ft (see figure). The dimensions of the cross section are $h = 10.5$ in., $b = 5.81$ in., $t_f = 0.510$ in., and $t_w = 0.300$ in.

(a) Calculate the maximum shear stress τ_{max} on cross section A–A located at distance $d = 2.5$ ft from the end of the beam.
(b) Calculate the shear stress τ at point B on the cross section. Point B is located at a distance $a = 1.5$ in. from the edge of the lower flange.



Probs. 6.8-1 and 6.8-2

Solution 6.8-1

SIMPLE BEAM:

$q = 3.0 \text{ k/ft} \qquad L = 12 \text{ ft}$

$R = \dfrac{qL}{2} \qquad R = 18.0 \text{ k} \qquad d = 2.5 \text{ ft}$

$V = |R - qd| \qquad V = 10.5 \text{ k}$

CROSS SECTION:

$h = 10.5 \text{ in.} \qquad b = 5.81 \text{ in.} \qquad t_f = 0.510 \text{ in.}$

$t_w = 0.30 \text{ in.}$

$I_z = \dfrac{t_w h^3}{12} + \dfrac{b t_f h^2}{2} \qquad I_z = 192.28 \text{ in.}^4$

(a) MAXIMUM SHEAR STRESS

$$\tau_{max} = \left(\frac{b t_f}{t_w} + \frac{h}{4}\right)\frac{Vh}{2I_z}$$

$\tau_{max} = 3584 \text{ psi} \qquad \longleftarrow$

(b) SHEAR STRESS AT POINT B

$a = 1.5 \text{ in.} \qquad \dfrac{b}{2} = 2.9 \text{ in.}$

$\tau_1 = \dfrac{bhV}{4I_z} \qquad \tau_1 = 832.8 \text{ psi}$

$\tau_B = \dfrac{a}{\frac{b}{2}}(\tau_1) \qquad \tau_B = 430 \text{ psi} \qquad \longleftarrow$

Problem 6.8-2 Solve the preceding problem for a W 250 × 44.8 wide-flange shape with the following data: $L = 3.5$ m, $q = 45$ kN/m, $h = 267$ mm, $b = 148$ mm, $t_f = 13$ mm, $t_w = 7.62$ mm, $d = 0.5$ m, and $a = 50$ mm.

Solution 6.8-2

SIMPLE BEAM:

$q = 45\ \text{kN/m} \quad L = 3.5\ \text{m}$

$R = \dfrac{qL}{2} \qquad R = 78.8\ \text{kN} \quad d = 0.5\ \text{m}$

$V = |R - qd| \qquad V = 56.3\ \text{kN}$

CROSS SECTION:

$h = 267\ \text{mm} \qquad b = 148\ \text{mm}$

$t_f = 13\ \text{mm} \qquad t_w = 7.62\ \text{mm}$

$I_z = \dfrac{t_w h^3}{12} + \dfrac{b t_f h^2}{2}$

$I_z = 80.667 \times 10^6\ \text{mm}^4$

(a) MAXIMUM SHEAR STRESS

$\tau_{max} = \left(\dfrac{b t_f}{t_w} + \dfrac{h}{4}\right)\dfrac{Vh}{2I_z}$

$\tau_{max} = 29.7\ \text{MPa} \quad \leftarrow$

(b) SHEAR STRESS AT POINT B

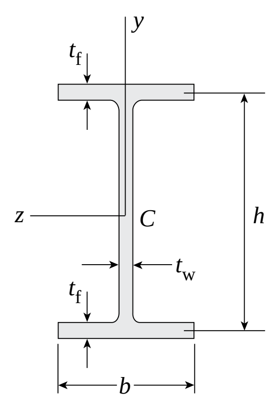
$a = 50\ \text{mm} \qquad b/2 = 74.0\ \text{mm}$

$\tau_1 = \dfrac{bhV}{4I_z} \qquad \tau_1 = 999.1\ \text{psi}$

$\tau_B = \dfrac{a}{b/2}(\tau_1) \qquad \tau_B = 4.65\ \text{MPa} \quad \leftarrow$

Problem 6.8-3 A beam of wide-flange shape, W 8 × 28, has the cross section shown in the figure. The dimensions are $b = 6.54$ in., $h = 8.06$ in., $t_w = 0.285$ in., and $t_f = 0.465$ in. The loads on the beam produce a shear force $V = 7.5$ k at the cross section under consideration.

(a) Using centerline dimensions, calculate the maximum shear stress τ_{max} in the web of the beam.
(b) Using the more exact analysis of Section 5.10 in Chapter 5, calculate the maximum shear stress in the web of the beam and compare it with the stress obtained in part a.



Probs. 6.8-3 and 6.8-4

Solution 6.8-3

$b = 6.54\ \text{in.} \qquad h = 8.06\ \text{in.} \qquad t_w = 0.285\ \text{in.}$

$t_f = 0.465\ \text{in.} \qquad V = 7.5\ \text{k}$

(a) CALCULATIONS BASED ON CENTERLINE DIMENSIONS

Moment of inertia:

$I_z = \dfrac{t_w h^3}{12} + \dfrac{b t_f h^2}{2}$

$I_z = 111.216\ \text{in.}^4$

Maximum shear stress in the web:

$\tau_{max} = \left(\dfrac{b t_f}{t_w} + \dfrac{h}{4}\right)\dfrac{Vh}{2I_z}$

$\tau_{max} = 3448\ \text{psi} \quad \leftarrow$

(b) CALCULATIONS BASED ON MORE EXACT ANALYSIS

$h_2 = h + t_f \qquad h_2 = 8.5\ \text{in.} \qquad h_1 = h - t_f$

$h_1 = 7.6\ \text{in.}$

Moment of inertia:

$$I = \frac{1}{12}(bh_2^3 - bh_1^3 + t_w h_1^3)$$

$$I = 109.295 \text{ in.}^4$$

Maximum shear stress in the web:

$$\tau_{max} = \frac{V}{8It_w}(bh_2^2 - bh_1^2 + t_w h_1^2)$$

$$\tau_{max} = 3446 \text{ psi} \quad \leftarrow$$

Problem 6.8-4 Solve the preceding problem for a W 200 × 41.7 shape with the following data: $b = 166$ mm, $h = 205$ mm, $t_w = 7.24$ mm, $t_f = 11.8$ mm, and $V = 38$ kN.

Solution 6.8-4

$b = 166$ mm $\quad h = 205$ mm $\quad t_w = 7.24$ mm

$t_f = 11.8$ mm $\quad V = 38$ kN

(a) CALCULATIONS BASED ON CENTERLINE DIMENSIONS

Moment of inertia:

$$I_z = \frac{t_w h^3}{12} + \frac{bt_f h^2}{2}$$

$$I_z = 46.357 \times 10^6 \text{ mm}^4$$

Maximum shear stress in the web:

$$\tau_{max} = \left(\frac{bt_f}{t_w} + \frac{h}{4}\right)\frac{Vh}{2I_z}$$

$$\tau_{max} = 27.04 \text{ MPa} \quad \leftarrow$$

(b) CALCULATIONS BASED ON MORE EXACT ANALYSIS

$h_2 = h + t_f \quad h_2 = 216.8$ mm $\quad h_1 = h - t_f$

$h_1 = 193.2$ mm

Moment of inertia:

$$I = \frac{1}{12}(bh_2^3 - bh_1^3 + t_w h_1^3)$$

$$I = 45.556 \times 10^6 \text{ mm}^4$$

Maximum shear stress in the web:

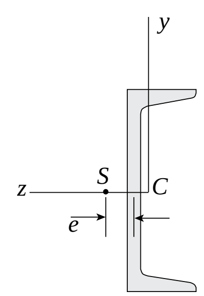
$$\tau_{max} = \frac{V}{8It_w}(bh_2^2 - bh_1^2 + t_w h_1^2)$$

$$\tau_{max} = 27.02 \text{ MPa} \quad \leftarrow$$

Shear Centers of Thin-Walled Open Sections

When locating the shear centers in the problems for Section 6.9, assume that the cross sections are thin-walled and use centerline dimensions for all calculations and derivations.

Problem 6.9-1 Calculate the distance e from the centerline of the web of a C 15 × 40 channel section to the shear center S (see figure). (*Note*: For purposes of analysis, consider the flanges to be rectangles with thickness t_f equal to the average flange thickness given in Table E-3a in Appendix E.)



Probs. 6.9-1 and 6.9-2

Solution 6.9-1

C 15×40 $\quad d = 15.0$ in. $\quad t_w = 0.520$ in. $\quad h = d - t_f \quad h = 14.350$ in.

$b_f = 3.520$ in. $\quad t_f = 0.650$ in. $\quad b = b_f - \dfrac{t_w}{2}$ $\quad e = \dfrac{3b^2 t_f}{ht_w + 6bt_f} \quad e = 1.027$ in. $\leftarrow$

$b = 3.260$ in.

Problem 6.9-2 Calculate the distance e from the centerline of the web of a C 310×45 channel section to the shear center S (see figure). (*Note*: For purposes of analysis, consider the flanges to be rectangles with thickness t_f equal to the average flange thickness given in Table E-3b in Appendix E.)

Solution 6.9-2

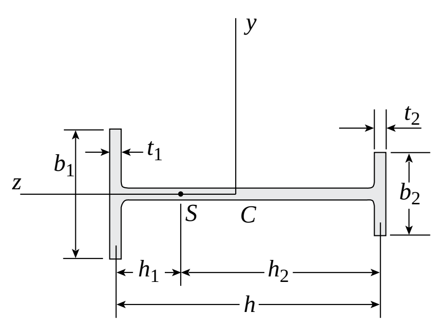
C 310×45 $\quad d = 305$ mm $\quad t_w = 13.0$ mm $\quad b = 74.0$ mm $\quad h = d - t_f \quad h = 292.3$ mm

$b_f = 80.5$ mm $\quad t_f = 12.7$ mm $\quad b = b_f - \dfrac{t_w}{2}$ $\quad e = \dfrac{3b^2 t_f}{ht_w + 6bt_f} \quad e = 22.1$ mm $\leftarrow$

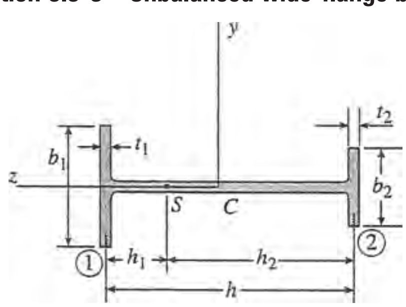
Problem 6.9-3 The cross section of an unbalanced wide-flange beam is shown in the figure. Derive the following formula for the distance h_1 from the centerline of one flange to the shear center S:

$$h_1 = \frac{t_2 b_2^3 h}{t_1 b_1^3 + t_2 b_2^3}$$

Also, check the formula for the special cases of a T-beam ($b_2 = t_2 = 0$) and a balanced wide-flange beam ($t_2 = t_1$ and $b_2 = b_1$).



Solution 6.9-3 Unbalanced wide-flange beam



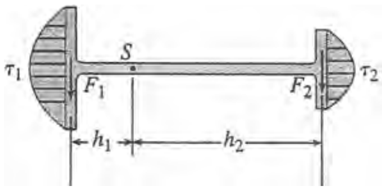
FLANGE 1:

$$\tau_1 = \frac{VQ}{I_z t_1}$$

$$Q = (b_1/2)(t_1)(b_1/4) = \frac{t_1 b_1^2}{8}$$

$$\tau_1 = \frac{Vb_1^2}{8I_z}$$

$$F_1 = \frac{2}{3}(\tau_1)(b_1)(t_1) = \frac{Vt_1 b_1^3}{12I_z}$$



FLANGE 2:

$$F_2 = \frac{Vt_2b_2^3}{12I_z}$$

Shear force V acts through the shear center S.

$$\therefore \sum M_S = F_1h_1 - F_2h_2 = 0$$

or $(t_1b_1^3)\,h_1 = (t_2b_2^3)\,h_2$ (1)

$h_1 + h_2 = h$ (2)

Solve Eqs. (1) and (2): $h_1 = \dfrac{t_2b_2^3h}{t_1b_1^3 + t_2b_2^3}$ ←

T-BEAM

$b_2 = t_2 = 0;$

$\therefore h_1 = 0$ ←

WIDE-FLANGE BEAM

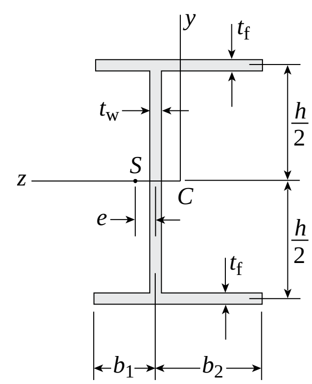
$t_2 = t_1$ and $b_2 = b_1$;

$\therefore h_1 = h/2$ ←

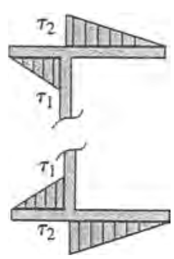
Problem 6.9-4 The cross section of an unbalanced wide-flange beam is shown in the figure. Derive the following formula for the distance e from the centerline of the web to the shear center S:

$$e = \frac{3t_f(b_2^2 - b_1^2)}{ht_w + 6t_f(b_1 + b_2)}$$

Also, check the formula for the special cases of a channel section ($b_1 = 0$ and $b_2 = b$) and a doubly symmetric beam ($b_1 = b_2 = b/2$).



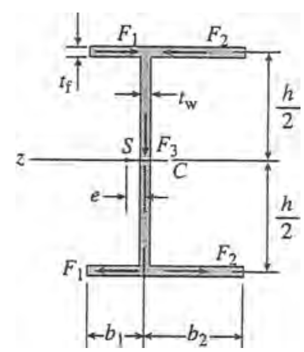
Solution 6.9-4 Unbalanced wide-flange beam



$$\tau_1 = \frac{VQ}{It_f} = \frac{b_1hV}{2I_z} \qquad \tau_2 = \frac{b_2hV}{2I_z}$$

$$F_1 = \frac{b_1\tau_1t_f}{2} = \frac{b_1^2ht_fV}{4I_z}$$

$$F_2 = \frac{b_2^2ht_fV}{4I_z} \qquad F_3 = V$$



Shear force V acts through the shear center S.

$$\therefore \sum M_S = -F_3e - F_1h + F_2h = 0$$

$$e = \frac{F_2h - F_1h}{F_3} = \frac{h^2t_f}{4I_z}(b_2^2 - b_1^2)$$

$$I_z = \frac{t_w h^3}{12} + 2(b_1 + b_2)(t_f)\left(\frac{h}{2}\right)^2$$

$$= \frac{h^2}{12}[ht_w + 6\,t_f(b_1 + b_2)]$$

$$e = \frac{3t_f(b_2^2 - b_1^2)}{ht_w + 6t_f(b_1 + b_2)} \quad \leftarrow$$

Channel section ($b_1 = 0$, $b_2 = b$)

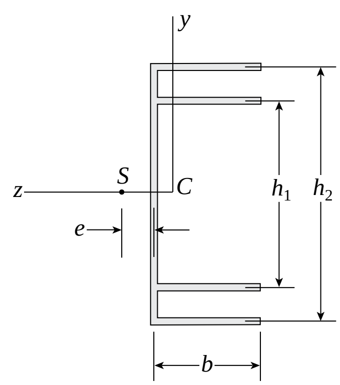
$$e = \frac{3b^2 t_f}{ht_w + 6bt_f} \quad \text{(Eq. 6-65)}$$

Doubly symmetric beam ($b_1 = b_2 = b/2$)
$e = 0$ (Shear center coincides with the centroid)

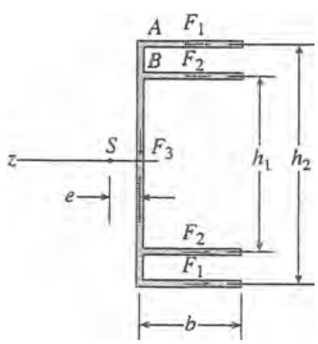
Problem 6.9-5 The cross section of a channel beam with double flanges and constant thickness throughout the section is shown in the figure.

Derive the following formula for the distance e from the centerline of the web of the shear center S:

$$e = \frac{3b^2(h_1^2 + h_2^2)}{h_2^3 + 6b(h_1^2 + h_2^2)}$$



Solution 6.9-5 Channel beam with double flanges



t = thickness

$$\tau_A = \frac{VQ_A}{I_z t} = \frac{V(bt)\left(\frac{h_2}{2}\right)}{I_z t} = \frac{bh_2 V}{2I_z}$$

$$F_1 = \frac{1}{2}\tau_A bt = \frac{b^2 h_2 tV}{4I_z}$$

$$\tau_B = \frac{bh_1 V}{2I_z} \qquad F_2 = \frac{b^2 h_1 tV}{4I_z}$$

$$F_3 = V$$

Shear force V acts through the shear center S.

$$\therefore \sum M_S = -F_3 e + F_1 h_2 + F_2 h_1 = 0$$

$$e = \frac{F_2 h_1 + F_1 h_2}{F_3} = \frac{b^2 t}{4I_z}(h_1^2 + h_2^2)$$

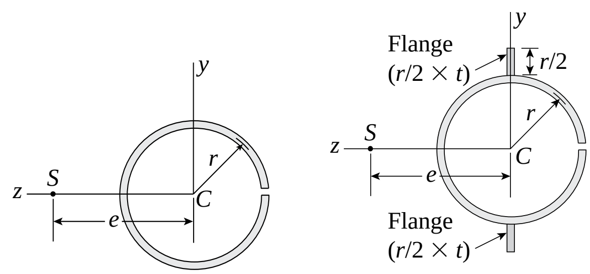
$$I_z = \frac{th_2^3}{12} + 2\,[bt(h_2/2)^2 + bt(h_1/2)^2]$$

$$= \frac{t}{12}[h_2^3 + 6b(h_1^2 + h_2^2)]$$

$$e = \frac{3b^2(h_1^2 + h_2^2)}{h_2^3 + 6b(h_1^2 + h_2^2)} \quad \leftarrow$$

Problem 6.9-6 The cross section of a slit circular tube of constant thickness is shown in the figure.

(a) Show that the distance e from the center of the circle to the shear center S is equal to $2r$ in the figure part a.
(b) Find an expression for e if flanges with the same thickness as that of the tube are added, as shown in the figure part b.



(a) (b)

Solution 6.9-6

(a) $Q_A = \int y\,dA = \int_0^{\theta} (r\,t\sin(\phi))\,d\phi$

$$Q_A = r^2 t(1 - \cos(\theta))$$

$$\tau_A = \frac{VQ_A}{I_z t} = \frac{Vr^2(1 - \cos(\theta))}{I_z}$$

$$I_z = \pi r^3 t$$

$$\tau_A = \frac{V(1 - \cos(\theta))}{\pi r t}$$

At point A: $dA = rtd\theta$

T_C = moment of shear stresses about center C.

$$T_C = \int \tau_A r\,dA = \int_0^{2\pi} \frac{Vr}{\pi}(1 - \cos(\theta))\,d\theta = 2Vr$$

Shear force V acts through the shear center S. Moment of the shear force V about any point must be equal to the moment of the shear stresses about that same point.

$$\Sigma M_C = Ve = T_C \qquad e = \frac{T_C}{V} = 2r \qquad \leftarrow$$

(b) $I_z = \pi r^3 t + 2\left[\frac{t\left(\frac{r}{2}\right)^3}{12} + t\frac{r}{2}\left(\frac{5r}{4}\right)^2\right]$

$$I_z = \pi r^3 t + \frac{19}{12}tr^3 = tr^3\left(\pi + \frac{19}{12}\right)$$

for $0 \le \theta < \frac{\pi}{2}$

$$Q_A = \int y\,dA = \int_0^{\theta} (rt\sin(\phi))\,d\phi$$

$$Q_A = r^2 t(1 - \cos(\theta))$$

$$\tau_A = \frac{VQ_A}{I_z t} = \frac{Vr^2(1 - \cos(\theta))}{I_z}$$

for $\frac{\pi}{2} \le \theta < \frac{3\pi}{2}$

$$Q_A = \int y\,dA = \int_0^{\theta} (rt\sin(\phi))\,d\phi + \frac{r}{2}t\frac{5r}{4}$$

$$Q_A = r^2 t\left(\frac{13}{8} - \cos(\theta)\right)$$

$$\tau_A = \frac{Vr^2\left(\frac{13}{8} - \cos(\theta)\right)}{I_z}$$

At point A: $dA = rtd\theta$

T_C = moment of shear stresses about center C.

$$T_C = \int \tau_A r\,dA = 2\int_0^{\frac{\pi}{2}} \frac{Vr^4 t}{I_z}(1 - \cos(\theta))\,d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{Vr^4 t}{I_z}\left(\frac{13}{8} - \cos(\theta)\right)d\theta$$

$$T_C = \int \tau_A r \, dA = Vr^4 t \frac{\pi - 2}{Iz} + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{Vr^4 t}{I_z}\left(\frac{13}{8} - \cos(\theta)\right) d\theta$$

$$T_C = Vr^4 t \frac{\pi - 2}{I_z} + \frac{1}{8} Vr^4 t \frac{13\pi + 16}{I_z}$$

$$= \frac{21 Vr^4 t\pi}{8 I_z}$$

Shear force V acts through the shear center S. Moment of the shear force V about any point must be equal to the moment of the shear stresses about that same point.

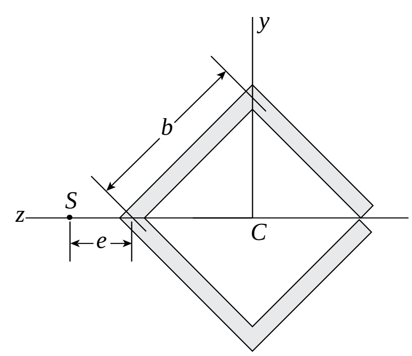
$$\Sigma M_C = Ve = T_C$$

$$e = \frac{T_C}{V} = \frac{21 r^4 t\pi}{8 I_z} = \frac{21 r^4 t\pi}{8\left[tr^3\left(\pi + \frac{19}{12}\right)\right]}$$

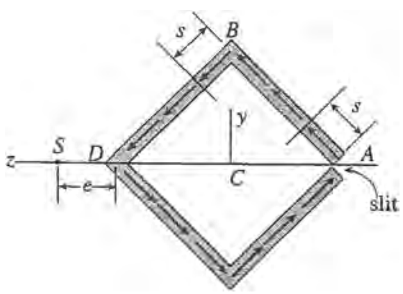
$$e = \frac{63\pi r}{24\pi + 38} = 1.745r \quad \leftarrow$$

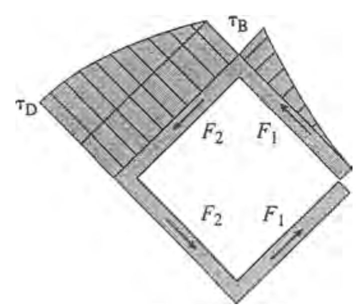
Problem 6.9-7 The cross section of a slit square tube of constant thickness is shown in the figure. Derive the following formula for the distance e from the corner of the cross section to the shear center S:

$$e = \frac{b}{2\sqrt{2}}$$



Solution 6.9-7 Slit square tube





b = length of each side

t = thickness

$$\tau = \frac{VQ}{I_z t}$$

FROM A TO B:

$$Q = \frac{ts^2}{2\sqrt{2}}$$

At A: $Q = 0 \quad \tau_A = 0$

At B: $Q = \dfrac{tb^2}{2\sqrt{2}}$

$$\tau_B = \frac{b^2V}{2\sqrt{2}I_z}$$

$$F_1 = \frac{\tau_B\, bt}{3} = \frac{b^3tV}{6\sqrt{2}I_z}$$

FROM B TO D:

$$Q = bt\left(\frac{b}{2\sqrt{2}}\right) + st\left(\frac{b}{\sqrt{2}} - \frac{S}{2\sqrt{2}}\right)$$

$$= \frac{tb^2}{2\sqrt{2}} + \frac{ts}{2\sqrt{2}}(2b - s)$$

$$\tau = \frac{VQ}{I_zt} = \frac{V}{I_z}\left[\frac{b^2}{2\sqrt{2}} + \frac{s}{2\sqrt{2}}(2b - s)\right]$$

At B: $\tau_B = \dfrac{b^2V}{2\sqrt{2}I_z}$ At D: $\tau_D = \dfrac{b^2V}{\sqrt{2}I_z}$

$$F_2 = \tau_B\, bt + \frac{2}{3}(\tau_D - \tau_B)\, bt = \frac{5tb^3V}{6\sqrt{2}I_z}$$

Shear force V acts through the shear center S.

$\therefore \sum M_s = 0$

$$2(F_1/\sqrt{2})(b\sqrt{2} + e) + 2(F_2/\sqrt{2})\,(e) = 0$$

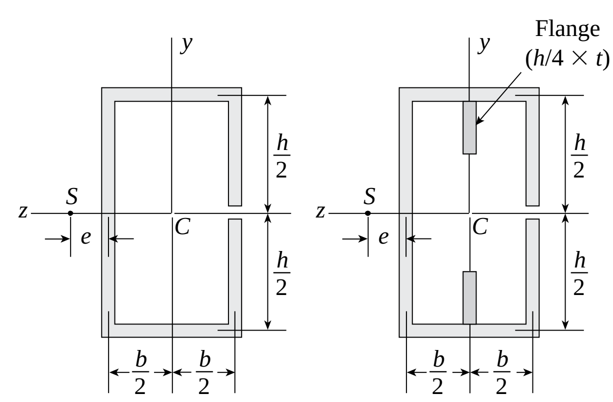
Substitute for F_1 and F_2 and solve for e:

$$e = \frac{b}{2\sqrt{2}} \quad \leftarrow$$

Problem 6.9-8 The cross section of a slit rectangular tube of constant thickness is shown in the figures.

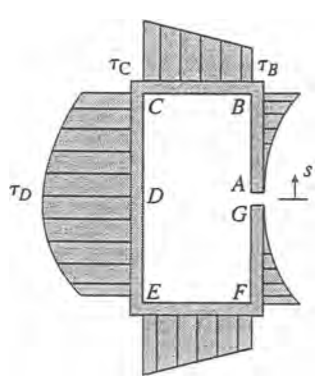
(a) Derive the following formula for the distance e from the centerline of the wall of the tube in figure part (a) to the shear center S: $e = \dfrac{b(2h + 3b)}{2(h + 3b)}$

(b) Find an expression for e if flanges with the same thickness as that of the tube are added as shown in figure part (b).



(a) (b)

Solution 6.9-8



(a) FROM A TO B: $Q = \dfrac{ts^2}{2}$ $\tau = \dfrac{VQ}{I_zt} = \dfrac{s^2V}{2I_z}$

$\tau_A = 0$ $\tau_B = \dfrac{h^2V}{8I_z}$

$$F_1 = \frac{\tau_B t}{3}\left(\frac{h}{2}\right) = \frac{th^3V}{48I_z}$$

FROM B TO C: $\tau_B = \dfrac{h^2V}{8I_z}$

$$Q_C = \frac{th}{2}\left(\frac{h}{4}\right) + bt\left(\frac{h}{2}\right) = \frac{th}{8}(h + 4b)$$

$$\tau_C = \frac{h(h + 4b)V}{8I_z}$$

$$F_2 = \frac{1}{2}(\tau_B + \tau_C)bt = \frac{bht(h + 2b)V}{8I_z}$$

$$\Sigma F_{VERT} = V \qquad F_3 - 2F_1 = V$$

$$F_3 = V\left(1 + \frac{th^3}{24I_z}\right)$$

Shear force V acts through the shear center S.

$$\Sigma M_s = 0 \qquad -F_3e + F_2h + 2F_1(b + e) = 0$$

solve for $e = \dfrac{bh^2t(2h + 3b)}{12I_z}$

$$I_z = 2\left[\frac{1}{12}th^3 + bt\left(\frac{h}{2}\right)^2\right] = \frac{th^2}{6}(h + 3b)$$

Therefore $e = \dfrac{b}{2}\left(\dfrac{2h + 3b}{h + 3b}\right)$ ←

(b) From A to B: $Q = \dfrac{ts^2}{2} \qquad \tau = \dfrac{VQ}{I_zt} = \dfrac{s^2V}{2I_z}$

$$\tau_A = 0 \qquad \tau_B = \frac{h^2V}{8I_z}$$

$$F_1 = \frac{\tau_Bt}{3}\left(\frac{h}{2}\right) = \frac{th^3V}{48I_z}$$

From B to C: $\tau_B = \dfrac{h^2 \cdot V}{8I_z}$

$$Q_C = \frac{th}{2}\left(\frac{h}{4}\right) + \frac{b}{2}t\left(\frac{h}{2}\right) = \frac{th}{8}(h + 2b)$$

$$\tau_C = \frac{h(h + 2b)V}{8I_z}$$

$$F_{BC} = \frac{1}{2}(\tau_B + \tau_C)\frac{b}{2}t = \frac{bht(h + b)V}{16I_z}$$

In flange:

$$Q_{C\_flange} = ts\left(\frac{s}{2} + \frac{h}{4}\right) = \frac{ts}{4}(2s + h)$$

$$F_{flange} = \int_0^{\frac{h}{4}}\left[\frac{s}{4}(2s+h)\right]\frac{tV}{I_z}\,ds = \frac{h^3Vt}{96I_z}$$

From C to D:

$$Q = \frac{th^2}{8} + t\frac{b}{2}\left(\frac{h}{2}\right) + t\frac{h}{4}\left(\frac{3h}{8}\right) + t\frac{h}{2}s$$

$$F_{CD} = \int_0^{\frac{b}{2}}\left[\frac{th^2}{8} + t\frac{b}{2}\left(\frac{h}{2}\right) + t\frac{h}{4}\left(\frac{3h}{8}\right) + t\frac{h}{2}s\right]\frac{Vt}{I_zt}\,ds = \frac{Vthb}{64I_z}(7h+12b)$$

$$\Sigma F_{VERT} = V \qquad F_3 - 2F_1 - 2F_{flange} = V$$

$$F_3 = V\left(1 + \frac{th^3}{16I_z}\right)$$

Shear force V acts through the shear center S.

$$\Sigma M_s = 0 \qquad -F_3e + (F_{BC} + F_{CD})h + 2F_1(b + e) + 2F_{flange}\left(\frac{b}{2} + e\right) = 0$$

$$e = \frac{61bth^3}{192I_z} + \frac{b^2h^2t}{4I_z}$$

$$e = \frac{th^2b}{192I_z}(43h + 48b)$$

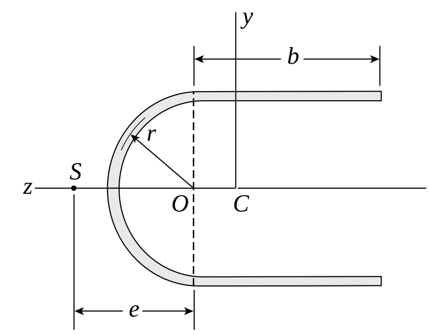
$$I_z = 2\left[\frac{1}{12}th^3 + bt\left(\frac{h}{2}\right)^2 + \frac{1}{12}t\left(\frac{h}{4}\right)^3 + t\left(\frac{h}{4}\right)\left(\frac{3h}{8}\right)^2\right] = \frac{th^2}{96}(23h + 48b)$$

$$e = \frac{b}{2}\left(\frac{43h + 48b}{23h + 48b}\right)$$ ←

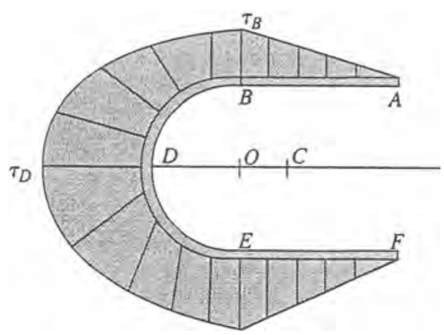
Problem 6.9-9 A U-shaped cross section of constant thickness is shown in the figure. Derive the following formula for the distance e from the center of the semicircle to the shear center S:

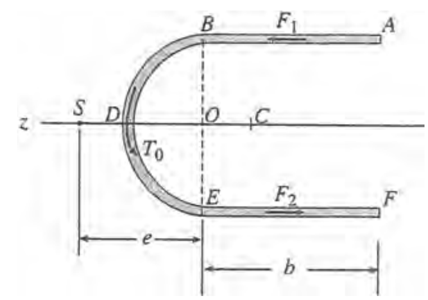
$$e = \frac{2(2r^2 + b^2 + \pi br)}{4b + \pi r}$$

Also, plot a graph showing how the distance e (expressed as the nondimensional ratio e/r) varies as a function of the ratio b/r. (Let b/r range from 0 to 2.)



Solution 6.9-9 U-shaped cross section





r = radius $\qquad F_1$ = force in AB

t = thickness $\qquad F_2$ = force in EF

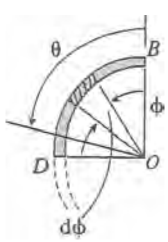
T_0 = moment in BDE

FROM A TO B: $\tau_A = 0 \qquad \tau_B = \dfrac{VQ}{I_z t} = \dfrac{V(btr)}{I_z t} = \dfrac{Vbr}{I_z}$

$$F_1 = \frac{bt\tau_B}{2} = \frac{Vb^2rt}{2I_z}$$

FROM B TO E: $Q_1 = \displaystyle\int y dA = \int_0^\theta (r\cos\phi)\, rtd\phi$

$$= r^2 t \sin\theta$$



$$Q_B = btr$$

$$Q_\theta = Q_B + Q_1 = btr + r^2 t\sin\theta$$

$$\tau_\theta = \frac{VQ_B}{I_z\, t} = \frac{Vr\,(b + r\sin\theta)}{I_z}$$

At angle θ: $dA = rtd\theta$

$$T_0 = \int \tau r dA = \int_0^\pi \tau r^2 t d\theta$$

$$= \int_0^\pi \frac{Vr^3 t\,(b + r\sin\theta) d\theta}{I_z}$$

$$= \frac{Vr^3 t}{I_z}(\pi b + 2r)$$

Shear force V acts through the shear center S. Moment of the shear force V about any point must be equal to the moment of the shear stresses about that same point.

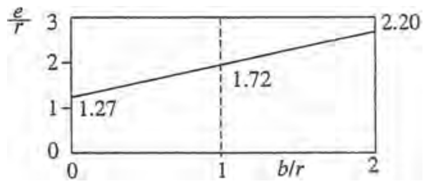
$$\therefore \sum M_0 = Ve = T_0 + F_1(2r)$$

$$e = \frac{T_0 + 2F_1 r}{V} = \frac{r^2 t}{I_z}(\pi br + 2r^2 + b^2)$$

$$I_z = \frac{\pi r^3 t}{2} + 2(btr^2) \qquad e = \frac{2(2r^2 + b^2 + \pi br)}{4b + \pi r} \quad \longleftarrow$$

GRAPH

$$\frac{e}{r} = \frac{2(2 + b^2/r^2 + \pi b/r)}{4b/r + \pi}$$



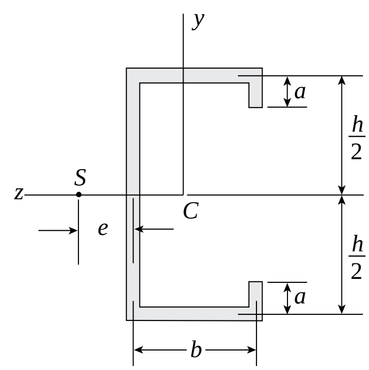
NOTE: When $b/r = 0$,

$$e/r = \frac{4}{\pi} \quad \text{(Eq. 6-73)}$$

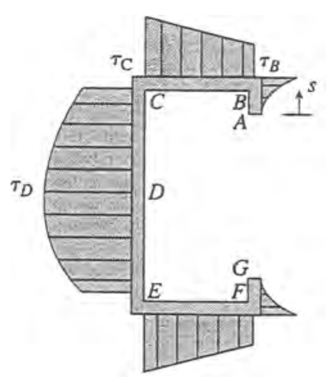
Problem 6.9-10 Derive the following formula for the distance e from the centerline of the wall to the shear center S for the C-section of constant thickness shown in the figure:

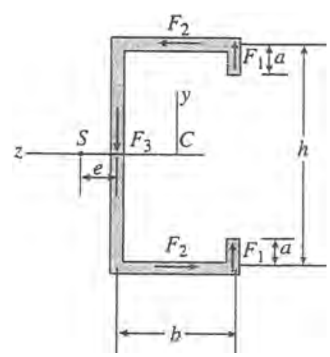
$$e = \frac{3bh^2(b + 2a) - 8ba^3}{h^2(h + 6b + 6a) + 4a^2(2a - 3h)}$$

Also, check the formula for the special cases of a channel section ($a = 0$) and a slit rectangular tube ($a = h/2$).



Solution 6.9-10 C-section of constant thickness





t = thickness

FROM A TO B:

$$Q = st\left(\frac{h}{2} - a + \frac{s}{2}\right) \qquad \tau = \frac{VQ}{I_z t} = s\left(\frac{h}{2} - a + \frac{s}{2}\right)\frac{V}{I_z}$$

$$\tau_A = 0 \quad \tau_B = \frac{a}{2}(h - a)\frac{V}{I_z}$$

$$F_1 = \int_0^a \tau t ds = \frac{tV}{I_z}\int_0^a s\left(\frac{h}{2} - a + \frac{s}{2}\right)ds$$

$$= \frac{a^2 t(3h - 4a)V}{12I_z}$$

From B to C:

$$\tau_B = \frac{a}{2}(h - a)\frac{V}{I_z} \quad Q_C = at\left(\frac{h}{2} - \frac{a}{2}\right) + bt\left(\frac{h}{2}\right)$$

$$= \frac{at}{2}(h - a) + \frac{bht}{2}$$

$$\tau_C = \left[\frac{a}{2}(h - a) + \frac{bh}{2}\right]\frac{V}{I_z}$$

$$F_2 = \frac{1}{2}(\tau_B + \tau_C)bt = \frac{bt}{4}[2a(h - a) + bh]\frac{V}{I_z}$$

From C to E:

$$\sum F_{\text{VERT}} = V \quad F_3 - 2F_1 = V$$

$$F_3 = V\left[1 + \frac{a^2t(3h - 4a)}{6I_z}\right]$$

Shear force V acts through the shear center S.

$$\therefore \sum M_s = 0 \qquad -F_3(e) + F_2h + 2F_1(b + e) = 0$$

Substitute for F_1, F_2, and F_3 and solve for e:

$$e = \frac{bt\,[3h^2(b + 2a) - 8a^3]}{12\,I_z}$$

$$I_z = 2\left(\frac{1}{12}th^3\right) + 2bt\left(\frac{h}{2}\right)^2 - \frac{6}{12}(h - 2a)^3$$

$$= \frac{t}{12}[h^2(h + 6b + 6a) + 4a^2(2a - 3h)]$$

$$e = \frac{3bh^2(b + 2a) - 8ba^3}{h^2(h + 6b + 6a) + 4a^2(2a - 3h)} \quad \leftarrow$$

Channel section ($a = 0$)

$$e = \frac{3b^2}{h + 6b} \quad \text{(agrees with Eq. 6-65 when } t_f = t_w\text{)}$$

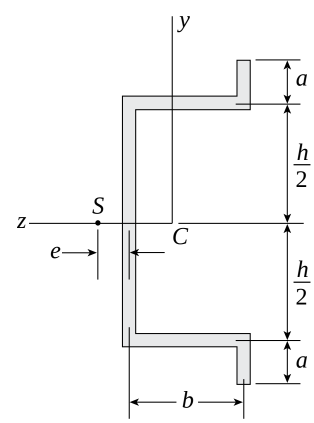
Slit rectangular tube ($a = h/2$)

$$e = \frac{b(2h + 3b)}{2(h + 3b)} \quad \text{(agrees with the result of Prob. 6.9-8)}$$

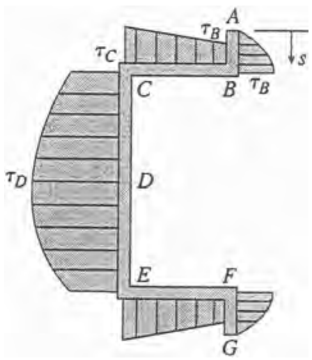
Problem 6.9-11 Derive the following formula for the distance e from the centerline of the wall to the shear center S for the hat section of constant thickness shown in the figure:

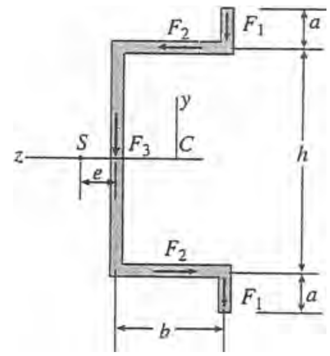
$$e = \frac{3bh^2(b + 2a) - 8ba^3}{h^2(h + 6b + 6a) + 4a^2(2a + 3h)}$$

Also, check the formula for the special case of a channel section ($a = 0$).



Solution 6.9-11 Hat section of constant thickness





t = thickness

FROM A TO B $\quad Q = st\left(\frac{h}{2} + a - \frac{s}{2}\right)$

$$\tau = \frac{VQ}{I_z t} = s\left(\frac{h}{2} + a - \frac{s}{2}\right)\frac{V}{I_z}$$

$$\tau_A = 0 \qquad \tau_B = \frac{a}{2}(h + a)\frac{V}{I_z}$$

$$F_1 = \int_0^a \tau t ds = \frac{tV}{I_z}\int_0^a s\left(\frac{h}{2} + a - \frac{s}{2}\right)ds$$

$$= \frac{a^2 t\,(3h + 4a)V}{12 I_z}$$

FROM B TO C $\quad \tau_B = \frac{a}{2}(h + a)\frac{V}{I_z}$

$$Q_C = at\left(\frac{h}{2} + \frac{a}{2}\right) + bt\left(\frac{h}{2}\right) = \frac{at}{2}(h + a) + \frac{bht}{2}$$

$$\tau_c = \left[\frac{a}{2}(h + a) + \frac{bh}{2}\right]\frac{V}{I_z}$$

$$F_2 = \frac{1}{2}(\tau_B + \tau_c)\,bt = \frac{bt}{4}[2a(h + a) + bh]\frac{V}{I_z}$$

FROM C TO E:

$$\Sigma F_{VERT} = V \qquad F_3 + 2F_1 = V$$

$$F_3 = V\left[1 - \frac{a^2 t(3h + 4a)}{6 I_z}\right]$$

Shear force V acts through the shear center S.

$$\therefore \Sigma M_S = 0 \quad -F_3 e + F_2 h - 2F_1(b + e) = 0$$

Substitute for F_1, F_2, and F_3 and solve for e:

$$e = \frac{bt[3h^2(b + 2a) - 8a^3]}{12 I_z}$$

$$I_z = \frac{1}{12}th^3 + 2bt\left(\frac{h}{2}\right)^2 + \frac{t}{12}(h + 2a)^3 - \frac{1}{12}th^3$$

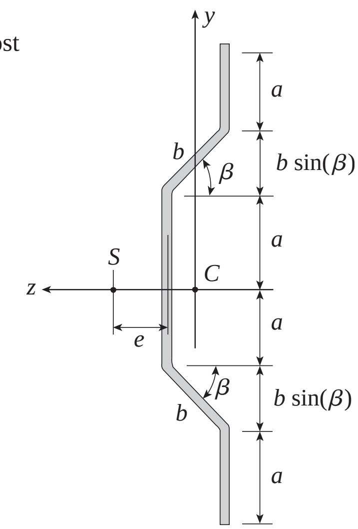
$$= \frac{t}{12}[h^2(h + 6b + 6a) + 4a^2(2a + 3h)]$$

$$e = \frac{3bh^2(b + 2a) - 8ba^3}{h^2(h + 6b + 6a) + 4a^2(2a + 3h)} \quad \leftarrow$$

CHANNEL SECTION ($a = 0$)

$$e = \frac{3b^2}{h + 6b} \quad \text{(agrees with Eq. 6-65 when } t_f = t_w\text{)}$$

Problem 6.9-12 The cross section of a sign post of constant thickness is shown in the figure. Derive the formula below for the distance e from the centerline of the wall of the post to the shear center S. Also, compare this formula with that given in Prob. 6.9-11 for the special case of $\beta = 0$ here and $a = h/2$ in both formulas.



Solution 6.9-12

From A to B

$$Q_{AB} = st\left(2a + b\sin(\beta) - \frac{s}{2}\right)$$

$$\tau_{AB} = \frac{VQ_{AB}}{I_z t}$$

$$F_{AB} = \int \tau dA = \int_0^a \frac{V\left[st\left(2a + b\sin(\beta) - \frac{s}{2}\right)\right]}{I_z t} t\,ds$$

$$F_{AB} = \frac{Vta^2(5a + 3b\sin(\beta))}{6I_z}$$

From B to C

$$Q_{BC} = at\left(2a + b\sin(\beta) - \frac{a}{2}\right) + ts\left(a + b\sin(\beta) - \frac{s}{2}\sin(\beta)\right)$$

$$Q_{BC} = \frac{3}{2}a^2t + atb\sin(\beta) + sta + stb\sin(\beta) - \frac{1}{2}s^2t\sin(\beta)$$

$$\tau_{BC} = \frac{VQ_{BC}}{I_z t}$$

$$F_{BC} = \int \tau dA = \int_0^b \frac{V\left(\frac{3}{2}a^2t + atb\sin(\beta) + sta + stb\sin(\beta) - \frac{1}{2}s^2t\sin(\beta)\right)}{I_z t} t\,ds$$

$$F_{BC} = \frac{Vtb\left(9a^2 + 6ab\sin(\beta) + 3ba + 2b^2\sin(\beta)\right)}{6I_z}$$

Shear force V acts through the shear center S.

$$\sum M_E = 0 \qquad Ve + 2F_{AB}b\cos(\beta) - F_{BC}\cos(\beta)(2a) = 0$$

$$e = 2\cos(\beta)\frac{(F_{BC}a - F_{AB}b)}{V}$$

$$e = \frac{tba\cos(\beta)}{3I_z}\left(4a^2 + 3ab\sin(\beta) + 3ab + 2b^2\sin(\beta)\right) \qquad \leftarrow$$

Now, compare this formula with that given in Prob. 6.9-11 for the special case of $\beta = 0$ here and $a = h/2$ in both formulas.

First modify above formula for $\beta = 0$ & $a = h/2$:

$$e = \frac{tb\frac{h}{2}\cos(0)}{3I_z}\left[4\left(\frac{h}{2}\right)^2 + 3\frac{h}{2}b\sin(0) + 3\frac{h}{2}b + 2b^2\sin(0)\right]$$

$$e = \frac{bht\left(h^2 + \frac{3bh}{2}\right)}{6I_z}$$

where I_z for the hat section of #6.9-11 is as follows:

$$I_z = \frac{th^3}{12} + 2bt\left(\frac{h}{2}\right)^2 + 2\frac{t\left(\frac{h}{2}\right)^3}{12} + 2t\frac{h}{2}\left(\frac{h}{2} + \frac{h}{4}\right)^2$$

$$I_z = \frac{h^2t(3b + 4h)}{6}$$

substituting expression for I_z & simplifying gives:

$$e = \frac{bht\left(h^2 + \frac{3bh}{2}\right)}{6\frac{h^2t(3b + 4h)}{6}}$$

$$e = \frac{b(3b + 2h)}{6b + 8h} \quad \text{for} \quad \beta = 0 \quad \text{and} \quad a = \frac{h}{2} \qquad \leftarrow$$

Now modify formula for e from #6.9-11 and compare to above

$$e = \frac{3bh^2(b + 2a) - 8ba^3}{h^2(h + 6b + 6a) + 4a^2(2a + 3h)}$$

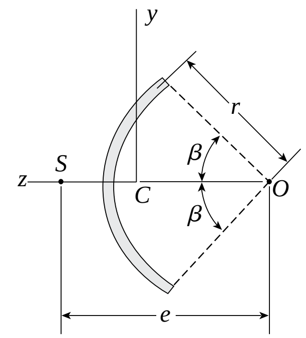
$$e = \frac{3bh^2\left(b + 2\frac{h}{2}\right) - 8b\left(\frac{h}{2}\right)^3}{h^2\left(h + 6b + 6\frac{h}{2}\right) + 4\left(\frac{h}{2}\right)^2\left(2\frac{h}{2} + 3h\right)}$$

$$e = \frac{b(3b + 2h)}{6b + 8h} \qquad \text{same expressions as that above from sign post solution} \qquad \leftarrow$$

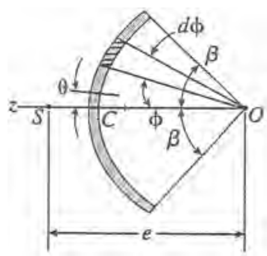
Problem 6.9-13 A cross section in the shape of a circular arc of constant thickness is shown in the figure. Derive the following formula for the distance e from the center of the arc to the shear center S:

$$e = \frac{2r(\sin\beta - \beta\cos\beta)}{\beta - \sin\beta\cos\beta}$$

in which β is in radians. Also, plot a graph showing how the distance e varies as β varies from 0 to π.



Solution 6.9-13 Circular arc



t = thickness $\qquad r$ = radius

At angle θ:

$$Q = \int y\,dA = \int_{\theta}^{\beta} (r\sin\phi)\,rt\,d\phi$$

$$= r^2 t(\cos\theta - \cos\beta)$$

$$\tau = \frac{VQ}{I_z t} = \frac{Vr^2(\cos\theta - \cos\beta)}{I_z}$$

$$I_z = \int y^2\,dA = \int_{-\beta}^{\beta} (r\sin\phi)^2 rt\,d\phi$$

$$= r^3 t(\beta - \sin\beta\cos\beta)$$

$$\tau = \frac{V(\cos\theta - \cos\beta)}{rt(\beta - \sin\beta\cos\beta)}$$

T_0 = moment of shear stresses

At angle θ, $dA = rt\,d\theta$

$$T_0 = \int \tau r\,dA = \int_{-\beta}^{\beta} \frac{V(\cos\theta - \cos\beta)}{t(\beta - \sin\beta\cos\beta)}\,rt\,d\theta$$

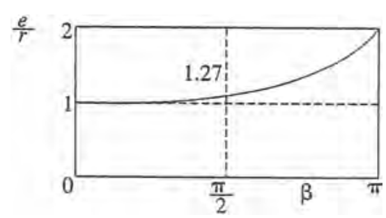
$$= \frac{2Vr(\sin\beta - \beta\cos\beta)}{\beta - \sin\beta\cos\beta}$$

Shear force V acts through the shear center S. Moment of the shear force V about any point must be equal to the moment of the shear stresses about that same point.

$$\therefore \sum M_0 = Ve = T_0 \qquad e = T_0/V$$

$$e = \frac{2r(\sin\beta - \beta\cos\beta)}{\beta - \sin\beta\cos\beta} \quad \leftarrow$$

GRAPH



$$\frac{e}{r} = \frac{2(\sin\beta - \beta\cos\beta)}{\beta - \sin\beta\cos\beta}$$

SEMICIRCULAR ARC ($\beta = \pi/2$):

$$\frac{e}{r} = \frac{4}{\pi} \text{ (Eq. 6-73)}$$

SLIT CIRCULAR ARC ($\beta = \pi$):

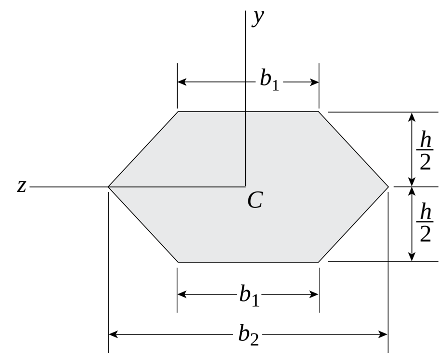
$$\frac{e}{r} = 2 \text{ (Prob. 6.9-6)}$$

Elastoplastic Bending

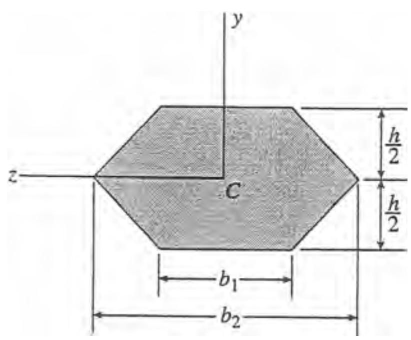
The problems for Section 6.10 are to be solved using the assumption that the material is elastoplastic with yield stress σ_Y.

Problem 6.10-1 Determine the shape factor f for a cross section in the shape of a double trapezoid having the dimensions shown in the figure.

Also, check your result for the special cases of a rhombus ($b_1 = 0$) and a rectangle ($b_1 = b_2$).



Solution 6.10-1 Double trapezoid



Neutral axis passes through the centroid C.

Use case 8, Appendix D.

SECTION MODULUS S

$$I_z = 2\left(\frac{h}{2}\right)^3 (3b_1 + b_2)/12$$

$$= \frac{h^3}{48}(3b_1 + b_2)$$

$$c = h/2 \qquad S = \frac{I}{c} = \frac{h^2}{24}(3b_1 + b_2)$$

PLASTIC MODULUS Z (EQ. 6-78)

$$A = 2\left(\frac{h}{2}\right)(b_1 + b_2)/2 = \frac{h}{2}(b_1 + b_2)$$

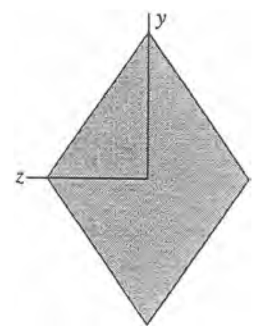
$$\bar{y}_1 = \bar{y}_2 = \frac{1}{3}\left(\frac{h}{2}\right)\left(\frac{2b_1 + b_2}{b_1 + b_2}\right)$$

$$z = \frac{A}{2}(\bar{y}_1 + \bar{y}_2) = \frac{h^2}{12}(2b_1 + b_2)$$

SHAPE FACTOR f (EQ. 6-79)

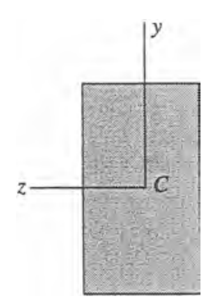
$$f = \frac{Z}{S} = \frac{2(2b_1 + b_2)}{3b_1 + b_2} \quad \leftarrow$$

SPECIAL CASE – RHOMBUS



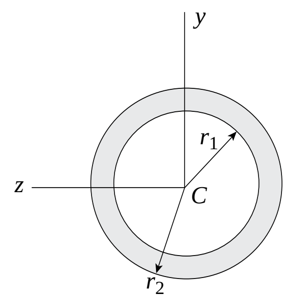
$b_1 = 0 \qquad f = 2$

SPECIAL CASE – RECTANGLE

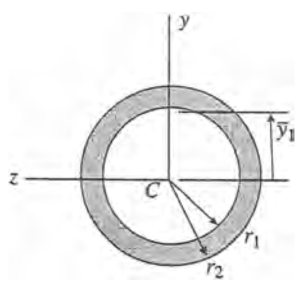


$b_1 = b_2 \qquad f = \frac{3}{2}$

Problem 6.10-2 (a) Determine the shape factor f for a hollow circular cross section having inner radius r_1 and outer radius r_2 (see figure).
(b) If the section is very thin, what is the shape factor?



Solution 6.10-2 Hollow circular cross sections



Neutral axis passes through the centroid C.

Use cases 9 and 10, Appendix D.

SECTION MODULUS S

$$I_z = \frac{\pi}{4}(r_2^4 - r_1^4) \quad c = r_2$$

$$S = \frac{I_z}{c} = \frac{\pi}{4r_2}(r_2^4 - r_1^4)$$

PLASTIC MODULUS Z (EQ. 6-78)

$$A = \pi(r_2^2 - r_1^2) \quad \text{For a semicircle,} \quad \bar{y} = \frac{4r}{3\pi}$$

$$\bar{y}_1 = \frac{\Sigma \bar{y}_i A_i}{\Sigma A_i} = \frac{\left(\frac{4r_2}{3\pi}\right)\left(\frac{\pi r_2^2}{2}\right) - \left(\frac{4r_1}{3\pi}\right)\left(\frac{\pi r_1^2}{2}\right)}{\pi/2(r_2^2 - r_1^2)}$$

$$= \frac{4}{3\pi}\left(\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2}\right)$$

$$\bar{y}_1 = \bar{y}_2 \quad Z = \frac{A}{2}(\bar{y}_1 + \bar{y}_2) = \frac{4}{3}(r_2^3 - r_1^3)$$

(a) SHAPE FACTOR f (EQ. 6-79)

$$f = \frac{Z}{S} = \frac{16r_2(r_2^3 - r_1^3)}{3\pi(r_2^4 - r_1^4)} \quad \leftarrow$$

(b) THIN SECTION ($r_1 \rightarrow r_2$)

Rewrite the expression for the shape factor f.

$$(r_2^3 - r_1^3) = (r_2 - r_1)(r_2^2 + r_1 r_2 + r_1^2)$$

$$(r_2^4 - r_1^4) = (r_2 - r_1)(r_2 + r_1)(r_2^2 + r_1^2)$$

$$f = \frac{16r_2}{3\pi}\left[\frac{r_2^2 + r_1 r_2 + r_1^2}{(r_2 + r_1)(r_2^2 + r_1^2)}\right]$$

$$= \frac{16}{3\pi}\left[\frac{1 + r_1/r_2 + (r_1/r_2)^2}{(1 + r_1/r_2)(1 + r_1^2/r_2^2)}\right]$$

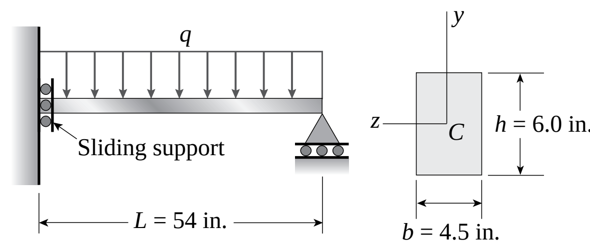
$$\text{Let } r_1/r_2 \rightarrow 1 \; f = \frac{16}{3\pi}\left(\frac{3}{4}\right) = \frac{4}{\pi} \approx 1.27 \quad \leftarrow$$

SPECIAL CASE OF A SOLID CIRCULAR CROSS SECTION

$$\text{Let } r_1 = 0 \quad f = \frac{16}{3\pi}\left(\frac{1}{1}\right) = \frac{16}{3\pi} \quad \text{(Eq. 6-90)}$$

Problem 6.10-3 A propped cantilever beam of length $L = 54$ in. with a sliding support supports a uniform load of intensity q (see figure). The beam is made of steel ($\sigma_Y = 36$ ksi) and has a rectangular cross section of width $b = 4.5$ in. and height $h = 6.0$ in.

What load intensity q will produce a fully plastic condition in the beam?



Solution 6.10-3

$L = 54$ in. $\quad \sigma_y = 36$ ksi $\quad b = 4.5$ in. $\quad h = 6.0$ in.

Maximum bending moment: $\quad M_{max} = \dfrac{qL^2}{2}$

Plastic Moment: $\quad M_P = \dfrac{\sigma_y b h^2}{4}$

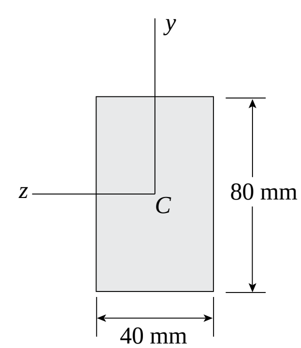
Let $\quad M_{max} = M_P \qquad$ gives $\quad q = \dfrac{\sigma_y b h^2}{2L^2}$

Therefore $\quad q = 1000$ lb/in. $\quad \leftarrow$

Problem 6.10-4 A steel beam of rectangular cross section is 40 mm wide and 80 mm high (see figure). The yield stress of the steel is 210 MPa.

(a) What percent of the cross-sectional area is occupied by the elastic core if the beam is subjected to a bending moment of 12.0 kN·m acting about the z axis?

(b) What is the magnitude of the bending moment that will cause 50% of the cross section to yield?



Solution 6.10-4

$\sigma_y = 210$ MPa $\quad b = 40$ mm $\quad h = 80$ mm

(a) Elastic core

$M = 12.0\ \text{kN}\cdot\text{m} \qquad M_y = \dfrac{\sigma_y b h^2}{6}$

$M_y = 9.0\ \text{kN}\cdot\text{m}$

$M_P = \dfrac{\sigma_y b h^2}{4} \qquad M_P = 13.4\ \text{kN}\cdot\text{m}$

M is between M_y and M_P

$e = h\sqrt{\dfrac{1}{2}\left(\dfrac{3}{2} - \dfrac{M}{M_y}\right)} \qquad e = 22.678$ mm

Percent of cross-sectional area is

$\dfrac{2e}{h} = 56.7\% \quad \leftarrow$

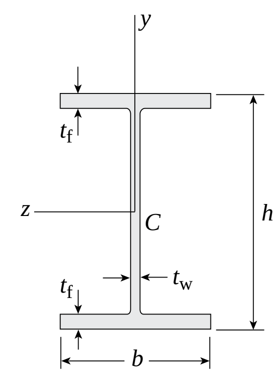
(b) Elastic core

$e = \dfrac{h}{4} \qquad e = 20$ mm

$M = M_y\left(\dfrac{3}{2} - \dfrac{2e^2}{h^2}\right)$

$M = 12.3\ \text{kN}\cdot\text{m} \quad \leftarrow$

Problem 6.10-5 Calculate the shape factor f for the wide-flange beam shown in the figure if
$h = 12.2$ in., $b = 8.08$ in., $t_f = 0.64$ in., and $t_w = 0.37$ in.



Probs. 6.10-5 and 6.10-6

Solution 6.10-5

$h = 12.2$ in. $\quad b = 8.08$ in.

$t_f = 0.64$ in. $\quad t_w = 0.37$ in.

SECTION MODULUS

$$I = \frac{1}{12}bh^3 - \frac{1}{12}(b - t_w)(h - 2t_f)^3$$

$$I = 386.0 \text{ in.}^4$$

$$c = \frac{h}{2} \qquad c = 6.1 \text{ in.} \qquad S = \frac{I}{c} \qquad S = 63.3 \text{ in.}^3$$

PLASTIC MODULUS

$$Z = \frac{1}{4}[bh^2 - (b - t_w)(h - 2t_f)^2]$$

$$Z = 70.8 \text{ in.}^3$$

SHAPE FACTOR

$$f = \frac{Z}{S} \qquad f = 1.12 \qquad \leftarrow$$

Problem 6.10-6 Solve the preceding problem for a wide-flange beam with $h = 404$ mm, $b = 140$ mm, $t_f = 11.2$ mm, and $t_w = 6.99$ mm.

Solution 6.10-6

$h = 404$ mm $\quad b = 140$ mm

$t_f = 11.2$ mm $\quad t_w = 6.99$ mm

SECTION MODULUS

$$I = \frac{1}{12}bh^3 - \frac{1}{12}(b - t_w)(h - 2t_f)^3$$

$$I = 153.4 \times 10^6 \text{ mm}^4$$

$$c = \frac{h}{2} \qquad c = 202.0 \text{ mm} \qquad S = \frac{I}{c}$$

$$S = 759.2 \times 10^3 \text{ mm}^3$$

PLASTIC MODULUS

$$Z = \frac{1}{4}[bh^2 - (b - t_w)(h - 2t_f)^2]$$

$$Z = 870.4 \times 10^3 \text{ mm}^3$$

SHAPE FACTOR

$$f = \frac{Z}{S} \qquad f = 1.15 \qquad \leftarrow$$

Problem 6.10-7 Determine the plastic modulus Z and shape factor f for a W 12 $\times$ 14 wide-flange beam. (*Note*: Obtain the cross-sectional dimensions and section modulus of the beam from Table E-1a in Appendix E.)

Solution 6.10-7

W 12 $\times$ 14

$h = 11.9$ in. $\quad b = 3.97$ in. $\quad t_f = 0.225$ in.

$t_w = 0.200$ in. $\quad S = 14.9$ in.^{3}

PLASTIC MODULUS

$$Z = \frac{1}{4}[bh^2 - (b - t_w)(h - 2t_f)^2]$$

$Z = 16.98$ in.^{3} $\quad\leftarrow$

SHAPE FACTOR

$f = \dfrac{Z}{S} \qquad f = 1.14 \quad\leftarrow$

Problem 6.10-8 Solve the preceding problem for a W 250 $\times$ 89 wide-flange beam. (*Note*: Obtain the cross-sectional dimensions and section modulus of the beam from Table E-1b in Appendix E.)

Solution 6.10-8

W 250 $\times$ 89

$h = 259$ mm $\quad b = 257$ mm $\quad t_f = 17.3$ mm

$t_w = 10.7$ mm $\quad S = 1090 \times 10^3$ mm^3

PLASTIC MODULUS

$$Z = \frac{1}{4}[bh^2 - (b - t_w)(h - 2t_f)^2]$$

$Z = 1.209 \times 10^6$ mm^3 $\quad\leftarrow$

SHAPE FACTOR

$f = \dfrac{Z}{S} \qquad f = 1.11 \quad\leftarrow$

Problem 6.10-9 Determine the yield moment M_Y, plastic moment M_P, and shape factor f for a W 16 $\times$ 100 wide-flange beam if $\sigma_Y = 36$ ksi. (*Note*: Obtain the cross-sectional dimensions and section modulus of the beam from Table E-1a in Appendix E.)

Solution 6.10-9

W 16 $\times$ 100

$h = 17.0$ in. $\quad b = 10.4$ in. $\quad t_f = 0.985$ in.

$t_w = 0.585$ in. $\quad S = 175$ in.^{3} $\quad \sigma_y = 36$ ksi

YIELD MOMENT

$M_y = \sigma_y S \qquad M_y = 525$ k-ft $\quad\leftarrow$

PLASTIC MODULUS

$Z = \frac{1}{4}[bh^2 - (b - t_w)(h - 2t_f)^2]$ $Z = 197.1\ \text{in.}^3$

SHAPE FACTOR

$f = \frac{Z}{S}$ $f = 1.13$ ←

PLASTIC MOMENT

$M_P = \sigma_y Z$ $M_P = 591\ \text{k-ft}$ ←

Problem 6.10-10 Solve the preceding problem for a W 410 × 85 wide-flange beam. Assume that $\sigma_Y = 250$ MPa. (*Note*: Obtain the cross-sectional dimensions and section modulus of the beam from Table E-1b in Appendix E.)

Solution 6.10-10

W 410 × 85

$h = 417$ mm $b = 181$ mm $t_f = 18.2$ mm

$t_w = 10.9$ mm $S = 1510 \cdot 10^3\ \text{mm}^3$

$\sigma_y = 250$ MPa

YIELD MOMENT

$M_y = \sigma_y S$ $M_y = 378\ \text{kN} \cdot \text{m}$ ←

PLASTIC MODULUS

$$Z = \frac{1}{4}\left[bh^2 - (b - t_w)(h - 2t_f)^2\right]$$

$$Z = 1.708 \times 10^6\ \text{mm}^3$$

PLASTIC MOMENT

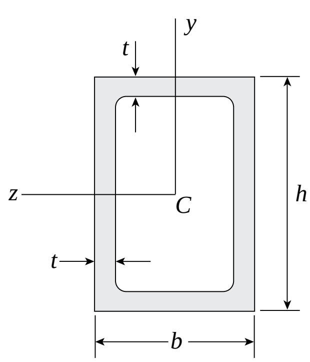
$M_P = \sigma_y Z$ $M_P = 427\ \text{kN} \cdot \text{m}$ ←

SHAPE FACTOR

$f = \frac{Z}{S}$ $f = 1.13$ ←

Problem 6.10-11 A hollow box beam with height $h = 16$ in., width $b = 8$ in., and constant wall thickness $t = 0.75$ in. is shown in the figure. The beam is constructed of steel with yield stress $\sigma_Y = 32$ ksi.

Determine the yield moment M_Y, plastic moment M_P, and shape factor f.



Probs. 6.10-11 and 6.10-12

Solution 6.10-11 Hollow box beam

$h = 16$ in. $\quad b = 8$ in.

$t = 0.75$ in. $\quad \sigma_Y = 32$ ksi

SECTION MODULUS ($S = I/c$)

$$I = \frac{1}{12} bh^3 - \frac{1}{12}(b - 2t)(h - 2t)^3$$
$$= 1079 \text{ in.}^4$$

$$c = \frac{h}{2} = 8.0 \text{ in.} \qquad S = \frac{I}{c} = 134.9 \text{ in.}^3$$

YIELD MOMENT (EQ. 6-74)

$$M_Y = \sigma_Y S = 4320 \text{ k-in.} \quad \leftarrow$$

PLASTIC MODULUS

Use (Eq. 6-86) with $t_w = 2t$ and $t_f = t$:

$$Z = \frac{1}{4}[bh^2 - (b - 2t)(h - 2t)^2]$$
$$= 170.3 \text{ in.}^3$$

PLASTIC MOMENT (EQ. 6-77)

$$M_p = \sigma_Y Z = 5450 \text{ k-in.} \quad \leftarrow$$

SHAPE FACTOR (EQ. 6-79)

$$f = \frac{M_P}{M_Y} = \frac{Z}{S} = 1.26 \quad \leftarrow$$

Problem 6.10-12 Solve the preceding problem for a box beam with dimensions $h = 0.5$ m, $b = 0.18$ m, and $t = 22$ mm. The yield stress of the steel is 210 MPa.

Solution 6.10-12

$h = 0.5$ m $\quad b = 0.18$ m $\quad t = 22$ mm

$\sigma_y = 210$ MPa

SECTION MODULUS

$$I = \frac{1}{12} bh^3 - \frac{1}{12}(b - 2t)(h - 2t)^3$$

$$I = 800.4 \times 10^6 \text{ mm}^4$$

$$c = \frac{h}{2} \qquad c = 250 \text{ mm}$$

$$S = \frac{I}{c} \qquad S = 3.202 \times 10^6 \text{ mm}^3$$

YIELD MOMENT

$$M_y = \sigma_y S \qquad M_y = 672 \text{ kN} \cdot \text{m} \quad \leftarrow$$

PLASTIC MODULUS

$$Z = \frac{1}{4}[bh^2 - (b - 2t)(h - 2t)^2]$$

$$Z = 4.180 \times 10^6 \text{ mm}^3$$

PLASTIC MOMENT

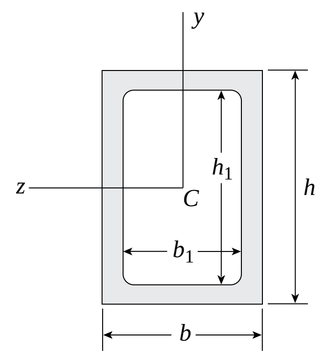
$$M_P = \sigma_y Z \qquad M_P = 878 \text{ kN} \cdot \text{m} \quad \leftarrow$$

SHAPE FACTOR

$$f = \frac{Z}{S} \qquad f = 1.31 \quad \leftarrow$$

Problem 6.10-13 A hollow box beam with height $h = 9.5$ in., inside height $h_1 = 8.0$ in., width $b = 5.25$ in., and inside width $b_1 = 4.5$ in. is shown in the figure.

Assuming that the beam is constructed of steel with yield stress $\sigma_Y = 42$ ksi, calculate the yield moment M_Y, plastic moment M_P, and shape factor f.



Probs. 6.10-13 through 6.10-16

Solution 6.10-13

$h = 9.5$ in. $\quad b = 5.25$ in. $\quad h_1 = 8.0$ in.

$b_1 = 4.5$ in. $\quad \sigma_y = 42$ ksi

SECTION MODULUS

$$I = \frac{1}{12}(bh^3 - b_1h_1^3) \qquad I = 183.10 \text{ in.}^4$$

$$c = \frac{h}{2} \qquad c = 4.75 \text{ in.} \qquad S = \frac{I}{c} \qquad S = 38.55 \text{ in.}^3$$

YIELD MOMENT

$$M_y = \sigma_y S \qquad M_y = 1619 \text{ k-in.} \qquad \leftarrow$$

PLASTIC MODULUS

$$Z = \frac{1}{4}(bh^2 - b_1h_1^2) \qquad Z = 46.5 \text{ in.}^3$$

PLASTIC MOMENT

$$M_P = \sigma_y Z \qquad M_P = 1951 \text{ k-in.} \qquad \leftarrow$$

SHAPE FACTOR

$$f = \frac{Z}{S} \qquad f = 1.21 \qquad \leftarrow$$

Problem 6.10-14 Solve the preceding problem for a box beam with dimensions $h = 200$ mm, $h_1 = 160$ mm, $b = 150$ mm, and $b_1 = 130$ mm. Assume that the beam is constructed of steel with yield stress $\sigma_Y = 220$ MPa.

Solution 6.10-14 Hollow box beam

$h = 200$ mm $\quad b = 150$ mm

$h_1 = 160$ mm $\quad b_1 = 130$ mm $\quad \sigma_Y = 220$ MPa

SECTION MODULUS ($S = I/c$)

$$I = \frac{1}{12}(bh^3 - b_1h_1^3) = 55.63 \times 10^6 \text{ mm}^4$$

$$c = \frac{h}{2} = 100 \text{ mm} \qquad S = \frac{I}{c} = 556.3 \times 10^3 \text{ mm}^3$$

YIELD MOMENT (EQ. 6-74)

$$M_Y = \sigma_Y S = 122 \text{ kN} \cdot \text{m} \qquad \leftarrow$$

PLASTIC MODULUS

Use (Eq. 6-86) with $b - t_w = b_1$ and $h - 2t_f = h_1$

$$Z = \frac{1}{4}(bh^2 - b_1h_1^2) = 668.0 \times 10^3 \text{ mm}^3$$

PLASTIC MOMENT (EQ. 6-77)

$$M_P = \sigma_Y Z = 147 \text{ kN} \cdot \text{m} \qquad \leftarrow$$

SHAPE FACTOR (EQ. 6-79)

$$f = \frac{M_P}{M_Y} = \frac{Z}{S} = 1.20 \qquad \leftarrow$$

Problem 6.10-15 The hollow box beam shown in the figure is subjected to a bending moment M of such magnitude that the flanges yield but the webs remain linearly elastic.

(a) Calculate the magnitude of the moment M if the dimensions of the cross section are $h = 15$ in., $h_1 = 12.75$ in., $b = 9$ in., and $b_1 = 7.5$ in. Also, the yield stress is $\sigma_Y = 33$ ksi.

Solution 6.10-15

$h = 15$ in. $\quad b = 9$ in. $\quad h_1 = 12.75$ in.

$b_1 = 7.5$ in. $\quad \sigma_y = 33$ ksi

ELASTIC CORE

$$S_1 = \frac{1}{6}(b - b_1)h_1^2 \qquad S_1 = 40.64 \text{ in.}^3$$

$$M_1 = \sigma_y S_1 \qquad M_1 = 1341 \text{ k-in.}$$

PLASTIC FLANGES

F = force in one flange

$$F = \sigma_y b\left(\frac{1}{2}\right)(h - h_1) \qquad F = 334.1 \text{ k}$$

$$M_2 = F\left(\frac{h + h_1}{2}\right) \qquad M_2 = 4636 \text{ k-in.}$$

(a) BENDING MOMENT

$$M = M_1 + M_2 \qquad M = 5977 \text{ k-in.} \quad \leftarrow$$

(b) PERCENT DUE TO ELASTIC CORE

$$\frac{M_1}{M} = 22.4\% \quad \leftarrow$$

Problem 6.10-16 Solve the preceding problem for a box beam with dimensions $h = 400$ mm, $h_1 = 360$ mm, $b = 200$ mm, and $b_1 = 160$ mm, and with yield stress $\sigma_Y = 220$ MPa.

Solution 6.10-16 Hollow box beam

$h = 400$ mm $\quad b = 200$ mm

$h_1 = 360$ mm $\quad b_1 = 160$ mm $\quad \sigma_Y = 220$ MPa

(see Figure 6-47, Example 6-9)

ELASTIC CORE

$$S_1 = \frac{1}{6}(b - b_1)h_1^2 = 864 \times 10^3 \text{ mm}^3$$

$$M_1 = \sigma_Y S_1 = 190.1 \text{ kN} \cdot \text{m}$$

PLASTIC FLANGES

F = force in one flange

$$F = \sigma_Y b\left(\frac{1}{2}\right)(h - h_1) = 880.0 \text{ kN}$$

$$M_2 = F\left(\frac{h + h_1}{2}\right) = 334.4 \text{ kN} \cdot \text{m}$$

(a) BENDING MOMENT

$$M = M_1 + M_2 = 524 \text{ kN·m} \quad \leftarrow$$

(b) PERCENT DUE TO ELASTIC CORE

$$\text{Percent} = \frac{M_1}{M}(100) = 36\% \quad \leftarrow$$

Problem 6.10-17 A W 10 × 60 wide-flange beam is subjected to a bending moment M of such magnitude that the flanges yield but the web remains linearly elastic.

(a) Calculate the magnitude of the moment M if the yield stress is $\sigma_Y = 36$ ksi.
(b) What percent of the moment M is produced by the elastic core?

Solution 6.10-17

W 10 × 60

$h = 10.2$ in. $\quad b = 10.1$ in. $\quad t_f = 0.680$ in.

$t_w = 0.420$ in. $\quad \sigma_y = 36$ ksi

ELASTIC CORE

$$S_1 = \frac{1}{6}(h - 2t_f)^2 t_w \qquad S_1 = 5.47 \text{ in.}^3$$

$$M_1 = \sigma_y S_1 \qquad M_1 = 196.9 \text{ k-in.}$$

PLASTIC FLANGES

F = force in one flange

$$F = \sigma_y b t_f \qquad F = 247.2 \text{ k}$$

$$M_2 = F(h - t_f) \qquad M_2 = 2354 \text{ k-in.}$$

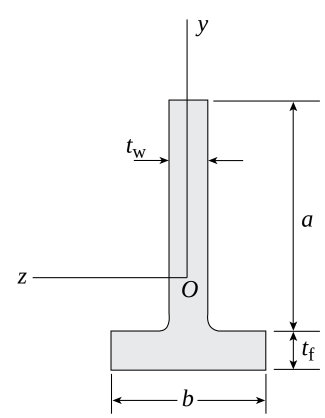
(a) BENDING MOMENT

$$M = M_1 + M_2 \qquad M = 2551 \text{ k-in.} \quad \leftarrow$$

(b) PERCENT DUE TO ELASTIC CORE

$$\frac{M_1}{M} = 7.7\% \quad \leftarrow$$

Problem 6.10-18 A singly symmetric beam of T-section (see figure) has cross-sectional dimensions $b = 140$ mm, $a = 190.8$ mm, $t_w = 6.99$ mm, and $t_f = 11.2$ mm.
Calculate the plastic modulus Z and the shape factor f.



Solution 6.10-18

$b = 140$ mm $\quad a = 190.8$ mm $\quad t_w = 6.99$ mm

$t_f = 11.2$ mm

ELASTIC BENDING

$$c_2 = \frac{\left(\frac{t_f}{2}\right) b t_f + \left(\frac{a}{2} + t_f\right) a t_w}{b t_f + a t_w} \qquad c_2 = 52.02 \text{ mm}$$

$$c_1 = a + t_f - c_2 \qquad c_1 = 149.98 \text{ mm}$$

$$I_z = \frac{1}{3} t_w c_1^3 + \frac{1}{3} b c_2^3 - \frac{1}{3}(b - t_w)(c_2 - t_f)^3$$

$$I_z = 11.41 \times 10^6 \text{ mm}^4$$

$$S = \frac{I_z}{c_1} \qquad S = 76.1 \times 10^3 \text{ mm}^3$$

Plastic bending

$A = bt_f + at_w \qquad A = 2902\ \text{mm}^2$

$h_2 = \dfrac{A}{2b} \qquad h_2 = 10.4\ \text{mm}$

$h_1 = a + t_f - h_2 \qquad h_1 = 191.6\ \text{mm}$

$y_{2\_bar} = h_2/2 \qquad y_{2\_bar} = 5.18\ \text{mm}$

$$y_{1\_bar} = \frac{\frac{1}{2}(b - t_w)(t_f - h_2)^2 + \frac{1}{2}t_w h_1^2}{A/2}$$

$y_{1\_bar} = 88.50\ \text{mm}$

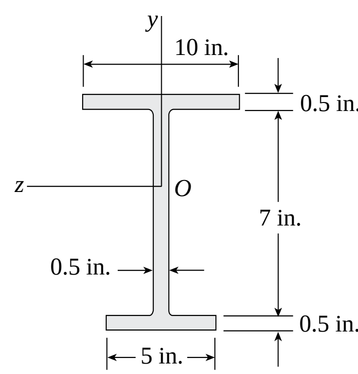
$Z = \dfrac{A}{2}(y_{1\_bar} + y_{2\_bar})$

$Z = 136 \times 10^3\ \text{mm}^3 \qquad \leftarrow$

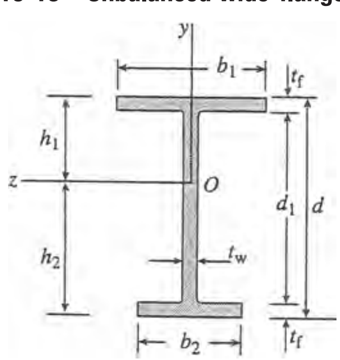
$f = \dfrac{Z}{S} \qquad f = 1.79 \qquad \leftarrow$

Problem 6.10-19 A wide-flange beam of unbalanced cross section has the dimensions shown in the figure.

Determine the plastic moment M_P if $\sigma_Y = 36$ ksi.



Solution 6.10-19 Unbalanced wide-flange beam



$\sigma_Y = 36$ ksi $\quad b_1 = 10$ in. $\quad b_2 = 5$ in.

$t_w = 0.5$ in. $\quad d = 8$ in. $\quad d_1 = 7$ in.

$t_f = 0.5$ in. $\quad A = b_1t_f + b_2t_f + d_1t_w = 11.0\ \text{in.}^2$

Neutral axis under fully plastic conditions

$$\frac{A}{2} = h_1t_w + (b_1 - t_w)t_f$$

from which we get $h_1 = 1.50$ in.

$h_2 = d - h_1 = 8.50$ in.

Plastic modulus

$$\bar{y}_1 = \frac{\sum y_iA_i}{A/2}$$

$$= \frac{(h_1/2)(t_w)(h_1) + (h_1 - t_f/2)(b_1 - t_w)(t_f)}{A/2}$$

$= 1.182$ in.

$$\bar{y}_2 = \frac{\Sigma\, y_i A_i}{A/2}$$

$$= \frac{(h_2/2)(t_w)(h_2) + (h_2 - t_f/2)(b_2 - t_w)(t_f)}{A/2}$$

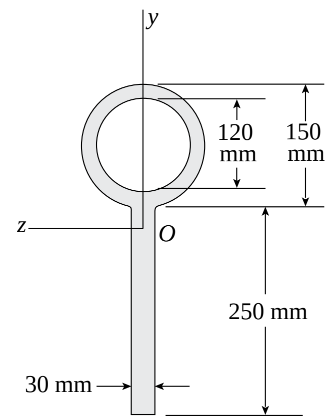
$$= 4.477 \text{ in.}$$

$$Z = \frac{A}{2}(\bar{y}_1 + \bar{y}_2) = 31.12 \text{ in.}^3$$

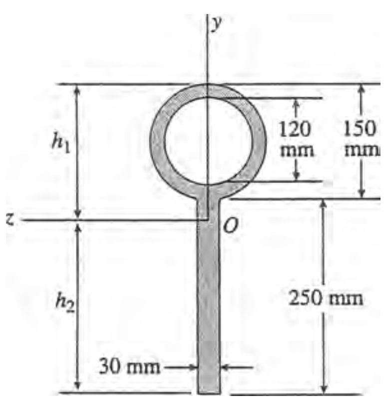
PLASTIC MOMENT

$$M_P = \sigma_y Z = 1120 \text{ k-in.} \quad \leftarrow$$

Problem 6.10-20 Determine the plastic moment M_P for a beam having the cross section shown in the figure if $\sigma_Y = 210$ MPa.



Solution 6.10-20 Cross section of beam



$\sigma_Y = 210$ MPa $\qquad d_2 = 150$ mm $\qquad d_1 = 120$ mm

NEUTRAL AXIS FOR FULLY PLASTIC CONDITIONS

Cross section is divided into two equal areas.

$$A = \frac{\pi}{4}[(150 \text{ mm})^2 - (120 \text{ mm})^2]$$

$$+ (250 \text{ mm})(30 \text{ mm}) = 13{,}862 \text{ mm}^2$$

$$\frac{A}{2} = 6931 \text{ mm}^2$$

$$(h_2)(30\text{ mm}) = \frac{A}{2} = 6931\text{ mm}^2$$

$$h_2 = 231.0\text{ mm}$$

$$h_1 = 150\text{ mm} + 250\text{ mm} - h_2 = 169.0\text{ mm}$$

PLASTIC MODULUS

$$\bar{y}_1 = \frac{\sum y_i A_i}{A/2} \text{ for upper half of cross section}$$

$$\bar{y}_2 = \frac{\sum y_i A_i}{A/2} \text{ for lower half of cross section}$$

$$Z = \frac{A}{2}(\bar{y}_1 + \bar{y}_2) = (\textstyle\sum y_i A_i)_{\text{upper}} + (\textstyle\sum y_i A_i)_{\text{lower}}$$

(Dimensions are in millimeters)

$$Z = (h_1 - 75)\left(\frac{\pi}{4}\right)(d_2^2 - d_1^2)$$

$$+ \left[\left(\frac{h_1 - 150}{2}\right)(30)(h_1 - 150)\right]$$

$$+ \left(\frac{h_2}{2}\right)(30)(h_2)$$

$$= 598{,}000 + 5{,}400 + 800{,}400$$

$$= 1404 \times 10^3\text{ mm}^3$$

PLASTIC MOMENT

$$M_P = \sigma_P Z = (210\text{ MPa})(1404 \times 10^3\text{ mm}^3)$$

$$= 295\text{ kN}\cdot\text{m} \quad \leftarrow$$

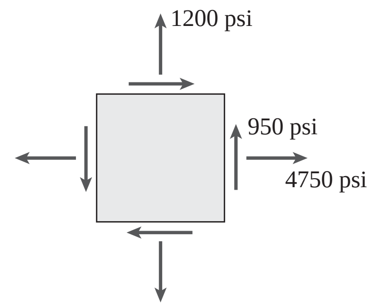
7

Analysis of Stress and Strain

Plane Stress

Problem 7.2-1 An element in *plane stress* is subjected to stresses $\sigma_x = 4750$ psi, $\sigma_y = 1200$ psi, and $\tau_{xy} = 950$ psi, as shown in the figure.

Determine the stresses acting on an element oriented at an angle $\theta = 60^\circ$ from the x axis, where the angle θ is positive when counterclockwise. Show these stresses on a sketch of an element oriented at the angle θ.



Solution 7.2-1

$\sigma_x = 4750 \text{ psi} \qquad \sigma_y = 1200 \text{ psi} \qquad \tau_{xy} = 950 \text{ psi}$

$\theta = 60°$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$$\sigma_{x1} = 2910 \text{ psi} \quad \leftarrow$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

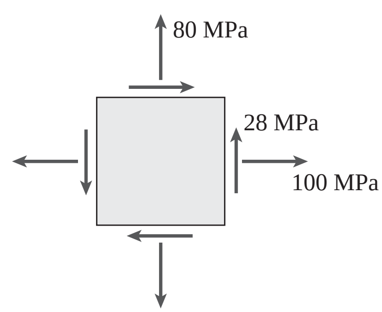
$$\tau_{x1y1} = -2012 \text{ psi} \quad \leftarrow$$

$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}$$

$$\sigma_{y1} = 3040 \text{ psi} \quad \leftarrow$$

Problem 7.2-2 Solve the preceding problem for an element in *plane stress* subjected to stresses $\sigma_x = 100$ MPa, $\sigma_y = 80$ MPa, and $\tau_{xy} = 28$ MPa, as shown in the figure.

Determine the stresses acting on an element oriented at an angle $\theta = 30^\circ$ from the x axis, where the angle θ is positive when counterclockwise. Show these stresses on a sketch of an element oriented at the angle θ.



Solution 7.2-2

$\sigma_x = 100 \text{ MPa} \qquad \sigma_y = 80 \text{ MPa} \qquad \tau_{xy} = 28 \text{ MPa}$

$\theta = 30°$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta)$$

$\sigma_{x1} = 119.2 \text{ MPa} \quad \leftarrow$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta)$$

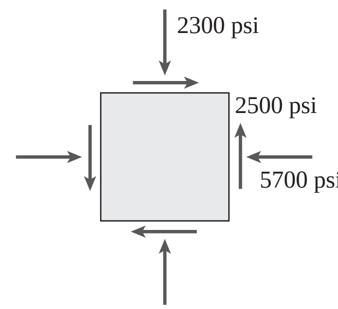
$\tau_{x1y1} = 5.30 \text{ MPa} \quad \leftarrow$

$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}$

$\sigma_{y1} = 60.8 \text{ MPa} \quad \leftarrow$

Problem 7.2-3 Solve Problem 7.2-1 for an element in *plane stress* subjected to stresses $\sigma_x = -5700$ psi, $\sigma_y = -2300$ psi, and $\tau_{xy} = 2500$ psi, as shown in the figure.

Determine the stresses acting on an element oriented at an angle $\theta = 50°$ from the x axis, where the angle θ is positive when counterclockwise. Show these stresses on a sketch of an element oriented at the angle θ.



Solution 7.2-3

$\sigma_x = -5700 \text{ psi} \qquad \sigma_y = -2300 \text{ psi} \qquad \tau_{xy} = 2500 \text{ psi}$

$\theta = 50°$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta)$$

$\sigma_{x1} = -1243 \text{ psi} \quad \leftarrow$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta)$$

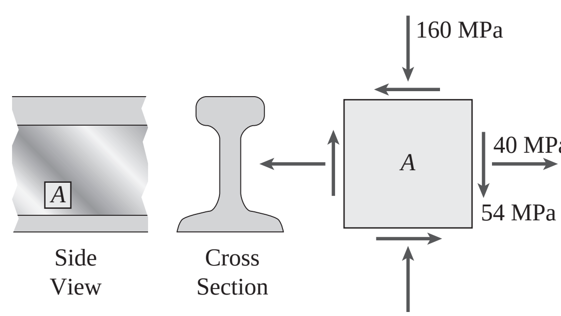
$\tau_{x1y1} = 1240 \text{ psi} \quad \leftarrow$

$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}$

$\sigma_{y1} = -6757 \text{ psi} \quad \leftarrow$

Problem 7.2-4 The stresses acting on element A in the web of a train rail are found to be 40 MPa tension in the horizontal direction and 160 MPa compression in the vertical direction (see figure). Also, shear stresses of magnitude 54 MPa act in the directions shown.

Determine the stresses acting on an element oriented at a counterclockwise angle of 52° from the horizontal. Show these stresses on a sketch of an element oriented at this angle.



Solution 7.2-4

$\sigma_x = 40 \text{ MPa} \qquad \sigma_y = -160 \text{ MPa} \qquad \tau_{xy} = -54 \text{ MPa}$

$\theta = 52°$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$\sigma_{x1} = -136.6 \text{ MPa} \quad \leftarrow$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

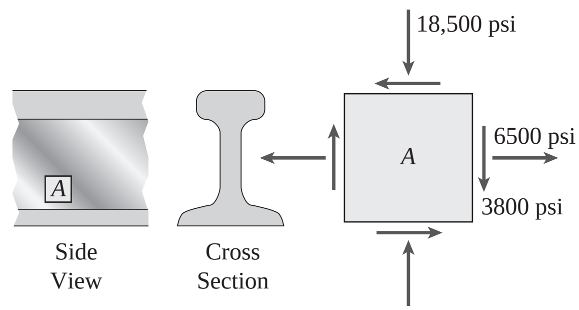
$\tau_{x1y1} = -84.0 \text{ MPa} \quad \leftarrow$

$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}$

$\sigma_{y1} = 16.6 \text{ MPa} \quad \leftarrow$

Problem 7.2-5 Solve the preceding problem if the normal and shear stresses acting on element *A* are 6500 psi, 18,500 psi, and 3800 psi (in the directions shown in the figure).

Determine the stresses acting on an element oriented at a counterclockwise angle of 30° from the horizontal. Show these stresses on a sketch of an element oriented at this angle.



Solution 7.2-5

$\sigma_x = 6500 \text{ psi} \qquad \sigma_y = -18500 \text{ psi} \qquad \tau_{xy} = -3800 \text{ psi}$

$\theta = 30°$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$\sigma_{x1} = -3041 \text{ psi} \quad \leftarrow$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

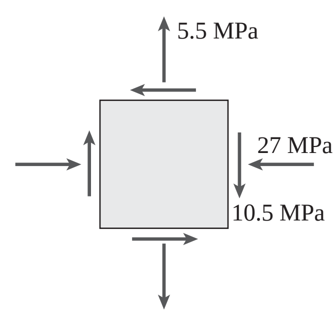
$\tau_{x1y1} = -12725 \text{ psi} \quad \leftarrow$

$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}$

$\sigma_{y1} = -8959 \text{ psi} \quad \leftarrow$

Problem 7.2-6 An element in *plane stress* from the fuselage of an airplane is subjected to compressive stresses of magnitude 27 MPa in the horizontal direction and tensile stresses of magnitude 5.5 MPa in the vertical direction (see figure). Also, shear stresses of magnitude 10.5 MPa act in the directions shown.

Determine the stresses acting on an element oriented at a clockwise angle of 35° from the horizontal. Show these stresses on a sketch of an element oriented at this angle.



Solution 7.2-6

$\sigma_x = -27 \text{ MPa} \qquad \sigma_y = 5.5 \text{ MPa} \qquad \tau_{xy} = -10.5 \text{ MPa}$

$\theta = -35°$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$\sigma_{x1} = -6.4 \text{ MPa} \quad \leftarrow$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

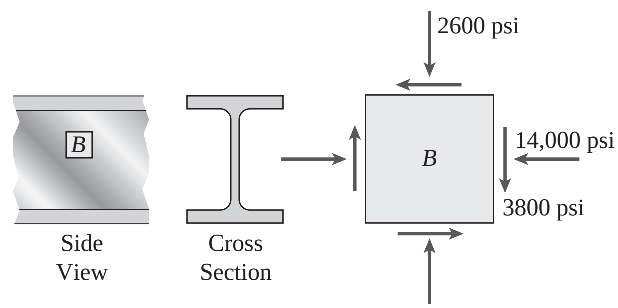
$\tau_{x1y1} = -18.9 \text{ MPa} \quad \leftarrow$

$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}$

$\sigma_{y1} = -15.1 \text{ MPa} \quad \leftarrow$

Problem 7.2-7 The stresses acting on element *B* in the web of a wide-flange beam are found to be 14,000 psi compression in the horizontal direction and 2600 psi compression in the vertical direction (see figure). Also, shear stresses of magnitude 3800 psi act in the directions shown.

Determine the stresses acting on an element oriented at a counterclockwise angle of 40° from the horizontal. Show these stresses on a sketch of an element oriented at this angle.



Solution 7.2-7

$\sigma_x = -14000 \text{ psi} \qquad \sigma_y = -2600 \text{ psi}$

$\tau_{xy} = -3800 \text{ psi}$

$\theta = 40°$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$\sigma_{x1} = -13032 \text{ psi} \quad \leftarrow$

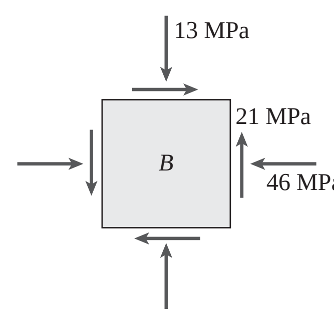
$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

$\tau_{x1y1} = 4954 \text{ psi} \quad \leftarrow$

$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}$

$\sigma_{y1} = -3568 \text{ psi} \quad \leftarrow$

Problem 7.2-8 Solve the preceding problem if the normal and shear stresses acting on element *B* are 46 MPa, 13 MPa, and 21 MPa (in the directions shown in the figure) and the angle is 42.5° (clockwise).



Solution 7.2-8

$\sigma_x = -46 \text{ MPa} \qquad \sigma_y = -13 \text{ MPa} \qquad \tau_{xy} = 21 \text{ MPa}$

$\theta = -42.5°$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$\sigma_{x1} = -51.9 \text{ MPa} \quad \longleftarrow$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

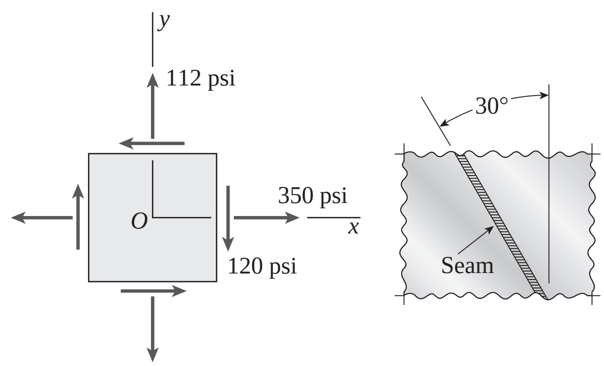
$\tau_{x1y1} = -14.6 \text{ MPa} \quad \longleftarrow$

$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}$

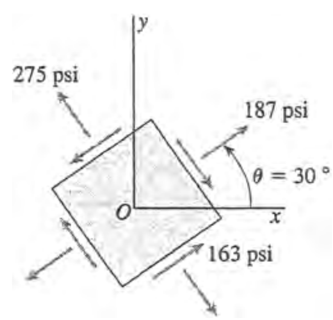
$\sigma_{y1} = -7.1 \text{ MPa} \quad \longleftarrow$

Problem 7.2-9 The polyethylene liner of a settling pond is subjected to stresses $\sigma_x = 350$ psi, $\sigma_y = 112$ psi, and $\tau_{xy} = -120$ psi, as shown by the plane-stress element in the first part of the figure.

Determine the normal and shear stresses acting on a seam oriented at an angle of 30° to the element, as shown in the second part of the figure. Show these stresses on a sketch of an element having its sides parallel and perpendicular to the seam.



Solution 7.2-9 Plane stress (angle θ)



$\sigma_x = 350 \text{ psi} \qquad \sigma_y = 112 \text{ psi} \qquad \tau_{xy} = -120 \text{ psi}$

$\theta = 30°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$= 187 \text{ psi} \quad \longleftarrow$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

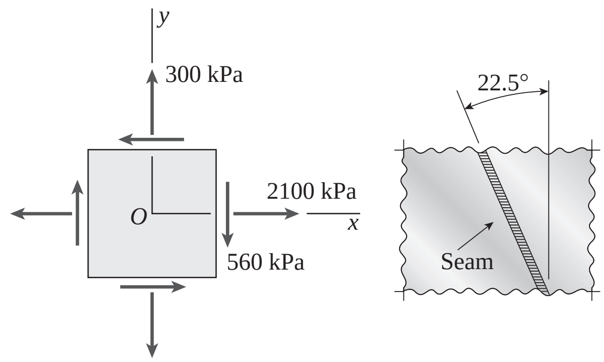
$= -163 \text{ psi} \quad \longleftarrow$

$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = 275 \text{ psi} \quad \longleftarrow$

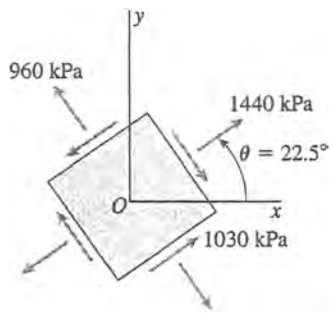
The normal stress on the seam equals 187 psi tension. $\longleftarrow$

The shear stress on the seam equals 163 psi, acting clockwise against the seam. $\longleftarrow$

Problem 7.2-10 Solve the preceding problem if the normal and shear stresses acting on the element are $\sigma_x = 2100$ kPa, $\sigma_y = 300$ kPa, and $\tau_{xy} = -560$ kPa, and the seam is oriented at an angle of 22.5° to the element (see figure).



Solution 7.2-10 Plane stress (angle θ)



$\sigma_x = 2100$ kPa $\quad \sigma_y = 300$ kPa $\quad \tau_{xy} = -560$ kPa

$\theta = 22.5°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= 1440 \text{ kPa} \quad \leftarrow$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$= -1030 \text{ kPa} \quad \leftarrow$$

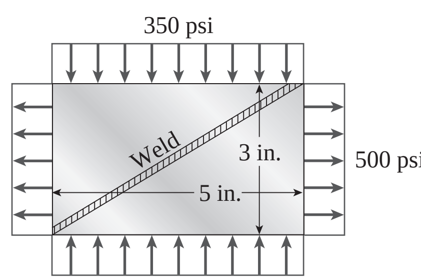
$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = 960 \text{ kPa} \quad \leftarrow$$

The normal stress on the seam equals 1440 kPa tension. $\leftarrow$

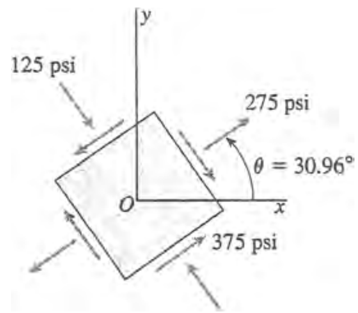
The shear stress on the seam equals 1030 kPa, acting clockwise against the seam. $\leftarrow$

Problem 7.2-11 A rectangular plate of dimensions 3.0 in. × 5.0 in. is formed by welding two triangular plates (see figure). The plate is subjected to a tensile stress of 500 psi in the long direction and a compressive stress of 350 psi in the short direction.

Determine the normal stress σ_w acting perpendicular to the line of the weld and the shear τ_w acting parallel to the weld. (Assume that the normal stress σ_w is positive when it acts in tension against the weld and the shear stress τ_w is positive when it acts counterclockwise against the weld.)



Solution 7.2-11 Biaxial stress (welded joint)



$$\sigma_x = 500 \text{ psi} \qquad \sigma_y = -350 \text{ psi} \qquad \tau_{xy} = 0$$

$$\theta = \arctan \frac{3 \text{ in.}}{5 \text{ in.}} = \arctan 0.6 = 30.96°$$

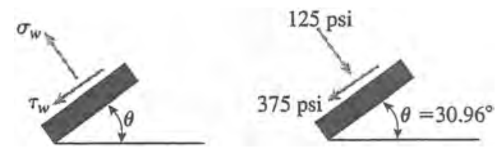
$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$= 275 \text{ psi}$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta = -375 \text{ psi}$$

$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = -125 \text{ psi}$$

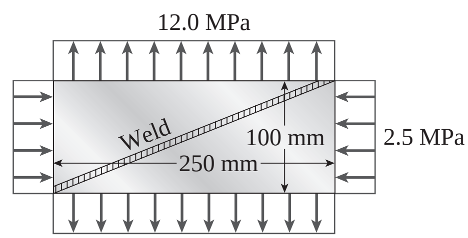
Stresses acting on the weld



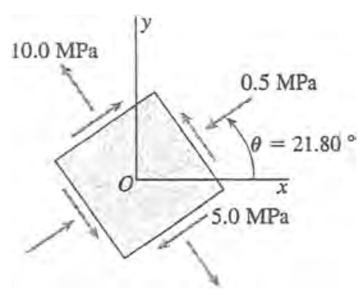
$$\sigma_w = -125 \text{ psi} \quad \longleftarrow$$

$$\tau_w = 375 \text{ psi} \quad \longleftarrow$$

Problem 7.2-12 Solve the preceding problem for a plate of dimensions 100 mm × 250 mm subjected to a compressive stress of 2.5 MPa in the long direction and a tensile stress of 12.0 MPa in the short direction (see figure).



Solution 7.2-12 Biaxial stress (welded joint)



$$\sigma_x = -2.5 \text{ MPa} \qquad \sigma_y = 12.0 \text{ MPa} \qquad \tau_{xy} = 0$$

$$\theta = \arctan \frac{100 \text{ mm}}{250 \text{ mm}} = \arctan 0.4 = 21.80°$$

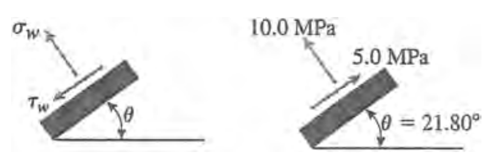
$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$= -0.5 \text{ MPa}$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta = 5.0 \text{ MPa}$$

$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = 10.0 \text{ MPa}$$

Stresses acting on the weld

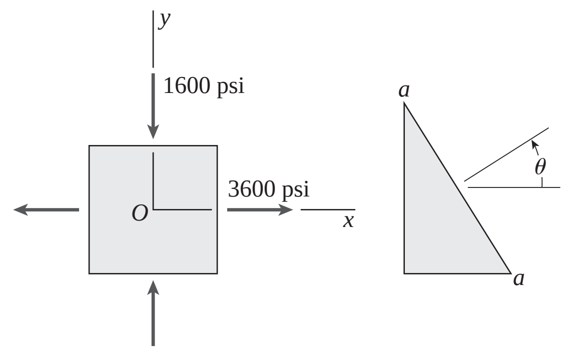


$$\sigma_w = 10.0 \text{ MPa} \quad \longleftarrow$$

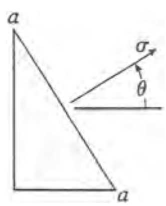
$$\tau_w = -5.0 \text{ MPa} \quad \longleftarrow$$

Problem 7.2-13 At a point on the surface of a machine the material is in *biaxial stress* with $\sigma_x = 3600$ psi, and $\sigma_y = -1600$ psi, as shown in the first part of the figure. The second part of the figure shows an inclined plane *aa* cut through the same point in the material but oriented at an angle θ.

Determine the value of the angle θ between zero and 90° such that no normal stress acts on plane *aa*. Sketch a stress element having plane *aa* as one of its sides and show all stresses acting on the element.



Solution 7.2-13 Biaxial stress



$$\sigma_x = 3600 \text{ psi}$$
$$\sigma_y = -1600 \text{ psi}$$
$$\tau_{xy} = 0$$

Find angle θ for $\sigma = 0$.

σ = normal stress on plane *a-a*

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$= 1000 + 2600\cos 2\theta \text{(psi)}$$

For $\sigma_{x_1} = 0$, we obtain $\cos 2\theta = -\dfrac{1000}{2600}$

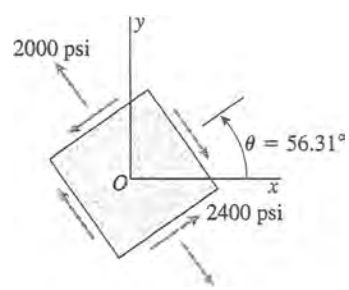
$\therefore \quad 2\theta = 112.62° \quad$ and $\quad \theta = 56.31°$

Stress element

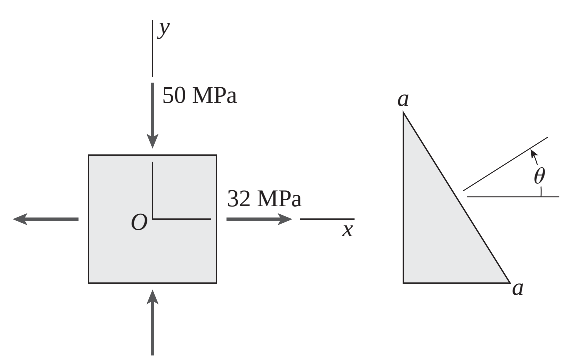
$\sigma_{x_1} = 0 \qquad \theta = 56.31°$

$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = 2000$ psi $\quad \leftarrow$

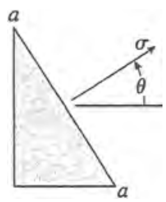
$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$
$$= -2400 \text{ psi}$$



Problem 7.2-14 Solve the preceding problem for $\sigma_x = 32$ MPa and $\sigma_y = -50$ MPa (see figure).



Solution 7.2-14 Biaxial stress



$$\sigma_x = 32 \text{ MPa}$$
$$\sigma_y = -50 \text{ MPa}$$
$$\tau_{xy} = 0$$

Find angles θ for $\sigma = 0$.

σ = normal stress on plane *a-a*

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= -9 + 41\cos 2\theta \text{ (MPa)}$$

For $\sigma_{x_1} = 0$, we obtain $\cos 2\theta = \dfrac{9}{41}$

$$\therefore \quad 2\theta = 77.32° \quad \text{and} \quad \theta = 38.66° \quad \leftarrow$$

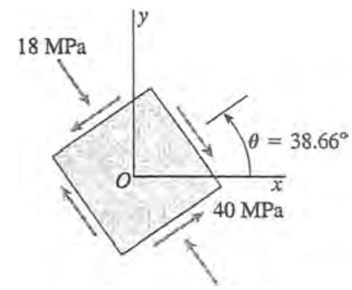
STRESS ELEMENT

$$\sigma_{x_1} = 0 \qquad \theta = 38.66°$$

$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = -18 \text{ MPa} \quad \leftarrow$$

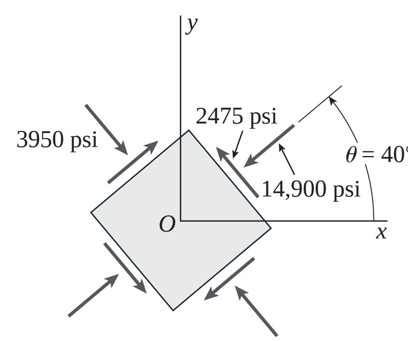
$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$= -40 \text{ MPa} \quad \leftarrow$$



Problem 7.2-15 An element in *plane stress* from the frame of a racing car is oriented at a known angle θ (see figure). On this inclined element, the normal and shear stresses have the magnitudes and directions shown in the figure.

Determine the normal and shear stresses acting on an element whose sides are parallel to the xy axes, that is, determine σ_x, σ_y, and τ_{xy}. Show the results on a sketch of an element oriented at $\theta = 0°$.



Solution 7.2-15

Transform from $\theta = 40°$ to $\theta = 0°$

$$\sigma_x = -14900 \text{ psi} \qquad \sigma_y = -3950 \text{ psi}$$
$$\tau_{xy} = 2475 \text{ psi}$$
$$\theta = -40°$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$$\sigma_{x1} = -12813 \text{ psi} \quad \leftarrow$$

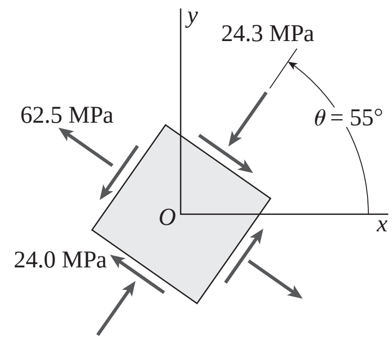
$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

$$\tau_{x1y1} = -4962 \text{ psi} \quad \leftarrow$$

$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}$$

$$\sigma_{y1} = -6037 \text{ psi} \quad \leftarrow$$

Problem 7.2-16 Solve the preceding problem for the element shown in the figure.



Solution 7.2-16

Transform from $\theta = 55°$ to $\theta = 0°$

$\sigma_x = -24.3$ MPa $\qquad \sigma_y = 62.5$ MPa

$\tau_{xy} = -24$ MPa

$\theta = -55°$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$\sigma_{x1} = 56.5$ MPa $\quad \leftarrow$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

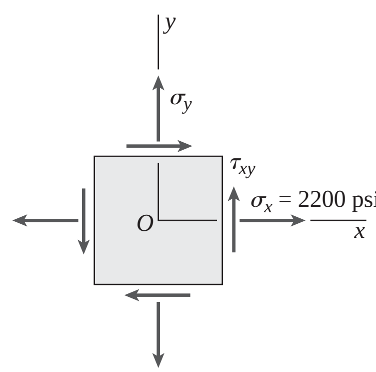
$\tau_{x1y1} = -32.6$ MPa $\quad \leftarrow$

$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}$

$\sigma_{y1} = -18.3$ MPa $\quad \leftarrow$

Problem 7.2-17 A plate in *plane stress* is subjected to normal stresses σ_x and σ_y and *shear stress* τ_{xy}, as shown in the figure. At counterclockwise angles $\theta = 35°$ and $\theta = 75°$ from the x axis, the normal stress is 4800 psi tension.

If the stress σ_x equals 2200 psi tension, what are the stresses σ_y and τ_{xy}?



Solution 7.2-17

$\sigma_x = 2200$ psi $\qquad \sigma_y$ unknown $\qquad \tau_{xy}$ unknown

At $\theta = 35°$ and $\theta = 75°$, $\sigma_{x1} = 4800$ psi

Find σ_y and τ_{xy}

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

For $\theta = 35°$

$\sigma_{x1} = 4800$ psi

$$4800 \text{ psi} = \frac{2200 \text{ psi} + \sigma_y}{2} + \frac{2200 \text{ psi} - \sigma_y}{2} \times \cos(70°) + \tau_{xy}\sin(70°)$$

or $\quad 0.32899\,\sigma_y + 0.93969\,\tau_{xy} = 3323.8$ psi $\qquad (1)$

For $\theta = 75°$:

$\sigma_{x1} = 4800$ psi

$$4800 \text{ psi} = \frac{2200 \text{ psi} + \sigma_y}{2} + \frac{2200 \text{ psi} - \sigma_y}{2} \times \cos(150°) + \tau_{xy}\sin(150°)$$

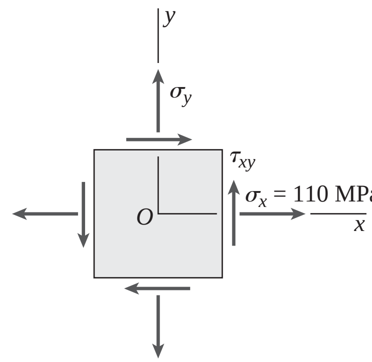
or $\quad 0.93301\sigma_y + 0.50000\,\tau_{xy} = 4652.6$ psi $\qquad$ (2)

Solve Eqs. (1) and (2):

$\sigma_y = 3805$ psi $\qquad \tau_{xy} = 2205$ psi $\quad \leftarrow$

Problem 7.2-18 The surface of an airplane wing is subjected to *plane stress* with normal stresses σ_x and σ_y and shear stress τ_{xy}, as shown in the figure. At a counterclockwise angle $\theta = 32°$ from the x axis, the normal stress is 37 MPa tension, and at an angle $\theta = 48°$, it is 12 MPa compression.

If the stress σ_x equals 110 MPa tension, what are the stresses σ_y and τ_{xy}?



Solution 7.2-18

$\sigma_x = 110$ MPa $\qquad \sigma_y$ unknown $\qquad \sigma_{xy}$ unknown

At $\theta = 32°$, $\sigma_{x1} = 37$ MPa $\qquad$ (tension)

At $\theta = 48°$, $\sigma_{x1} = -12$ MPa $\qquad$ (compression)

Find σ_y and τ_{xy}

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

For $\theta = 32°$

$\sigma_{x1} = 37$ MPa

$$37 \text{ MPa} = \frac{110 \text{ MPa} + \sigma_y}{2} + \frac{110 \text{ MPa} - \sigma_y}{2} \times \cos(64°) + \tau_{xy}\sin(64°)$$

or $\quad 0.28081\sigma_y + 0.89879\tau_{xy} = -42.11041$ MPa $\quad$ (1)

For $\theta = 48°$:

$\sigma_{x1} = -12$ MPa

$$-12 \text{ MPa} = \frac{110 \text{ MPa} + \sigma_y}{2} + \frac{110 \text{ MPa} - \sigma_y}{2} \times \cos(96°) + \tau_{xy}\sin(96°)$$

or $\quad 0.55226\sigma_y + 0.99452\tau_{xy} = -61.25093$ MPa (2)

Solve Eqs. (1) and (2):

$\sigma_y = -60.7$ MPa $\qquad \tau_{xy} = -27.9$ MPa $\quad \leftarrow$

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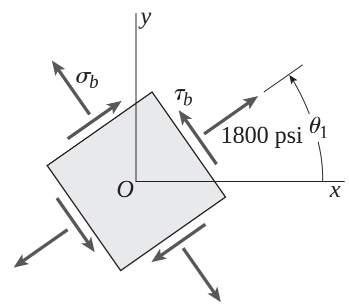
Problem 7.2-19 At a point in a structure subjected to *plane stress*, the stresses are $\sigma_x = -4100$ psi, $\sigma_y = 2200$ psi, and $\tau_{xy} = 2900$ psi (the sign convention for these stresses is shown in Fig. 7-1). A stress element located at the same point in the structure (but oriented at a counterclockwise angle θ_1 with respect to the x axis) is subjected to the stresses shown in the figure (σ_b, τ_b, and 1800 psi).

Assuming that the angle θ_1 is between zero and 90°, calculate the normal stress σ_b, the shear stress τ_b, and the angle θ_1



Solution 7.2-19

$\sigma_x = -4100$ psi $\qquad \sigma_y = 2200$ psi

$\tau_{xy} = 2900$ psi

For $\theta = \theta_1$:

$\sigma_{x1} = 1800$ psi $\qquad \sigma_{y1} = \sigma_b \qquad \tau_{x1y1} = \tau_b$

Find σ_b, τ_b, and θ_1

Stress σ_b

$\sigma_b = \sigma_x + \sigma_y - 1800$ psi $\qquad \sigma_b = -3700$ psi $\quad \longleftarrow$

Angle θ_1

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$$1800\text{ psi} = -950\text{ psi} - 3150\text{ psi}\cos(2\theta_1) + 2900\text{ psi}\sin(2\theta_1)$$

SOLVE NUMERICALLY:

$2\theta_1 = 87.32° \qquad \theta_1 = 43.7° \quad \longleftarrow$

Shear Stress τ_b

$$\tau_b = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta_1) + \tau_{xy}\cos(2\theta_1)$$

$\tau_b = 3282$ psi $\quad \longleftarrow$

Principal Stresses and Maximum Shear Stresses

When solving the problems for Section 7.3, consider only the in-plane stresses (the stresses in the xy plane).

Problem 7.3-1 An element in plane stress is subjected to stresses $\sigma_x = 4750$ psi, $\sigma_y = 1200$ psi, and $\tau_{xy} = 950$ psi (see the figure for Problem 7.2-1).

Determine the principal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-1

$\sigma_x = 4750$ psi $\qquad \sigma_y = 1200$ psi $\qquad \tau_{xy} = 950$ psi

PRINCIPAL STRESSES

$$\theta_{p1} = \frac{\operatorname{atan}\left(\dfrac{2\,\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$\theta_{p1} = 14.08°$

$\theta_{p2} = \theta_{p1} + 90° \qquad \theta_{p2} = 104.08°$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p1}) + \tau_{xy}\sin(2\theta_{p1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p2}) + \tau_{xy}\sin(2\theta_{p2})$$

$\sigma_1 = 4988$ psi $\quad \longleftarrow$

$\sigma_2 = 962$ psi $\quad \longleftarrow$

Problem 7.3-2 An element in *plane stress* is subjected to stresses $\sigma_x = 100$ MPa, $\sigma_y = 80$ MPa, and $\tau_{xy} = 28$ MPa (see the figure for Problem 7.2-2).

Determine the principal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-2

$\sigma_x = 100 \text{ MPa} \qquad \sigma_y = 80 \text{ MPa} \qquad \tau_{xy} = 28 \text{ MPa}$

PRINCIPAL STRESSES

$$\theta_{p1} = \frac{\operatorname{atan}\left(\dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$\theta_{p1} = 35.2°$

$\theta_{p2} = \theta_{p1} + 90° \qquad \theta_{p2} = 125.17°$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p1}) + \tau_{xy}\sin(2\theta_{p1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p2}) + \tau_{xy}\sin(2\theta_{p2})$$

$\sigma_1 = 120 \text{ MPa} \quad \leftarrow$

$\sigma_2 = 60 \text{ MPa} \quad \leftarrow$

Problem 7.3-3 An element in *plane stress* is subjected to stresses $\sigma_x = -5700$ psi, $\sigma_y = -2300$ psi, and $\tau_{xy} = 2500$ psi (see the figure for Problem 7.2-3).

Determine the principal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-3

$\sigma_x = -5700 \text{ psi} \qquad \sigma_y = -2300 \text{ psi} \qquad \tau_{xy} = 2500 \text{ psi}$

PRINCIPAL STRESSES

$$\theta_{p2} = \frac{\operatorname{atan}\left(\dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$\theta_{p2} = -27.89°$

$\theta_{p1} = \theta_{p2} + 90° \qquad \theta_{p1} = 62.1°$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p1}) + \tau_{xy}\sin(2\theta_{p1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p2}) + \tau_{xy}\sin(2\theta_{p2})$$

$\sigma_1 = -977 \text{ psi} \quad \leftarrow$

$\sigma_2 = -7023 \text{ psi} \quad \leftarrow$

Problem 7.3-4 The stresses acting on element *A* in the web of a train rail are found to be 40 MPa tension in the horizontal direction and 160 MPa compression in the vertical direction (see figure). Also, shear stresses of magnitude 54 MPa act in the directions shown (see the figure for Problem 7.2-4).

Determine the principal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-4

$\sigma_x = 40\text{ MPa} \qquad \sigma_y = -160\text{ MPa} \qquad \tau_{xy} = -54\text{ MPa}$

PRINCIPAL STRESSES

$$\theta_{p1} = \frac{\operatorname{atan}\left(\dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$\theta_{p1} = -14.2°$

$\theta_{p2} = \theta_{p1} + 90° \qquad \theta_{p2} = 75.8°$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p_1}) + \tau_{xy}\sin(2\theta_{p_1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p2}) + \tau_{xy}\sin(2\theta_{p2})$$

$\sigma_1 = 53.6\text{ MPa} \quad \leftarrow$

$\sigma_2 = -173.6\text{ MPa} \quad \leftarrow$

Problem 7.3-5 The normal and shear stresses acting on element A are 6500 psi, 18,500 psi, and 3800 psi (in the directions shown in the figure) (see the figure for Problem 7.2-5).

Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-5

$\sigma_x = 6500\text{ psi} \qquad \sigma_y = -18500\text{ psi} \qquad \tau_{xy} = -3800\text{ psi}$

PRINCIPAL ANGLES

$$\theta_{p1} = \frac{\operatorname{atan}\left(\dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$\theta_{p1} = -8.45°$

$\theta_{p2} = \theta_{p1} + 90° \qquad \theta_{p2} = 81.55°$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p_1}) + \tau_{xy}\sin(2\theta_{p_1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p2}) + \tau_{xy}\sin(2\theta_{p2})$$

$\sigma_1 = 7065\text{ psi}$

$\sigma_2 = -19065\text{ psi}$

MAXIMUM SHEAR STRESSES

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} \qquad \tau_{\text{max}} = 13065\text{ psi} \quad \leftarrow$$

$\theta_{s1} = \theta_{p1} - 45° \qquad \theta_{s1} = -53.4° \quad \leftarrow$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{\text{aver}} = -6000\text{ psi} \quad \leftarrow$$

Problem 7.3-6 An element in *plane stress* from the fuselage of an airplane is subjected to compressive stresses of magnitude 27 MPa in the horizontal direction and tensile stresses of magnitude 5.5 MPa in the vertical direction. Also, shear stresses of magnitude 10.5 MPa act in the directions shown (see the figure for Problem 7.2-6).

Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-6

$\sigma_x = -27\text{ MPa} \qquad \sigma_y = 5.5\text{ MPa} \qquad \tau_{xy} = -10.5\text{ MPa}$

PRINCIPAL ANGLES

$$\theta_{p2} = \frac{\operatorname{atan}\left(\dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$\theta_{p2} = 16.43°$

$\theta_{p1} = \theta_{p2} + 90° \qquad \theta_{p1} = 106.43°$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p_1}) + \tau_{xy}\sin(2\theta_{p_1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p2}) + \tau_{xy}\sin(2\theta_{p2})$$

$\sigma_1 = 8.6\text{ MPa}$

$\sigma_2 = -30.1\text{ MPa}$

MAXIMUM SHEAR STRESSES

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + {\tau_{xy}}^2}$$

$\tau_{max} = 19.3\text{ MPa} \quad \longleftarrow$

$\theta_{s1} = \theta_{p1} - 45° \qquad \theta_{s1} = 61.4°$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{aver} = -10.8\text{ MPa} \quad \longleftarrow$$

Problem 7.3-7 The stresses acting on element B in the web of a wide-flange beam are found to be 14,000 psi compression in the horizontal direction and 2600 psi compression in the vertical direction. Also, shear stresses of magnitude 3800 psi act in the directions shown (see the figure for Problem 7.2-7).

Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-7

$\sigma_x = -14000\text{ psi} \qquad \sigma_y = -2600\text{ psi}$

$\tau_{xy} = -3800\text{ psi}$

PRINCIPAL ANGLES

$$\theta_{p2} = \frac{\operatorname{atan}\left(\dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$\theta_{p2} = 16.85°$

$\theta_{p1} = \theta_{p2} + 90° \qquad \theta_{p1} = 106.85°$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p_1}) + \tau_{xy}\sin(2\theta_{p_1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p2}) + \tau_{xy}\sin(2\theta_{p2})$$

$\sigma_1 = -1449\text{ psi}$

$\sigma_2 = -15151\text{ psi}$

MAXIMUM SHEAR STRESSES

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + {\tau_{xy}}^2} \qquad \tau_{max} = 6851\text{ psi} \quad \longleftarrow$$

$\theta_{s1} = \theta_{p1} - 45° \qquad \theta_{s1} = 61.8° \quad \longleftarrow$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{aver} = -8300\text{ psi} \quad \longleftarrow$$

Problem 7.3-8 The normal and shear stresses acting on element B are $\sigma_x = -46$ MPa, $\sigma_y = -13$ MPa, and $\tau_{xy} = 21$ MPa (see figure for Problem 7.2-8).

Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-8

$\sigma_x = -46\text{ MPa} \qquad \sigma_y = -13\text{ MPa} \qquad \tau_{xy} = 21\text{ MPa}$

PRINCIPAL ANGLES

$$\theta_{p2} = \frac{\operatorname{atan}\left(\dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$\theta_{p2} = -25.92°$

$\theta_{p1} = \theta_{p2} + 90° \qquad \theta_{p1} = 64.08°$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p_1}) + \tau_{xy}\sin(2\theta_{p_1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p2}) + \tau_{xy}\sin(2\theta_{p2})$$

$\sigma_1 = -2.8\text{ MPa}$

$\sigma_2 = -56.2\text{ MPa}$

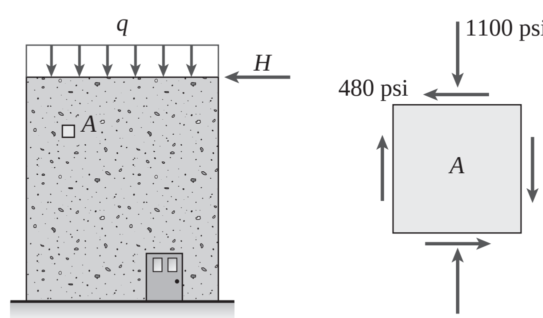
MAXIMUM SHEAR STRESSES

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \tau_{max} = 26.7\text{ MPa} \qquad \longleftarrow$$

$\theta_{s1} = \theta_{p1} - 45° \qquad \theta_{s1} = 19.08° \qquad \longleftarrow$

$$\sigma_{aver} = \frac{\sigma_x - \sigma_y}{2} \qquad \sigma_{aver} = -29.5\text{ MPa} \qquad \longleftarrow$$

Problem 7.3-9 A shear wall in a reinforced concrete building is subjected to a vertical uniform load of intensity q and a horizontal force H, as shown in the first part of the figure. (The force H represents the effects of wind and earthquake loads.) As a consequence of these loads, the stresses at point A on the surface of the wall have the values shown in the second part of the figure (compressive stress equal to 1100 psi and shear stress equal to 480 psi).



(a) Determine the principal stresses and show them on a sketch of a properly oriented element.
(b) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-9 Shear wall

$\sigma_x = 0 \qquad \sigma_y = -1100\text{ psi} \qquad \tau_{xy} = -480\text{ psi}$

(a) PRINCIPAL STRESSES

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -0.87273$$

$2\theta_p = -41.11°$ and $\theta_p = -20.56°$

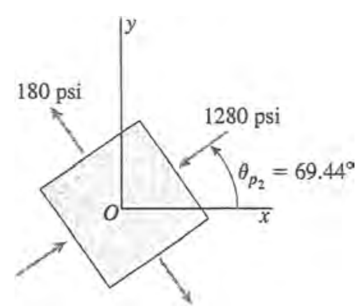
$2\theta_p = 138.89°$ and $\theta_p = 69.44°$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

For $2\theta_p = -41.11°$: $\qquad \sigma_{x_1} = 180\text{ psi}$

For $2\theta_p = 138.89°$: $\qquad \sigma_{x_1} = -1280\text{ psi}$

$$\left.\begin{aligned}\text{Therefore, } \sigma_1 &= 180\text{ psi and } \theta_{p_1} = -20.56°\\ \sigma_2 &= -1280\text{ psi and } \theta_{p_2} = 69.44°\end{aligned}\right\} \longleftarrow$$

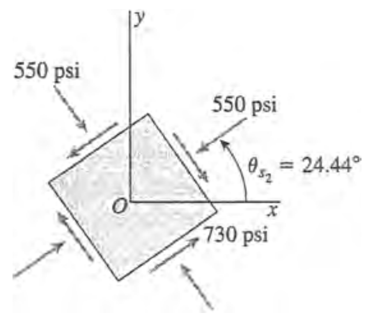


(b) Maximum shear stresses

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 730 \text{ psi}$$

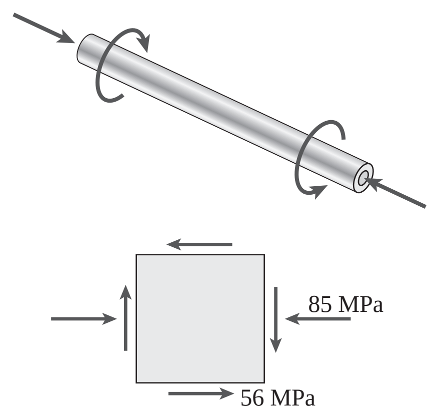
$$\left.\begin{aligned} \theta_{s_1} &= \theta_{p_1} - 45° = -65.56° \text{ and } \tau = 730 \text{ psi} \\ \theta_{s_2} &= \theta_{p_1} + 45° = 24.44° \quad \text{ and } \tau = -730 \text{ psi} \end{aligned}\right\} \longleftarrow$$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} = -550 \text{ psi} \quad \longleftarrow$$



Problem 7.3-10 A propeller shaft subjected to combined torsion and axial thrust is designed to resist a shear stress of 56 MPa and a compressive stress of 85 MPa (see figure).

(a) Determine the principal stresses and show them on a sketch of a properly oriented element.
(b) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.



Solution 7.3-10

$\sigma_x = -85$ MPa $\qquad \sigma_y = 0$ MPa $\qquad T_{xy} = -56$ MPa

(a) Principal stresses

$$\theta_{p2} = \frac{\operatorname{atan}\left(\dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$$\theta_{p2} = 26.4°$$

$$\theta_{p1} = \theta_{p2} + 90° \qquad \theta_{p1} = 116.4° \quad \longleftarrow$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p_1}) + \tau_{xy}\sin(2\theta_{p_1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p2}) + \tau_{xy}\sin(2\theta_{p2})$$

$$\sigma_1 = 27.8 \text{ MPa} \quad \longleftarrow$$

$$\sigma_2 = -112.8 \text{ MPa} \quad \longleftarrow$$

(b) Maximum shear stresses

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

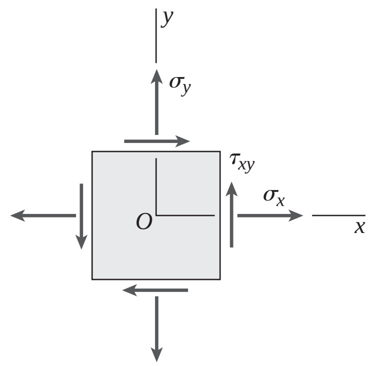
$$\tau_{max} = 70.3 \text{ MPa} \quad \longleftarrow$$

$$\theta_{s1} = \theta_{p1} - 45° \qquad \theta_{s1} = 71.4° \quad \longleftarrow$$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{aver} = -42.5 \text{ MPa} \quad \longleftarrow$$

Problems 7.3-11 $\sigma_x = 2500$ psi, $\sigma_y = 1020$ psi, $\tau_{xy} = -900$ psi

(a) Determine the principal stresses and show them on a sketch of a properly oriented element.
(b) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.



Probs. 7.3-11 through 7.3-16

Solution 7.3-11

$\sigma_x = 2500$ psi $\qquad \sigma_y = 1020$ psi $\qquad \tau_{xy} = -900$ psi

(a) Principal stresses

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \theta_{p1} = \frac{\operatorname{atan}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$\theta_{p1} = 25.29°$

$\theta_{p2} = 90° + \theta_{p1} \qquad \theta_{p2} = 64.71°$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p_1}) + \tau_{xy}\sin(2\theta_{p_1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p2}) + \tau_{xy}\sin(2\theta_{p2})$$

Therefore,

For $\theta_{p1} = -25.3°$: $\quad \sigma_1 = 2925$ psi $\quad \longleftarrow$

For $\theta_{p2} = 64.7°$: $\quad \sigma_2 = 595$ psi $\quad \longleftarrow$

(b) Maximum Shear Stresses

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\;2}}$$

$\tau_{\max} = 1165$ psi

$\theta_{s1} = \theta_{p1} - 45° \qquad \theta_{s1} = -70.3°$ and
$\tau_1 = 1165$ psi $\quad \longleftarrow$

$\theta_{s2} = \theta_{p1} - 45° \qquad \theta_{s2} = 19.71°$ and
$\tau_2 = -1165$ psi $\quad \longleftarrow$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{\text{aver}} = 1760 \text{ psi} \quad \longleftarrow$$

Problems 7.3-12 $\sigma_x = 2150$ kPa, $\sigma_y = 375$ kPa, $\tau_{xy} = -460$ kPa

(a) Determine the principal stresses and show them on a sketch of a properly oriented element.
(b) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-12

$\sigma_x = 2150$ kPa $\qquad \sigma_y = 375$ kPa $\qquad \tau_{xy} = -460$ kPa

(a) Principal Stresses

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \theta_{p1} = \frac{\operatorname{atan}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$\theta_{p1} = -13.70°$

$\theta_{p2} = 90° + \theta_{p1} \qquad \theta_{p2} = 76.30°$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p_1}) + \tau_{xy}\sin(2\theta_{p_1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p2}) + \tau_{xy}\sin(2\theta_{p2})$$

Therefore,

For $\theta_{p1} = -13.70°$ $\quad \sigma_1 = 2262 \text{ kPa}$ $\leftarrow$

For $\theta_{p2} = 76.3°$ $\quad \sigma_2 = 263 \text{ kPa}$ $\leftarrow$

(b) Maximum Shear Stresses

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$\tau_{max} = 145 \text{ psi}$

$\theta_{s1} = \theta_{p1} - 45°$ $\quad \theta_{s1} = -58.7°$ $\leftarrow$
and $\tau_1 = 1000 \text{ kPa}$

$\theta_{s2} = \theta_{p1} + 45°$ $\quad \theta_{s2} = 31.3°$ $\leftarrow$
and $\tau_2 = -1000 \text{ kPa}$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{aver} = 1263 \text{ kPa}$$

Problems 7.3-13 $\sigma_x = 14{,}500$ psi, $\sigma_y = 1070$ psi, $\tau_{xy} = 1900$ psi

(a) Determine the principal stresses and show them on a sketch of a properly oriented element.
(b) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-13

$\sigma_x = 14500 \text{ psi}$ $\quad \sigma_y = 1070 \text{ psi}$ $\quad \tau_{xy} = 1900 \text{ psi}$

(a) Principal Stresses

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \theta_{p1} = \frac{\operatorname{atan}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$\theta_{p1} = 7.90°$

$\theta_{p2} = 90° + \theta_{p1}$ $\quad \theta_{p2} = 97.90°$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p_1}) + \tau_{xy}\sin(2\theta_{p_1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p2}) + \tau_{xy}\sin(2\theta_{p2})$$

Therefore,

For $\theta_{p1} = 7.90°$ $\quad \sigma_1 = 14764 \text{ psi}$ $\leftarrow$

For $\theta_{p2} = 97.9°$ $\quad \sigma_2 = 806 \text{ psi}$ $\leftarrow$

(b) Maximum Shear Stresses

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$\tau_{max} = 6979 \text{ psi}$

$\theta_{s1} = \theta_{p1} - 45°$ $\quad \theta_{s1} = -37.1°$ $\leftarrow$
and $\tau_1 = 6979 \text{ psi}$

$\theta_{s2} = \theta_{p1} + 45°$ $\quad \theta_{s2} = 52.9°$ $\leftarrow$
and $\tau_2 = -6979 \text{ psi}$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{aver} = 7785 \text{ psi}$$

Problems 7.3-14 $\sigma_x = 16.5$ MPa, $\sigma_v = -91$ MPa, $\tau_{xy} = -39$ MPa

(a) Determine the principal stresses and show them on a sketch of a properly oriented element.
(b) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-14

$\sigma_x = 16.5\,\text{MPa} \qquad \sigma_y = -91\,\text{MPa} \qquad \tau_{xy} = -39\,\text{MPa}$

(a) Principal Stresses

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \theta_{p1} = \frac{\operatorname{atan}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$\theta_{p1} = -17.98°$

$\theta_{p2} = 90° + \theta_{p1} \qquad \theta_{p2} = 72.02°$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p_1}) + \tau_{xy}\sin(2\theta_{p_1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p2}) + \tau_{xy}\sin(2\theta_{p2})$$

Therefore,

For $\theta_{p1} = -17.98° \qquad \sigma_1 = 29.2\ \text{MPa}$

For $\theta_{p2} = 72.0° \qquad \sigma_2 = -103.7\ \text{MPa}$

(b) Maximum Shear Stresses

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

$\tau_{max} = 9631.7\ \text{psi}$

$\theta_{s1} = \theta_{p1} - 45° \qquad \theta_{s1} = -63.0° \quad \leftarrow$

and $\tau_1 = 66.4\ \text{MPa}$

$\theta_{s2} = \theta_{p1} + 45° \qquad \theta_{s2} = 27.0° \quad \leftarrow$

and $\tau_2 = -66.4\ \text{MPa}$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{aver} = -37.3\ \text{MPa}$$

Problems 7.3-15 $\sigma_x = -3300$ psi, $\sigma_y = -11{,}000$ psi, $\tau_{xy} = 4500$ psi

(a) Determine the principal stresses and show them on a sketch of a properly oriented element.
(b) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-15

$\sigma = -3300\ \text{psi} \qquad \sigma_y = -11000\ \text{psi} \qquad \tau_{xy} = 4500\ \text{psi}$

(a) Principal Stresses

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \theta_{p1} = \frac{\operatorname{atan}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$\theta_{p1} = 24.73°$

$\theta_{p2} = 90° + \theta_{p1} \qquad \theta_{p2} = 114.73°$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p1}) + \tau_{xy}\sin(2\theta_{p1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p2}) + \tau_{xy}\sin(2\theta_{p2})$$

Therefore,

For $\theta_{p1} = 24.7° \qquad \sigma_1 = -1228\ \text{psi}$

For $\theta_{p2} = 114.7° \qquad \sigma_2 = -13072\ \text{psi}$

(b) Maximum Shear Stresses

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

$\tau_{max} = 5922\ \text{psi}$

$\theta_{s1} = \theta_{p1} - 45° \qquad \theta_{s1} = -20.3° \quad \leftarrow$

and $\tau_1 = 5922\ \text{psi}$

$\theta_{s2} = \theta_{p1} + 45° \qquad \theta_{s2} = 69.7° \quad \leftarrow$

and $\tau_2 = -5922\ \text{psi}$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{aver} = -7150\ \text{psi}$$

Problems 7.3-16 $\sigma_x = -108$ MPa, $\sigma_y = 58$ MPa, $\tau_{xy} = -58$ MPa

(a) Determine the principal stresses and show them on a sketch of a properly oriented element.
(b) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-16

$\sigma_x = -108$ MPa $\quad \sigma_y = 58$ MPa $\quad \tau_{xy} = -58$ MPa

(a) Principal Stresses

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \theta_{p2} = \frac{\operatorname{atan}\left(\dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$\theta_{p2} = 17.47°$

$\theta_{p1} = 90° + \theta_{p2} \qquad \theta_{p1} = 107.47°$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p_1}) + \tau_{xy}\sin(2\theta_{p_1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p2}) + \tau_{xy}\sin(2\theta_{p2})$$

Therefore,

For $\theta_{p1} = 107.47°$ $\quad \sigma_1 = 76.3$ MPa $\quad \leftarrow$

For $\theta_{p2} = 17.47°$ $\quad \sigma_2 = -126.3$ MPa $\quad \leftarrow$

(b) Maximum Shear Stresses

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

$\tau_{max} = 14686.1$ psi

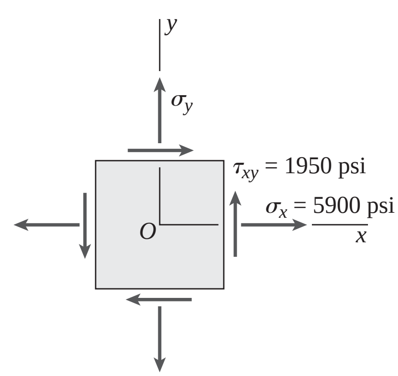
$\theta_{s1} = \theta_{p1} - 45° \qquad \theta_{s1} = 62.47° \quad \leftarrow$
and $\tau_1 = 101.3$ MPa

$\theta_{s2} = \theta_{p1} + 45° \qquad \theta_{s2} = 152.47° \quad \leftarrow$
and $\tau_2 = -101.3$ MPa

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{aver} = -25.0 \text{ MPa}$$

Problem 7.3-17 At a point on the surface of a machine component, the stresses acting on the x face of a stress element are $\sigma_x = 5900$ psi and $\tau_{xy} = 1950$ psi (see figure).

What is the allowable range of values for the stress σ_y if the maximum shear stress is limited to $\tau_0 = 2500$ psi?



Solution 7.3-17

$\sigma_x = 5900$ psi $\quad \sigma_y$ unknown $\quad \tau_{xy} = 1950$ psi

Find the allowable range of values for σ_y if the maximum allowable shear stresses is $\tau_{max} = 2500$ psi

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}} \qquad (1)$$

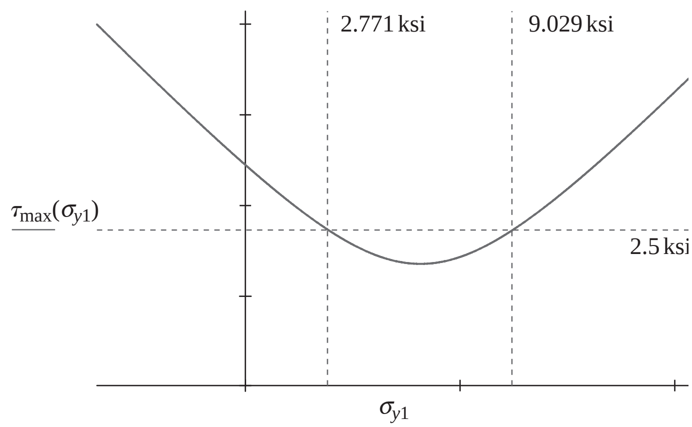
Solve for σ_y

$$\sigma_y = \sigma_x + \begin{bmatrix} 2\sqrt{\tau_{max}^{\ 2} - \tau_{xy}^{\ 2}} \\ -\left(2\sqrt{\tau_{max}^{\ 2} - \tau_{xy}^{\ 2}}\right) \end{bmatrix} \qquad \sigma_y = \begin{pmatrix} 9029 \\ 2771 \end{pmatrix} \text{psi}$$

Therefore, 2771 psi $\leq \sigma_y \leq$ 9029 psi

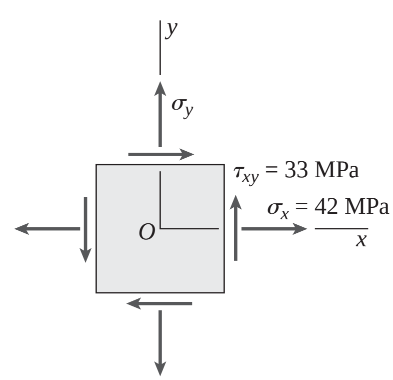
From Eq. (1):

$$\tau_{max}(\sigma_{y1}) = \sqrt{\left(\frac{\sigma_x - \sigma_{y1}}{2}\right)^2 + \tau_{xy}^{\ 2}}$$



Problem 7.3-18 At a point on the surface of a machine component the stresses acting on the x face of a stress element are $\sigma_x = 42$ MPa and $\tau_{xy} = 33$ MPa (see figure).

What is the allowable range of values for the stress σ_y if the maximum shear stress is limited to $\tau_0 = 35$ MPa?



Solution 7.3-18

$\sigma_x = 42$ MPa $\quad \sigma_y$ unknown $\quad \tau_{xy} = 33$ MPa

Find the allowable range of values for σ_y if the maximum allowable shear stresses is $\tau_{max} = 35$ MPa

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}} \qquad (1)$$

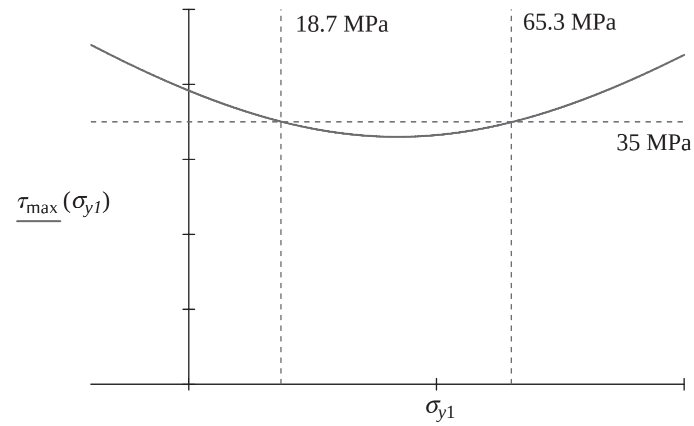
Solve for σ_y

$$\sigma_y = \sigma_x + \begin{bmatrix} 2\sqrt{\tau_{max}{}^2 - \tau_{xy}{}^2} \\ -\left(2\sqrt{\tau_{max}{}^2 - \tau_{xy}{}^2}\right) \end{bmatrix} \qquad \sigma_y = \begin{pmatrix} 65.3 \\ 18.7 \end{pmatrix} \text{MPa}$$

From Eq. (1):

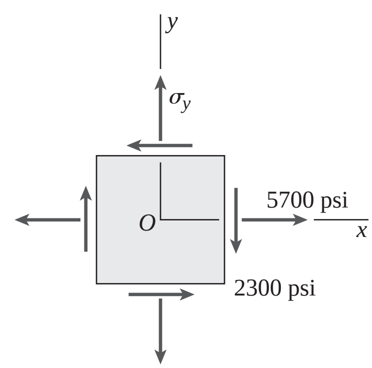
$$\tau_{max}(\sigma_{y1}) = \sqrt{\left(\frac{\sigma_x - \sigma_{y1}}{2}\right)^2 + \tau_{xy}{}^2}$$

Therefore, 18.7 MPa $\leq \sigma_y \leq$ 65.3 MPa



Problem 7.3-19 An element in *plane stress* is subjected to stresses σ_x = 5700 psi and τ_{xy} = −2300 psi (see figure). It is known that one of the principal stresses equals 6700 psi in tension.

(a) Determine the stress σ_y.
(b) Determine the other principal stress and the orientation of the principal planes, then show the principal stresses on a sketch of a properly oriented element.



Solution 7.3-19

σ_x = 5700 psi $\quad \sigma_y$ unknown $\quad \tau_{xy}$ = −2300 psi

(a) Stress σ_y

Because σ_y is smaller than a given principal stress, we know that the given stress is the larger principal stress.

σ_1 = 6700 psi

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

Solve for σ_y $\qquad \sigma_y$ = 1410 psi $\quad \leftarrow$

(b) Principal stresses

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \theta_{p1} = \frac{\operatorname{atan}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

θ_{p1} = −23.50°

$\theta_{p2} = 90° + \theta_{p1} \qquad \theta_{p2}$ = 66.50°

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p_1}) + \tau_{xy}\sin(2\theta_{p_1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p2}) + \tau_{xy}\sin(2\theta_{p2})$$

Therefore,

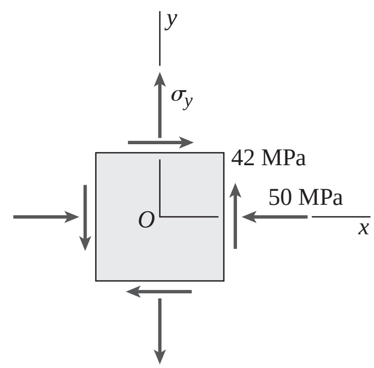
For $\theta_{p1} = -23.5°$: $\sigma_1 = 6700$ psi $\leftarrow$

For $\theta_{p2} = 66.5°$: $\sigma_2 = 410$ psi $\leftarrow$

Problem 7.3-20 An element in *plane stress* is subjected to stresses $\sigma_x = -50$ MPa and $\tau_{xy} = 42$ MPa (see figure). It is known that one of the principal stresses equals 33 MPa in tension.

(a) Determine the stress σ_y.

(b) Determine the other principal stress and the orientation of the principal planes, then show the principal stresses on a sketch of a properly oriented element.



Solution 7.3-20

$\sigma_x = -50$ MPa $\quad \sigma_y$ unknown $\quad \tau_{xy} = 42$ MPa

(a) Stress σ_y

Because σ_y is smaller than a given principal stress, we know that the given stress is the larger principal stress.

$\sigma_1 = 33$ MPa

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\,2}}$$

Solve for σ_y $\quad \sigma_y = 11.7$ MPa $\leftarrow$

(b) Principal stresses

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \theta_{p2} = \frac{\operatorname{atan}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$\theta_{p2} = -26.85°$

$\theta_{p1} = 90° + \theta_{p2} \qquad \theta_{p1} = 63.15°$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p_1}) + \tau_{xy}\sin(2\theta_{p_1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_{p2}) + \tau_{xy}\sin(2\theta_{p2})$$

Therefore,

For $\theta_{p1} = 63.2°$: $\sigma_1 = 33.0$ MPa $\leftarrow$

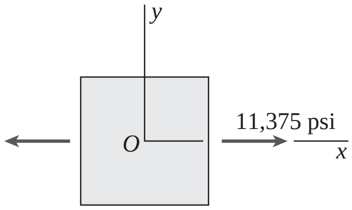
For $\theta_{p2} = -26.8°$: $\sigma_2 = -71.3$ MPa $\leftarrow$

Mohr's Circle

The problems for Section 7.4 are to be solved using Mohr's circle. Consider only the in-plane stresses (the stresses in the xy *plane).*

Problem 7.4-1 An element in *uniaxial stress* is subjected to tensile stresses $\sigma_x = 11{,}375$ psi, as shown in the figure. Using Mohr's circle, determine:

(a) The stresses acting on an element oriented at a counterclockwise angle $\theta = 24°$ from the x axis.
(b) The maximum shear stresses and associated normal stresses.
Show all results on sketches of properly oriented elements.



Solution 7.4-1

$\sigma_x = 11375$ psi $\quad \sigma_y = 0$ psi $\quad \tau_{xy} = 0$ psi

(a) Element at $\quad \theta = 24° \quad \leftarrow$

$2\theta = 48° \quad R = \dfrac{\sigma_x}{2} \quad R = 5688$ psi

Point C: $\quad \sigma_c = R \quad \sigma_c = 5688$ psi

Point D: $\quad \sigma_{x1} = R + R\cos(2\theta)$

$\sigma_{x1} = 9493$ psi $\quad \leftarrow$

$\tau_{x1y1} = -R\sin(2\theta)$

$\tau_{x1y1} = -4227$ psi $\quad \leftarrow$

PointD$'$: $\quad \sigma_{y1} = R - R\cos(2\theta)$

$\sigma_{y1} = 1882$ psi $\quad \leftarrow$

(b) Maximum shear stresses

Point S1: $\theta_{s1} = \dfrac{-90°}{2} \quad \theta_{s1} = -45° \quad \leftarrow$

$\tau_{max} = R \quad \tau_{max} = 5688$ psi $\quad \leftarrow$

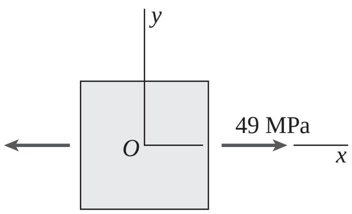
Point S2: $\quad \theta_{s2} = \dfrac{90°}{2} \quad \theta_{s2} = 45° \quad \leftarrow$

$\tau_{max} = -R \quad \tau_{max} = -5688$ psi $\quad \leftarrow$

$\sigma_{aver} = R \quad \sigma_{aver} = 5688$ psi $\quad \leftarrow$

Problem 7.4-2 An element in *uniaxial stress* is subjected to tensile stresses $\sigma_x = 49$ MPa, as shown in the figure Using Mohr's circle, determine:

(a) The stresses acting on an element oriented at an angle $\theta = -27°$ from the x axis (minus means clockwise).
(b) The maximum shear stresses and associated normal stresses.
Show all results on sketches of properly oriented elements.



Solution 7.4-2

$\sigma_x = 49$ MPa $\quad \sigma_y = 0$ MPa $\quad \tau_{xy} = 0$ MPa

(a) Element at $\quad \theta = -27°$

$2\theta = -54.0° \quad R = \dfrac{\sigma_x}{2} \quad R = 24.5$ MPa

Point C: $\quad \sigma_c = R \quad \sigma_c = 24.5$ MPa

Point D: $\quad \sigma_{x1} = R + R\cos(|2\theta|)$

$\sigma_{x1} = 38.9$ MPa $\quad \leftarrow$

$\tau_{x1y1} = -R\sin(2\theta)$

$\tau_{x1y1} = 19.8$ MPa $\quad \leftarrow$

Point D$'$ $\sigma_{y1} = R - R\cos(|2\theta|)$

$\sigma_{y1} = 10.1\ MPa \quad \leftarrow$

(b) Maximum shear stresses

Point S1: $\theta_{s1} = \dfrac{-90°}{2}$

$\theta_{s1} = -45.0°$ ←

$\tau_{max} = R \qquad \tau_{max} = 24.5 \text{ MPa}$ ←

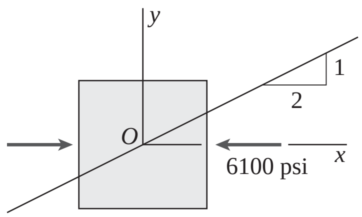
Point S2: $\theta_{s2} = \dfrac{90°}{2} \qquad \theta_{s2} = 45.0°$ ←

$\tau_{max} = -R \qquad \tau_{max} = -24.5 \text{ MPa}$ ←

$\sigma_{aver} = R \qquad \sigma_{aver} = 24.5 \text{ MPa}$ ←

Problem 7.4-3 An element in *uniaxial stress* is subjected to compressive stresses of magnitude 6100 psi, as shown in the figure. Using Mohr's circle, determine:

(a) The stresses acting on an element oriented at a slope of 1 on 2 (see figure).
(b) The maximum shear stresses and associated normal stresses.
Show all results on sketches of properly oriented elements.



Solution 7.4-3

$\sigma_x = -6100 \text{ psi} \quad \sigma_y = 0 \text{ psi} \qquad \tau_{xy} = 0 \text{ psi}$

(a) Element at a slope of 1 on 2

$\theta = \text{atan}\left(\dfrac{1}{2}\right) \qquad \theta = 26.565°$ ←

$2\theta = 53.130° \quad R = \dfrac{\sigma_x}{2} \quad R = -3050 \text{ psi}$

Point C: $\sigma_c = R \qquad \sigma_c = -3050 \text{ psi}$

Point D: $\sigma_{x1} = R + R\cos(2\theta)$

$\sigma_{x1} = -4880 \text{ psi}$ ←

$\tau_{x1y1} = -R\sin(2\theta) \qquad \tau_{x1y1} = 2440 \text{ psi}$ ←

Point D′: $\sigma_{y1} = R - R\cos(2\theta)$

$\sigma_{y1} = -1220 \text{ psi}$

(b) Maximum shear stresses

Point S1: $\theta_{s1} = \dfrac{90°}{2} \qquad \theta_{s1} = 45°$ ←

$\tau_{max} = -R \qquad \tau_{max} = 3050 \text{ psi}$ ←

Point S2: $\theta_{s2} = \dfrac{-90°}{2} \qquad \theta_{s2} = -45°$ ←

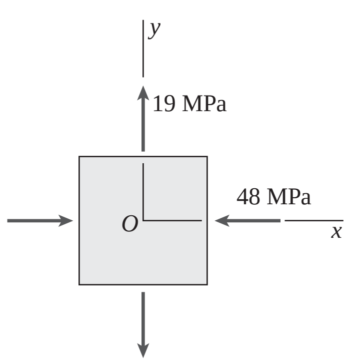
$\tau_{max} = R \qquad \tau_{max} = -3050 \text{ psi}$ ←

$\sigma_{aver} = R \qquad \sigma_{aver} = -3050 \text{ psi}$ ←

Problem 7.4-4 An element in *biaxial stress* is subjected to stresses $\sigma_x = -48$ MPa and $\sigma_y = 19$ MPa, as shown in the figure. Using Mohr's circle, determine:

(a) The stresses acting on an element oriented at a counterclockwise angle $\theta = 25°$ from the x axis.
(b) The maximum shear stresses and associated normal stresses.

Show all results on sketches of properly oriented elements.



Solution 7.4-4

$\sigma_x = -48$ MPa $\qquad \sigma_y = 19$ MPa $\qquad \tau_{xy} = 0$ MPa

(a) ELEMENT AT $\theta = 25°$ $\leftarrow$

$2\theta = 50.0$ deg $\quad R = \dfrac{|\sigma_x| + |\sigma_y|}{2} \quad R = 33.5$ MPa

Point C: $\sigma_x = \sigma_x + R \qquad \sigma_c = -14.5$ MPa

Point D: $\sigma_{x1} = \sigma_c - R\cos(2\theta)$

$\sigma_{x1} = -36.0$ MPa $\leftarrow$

$\tau_{x1y1} = -R\sin(2\theta)$

$\tau_{x1y1} = 25.7$ MPa $\leftarrow$

Point D′: $\sigma_{y1} = \sigma_c + R\cos(2\theta)$

$\sigma_{y1} = 7.0$ MPa

(b) MAXIMUM SHEAR STRESSES

Point S1: $\theta_{s1} = \dfrac{90°}{2}$

$\theta_{s1} = 45.0°$ $\leftarrow$

$\tau_{max} = R \qquad \tau_{max} = 33.5$ MPa $\leftarrow$

Point S2: $\theta_{s2} = \dfrac{-90°}{2}$

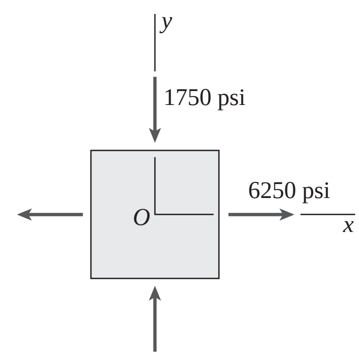
$\theta_{s2} = -45.0°$ $\leftarrow$

$\tau_{max} = -R \qquad \tau_{max} = -33.5$ MPa $\leftarrow$

$\sigma_{aver} = \sigma_c \qquad \sigma_{aver} = -14.5$ MPa $\leftarrow$

Problem 7.4-5 An element in *biaxial stress* is subjected to stresses $\sigma_x = 6250$ psi and $\sigma_y = -1750$ psi, as shown in the figure. Using Mohr's circle, determine:

(a) The stresses acting on an element oriented at a counterclockwise angle $\theta = 55°$ from the x axis.
(b) The maximum shear stresses and associated normal stresses.
Show all results on sketches of properly oriented elements.



Solution 7.4-5

$\sigma_x = 6250$ psi $\qquad \sigma_y = -1750$ psi $\qquad \tau_{xy} = 0$ psi

(a) ELEMENT AT $\theta = 60°$

$2\theta = 120° \quad R = \dfrac{|\sigma_x| + |\sigma_y|}{2} \, R = 4000$ psi

Point C: $\sigma_c = \sigma_x - R \qquad \sigma_c = 2250$ psi

Point D: $\sigma_{x1} = \sigma_c + R\cos(2\theta)$

$\sigma_{x1} = 250$ psi $\leftarrow$

$\tau_{x1y1} = -R\sin(2\theta)$

$\tau_{x1y1} = -3464$ psi $\leftarrow$

Point D′: $\sigma_{y1} = \sigma_c - R\cos(2\theta) \qquad \sigma_{y1} = 4250$ psi

(b) MAXIMUM SHEAR STRESSES

Point S1: $\theta_{s1} = \dfrac{-90°}{2}$

$\theta_{s1} = -45°$ $\leftarrow$

$\tau_{max} = R \qquad \tau_{max} = 4000$ psi $\leftarrow$

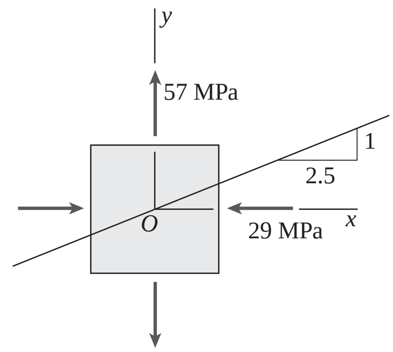
Point S2: $\theta_{s2} = \dfrac{90°}{2} \qquad \theta_{s2} = 45°$ $\leftarrow$

$\tau_{max} = R \qquad \tau_{max} = 4000$ psi $\leftarrow$

$\sigma_{aver} = \sigma_c \qquad \sigma_{aver} = 2250$ psi $\leftarrow$

Problem 7.4-6 An element in *biaxial stress* is subjected to stresses $\sigma_x = -29$ MPa and $\sigma_y = 57$ MPa, as shown in the figure. Using Mohr's circle, determine:

(a) The stresses acting on an element oriented at a slope of 1 on 2.5 (see figure).
(b) The maximum shear stresses and associated normal stresses.
Show all results on sketches of properly oriented elements.



Solution 7.4-6

$\sigma_x = -29 \text{ MPa} \qquad \sigma_y = 57 \text{ MPa} \qquad \tau_{xy} = 0 \text{ MPa}$

(a) Element at a slope of 1 on 2.5

$\theta = \text{atan}\left(\frac{1}{2.5}\right) \qquad \theta = 21.801° \quad \leftarrow$

$2\theta = 43.603° \quad R = \frac{|\sigma_x| + |\sigma_y|}{2} \quad R = 43.0 \text{ MPa}$

Point C: $\sigma_c = \sigma_x + R \qquad \sigma_c = 14.0 \text{ MPa}$

Point D: $\sigma_{x1} = \sigma_c - R\cos(2\theta)$

$\sigma_{x1} = -17.1 \text{ MPa} \quad \leftarrow$

$\tau_{x1y1} = R\sin(2\theta) \qquad \tau_{x1y1} = 29.7 \text{ MPa} \quad \leftarrow$

Point D′: $\sigma_{y1} = \sigma_c + R\cos(2\theta)$

$\sigma_{y1} = 45.1 \text{ MPa}$

(b) Maximum shear stresses

Point S1: $\theta_{s1} = \frac{90°}{2} \qquad \theta_{s1} = 45.0° \quad \leftarrow$

$\tau_{max} = R \qquad \tau_{max} = 43.0 \text{ MPa} \quad \leftarrow$

Point S2: $\theta_{s2} = \frac{-90°}{2}$

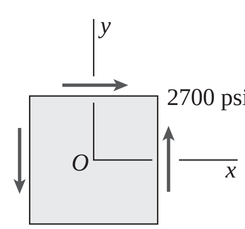
$\theta_{s2} = -45.0° \quad \leftarrow$

$\tau_{max} = -R \qquad \tau_{max} = -43.0 \text{ MPa} \quad \leftarrow$

$\sigma_{aver} = \sigma_c \qquad \sigma_{aver} = 14.0 \text{ MPa} \quad \leftarrow$

Problem 7.4-7 An element in *pure shear* is subjected to stresses $\tau_{xy} = 2700$ psi, as shown in the figure. Using Mohr's circle, determine:

(a) The stresses acting on an element oriented at a counterclockwise angle $\theta = 52°$ from the x axis.
(b) The principal stresses.
Show all results on sketches of properly oriented elements.



Solution 7.4-7

$\sigma_x = 0$ psi $\quad \sigma_y = 0$ psi $\quad \tau_{xy} = 2700$ psi

(a) Element at $\theta = 52°$

$2\theta = 104.0° \quad R = \tau_{xy} \quad R = 2700$ psi

Point D: $\sigma_{x1} = R\cos(2\theta - 90°)$

$\sigma_{x1} = 2620$ psi $\leftarrow$

$\tau_{x1y1} = -R\sin(2\theta - 90°)$

$\tau_{x1y1} = -653$ psi $\leftarrow$

Point D′: $\sigma_{y1} = -R\cos(2\theta - 90°)$

$\sigma_{y1} = -2620$ psi $\leftarrow$

(b) Principal stresses

Point P1: $\theta_{p1} = \dfrac{90°}{2} \quad \theta_{p1} = 45°$ $\leftarrow$

$\sigma_1 = R \quad \sigma_1 = 2700$ psi $\leftarrow$

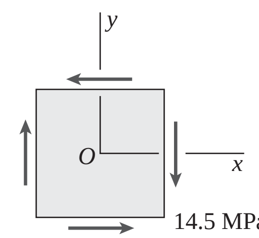
Point P2: $\theta_{p2} = \dfrac{-90°}{2}$

$\theta_{p2} = -45°$ $\leftarrow$

$\sigma_2 = -R \quad \sigma_2 = -2700$ psi $\leftarrow$

Problem 7.4-8 An element in *pure shear* is subjected to stresses $\tau_{xy} = -14.5$ MPa, as shown in the figure. Using Mohr's circle, determine:

(a) The stresses acting on an element oriented at a counterclockwise angle $\theta = 22.5°$ from the x axis
(b) The principal stresses.
Show all results on sketches of properly oriented elements.



Solution 7.4-8

$\sigma_x = 0$ MPa $\quad \sigma_y = 0$ MPa $\quad \tau_{xy} = -14.5$ MPa

(a) Element at $\theta = 22.5°$

$2\theta = 45.00°$

$R = |\tau_{xy}| \quad R = 14.50$ MPa

Point D: $\sigma_{x1} = -R\cos(2\theta - 90°)$

$\sigma_{x1} = -10.25$ MPa $\leftarrow$

$\tau_{x1y1} = R\sin(2\theta - 90°)$

$\tau_{x1y1} = -10.25$ MPa $\leftarrow$

Point D′: $\sigma_{y1} = R\cos(2\theta - 90°)$

$\sigma_{y1} = 10.25$ MPa $\leftarrow$

(b) Principal stresses

Point P1: $\theta_{p1} = \dfrac{270°}{2} \; \theta_{p1} = 135.0°$ $\leftarrow$

$\sigma_1 = R \quad \sigma_1 = 14.50$ MPa $\leftarrow$

Point P2: $\theta_{p2} = \dfrac{-270°}{2}$

$\theta_{p2} = -135.0°$ $\leftarrow$

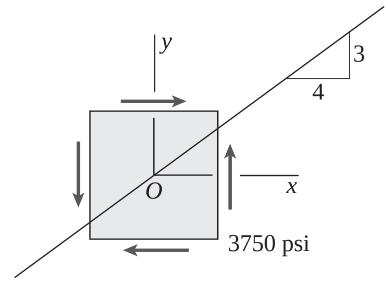
$\sigma_2 = -R$

$\sigma_2 = -14.50$ MPa $\leftarrow$

Problem 7.4-9 An element in *pure shear* is subjected to stresses $\tau_{xy} = 3750$ psi, as shown in the figure. Using Mohr's circle, determine:

(a) The stresses acting on an element oriented at a slope of 3 on 4 (see figure).
(b) The principal stresses.

Show all results on sketches of properly oriented elements.



Solution 7.4-9

$\sigma_x = 0$ psi $\quad \sigma_y = 0$ psi $\quad \tau_{xy} = 3750$ psi

(a) Element at a slope of 3 on 4

$$\theta = \mathrm{atan}\left(\frac{3}{4}\right) \qquad \theta = 36.870°$$

$$2\theta = 73.740° \qquad R = \tau_{xy} \qquad R = 3750 \text{ psi}$$

Point D: $\sigma_{x1} = R\cos(2\theta - 90°)$

$\sigma_{x1} = 3600$ psi $\quad \longleftarrow$

$\tau_{x1y1} = -R\sin(2\theta - 90°)$

$\tau_{x1y1} = 1050$ psi $\quad \longleftarrow$

Point D′: $\sigma_{y1} = -R\cos(2\theta - 90°)$

$\sigma_{y1} = -3600$ psi $\quad \longleftarrow$

(b) Principal stresses

Poin P1: $\theta_{p1} = \dfrac{90°}{2} \qquad \theta_{p1} = 45° \quad \longleftarrow$

$\sigma_1 = R \qquad \sigma_1 = 3750$ psi $\quad \longleftarrow$

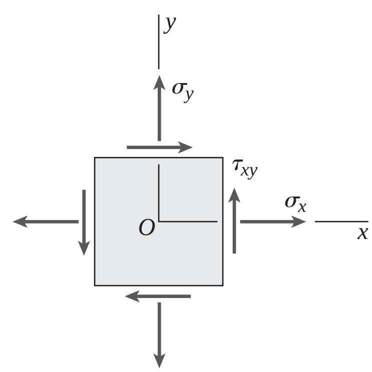
Point P2: $\theta_{p2} = \dfrac{-90°}{2}$

$\theta_{p2} = -45° \quad \longleftarrow$

$\sigma_2 = -R \qquad \sigma_2 = -3750$ psi $\quad \longleftarrow$

Problem 7.4-10 $\sigma_x = 27$ MPa, $\sigma_y = 14$ MPa, $\tau_{xy} = 6$ MPa, $\theta = 40°$

Using Mohr's circle, determine the stresses acting on an element oriented at an angle θ from the x axis. Show these stresses on a sketch of an element oriented at the angle θ. (*Note:* The angle θ is positive when counterclockwise and negative when clockwise.)



Probs. 7.4-10 through 7.4-15

Solution 7.4-10

$\sigma_x = 27 \text{ MPa} \qquad \sigma_y = 14 \text{ MPa} \qquad \tau_{xy} = 6 \text{ MPa}$

$\theta = 40°$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{aver} = 20.50 \text{ MPa}$$

$$R = \sqrt{(\sigma_x - \sigma_{aver})^2 + \tau_{xy}^{\,2}} \qquad R = 8.8459 \text{ MPa}$$

$$\alpha = \text{atan}\left(\frac{\tau_{xy}}{\sigma_x - \sigma_{aver}}\right) \qquad \alpha = 42.71°$$

$\beta = 2\theta - \alpha \qquad \beta = 37.29°$

Point D: $\sigma_{x1} = \sigma_{aver} + R\cos(\beta)$

$\sigma_{x1} = 27.5 \text{ MPa} \quad \leftarrow$

$\tau_{x1y1} = -R\sin(\beta)$

$\tau_{x1y1} = -5.36 \text{ MPa} \quad \leftarrow$

Point D′: $\sigma_{y1} = \sigma_{aver} - R\cos(\beta)$

$\sigma_{y1} = 13.46 \text{ MPa} \quad \leftarrow$

Problem 7.4-11 $\sigma_x = 3500$ psi, $\sigma_y = 12{,}200$ psi, $\tau_{xy} = -3300$ psi, $\theta = -51°$

Using Mohr's circle, determine the stresses acting on an element oriented at an angle θ from the x axis. Show these stresses on a sketch of an element oriented at the angle θ. (*Note:* The angle θ is positive when counterclockwise and negative when clockwise.)

Solution 7.4-11

$\sigma_x = 3500 \text{ psi} \qquad \sigma_y = 12200 \text{ psi} \qquad \tau_{xy} = -3300 \text{ psi}$

$\theta = -51° \quad \leftarrow$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{aver} = 7850 \text{ psi}$$

$$R = \sqrt{(\sigma_x - \sigma_{aver})^2 + \tau_{xy}^{\,2}} \qquad R = 5460 \text{ psi}$$

$$\alpha = \text{atan}\left(\frac{\tau_{xy}}{\sigma_x - \sigma_{aver}}\right) \qquad \alpha = 37.18°$$

$\beta = 180° + 2\theta - \alpha \qquad \beta = 40.82°$

Point D: $\sigma_{x1} = \sigma_{aver} + R\cos(\beta)$

$\sigma_{x1} = 11982 \text{ psi} \quad \leftarrow$

$\tau_{x1y1} = -R\sin(\beta)$

$\tau_{x1y1} = -3569 \text{ psi} \quad \leftarrow$

Point D′: $\sigma_{y1} = \sigma_{aver} - R\cos(\beta)$

$\sigma_{y1} = 3718 \text{ psi} \quad \leftarrow$

Problem 7.4-12 $\sigma_x = -47$ MPa, $\sigma_y = -186$ MPa, $\tau_{xy} = -29$ MPa, $\theta = -33°$

Using Mohr's circle, determine the stresses acting on an element oriented at an angle θ from the x axis. Show these stresses on a sketch of an element oriented at the angle θ. (*Note:* The angle θ is positive when counterclockwise and negative when clockwise.)

Solution 7.4-12

$\sigma_x = -47\text{MPa} \qquad \sigma_y = -186\text{MPa} \qquad \tau_{xy} = -29\text{MPa}$

$\theta = -33°$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{aver} = -116.50 \text{ MPa}$$

$$R = \sqrt{(\sigma_x - \sigma_{aver})^2 + \tau_{xy}^{\,2}} \qquad R = 75.3077 \text{ MPa}$$

$$\alpha = \text{atan}\left(\left|\frac{\tau_{xy}}{\sigma_x - \sigma_{aver}}\right|\right) \qquad \alpha = 22.65°$$

$\beta = -2\theta - \alpha \qquad \beta = 43.35°$

Point D: $\sigma_{x1} = \sigma_{aver} + R\cos(\beta)$

$\sigma_{x1} = -61.7 \text{ MPa} \quad \leftarrow$

$\tau_{x1y1} = -R\sin(\beta)$

$\tau_{x1y1} = -51.7 \text{ MPa} \quad \leftarrow$

Point D′: $\sigma_{y1} = \sigma_{aver} - R\cos(\beta)$

$\sigma_{y1} = -171.3 \text{ MPa} \quad \leftarrow$

Problem 7.4-13 $\sigma_x = -1720$ psi, $\sigma_y = -680$ psi, $\tau_{xy} = 320$ psi, $\theta = 14°$

Using Mohr's circle, determine the stresses acting on an element oriented at an angle θ from the x axis. Show these stresses on a sketch of an element oriented at the angle θ. (*Note:* The angle θ is positive when counterclockwise and negative when clockwise.)

Solution 7.4-13

$\sigma_x = -1720 \text{ psi} \qquad \sigma_y = -680 \text{ psi} \qquad \tau_{xy} = 380 \text{ psi}$

$\theta = 14°$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{aver} = -1200 \text{ psi}$$

$$R = \sqrt{(\sigma_x - \sigma_{aver})^2 + \tau_{xy}{}^2} \qquad R = 644.0 \text{ psi}$$

$$\alpha = \text{atan}\left(\frac{\tau_{xy}}{|\sigma_x - \sigma_{aver}|}\right) \qquad \alpha = 36.16°$$

$$\beta = 2\theta + \alpha \qquad \beta = 64.16°$$

Point D: $\sigma_{x1} = \sigma_{aver} - R\cos(\beta)$

$\sigma_{x1} = -1481 \text{ psi} \quad \leftarrow$

$\tau_{x1y1} = R\sin(\beta) \quad \tau_{x1y1} = 580 \text{ psi} \quad \leftarrow$

Point D′: $\sigma_{y1} = \sigma_{aver} + R\cos(\beta)$

$\sigma_{y1} = -919 \text{ psi} \quad \leftarrow$

Problem 7.4-14 $\sigma_x = 33$ MPa, $\sigma_y = -9$ MPa, $\tau_{xy} = 29$ MPa, $\theta = 35°$

Using Mohr's circle, determine the stresses acting on an element oriented at an angle θ from the x axis. Show these stresses on a sketch of an element oriented at the angle θ. (*Note:* The angle θ is positive when counterclockwise and negative when clockwise.)

Solution 7.4-14

$\sigma_x = 33 \text{ MPa} \quad \sigma_y = -9 \text{ MPa} \quad \tau_{xy} = 29 \text{ MPa}$

$\theta = 35°$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{aver} = 12.00 \text{ MPa}$$

$$R = \sqrt{(\sigma_x - \sigma_{aver})^2 + \tau_{xy}{}^2} \qquad R = 35.8050 \text{ MPa}$$

$$\alpha = \text{atan}\left(\left|\frac{\tau_{xy}}{\sigma_x - \sigma_{aver}}\right|\right) \qquad \alpha = 54.09°$$

$$\beta = 2\theta - \alpha \qquad \beta = 15.91°$$

Point D: $\sigma_{x1} = \sigma_{aver} + R\cos(\beta)$

$\sigma_{x1} = 46.4 \text{ MPa} \quad \leftarrow$

$\tau_{x1y1} = -R\sin(\beta)$

$\tau_{x1y1} = -9.81 \text{ MPa} \quad \leftarrow$

Point D′ : $\sigma_{y1} = \sigma_{aver} - R\cos(\beta)$

$\sigma_{y1} = -22.4 \text{ MPa} \quad \leftarrow$

Problem 7.4-15 $\sigma_x = -5700$ psi, $\sigma_y = 950$ psi, $\tau_{xy} = -2100$ psi, $\theta = 65°$

Using Mohr's circle, determine the stresses acting on an element oriented at an angle θ from the x axis. Show these stresses on a sketch of an element oriented at the angle θ. (*Note:* The angle θ is positive when counterclockwise and negative when clockwise.)

Solution 7.4-15

$\sigma_x = -5700 \text{ psi} \quad \sigma_y = 950 \text{ psi} \quad \tau_{xy} = -2100 \text{ psi}$

$\theta = 65°$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{aver} = -2375 \text{ psi}$$

$$R = \sqrt{(\sigma_x - \sigma_{aver})^2 + \tau_{xy}{}^2} \qquad R = 3933 \text{ psi}$$

$$\alpha = \text{atan}\left(\frac{|\tau_{xy}|}{|\sigma_x - \sigma_{aver}|}\right) \qquad \alpha = 32.28°$$

$$\beta = 180° - 2\theta + \alpha \qquad \beta = 82.28°$$

Point D: $\sigma_{x1} = \sigma_{aver} + R\cos(\beta)$

$\sigma_{x1} = -1846 \text{ psi} \quad \leftarrow$

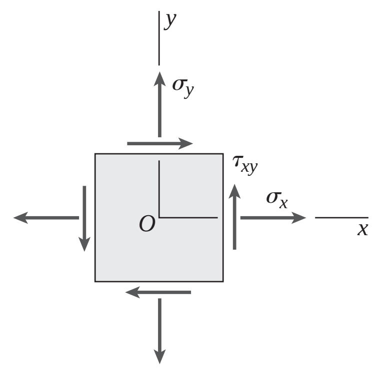
$\tau_{x1y1} = R\sin(\beta) \qquad \tau_{x1y1} = 3897 \text{ psi} \quad \leftarrow$

Point D′: $\sigma_{y1} = \sigma_{aver} - R\cos(\beta)$

$\sigma_{y1} = -2904 \text{ psi} \quad \leftarrow$

Problems 7.4-16 $\sigma_x = -29.5$ MPa, $\sigma_y = 29.5$ MPa, $\tau_{xy} = 27$ MPa

Using Mohr's circle, determine (a) the principal stresses and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.



Probs. 7.4-16 through 7.4-23

Solution 7.4-16

$\sigma_x = -29.5 \text{ MPa} \quad \sigma_y = 29.5 \text{ MPa} \quad \tau_{xy} = 27 \text{ MPa}$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{aver} = 0 \text{ MPa}$$

$$R = \sqrt{(\sigma_x - \sigma_{aver})^2 + \tau_{xy}{}^2} \qquad R = 39.9906 \text{ MPa}$$

$$\alpha = \text{atan}\left(\left|\frac{\tau_{xy}}{\sigma_x - \sigma_{aver}}\right|\right) \qquad \alpha = 42.47°$$

(a) Principal stresses

$$\theta_{p1} = \frac{180° - \alpha}{2} \qquad \theta_{p1} = 68.8° \quad \leftarrow$$

$$\theta_{p2} = \theta_{p1} - 90° \qquad \theta_{p2} = -21.2° \quad \leftarrow$$

Point P1: $\sigma_1 = R \qquad \sigma_1 = 40.0 \text{ MPa} \quad \leftarrow$

Point P2: $\sigma_2 = -R \qquad \sigma_2 = -40.0 \text{ MPa} \quad \leftarrow$

(b) Maximum shear stresses

$$\theta_{s1} = \frac{90° - \alpha}{2} \qquad \theta_{s1} = 23.8° \quad \leftarrow$$

$$\theta_{s2} = 90° + \theta_{s1} \qquad \theta_{s2} = 113.8° \quad \leftarrow$$

Point S1: $\sigma_{aver} = 0 \text{ MPa} \quad \leftarrow$

$\tau_{max} = R \qquad \tau_{max} = 40.0 \text{ MPa} \quad \leftarrow$

Problems 7.4-17 $\sigma_x = 7300$ psi, $\sigma_y = 0$ psi, $\tau_{xy} = 1300$ psi

Using Mohr's circle, determine (a) the principal stresses and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.

Solution 7.4-17

$\sigma_x = 7300\text{ psi} \quad \sigma_y = 0\text{ psi} \quad \tau_{xy} = 1300\text{ psi}$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{aver} = 3650\text{ psi}$$

$$R = \sqrt{(\sigma_x - \sigma_{aver})^2 + \tau_{xy}{}^2} \qquad R = 3875\text{ psi}$$

$$\alpha = \text{atan}\left(\left|\frac{\tau_{xy}}{\sigma_x - \sigma_{aver}}\right|\right) \qquad \alpha = 19.60°$$

(a) Principal stresses

$$\theta_{p1} = \frac{\alpha}{2} \qquad \theta_{p1} = 9.80° \quad \leftarrow$$

$$\theta_{p2} = \frac{\alpha + 180°}{2} \qquad \theta_{p2} = 99.8°$$

Point P1: $\sigma_1 = R + \sigma_{aver}$

$\sigma_1 = 7525\text{ psi} \quad \leftarrow$

Point P2: $\sigma_2 = -R + \sigma_{aver}$

$\sigma_2 = -225\text{ psi}$

(b) Maximum shear stresses

$$\theta_{s1} = \frac{-90° + \alpha}{2} \qquad \theta_{s1} = -35.2°$$

$\theta_{s2} = 90° + \theta_{s1} \qquad \theta_{s2} = 54.8°$

Point S1: $\sigma_{aver} = 3650\text{ psi} \quad \leftarrow$

$\tau_{max} = R \qquad \tau_{max} = 3875\text{ psi} \quad \leftarrow$

Problems 7.4-18 $\sigma_x = 0$ MPa, $\sigma_y = -23.4$ MPa, $\tau_{xy} = -9.6$ MPa

Using Mohr's circle, determine (a) the principal stresses and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.

Solution 7.4-18

$\sigma_x = 0\text{ MPa} \quad \sigma_y = -23.4\text{ MPa} \quad \tau_{xy} = -9.6\text{ MPa}$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{aver} = -11.70\text{ MPa}$$

$$R = \sqrt{(\sigma_x - \sigma_{aver})^2 + \tau_{xy}{}^2} \qquad R = 15.1344\text{ MPa}$$

$$\alpha = \text{atan}\left(\left|\frac{\tau_{xy}}{\sigma_x - \sigma_{aver}}\right|\right) \qquad \alpha = 39.37°$$

(a) Principal stresses

$$\theta_{p1} = \frac{-\alpha}{2} \qquad \theta_{p1} = -19.68° \quad \leftarrow$$

$\theta_{p2} = \theta_{p1} + 90° \qquad \theta_{p2} = 70.32° \quad \leftarrow$

Point P1: $\sigma_1 = R + \sigma_{aver}$

$\sigma_1 = 3.43\text{ MPa} \quad \leftarrow$

Point P2: $\sigma_2 = -R + \sigma_{aver}$

$\sigma_2 = -26.8\text{ MPa} \quad \leftarrow$

(b) Maximum shear stresses

$$\theta_{s1} = \frac{-90° - \alpha}{2} \qquad \theta_{s1} = -64.7° \quad \leftarrow$$

$\theta_{s2} = 90° + \theta_{s1} \qquad \theta_{s2} = 25.3°$

Point S1: $\sigma_{aver} = -11.70\text{ MPa}$

$\tau_{max} = R \qquad \tau_{max} = 15.13\text{ MPa} \quad \leftarrow$

Problems 7.4-19 $\sigma_x = 2050$ psi, $\sigma_y = 6100$ psi, $\tau_{xy} = 2750$ psi

Using Mohr's circle, determine (a) the principal stresses and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.

Solution 7.4-19

$\sigma_x = 2050 \text{ psi} \quad \sigma_y = 6100 \text{ psi} \quad \tau_{xy} = 2750 \text{ psi}$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{\text{aver}} = 4075 \text{ psi}$$

$$R = \sqrt{(\sigma_x - \sigma_{\text{aver}})^2 + \tau_{xy}^{\,2}} \qquad R = 3415 \text{ psi}$$

$$\alpha = \text{atan}\left(\left|\frac{\tau_{xy}}{\sigma_x - \sigma_{\text{aver}}}\right|\right) \qquad \alpha = 53.63°$$

(a) Principal stresses

$$\theta_{p1} = \frac{180° - \alpha}{2} \qquad \theta_{p1} = 63.2° \quad \leftarrow$$

$$\theta_{p2} = \frac{-\alpha}{2} \qquad \theta_{p2} = -26.8° \quad \leftarrow$$

Point P1: $\sigma_1 = R + \sigma_{\text{aver}}$

$\sigma_1 = 7490 \text{ psi} \quad \leftarrow$

Point P2: $\sigma_2 = -R + \sigma_{\text{aver}}$

$\sigma_2 = 660 \text{ psi} \quad \leftarrow$

(b) Maximum shear stresses

$$\theta_{s1} = \frac{-90° + \alpha}{2} \qquad \theta_{s1} = -18.2° \quad \leftarrow$$

$\theta_{s2} = 90° + \theta_{s1} \qquad \theta_{s2} = 71.8°$

Point S1: $\sigma_{\text{aver}} = 4075 \text{ psi} \quad \leftarrow$

$\tau_{\max} = R \qquad \tau_{\max} = 3415 \text{ psi} \quad \leftarrow$

Problems 7.4-20 $\sigma_x = 2900$ kPa, $\sigma_y = 9100$ kPa, $\tau_{xy} = -3750$ kPa

Using Mohr's circle, determine (a) the principal stresses and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.

Solution 7.4-20

$\sigma_x = 2900 \text{ kPa} \quad \sigma_y = 9100 \text{ kPa} \quad \tau_{xy} = -3750 \text{ kPa}$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{\text{aver}} = 6000 \text{ kPa}$$

$$R = \sqrt{(\sigma_x - \sigma_{\text{aver}})^2 + \tau_{xy}^{\,2}} \qquad R = 4865.4393 \text{ kPa}$$

$$\alpha = \text{atan}\left(\left|\frac{\tau_{xy}}{\sigma_x - \sigma_{\text{aver}}}\right|\right) \qquad \alpha = 50.42°$$

(a) Principal stresses

$$\theta_{p1} = \frac{\alpha + 180°}{2} \qquad \theta_{p1} = 115.2° \quad \leftarrow$$

$$\theta_{p2} = \frac{\alpha}{2} \qquad \theta_{p2} = 25.2° \quad \leftarrow$$

Point P1: $\sigma_1 = R + \sigma_{\text{aver}}$

$\sigma_1 = 10865 \text{ KPa} \quad \leftarrow$

Point P2: $\sigma_2 = -R + \sigma_{\text{aver}}$

$\sigma_2 = 1135 \text{ kPa} \quad \leftarrow$

(b) Maximum shear stresses

$$\theta_{s1} = \frac{90° + \alpha}{2} \qquad \theta_{s1} = 70.2° \quad \leftarrow$$

$\theta_{s2} = 90° + \theta_{s1} \qquad \theta_{s2} = 160.2° \quad \leftarrow$

Point S1: $\sigma_{\text{aver}} = 6000 \text{ kPa} \quad \leftarrow$

$\tau_{\max} = R \qquad \tau_{\max} = 4865 \text{ kPa} \quad \leftarrow$

Problems 7.4-21 $\sigma_x = -11{,}500$ psi, $\sigma_y = -18{,}250$ psi, $\tau_{xy} = -7200$ psi

Using Mohr's circle, determine (a) the principal stresses and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.

Solution 7.4-21

$\sigma_x = -11500 \text{ psi} \qquad \sigma_y = -18250 \text{ psi}$

$\tau_{xy} = -7200 \text{ psi}$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{aver} = -14875 \text{ psi}$$

$$R = \sqrt{(\sigma_x - \sigma_{aver})^2 + \tau_{xy}{}^2} \qquad R = 7952 \text{ psi}$$

$$\alpha = \operatorname{atan}\left(\left|\frac{\tau_{xy}}{\sigma_x - \sigma_{aver}}\right|\right) \qquad \alpha = 64.89°$$

(a) Principal stresses

$$\theta_{p1} = \frac{-\alpha}{2} \qquad \theta_{p1} = -32.4° \quad \longleftarrow$$

$$\theta_{p2} = \frac{180° - \alpha}{2} \qquad \theta_{p2} = 57.6° \quad \longleftarrow$$

Point P1: $\sigma_1 = R + \sigma_{aver}$

$\sigma_1 = -6923 \text{ psi} \quad \longleftarrow$

Point P2: $\sigma_2 = -R + \sigma_{aver}$

$\sigma_2 = -22827 \text{ psi} \quad \longleftarrow$

(b) Maximum shear stresses

$$\theta_{s1} = \frac{270° - \alpha}{2} \qquad \theta_{s1} = 102.6° \quad \longleftarrow$$

$$\theta_{s2} = 90° + \theta_{s1} \qquad \theta_{s2} = 192.6°$$

Point S1: $\sigma_{aver} = -14875 \text{ psi} \quad \longleftarrow$

$\tau_{max} = R \qquad \tau_{max} = 7952 \text{ psi} \quad \longleftarrow$

Problems 7.4-22 $\sigma_x = -3.3$ MPa, $\sigma_y = 8.9$ MPa, $\tau_{xy} = -14.1$ MPa

Using Mohr's circle, determine (a) the principal stresses and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.

Solution 7.4-22

$\sigma_x = -3.3 \text{ MPa} \qquad \sigma_y = 8.9 \text{ MPa}$

$\tau_{xy} = -14.1 \text{ MPa}$

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{aver} = 2.8 \text{ MPa}$$

$$R = \sqrt{(\sigma_x - \sigma_{aver})^2 + \tau_{xy}{}^2} \qquad R = 15.4 \text{ MPa}$$

$$\alpha = \operatorname{atan}\left(\left|\frac{\tau_{xy}}{\sigma_x - \sigma_{aver}}\right|\right) \qquad \alpha = 66.6°$$

(a) Principal stresses

$$\theta_{p1} = \frac{\alpha + 180°}{2} \qquad \theta_{p1} = 123.3° \quad \longleftarrow$$

$$\theta_{p2} = \frac{\alpha}{2} \qquad \theta_{p2} = 33.3°$$

Point P1: $\sigma_1 = R + \sigma_{aver}$

$\sigma_1 = 18.2 \text{ MPa} \quad \longleftarrow$

Point P2: $\sigma_2 = -R + \sigma_{aver}$

$\sigma_2 = -12.6\text{ MPa}$ ←

(b) MAXIMUM SHEAR STRESSES

$\theta_{s1} = \dfrac{90° + \alpha}{2}$ $\theta_{s1} = 78.3°$

$\theta_{s2} = 90° + \theta_{s1}$ $\theta_{s2} = 168.3°$

Point S1: $\sigma_{aver} = 2.8\text{ MPa}$ ←

$\tau_{max} = R$ $\tau_{max} = 15.4\text{ MPa}$ ←

Problems 7.4-23 $\sigma_x = 800\text{ psi}, \sigma_y = -2200\text{ psi}, \tau_{xy} = 2900\text{ psi}$

Using Mohr's circle, determine (a) the principal stresses and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.

Solution 7.4-23

$\sigma_x = 800\text{ psi}$ $\sigma_y = -2200\text{ psi}$ $\tau_{xy} = 2900\text{ psi}$

$\sigma_{aver} = \dfrac{\sigma_x + \sigma_y}{2}$ $\sigma_{aver} = -700\text{ psi}$

$R = \sqrt{(\sigma_x - \sigma_{aver})^2 + \tau_{xy}{}^2}$ $R = 3265\text{ psi}$

$\alpha = \operatorname{atan}\left(\left|\dfrac{\tau_{xy}}{\sigma_x - \sigma_{aver}}\right|\right)$ $\alpha = 62.65°$

(a) PRINCIPAL STRESSES

$\theta_{p1} = \dfrac{\alpha}{2}$ $\theta_{p1} = 31.3°$ ←

$\theta_{p2} = \dfrac{180° + \alpha}{2}$ $\theta_{p2} = 121.3°$ ←

Point P1: $\sigma_1 = R + \sigma_{aver}$

$\sigma_1 = 2565\text{ psi}$ ←

Point P2: $\sigma_2 = -R + \sigma_{aver}$

$\sigma_2 = -3965\text{ psi}$ ←

(b) MAXIMUM SHEAR STRESSES

$\theta_{s1} = \dfrac{-90° + \alpha}{2}$ $\theta_{s1} = -13.7°$ ←

$\theta_{s2} = 90° + \theta_{s1}$ $\theta_{s2} = 76.3°$ ←

Point S1: $\sigma_{aver} = -700\text{ psi}$ ←

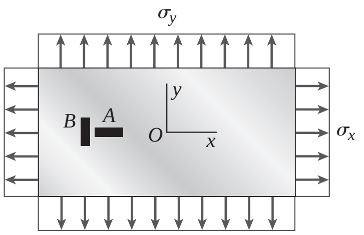
$\tau_{max} = R$ $\tau_{max} = 3265\text{ psi}$

Hooke's Law for Plane Stress

When solving the problems for Section 7.5, assume that the material is linearly elastic with modulus of elasticity E and Poisson's ratio ν.

Problem 7.5-1 A rectangular steel plate with thickness $t = 0.25$ in. is subjected to uniform normal stresses σ_x and σ_y, as shown in the figure. Strain gages A and B, oriented in the x and y directions, respectively, are attached to the plate. The gage readings give normal strains $\epsilon_x = 0.0010$ (elongation) and $\epsilon_y = -0.0007$ (shortening).

Knowing that $E = 30 \times 10^6$ psi and $\nu = 0.3$, determine the stresses σ_x and σ_y and the change Δt in the thickness of the plate.



Probs. 7.5-1 and 7.5-2

Solution 7.5-1 Rectangular plate in biaxial stress

$t = 0.25$ in. $\quad \varepsilon_x = 0.0010 \quad \varepsilon_y = -0.0007$

$E = 30 \times 10^6$ psi $\quad \nu = 0.3$

SUBSTITUTE NUMERICAL VALUES:

Eq. (7-40a):

$$\sigma_x = \frac{E}{(1-\nu)^2}(\varepsilon_x + \nu\varepsilon_y) = 26{,}040 \text{ psi} \quad \longleftarrow$$

Eq. (7-40b):

$$\sigma_y = \frac{E}{(1-\nu)^2}(\varepsilon_y + \nu\varepsilon_x) = -13{,}190 \text{ psi} \quad \longleftarrow$$

Eq. (7-39c):

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) = -128.5 \times 10^{-6}$$

$$\Delta t = \varepsilon_z t = -32.1 \times 10^{-6} \text{ in.} \quad \longleftarrow$$

(Decrease in thickness)

Problem 7.5-2 Solve the preceding problem if the thickness of the steel plate is t = 10 mm, the gage readings are $\epsilon_x = 480 \times 10^{-6}$ (elongation) and $\epsilon_y = 130 \times 10^{-6}$ (elongation), the modulus is $E = 200$ GPa, and Poisson's ratio is $\nu = 0.30$.

Solution 7.5-2 Rectangular plate in biaxial stress

$t = 10$ mm $\quad \varepsilon_x = 480 \times 10^{-6}$

$\varepsilon_y = 130 \times 10^{-6}$

$E = 200$ GPa $\quad \nu = 0.3$

SUBSTITUTE NUMERICAL VALUES:

Eq. (7-40a):

$$\sigma_x = \frac{E}{(1-\nu)^2}(\varepsilon_x + \nu\varepsilon_y) = 114.1 \text{ MPa} \quad \longleftarrow$$

Eq. (7-40b):

$$\sigma_y = \frac{E}{(1-\nu)^2}(\varepsilon_y + \nu\varepsilon_x) = 60.2 \text{ MPa} \quad \longleftarrow$$

Eq. (7-39c):

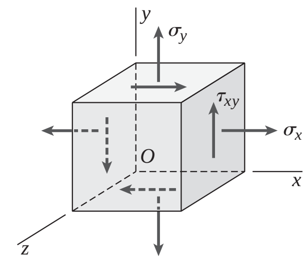
$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) = -261.4 \times 10^{-6}$$

$$\Delta t = \varepsilon_z t = -2610 \times 10^{-6} \text{ mm} \quad \longleftarrow$$

(Decrease in thickness)

Problem 7.5-3 Assume that the normal strains ϵ_x and ϵ_y for an element in *plane stress* (see figure) are measured with strain gages.

(a) Obtain a formula for the normal strain ϵ_z in the z direction in terms of ϵ_x, ϵ_y, and Poisson's ratio ν.
(b) Obtain a formula for the dilatation e in terms of ϵ_x, ϵ_y, and Poisson's ratio ν.



Solution 7.5-3 Plane stress

Given: $\varepsilon_x, \varepsilon_y, \nu$

(a) Normal strain ε_z

Eq. (7-34c): $\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$

Eq. (7-36a): $\sigma_x = \frac{E}{(1 - \nu^2)}(\varepsilon_x + \nu\varepsilon_y)$

Eq. (7-36b): $\sigma_y = \frac{E}{(1 - \nu^2)}(\varepsilon_y + \nu\varepsilon_x)$

Substitute σ_x and σ_y into the first equation and simplify:

$$\varepsilon_z = -\frac{\nu}{1 - \nu}(\varepsilon_x + \varepsilon_y) \quad \leftarrow$$

(b) Dilatation

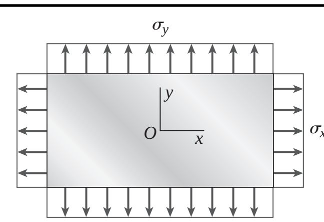
Eq. (7-47): $e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y)$

Substitute σ_x and σ_y from above and simplify:

$$e = \frac{1 - 2\nu}{1 - \nu}(\varepsilon_x + \varepsilon_y) \quad \leftarrow$$

Problem 7.5-4 A magnesium plate in *biaxial stress* is subjected to tensile stresses σ_x= 24 MPa and σ_y= 12 MPa (see figure). The corresponding strains in the plate are $\varepsilon_x = 440 \times 10^{-6}$ and $\varepsilon_y = 80 \times 10^{-6}$.

Determine Poisson's ratio ν and the modulus of elasticity E for the material.



Probs. 7.5-4 through 7.5-7

Solution 7.5-4 Biaxial stress

$\sigma_x = 24$ MPa $\quad \sigma_y = 12$ MPa

$\varepsilon_x = 440 \times 10^{-6} \quad \varepsilon_y = 80 \times 10^{-6}$

Poisson's ration and modulus of elasticity

Eq. (7-39a): $\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$

Eq. (7-39b): $\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$

Substitute numerical values:

$E(440 \times 10^{-6}) = 24 \text{ MPa} - \nu(12 \text{ MPa})$

$E(80 \times 10^{-6}) = 12 \text{ MPa} - \nu(24 \text{ MPa})$

Solve simultaneously:

$\nu = 0.35 \quad E = 45 \text{ GPa} \quad \leftarrow$

Problem 7.5-5 Solve the preceding problem for a steel plate with $\sigma_x = 10{,}800$ psi (tension), $\sigma_y = -5400$ psi (compression), $\epsilon_x = 420 \times 10^{-6}$ (elongation), and $\epsilon_y = -300 \times 10^{-6}$ (shortening).

Solution 7.5-5 Biaxial stress

$\sigma_x = 10{,}800$ psi $\quad \sigma_y = -5400$ psi

$\varepsilon_x = 420 \times 10^{-6} \quad \varepsilon_y = -300 \times 10^{-6}$

POISSON'S RATIO AND MODULUS OF ELASTICITY

$$\text{Eq. (7-39a): } \varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$

$$\text{Eq. (7-39b): } \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

Substitute numerical values:

$E\,(420 \times 10^{-6}) = 10{,}800 \text{ psi} - \nu\,(-5400 \text{ psi})$

$E\,(-300 \times 10^{-6}) = -5400 \text{ psi} - \nu\,(10{,}800 \text{ psi})$

Solve simultaneously:

$\nu = 1/3 \quad E = 30 \times 10^6$ psi $\quad \leftarrow$

Problem 7.5-6 A rectangular plate in *biaxial stress* (see figure) is subjected to normal stresses $\sigma_x = 90$ MPa (tension) and $\sigma_y = -20$ MPa (compression). The plate has dimensions $400 \times 800 \times 20$ mm and is made of steel with $E = 200$ GPa and $\nu = 0.30$.

(a) Determine the maximum in-plane shear strain γ_{max} in the plate.
(b) Determine the change Δt in the thickness of the plate.
(c) Determine the change ΔV in the volume of the plate.

Solution 7.5-6 Biaxial stress

$\sigma_x = 90$ MPa $\quad \sigma_y = -20$ MPa

$E = 200$ GPa $\quad \nu = 0.30$

Dimensions of Plate: 400 mm $\times$ 800 mm $\times$ 20 mm
Shear Modulus (Eq. 7-38):

$$G = \frac{E}{2(1 + \nu)} = 76.923 \text{ GPa}$$

(a) MAXIMUM IN-PLANE SHEAR STRAIN

Principal stresses: $\sigma_1 = 90$ MPa $\quad \sigma_2 = -20$ MPa

$$\text{Eq. (7-26): } \tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = 55.0 \text{ MPa}$$

$$\text{Eq. (7-35): } \gamma_{max} = \frac{\tau_{max}}{G} = 715 \times 10^{-6} \quad \leftarrow$$

(b) CHANGE IN THICKNESS

$$\text{Eq. (7-39c): } \varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) = -105 \times 10^{-6}$$

$$\Delta t = \varepsilon_z t = -2100 \times 10^{-6} \text{ mm} \quad \leftarrow$$

(Decrease in thickness)

(c) CHANGE IN VOLUME

$$\text{From Eq. (7-47): } \Delta V = V_0\left(\frac{1 - 2\nu}{E}\right)(\sigma_x + \sigma_y)$$

$$V_0 = (400)(800)(20) = 6.4 \times 10^6 \text{ mm}^3$$

$$\text{Also, } \left(\frac{1 - 2\nu}{E}\right)(\sigma_x + \sigma_y) = 140 \times 10^{-6}$$

$$\therefore \Delta V = (6.4 \times 10^6 \text{ mm}^3)(140 \times 10^{-6}) = 896 \text{ mm}^3 \quad \leftarrow$$

(Increase in volume)

Problem 7.5-7 Solve the preceding problem for an aluminum plate with $\sigma_x = 12{,}000$ psi (tension), $\sigma_y = -3{,}000$ psi (compression), dimensions $20 \times 30 \times 0.5$ in., $E = 10.5 \times 10^6$ psi, and $\nu = 0.33$.

Solution 7.5-7 Biaxial stress

$\sigma_x = 12{,}000 \text{ psi} \quad \sigma_y = -3{,}000 \text{ psi}$

$E = 10.5 \times 10^6 \text{ psi} \quad \nu = 0.33$

Dimensions of Plate: 20 in. $\times$ 30 in. $\times$ 0.5 in.
Shear Modulus (Eq. 7-38):

$$G = \frac{E}{2(1 + \nu)} = 3.9474 \times 10^6 \text{ psi}$$

(a) MAXIMUM IN-PLANE SHEAR STRAIN

Principal stresses: $\sigma_1 = 12{,}000$ psi

$\sigma_2 = -3{,}000$ psi

$$\text{Eq. (7-26): } \tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = 7{,}500 \text{ psi}$$

$$\text{Eq. (7-35): } \gamma_{max} = \frac{\tau_{max}}{G} = 1{,}900 \times 10^{-6} \quad \leftarrow$$

(b) CHANGE IN THICKNESS

$$\text{Eq. (7-39c): } \varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

$$= -282.9 \times 10^{-6}$$

$$\Delta t = \varepsilon_z t = -141 \times 10^{-6} \text{ in.} \quad \leftarrow$$

(Decrease in thickness)

(c) CHANGE IN VOLUME

$$\text{From Eq. (7-47): } \Delta V = V_0\left(\frac{1 - 2\nu}{E}\right)(\sigma_x + \sigma_y)$$

$$V_0 = (20)(30)(0.5) = 300 \text{ in.}^3$$

$$\text{Also, } \left(\frac{1 - 2\nu}{E}\right)(\sigma_x + \sigma_y) = 291.4 \times 10^{-6}$$

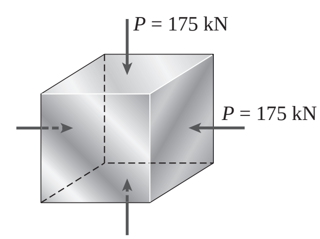
$$\therefore \Delta V = (300 \text{ in.}^3)(291.4 \times 10^{-6})$$

$$= 0.0874 \text{ in.}^3 \quad \leftarrow$$

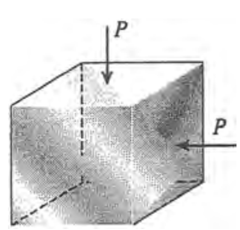
(Increase in volume)

Problem 7.5-8 A brass cube 50 mm on each edge is compressed in two perpendicular directions by forces $P = 175$ kN (see figure).

Calculate the change ΔV in the volume of the cube and the strain energy U stored in the cube, assuming $E = 100$ GPa and $\nu = 0.34$.



Solution 7.5-8 Biaxial stress-cube



Side $b = 50$ mm $\quad P = 175$ kN

$E = 100$ GPa $\quad \nu = 0.34$ (Brass)

$$\sigma_x = \sigma_y = -\frac{P}{b^2} = -\frac{(175 \text{ kN})}{(50 \text{ mm})^2} = -70.0 \text{ MPa}$$

CHANGE IN VOLUME

$$\text{Eq. (7-47): } e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y) = -448 \times 10^{-6}$$

$$V_0 = b^3 = (50 \text{ mm})^3 = 125 \times 10^3 \text{mm}^3$$

$$\Delta V = eV_0 = -56 \text{ mm}^3 \quad \leftarrow$$

(Decrease in volume)

STRAIN ENERGY

Eq. (7-50): $u = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y)$

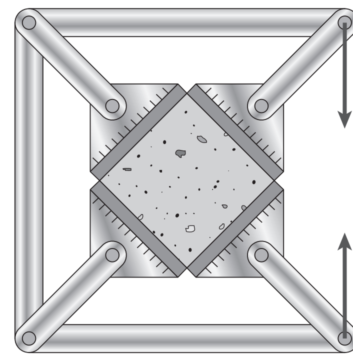
$= 0.03234 \text{ MPa}$

$U = uV_0 = (0.03234 \text{ MPa})(125 \times 10^3 \text{ mm}^3)$

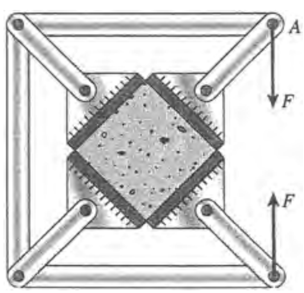
$= 4.04 \text{ J} \quad \leftarrow$

Problem 7.5-9 A 4.0-inch cube of concrete ($E = 3.0 \times 10^6$ psi, $\nu = 0.1$) is compressed in *biaxial stress* by means of a framework that is loaded as shown in the figure.

Assuming that each load F equals 20 k, determine the change ΔV in the volume of the cube and the strain energy U stored in the cube.



Solution 7.5-9 Biaxial stress – concrete cube



$b = 4$ in.

$E = 3.0 \times 10^6$ psi

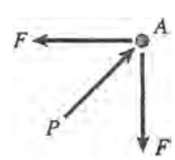
$\nu = 0.1$

$F = 20$ kips

Joint A:

$P = F\sqrt{2}$

$= 28.28$ kips



$\sigma_x = \sigma_y = -\frac{P}{b^2} = -1768 \text{ psi}$

CHANGE IN VOLUME

Eq. (7-47): $e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y) = -0.0009429$

$V_0 = b^3 = (4 \text{ in.})^3 = 64 \text{ in.}^3$

$\Delta V = eV_0 = -0.0603 \text{ in.}^3 \quad \leftarrow$

(Decrease in volume)

STRAIN ENERGY

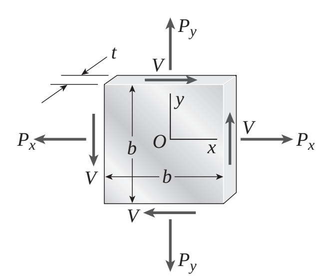
Eq. (7-50): $u = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y)$

$= 0.9377 \text{ psi}$

$U = uV_0 = 60.0 \text{ in.-lb} \quad \leftarrow$

Problem 7.5-10 A square plate of width b and thickness t is loaded by normal forces P_x and P_y, and by shear forces V, as shown in the figure. These forces produce uniformly distributed stresses acting on the side faces of the place.

Calculate the change ΔV in the volume of the plate and the strain energy U stored in the plate if the dimensions are $b = 600$ mm and $t = 40$ mm, the plate is made of magnesium with $E = 45$ GPa and $v = 0.35$, and the forces are $P_x = 480$ kN, $P_y = 180$ kN, and $V = 120$ kN.



Probs. 7.5-10 and 7.5-11

Solution 7.5-10 Square plate in plane stress

$b = 600$ mm $\quad t = 40$ mm

$E = 45$ GPa $\quad \nu = 0.35$ (magnesium)

$P_x = 480$ kN $\quad \sigma_x = \dfrac{P_x}{bt} = 20.0$ MPa

$P_y = 180$ kN $\quad \sigma_y = \dfrac{P_y}{bt} = 7.5$ MPa

$V = 120$ kN $\quad \tau_{xy} = \dfrac{V}{bt} = 5.0$ MPa

CHANGE IN VOLUME

Eq. (7-47): $e = \dfrac{1 - 2\nu}{E}(\sigma_x + \sigma_y) = 183.33 \times 10^{-6}$

$V_0 = b^2 t = 14.4 \times 10^6\ \text{mm}^3$

$\Delta V = eV_0 = 2640\ \text{mm}^3 \quad \leftarrow$

(Increase in volume)

STRAIN ENERGY

Eq. (7-50): $u = \dfrac{1}{2E}(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y) + \dfrac{\tau_{xy}^2}{2G}$

$$G = \frac{E}{2(1 + \nu)} = 16.667\ \text{GPa}$$

Substitute numerical values:

$u = 4653$ Pa

$U = uV_0 = 67.0\ \text{N} \cdot \text{m} = 67.0\ \text{J} \quad \leftarrow$

Problem 7.5-11 Solve the preceding problem for an aluminum plate with $b = 12$ in., $t = 1.0$ in., $E = 10{,}600$ ksi, $\nu = 0.33$, $P_x = 90$ k, $P_y = 20$ k, and $V = 15$ k.

Solution 7.5-11 Square plate in plane stress

$b = 12.0$ in. $\quad t = 1.0$ in.

$E = 10{,}600$ ksi $\quad \nu = 0.33$ (aluminum)

$P_x = 90$ k $\quad \sigma_x = \dfrac{P_x}{bt} = 7500$ psi

$P_y = 20$ k $\quad \sigma_y = \dfrac{P_y}{bt} = 1667$ psi

$V = 15$ k $\quad \tau_{xy} = \dfrac{V}{bt} = 1250$ psi

CHANGE IN VOLUME

Eq. (7-47): $e = \dfrac{1 - 2\nu}{E}(\sigma_x + \sigma_y) = 294 \times 10^{-6}$

$V_0 = b^2 t = 144\ \text{in.}^3$

$\Delta V = eV_0 = 0.0423\ \text{in.}^3 \quad \leftarrow$

(Increase in volume)

STRAIN ENERGY

Eq. (7-50): $u = \dfrac{1}{2E}(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y) + \dfrac{\tau_{xy}^2}{2G}$

$$G = \frac{E}{2(1 + \nu)} = 3985\ \text{ksi}$$

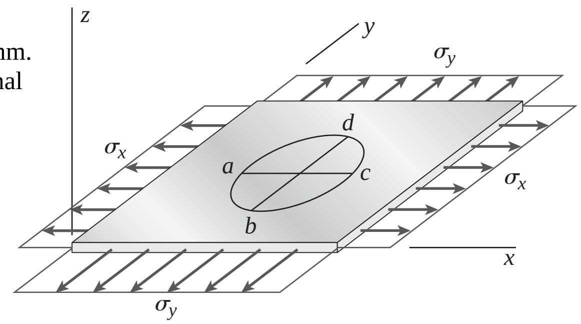
Substitute numerical values:

$u = 2.591$ psi

$U = uV_0 = 373$ in.-lb $\quad \leftarrow$

Problem 7.5-12 A circle of diameter $d = 200$ mm is etched on a brass plate (see figure). The plate has dimensions $400 \times 400 \times 20$ mm. Forces are applied to the plate, producing uniformly distributed normal stresses $\sigma_x = 42$ MPa and $\sigma_y = 14$ MPa.

Calculate the following quantities: (a) the change in length Δac of diameter ac; (b) the change in length Δbd of diameter bd; (c) the change Δt in the thickness of the plate; (d) the change ΔV in the volume of the plate, and (e) the strain energy U stored in the plate. (Assume $E = 100$ GPa and $\nu = 0.34$.)



Solution 7.5-12 Plate in biaxial stress

$\sigma_x = 42$ MPa $\quad \sigma_y = 14$ MPa

Dimensions: $400 \times 400 \times 20$ (mm)
Diameter of circle: $d = 200$ mm

$E = 100$ GPa $\quad \nu = 0.34$ (Brass)

(a) CHANGE IN LENGTH OF DIAMETER IN x DIRECTION

$$\text{Eq. (7-39a): } \varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = 372.4 \times 10^{-6}$$

$$\Delta ac = \varepsilon_x d = 0.0745 \text{ mm} \quad \leftarrow$$
(increase)

(b) CHANGE IN LENGTH OF DIAMETER IN y DIRECTION

$$\text{Eq. (7-39b): } \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = -2.80 \times 10^{-6}$$

$$\Delta bd = \varepsilon_y d$$
$$= -560 \times 10^{-6} \text{ mm} \quad \leftarrow$$
(decrease)

(c) CHANGE IN THICKNESS

$$\text{Eq. (7-39c): } \varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$
$$= -190.4 \times 10^{-6}$$

$$\Delta t = \varepsilon_z t = -0.00381 \text{ mm} \quad \leftarrow$$
(decrease)

(d) CHANGE IN VOLUME

Eq. (7-47):

$$e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y) = 179.2 \times 10^{-6}$$

$$V_0 = (400)(400)(20) = 3.2 \times 10^6 \text{ mm}^3$$

$$\Delta V = eV_0 = 573 \text{ mm}^3 \quad \leftarrow$$
(increase)

(e) STRAIN ENERGY

$$\text{Eq. (7-50): } u = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y)$$
$$= 7.801 \times 10^{-3} \text{ MPa}$$

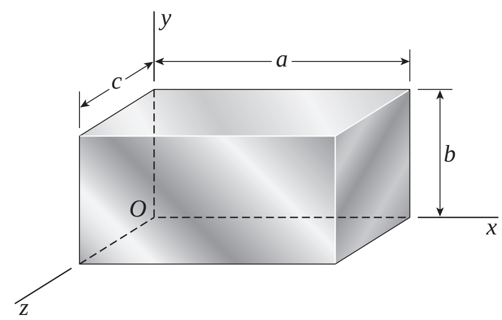
$$U = uV_0 = 25.0 \text{ N}\cdot\text{m} = 25.0 \text{ J} \quad \leftarrow$$

Triaxial Stress

When solving the problems for Section 7.6, assume that the material is linearly elastic with modulus of elasticity E and Poisson's ratio ν.

Problem 7.6-1 An element of aluminum in the form of a rectangular parallelepiped (see figure) of dimensions $a = 6.0$ in., $b = 4.0$ in, and $c = 3.0$ in. is subjected to *triaxial stresses* $\sigma_x = 12{,}000$ psi, $\sigma_y = -4{,}000$ psi, and $\sigma_z = -1{,}000$ psi acting on the x, y, and z faces, respectively.

Determine the following quantities: (a) the maximum shear stress τ_{max} in the material; (b) the changes Δa, Δb, and Δc in the dimensions of the element; (c) the change ΔV in the volume; and (d) the strain energy U stored in the element. (Assume $E = 10{,}400$ ksi and $\nu = 0.33$.)



Probs. 7.6-1 and 7.6-2

Solution 7.6-1 Triaxial stress

$\sigma_x = 12{,}000$ psi $\quad \sigma_y = -4{,}000$ psi

$\sigma_z = -1{,}000$ psi

$a = 6.0$ in. $\quad b = 4.0$ in. $\quad c = 3.0$ in.

$E = 10{,}400$ ksi $\quad \nu = 0.33 \quad$ (aluminum)

(a) MAXIMUM SHEAR STRESS

$\sigma_1 = 12{,}000$ psi $\quad \sigma_2 = -1{,}000$ psi

$\sigma_3 = -4{,}000$ psi

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = 8{,}000 \text{ psi} \quad \longleftarrow$$

(b) CHANGES IN DIMENSIONS

$$\text{Eq. (7-53 a): } \varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z)$$

$$= 1312.5 \times 10^{-6}$$

$$\text{Eq. (7-53 b): } \varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_z + \sigma_x)$$

$$= -733.7 \times 10^{-6}$$

$$\text{Eq. (7-53 c): } \varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y)$$

$$= -350.0 \times 10^{-6}$$

$$\left.\begin{aligned} \Delta a &= a\varepsilon_x = 0.0079 \text{ in.} \quad (\text{increase}) \\ \Delta b &= b\varepsilon_y = -0.0029 \text{ in.} \quad (\text{decrease}) \\ \Delta c &= c\varepsilon_z = -0.0011 \text{ in.} \quad (\text{decrease}) \end{aligned}\right\} \quad \longleftarrow$$

(c) CHANGE IN VOLUME

Eq. (7-56):

$$e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) = 228.8 \times 10^{-6}$$

$V = abc$

$$\Delta V = e\,(abc) = 0.0165 \text{ in.}^3 \ (\text{increase}) \quad \longleftarrow$$

(d) STRAIN ENERGY

$$\text{Eq. (7-57a): } u = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z)$$

$$= 9.517 \text{ psi}$$

$$U = u\,(abc) = 685 \text{ in.-lb} \quad \longleftarrow$$

Problem 7.6-2 Solve the preceding problem if the element is steel (E = 200 GPA, $\nu = 0.30$) with dimensions a = 300 mm, b = 150 mm, and c = 150 mm and the stresses are $\sigma_x = -60$ MPa, $\sigma_y = -40$ MPa, and $\sigma_z = -40$ MPa.

Solution 7.6-2 Triaxial stress

$\sigma_x = -60$ MPa $\quad \sigma_y = -40$ MPa

$\sigma_z = -40$ MPa

$a = 300$ mm $\quad b = 150$ mm $\quad c = 150$ mm

$E = 200$ GPa $\quad \nu = 0.30$ (steel)

(a) MAXIMUM SHEAR STRESS

$\sigma_1 = -40$ MPa $\quad \sigma_2 = -40$ MPa

$\sigma_3 = -60$ MPa

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = 10.0 \text{ MPa} \quad \leftarrow$$

(b) CHANGES IN DIMENSIONS

$$\text{Eq. (7-53 a): } \varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z) = -180.0 \times 10^{-6}$$

$$\text{Eq. (7-53 b): } \varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_z + \sigma_x) = -50.0 \times 10^{-6}$$

$$\text{Eq. (7-53 c): } \varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y) = -50.0 \times 10^{-6}$$

$$\left.\begin{aligned} \Delta a &= a\varepsilon_x = -0.0540 \text{ mm} \quad \text{(decrease)} \\ \Delta b &= b\varepsilon_y = -0.0075 \text{ mm} \quad \text{(decrease)} \\ \Delta c &= c\varepsilon_z = -0.0075 \text{ mm.} \quad \text{(decrease)} \end{aligned}\right\} \quad \leftarrow$$

(c) CHANGE IN VOLUME

Eq. (7-56):

$$e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) = -280.0 \times 10^{-6}$$

$V = abc$

$$\Delta V = e(abc) = -1890 \text{ mm}^3 \text{ (decrease)} \quad \leftarrow$$

(d) STRAIN ENERGY

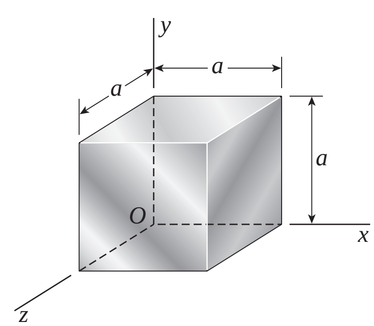
$$\text{Eq. (7-57 a): } u = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z)$$

$$= 0.00740 \text{ MPa}$$

$$U = u(abc) = 50.0 \text{ N} \cdot \text{m} = 50.0 \text{ J} \quad \leftarrow$$

Problem 7.6-3 A cube of cast iron with sides of length a = 4.0 in. (see figure) is tested in a laboratory under *triaxial stress*. Gages mounted on the testing machine show that the compressive strains in the material are $\epsilon_x = -225 \times 10^{-6}$ and $\epsilon_y = \epsilon_z = -37.5 \times 10^{-6}$.

Determine the following quantities: (a) the normal stresses σ_x, σ_y, and σ_z acting on the x, y, and z faces of the cube; (b) the maximum shear stress τ_{max} in the material; (c) the change ΔV in the volume of the cube; and (d) the strain energy U stored in the cube. (Assume E = 14,000 ksi and $\nu = 0.25$.)



Probs. 7.6-3 and 7.6-4

Solution 7.6-3 Triaxial stress (cube)

$\varepsilon_x = -225 \times 10^{-6} \quad \varepsilon_y = -37.5 \times 10^{-6}$

$\varepsilon_z = -37.5 \times 10^{-6} \quad a = 4.0$ in.

$E = 14{,}000$ ksi $\quad \nu = 0.25 \quad$ (cast iron)

(a) NORMAL STRESSES

Eq. (7-54a):

$$\sigma_x = \frac{E}{(1 + \nu)(1 - 2\nu)}[(1 - \nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)]$$

$$= -4200 \text{ psi} \quad \leftarrow$$

In a similar manner, Eqs. (7-54 b and c) give
$\sigma_y = -2100$ psi $\quad \sigma_z = -2100$ psi $\quad \leftarrow$

(b) MAXIMUM SHEAR STRESS

$\sigma_1 = -2100$ psi $\quad \sigma_2 = -2100$ psi

$\sigma_3 = -4200$ psi

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = 1050 \text{ psi} \quad \leftarrow$$

(c) CHANGE IN VOLUME

Eq. (7-55): $e = \varepsilon_x + \varepsilon_y + \varepsilon_z = -0.000300$

$V = a^3$

$\Delta V = ea^3 = -0.0192$ in.^{3} (decrease) $\quad \leftarrow$

(d) STRAIN ENERGY

$$\text{Eq. (7-57a): } u = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z)$$

$$= 0.55125 \text{ psi}$$

$U = ua^3 = 35.3$ in.-lb $\quad \leftarrow$

Problem 7.6-4 Solve the preceding problem if the cube is granite ($E = 60$ GPa, $\nu = 0.25$) with dimensions $a = 75$ mm and compressive strains $\epsilon_x = -720 \times 10^{-6}$ and $\epsilon_y = \epsilon_z = -270 \times 10^{-6}$.

Solution 7.6-4 Triaxial stress (cube)

$\varepsilon_x = -720 \times 10^{-6} \quad \varepsilon_y = -270 \times 10^{-6}$

$\varepsilon_z = -270 \times 10^{-6} \quad a = 75$ mm $\quad E = 60$ GPa

$\nu = 0.25 \quad$ (Granite)

(a) NORMAL STRESSES

Eq.(7-54a):

$$\sigma_x = \frac{E}{(1 + \nu)(1 - 2\nu)}[(1 - \nu)\varepsilon_x + \nu(\varepsilon_x + \varepsilon_z)]$$

$$= -64.8 \text{ MPa} \quad \leftarrow$$

In a similar manner, Eqs. (7-54 b and c) give
$\sigma_y = -43.2$ MPa $\quad \sigma_z = -43.2$ MPa $\quad \leftarrow$

(b) MAXIMUM SHEAR STESS

$\sigma_1 = -43.2$ MPa $\quad \sigma_2 = -43.2$ MPa

$\sigma_3 = -64.8$ MPa

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = 10.8 \text{ MPa} \quad \leftarrow$$

(c) CHANGE IN VOLUME

Eq. (7-55): $e = \varepsilon_x + \varepsilon_y + \varepsilon_z = -1260 \times 10^{-6}$

$V = a^3$

$\Delta V = ea^3 = -532$ mm^3 (decrease) $\quad \leftarrow$

(d) STRAIN ENERGY

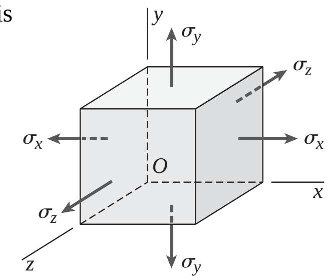
$$\text{Eq. (7-57 a): } u = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z)$$

$$= 0.03499 \text{ MPa} = 34.99 \text{ kPa}$$

$U = ua^3 = 14.8$ N·m $= 14.8$ J $\quad \leftarrow$

Problem 7.6-5 An element of aluminum in *triaxial stress* (see figure) is subjected to stresses $\sigma_x = 5200$ psi (tension), $\sigma_y = -4750$ psi (compression), and $\sigma_z = -3090$ psi (compression). It is also known that the normal strains in the x and y directions are $\epsilon_x = 7138.8 \times 10^{-6}$ (elongation) and $\epsilon_y = -502.3 \times 10^{-6}$ (shortening).

What is the bulk modulus K for the aluminum?



Probs. 7.6-5 and 7.6-6

Solution 7.6-5 Triaxial stress (bulk modulus)

$\sigma_x = 5200$ psi $\quad \sigma_y = -4750$ psi

$\sigma_z = -3090$ psi $\quad \varepsilon_x = 713.8 \times 10^{-6}$

$\varepsilon_y = -502.3 \times 10^{-6}$

Find K.

Eq. (7-53 a): $\varepsilon_x = \dfrac{\sigma_x}{E} - \dfrac{\nu}{E}(\sigma_y + \sigma_z)$

Eq. (7-53 b): $\varepsilon_y = \dfrac{\sigma_y}{E} - \dfrac{\nu}{E}(\sigma_x + \sigma_y)$

Substitute numerical values and rearrange:

$(713.8 \times 10^{-6})\, E = 5200 + 7840\, \nu \qquad (1)$

$(-502.3 \times 10^{-6})\, E = -4750 - 2110\, \nu \qquad (2)$

Units: E = psi

Solve simultaneously Eqs. (1) and (2):

$E = 10.801 \times 10^6$ psi $\quad \nu = 0.3202$

Eq. (7-16): $K = \dfrac{E}{3(1 - 2\nu)} = 10.0 \times 10^{-6}$ psi $\quad \leftarrow$

Problem 7.6-6 Solve the preceding problem if the material is nylon subjected to compressive stresses $\sigma_x = -4.5$ MPa, $\sigma_y = -3.6$ MPa, and $\sigma_z = -2.1$ MPa, and the normal strains are $\epsilon_x = -740 \times 10^{-6}$ and $\epsilon_y = -320 \times 10^{-6}$ (shortenings).

Solution 7.6-6 Triaxial stress (bulk modulus)

$\sigma_x = -4.5$ MPa $\quad \sigma_y = -3.6$ MPa

$\sigma_z = -2.1$ MPa $\quad \varepsilon_x = -740 \times 10^{-6}$

$\varepsilon_y = -320 \times 10^{-6}$

Find K.

Eq. (7-53 a): $\varepsilon_x = \dfrac{\sigma_x}{E} - \dfrac{\nu}{E}(\sigma_y + \sigma_z)$

Eq. (7-53 b): $\varepsilon_y = \dfrac{\sigma_y}{E} - \dfrac{\nu}{E}(\sigma_z + \sigma_x)$

Substitute numerical values and rearrange:

$(-740 \times 10^{-6})\, E = -4.5 + 5.7\, \nu \qquad (1)$

$(-320 \times 10^{-6})\, E = -3.6 + 6.6\, \nu \qquad (2)$

Units: E = MPa

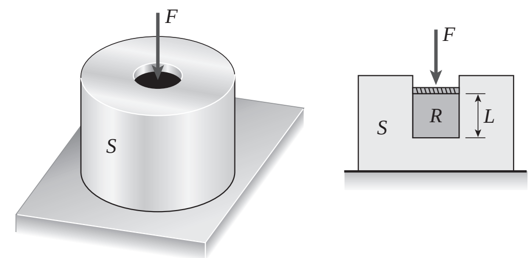
Solve simultaneously Eqs. (1) and (2):

$E = 3{,}000$ MPa $= 3.0$ GPa $\quad \nu = 0.40$

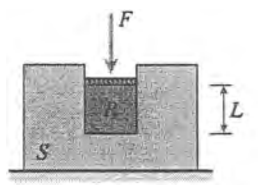
Eq. (7-16): $K = \dfrac{E}{3(1 - 2\nu)} = 5.0$ GPa $\quad \leftarrow$

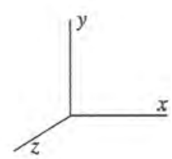
Problem 7.6-7 A rubber cylinder R of length L and cross-sectional area A is compressed inside a steel cylinder S by a force F that applies a uniformly distributed pressure to the rubber (see figure).

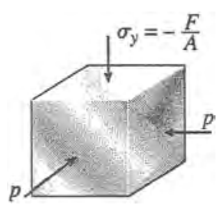
(a) Derive a formula for the lateral pressure p between the rubber and the steel. (Disregard friction between the rubber and the steel, and assume that the steel cylinder is rigid when compared to the rubber.)
(b) Derive a formula for the shortening δ of the rubber cylinder.



Solution 7.6-7 Rubber cylinder







$$\sigma_x = -p \quad \sigma_y = -\frac{F}{A}$$

$$\sigma_z = -p$$

$$\varepsilon_x = \varepsilon_z = 0$$

(a) LATERAL PRESSURE

$$\text{Eq. (7-53 a): } \varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z)$$

$$\text{or} \quad 0 = -p - \nu\left(-\frac{F}{A} - p\right)$$

$$\text{Solve for } p\text{:} \quad p = \frac{\nu}{1-\nu}\left(\frac{F}{A}\right) \quad \leftarrow$$

(b) SHORTENING

$$\text{Eq. (7-53 b): } \varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_z + \sigma_x)$$

$$= -\frac{F}{EA} - \frac{\nu}{E}(-2p)$$

Substitute for p and simplify:

$$\varepsilon_y = \frac{F}{EA}\,\frac{(1+\nu)(-1+2\nu)}{1-\nu}$$

(Positive ε_y represents an increase in strain, that is, elongation.)

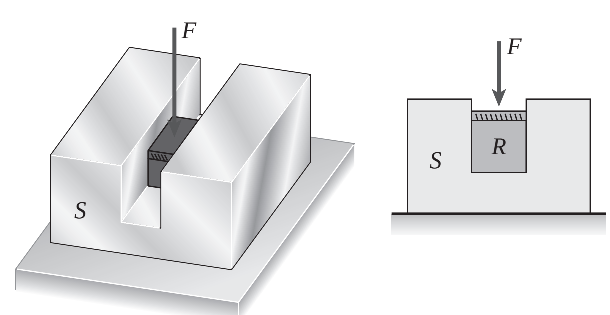
$$\delta = -\varepsilon_y L$$

$$\delta = \frac{(1+\nu)(1-2\nu)}{(1-\nu)}\left(\frac{FL}{EA}\right) \quad \leftarrow$$

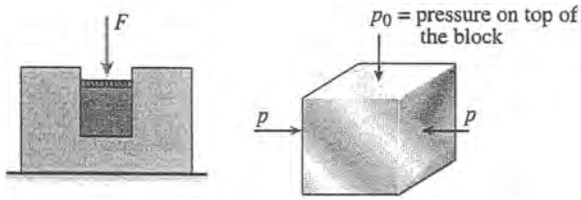
(Positive δ represents a shortening of the rubber cylinder.)

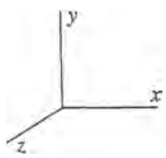
Problem 7.6-8 A block R of rubber is confined between plane parallel walls of a steel block S (see figure). A uniformly distributed pressure p_0 is applied to the top of the rubber block by a force F.

(a) Derive a formula for the lateral pressure p between the rubber and the steel. (Disregard friction between the rubber and the steel, and assume that the steel block is rigid when compared to the rubber.)
(b) Derive a formula for the dilatation e of the rubber.
(c) Derive a formula for the strain-energy density u of the rubber.



Solution 7.6-8 Block of rubber





$$\sigma_x = -p$$
$$\sigma_y = -p_0 \quad \sigma_z = 0$$
$$\varepsilon_x = 0 \quad \varepsilon_y \neq 0 \quad \varepsilon_z \neq 0$$

(a) LATERAL PRESSURE

Eq. (7-53 a): $\varepsilon_x = \dfrac{\sigma_x}{E} - \dfrac{\nu}{E}(\sigma_y + \sigma_z)$

OR $0 = -p - \nu(-p_0) \qquad \therefore\ p = \nu p_0 \quad \leftarrow$

(b) DILATATION

Eq. (7-56): $e = \dfrac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z)$

$$= \frac{1 - 2\nu}{E}(-p - p_0)$$

Substitute for p:

$$e = -\frac{(1 + \nu)(1 - 2\nu)p_0}{E} \quad \leftarrow$$

(c) STRAIN ENERGY DENSITY

Eq. (7-57b):

$$u = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\nu}{E}(\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z)$$

Substitute for σ_x, σ_y, σ_z, and p:

$$u = \frac{(1 - \nu^2)p_0^2}{2E} \quad \leftarrow$$

Problem 7.6-9 A solid spherical ball of brass ($E = 15 \times 10^6$ psi $\nu = 0.34$) is lowered into the ocean to a depth of 10,000 ft. The diameter of the ball is 11.0 in.

Determine the decrease Δd in diameter, the decrease ΔV in volume, and the strain energy U of the ball.

Solution 7.6-9 Brass sphere

$E = 15 \times 10^{-6}$ psi $\quad \nu = 0.34$

Lowered in the ocean to depth $h = 10{,}000$ ft

Diameter $d = 11.0$ in.

Sea water: $\gamma = 63.8$ lb/ft^3

Pressure: $\sigma_0 = \gamma h = 638{,}000$ lb/ft^2 $= 4431$ psi

DECREASE IN DIAMETER

Eq. (7-59): $\varepsilon_0 = \dfrac{\sigma_0}{E}(1 - 2\nu) = 94.53 \times 10^{-6}$

$$\Delta d = \varepsilon_0 d = 1.04 \times 10^{-3} \text{ in.} \quad \leftarrow$$
(decrease)

DECREASE IN VOLUME

Eq. (7-60): $e = 3\varepsilon_0 = 283.6 \times 10^{-6}$

$$V_0 = \frac{4}{3}\pi r^3 = \frac{4}{3}(\pi)\left(\frac{11.0 \text{ in.}}{2}\right)^3 = 696.9 \text{ in.}^3$$

$$\Delta V = eV_0 = 0.198 \text{ in.}^3 \quad \leftarrow$$
(decrease)

STRAIN ENERGY

Use Eq. (7-57 b) with $\sigma_x = \sigma_y = \sigma_z = \sigma_0$:

$$u = \frac{3(1 - 2\nu)\sigma_0^2}{2E} = 0.6283 \text{ psi}$$

$$U = uV_0 = 438 \text{ in.-lb} \quad \leftarrow$$

Problem 7.6-10 A solid steel sphere ($E = 210$ GPa, $\nu = 0.3$) is subjected to hydrostatic pressure p such that its volume is reduced by 0.4%.

(a) Calculate the pressure p.
(b) Calculate the volume modulus of elasticity K for the steel.
(c) Calculate the strain energy U stored in the sphere if its diameter is $d = 150$ mm

Solution 7.6-10 Steel sphere

$E = 210$ GPa $\quad \nu = 0.3$

Hydrostatic Pressure. V_0 = Initial volume

$\Delta V = 0.004\, V_0$

Dilatation: $e = \dfrac{\Delta V}{V_0} = 0.004$

(a) PRESSURE

$$\text{Eq.(7-60): } e = \frac{3\sigma_0(1 - 2\nu)}{E}$$

$$\text{or} \qquad \sigma_0 = \frac{Ee}{3(1 - 2\nu)} = 700 \text{ MPa}$$

Pressure $p = \sigma_0 = 700$ MPa $\quad \leftarrow$

(b) VOLUME MODULUS OF ELASTICITY

$$\text{Eq. (7-63): } K = \frac{\sigma_0}{E} = \frac{700 \text{ MPa}}{0.004} = 175 \text{ GPa} \quad \leftarrow$$

(c) STRAIN ENERGY (d = diameter)

$d = 150$ mm $\quad$ r = 75 mm

From Eq. (7-57b) with $\sigma_x = \sigma_y = \sigma_z = \sigma_0$:

$$u = \frac{3(1 - 2\nu)\sigma_0^2}{2E} = 1.40 \text{ MPa}$$

$$V_0 = \frac{4\pi r^3}{3} = 1767 \times 10^{-6} \text{ m}^3$$

$$U = uV_0 = 2470 \text{ N} \cdot \text{m} = 2470 \text{ J} \quad \leftarrow$$

Problem 7.6-11 A solid bronze sphere (volume modulus of elasticity $K = 14.5 \times 10^6$ psi) is suddenly heated around its outer surface. The tendency of the heated part of the sphere to expand produces uniform tension in all directions at the center of the sphere.

If the stress at the center is 12,000 psi, what is the strain? Also, calculate the unit volume change e and the strain-energy density u at the center.

Solution 7.6-11 Bronze sphere (heated)

$K = 14.5 \times 10^6$ psi
$\sigma_0 = 12{,}000$ psi (tension at the center)

STRAIN AT THE CENTER OF THE SPHERE

$$\text{Eq. (7-59): } \varepsilon_0 = \frac{\sigma_0}{E}(1 - 2\nu)$$

$$\text{Eq. (7-61): } K = \frac{E}{3(1 - 2\nu)}$$

Combine the two equations:

$$\varepsilon_0 = \frac{\sigma_0}{3K} = 276 \times 10^{-6} \quad \leftarrow$$

UNIT VOLUME CHANGE AT THE CENTER

$$\text{Eq. (7-62): } e = \frac{\sigma_0}{K} = 828 \times 10^{-6} \quad \leftarrow$$

STRAIN ENERGY DENSITY AT THE CENTER

Eq. (7-57b) with $\sigma_x = \sigma_y = \sigma_z = \sigma_0$:

$$u = \frac{3(1 - 2\nu)\sigma_0^2}{2E} = \frac{\sigma_0^2}{2K}$$

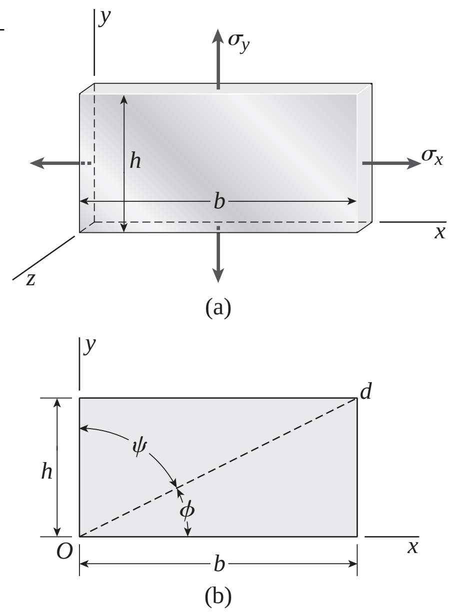
$u = 4.97$ psi $\quad \leftarrow$

Plane Strain

When solving the problems for Section 7.7, consider only the in-plane strains (the strains in the xy plane) unless stated otherwise. Use the transformation equations of plane strain except when Mohr's circle is specified (Problems 7.7-23 through 7.7-28).

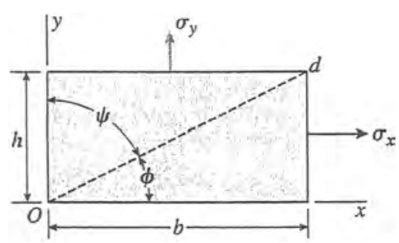
Problem 7.7-1 A thin rectangular plate in *biaxial stress* is subjected to stresses σ_x and σ_y, as shown in part (a) of the figure on the next page. The width and height of the plate are $b = 8.0$ in. and $h = 4.0$ in., respectively. Measurements show that the normal strains in the x and y directions are $\epsilon_x = 195 \times 10^{-6}$ and $\epsilon_y = -125 \times 10^{-6}$, respectively.

With reference to part (b) of the figure, which shows a two-dimensional view of the plate, determine the following quantities: (a) the increase Δd in the length of diagonal Od; (b) the change $\Delta\phi$ in the angle ϕ between diagonal Od and the x axis; and (c) the change Δc in the angle c between diagonal Od and the y axis.



Probs. 7.7-1 and 7.7-2

Solution 7.7-1 Plate in biaxial stress



$b = 8.0$ in. $\quad h = 4.0$ in. $\quad \varepsilon_x = 195 \times 10^{-6}$

$\varepsilon_y = -125 \times 10^{-6} \quad \gamma_{xy} = 0$

$$\phi = \arctan\frac{h}{b} = 26.57°$$

$$L_d = \sqrt{b^2 + h^2} = 8.944 \text{ in.}$$

(a) INCREASE IN LENGTH OF DIAGONAL

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

For $\theta = \phi = 26.57°$, $\varepsilon_{x_1} = 130.98 \times 10^{-6}$

$\Delta d = \varepsilon_{x_1} L_d = 0.00117$ in. $\quad\leftarrow$

(b) CHANGE IN ANGLE ϕ

Eq. (7-68): $\alpha = -(\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta - \gamma_{xy}\sin^2\theta$

For $\theta = \phi = 26.57°$: $\alpha = -128.0 \times 10^{-6}$ rad

Minus sign means line Od rotates clockwise (angle ϕ decreases).

$\Delta\phi = 128 \times 10^{-6}$ rad (decrease) $\quad\leftarrow$

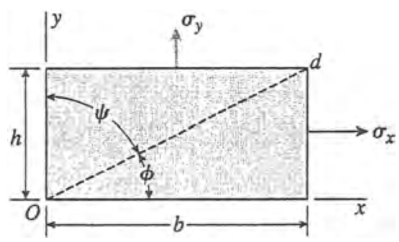
(c) CHANGE IN ANGLE c

Angle c increases the same amount that ϕ decreases.

Δc $= 128 \times 10^{-6}$ rad (increase) $\quad\leftarrow$

Problem 7.7-2 Solve the preceding problem if $b = 160$ mm, $h = 60$ mm, $\epsilon_x = 410 \times 10^{-6}$, and $\epsilon_y = -320 \times 10^{-6}$.

Solution 7.7-2 Plate in biaxial stress



$b = 160$ mm $\quad h = 60$ mm $\quad \varepsilon_x = 410 \times 10^{-6}$

$\varepsilon_y = -320 \times 10^{-6} \qquad \gamma_{xy} = 0$

$$\phi = \arctan \frac{h}{b} = 20.56°$$

$$L_d = \sqrt{b^2 + h^2} = 170.88 \text{ mm}$$

(a) Increase in length of diagonal

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

For $\theta = \phi = 20.56°$: $\varepsilon_{x1} = 319.97 \times 10^{-6}$

$\Delta d = \varepsilon_{x_1} L_d = 0.0547$ mm $\quad\leftarrow$

(b) Change in angle ϕ

Eq. (7-68): $\alpha = -(\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta - \gamma_{xy}\sin^2\theta$

For $\theta = \phi = 20.56°$: $\alpha = -240.0 \times 10^{-6}$ rad

Minus sign means line Od rotates clockwise (angle ϕ decreases.)

$\Delta\phi = 240 \times 10^{-6}$ rad (decrease) $\quad\leftarrow$

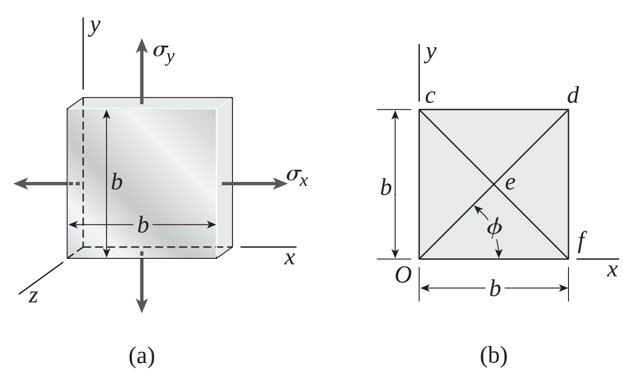
(c) Change in angle c

Angle c increases the same amount that ϕ decreases.

$\Delta c = 240 \times 10^{-6}$ rad (increase) $\quad\leftarrow$

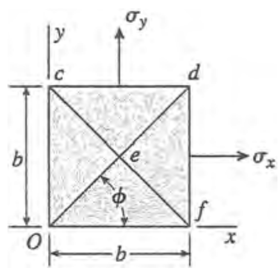
Problem 7.7-3 A thin square plate in *biaxial stress* is subjected to stresses σ_x and σ_y, as shown in part (a) of the figure . The width of the plate is $b = 12.0$ in. Measurements show that the normal strains in the x and y directions are $\epsilon_x = 427 \times 10^{-6}$ and $\epsilon_y = 113 \times 10^{-6}$, respectively.

With reference to part (b) of the figure, which shows a two-dimensional view of the plate, determine the following quantities: (a) the increase Δd in the length of diagonal Od; (b) the change $\Delta\phi$ in the angle ϕ between diagonal Od and the x axis; and (c) the shear strain γ associated with diagonals Od and cf (that is, find the decrease in angle ced).



PROBS. 7.7-3 and 7.7-4

Solution 7.7-3 Square plate in biaxial stress



$b = 12.0 \text{ in.} \qquad \varepsilon_x = 427 \times 10^{-6}$

$\varepsilon_y = 113 \times 10^{-6}$

$\phi = 45° \qquad \gamma_{xy} = 0$

$L_d = b\sqrt{2} = 16.97 \text{ in.}$

(a) Increase in length of diagonal

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

For $\theta = \phi = 45°$: $\varepsilon_{x_1} = 270 \times 10^{-6}$

$\Delta d = \varepsilon_{x_1} L_d = 0.00458 \text{ in.} \quad \longleftarrow$

(b) Change in angle ϕ

Eq. (7-68): $\alpha = -(\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta - \gamma_{xy}\sin^2\theta$

For $\theta = \phi = 45°$: $\alpha = -157 \times 10^{-6}$ rad

Minus sign means line *Od* rotates clockwise (angle ϕ decreases.)

$\Delta\phi = 157 \times 10^{-6}$ rad (decrease) $\quad \longleftarrow$

(c) Shear strain between diagonals

Eq. (7-71b): $$\frac{\gamma_{x_1y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

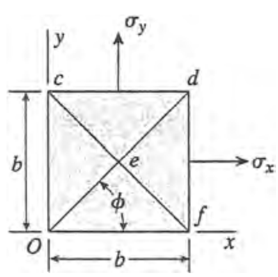
For $\theta = \phi = 45°$: $\gamma_{x_1y_1} = -314 \times 10^{-6}$ rad

(Negative strain means angle *ced* increases)

$\gamma = -314 \times 10^{-6}$ rad $\quad \longleftarrow$

Problem 7.7-4 Solve the preceding problem if $b = 225$ mm, $\epsilon_x = 845 \times 10^{-6}$, and $\epsilon_y = 211 \times 10^{-6}$.

Solution 7.7-4 Square plate in biaxial stress



$b = 225 \text{ mm} \qquad \varepsilon_x = 845 \times 10^{-6}$

$\varepsilon_y = 211 \times 10^{-6} \qquad \phi = 45° \qquad \gamma_{xy} = 0$

$L_d = b\sqrt{2} = 318.2 \text{ mm}$

(a) Increase in length of diagonal

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

For $\theta = \phi = 45°$: $\varepsilon_{x_1} = 528 \times 10^{-6}$

$\Delta d = \varepsilon_{x_1} L_d = 0.168 \text{ mm} \quad \longleftarrow$

(b) Change in angle ϕ

Eq. (7-68): $\alpha = -(\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta - \gamma_{xy}\sin^2\theta$

For $\theta = \phi = 45°$: $\alpha = -317 \times 10^{-6}$ rad

Minus sign means line *Od* rotates clockwise (angle ϕ decreases.)

$\Delta\phi = 317 \times 10^{-6}$ rad (decrease) $\quad \longleftarrow$

(c) SHEAR STRAIN BETWEEN DIAGONALS

Eq. (7-71b): $\dfrac{\gamma_{x_1y_1}}{2} = -\dfrac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \dfrac{\gamma_{xy}}{2}\cos 2\theta$

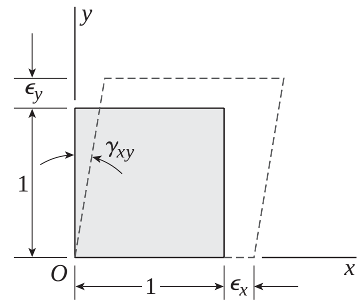
For $\theta = \phi = 45°$: $\gamma_{x_1y_1} = -634 \times 10^{-6}$ rad

(Negative strain means angle *ced* increases)

$\gamma = -634 \times 10^{-6}$ rad

Problem 7.7-5 An element of material subjected to *plane strain* (see figure) has strains as follows: $\epsilon_x = 220 \times 10^{-6}$, $\epsilon_y = 480 \times 10^{-6}$, and $\gamma_{xy} = 180 \times 10^{-6}$.

Calculate the strains for an element oriented at an angle $\theta = 50°$ and show these strains on a sketch of a properly oriented element.



Probs. 7.7-5 through 7.7-10

Solution 7.7-5 Element in plane strain

$\varepsilon_x = 220 \times 10^{-6} \qquad \varepsilon_y = 480 \times 10^{-6}$

$\gamma_{xy} = 180 \times 10^{-6}$

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

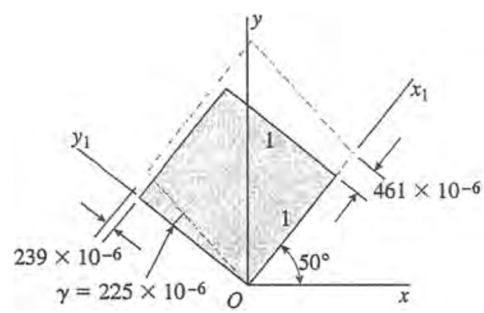
$$\frac{\gamma_{x_1y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$\varepsilon_{y_1} = \varepsilon_x + \varepsilon_y - \varepsilon_{x_1}$

For $\theta = 50°$:

$\varepsilon_{x_1} = 461 \times 10^{-6} \qquad \gamma_{x_1y_1} = 225 \times 10^{-6}$

$\varepsilon_{y_1} = 239 \times 10^{-6}$



Problem 7.7-6 Solve the preceding problem for the following data: $\epsilon_x = 420 \times 10^{-6}$, $\epsilon_y = -170 \times 10^{-6}$, $\gamma_{xy} = 310 \times 10^{-6}$, and $\theta = 37.5°$.

Solution 7.7-6 Element in plane strain

$$\varepsilon_x = 420 \times 10^{-6} \qquad \varepsilon_y = -170 \times 10^{-6}$$

$$\gamma_{xy} = 310 \times 10^{-6}$$

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

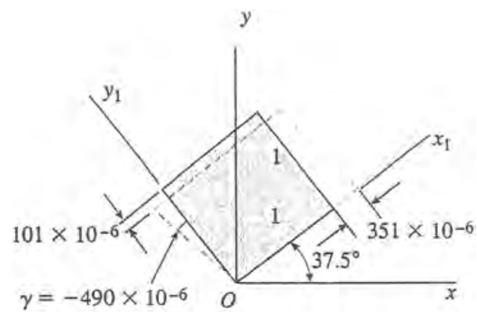
$$\frac{\gamma_{x_1y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$\varepsilon_{y_1} = \varepsilon_x + \varepsilon_y - \varepsilon_{x_1}$$

For $\theta = 37.5°$:

$$\varepsilon_{x_1} = 351 \times 10^{-6} \qquad \gamma_{x_1y_1} = -490 \times 10^{-6}$$

$$\varepsilon_{y_1} = -101 \times 10^{-6}$$



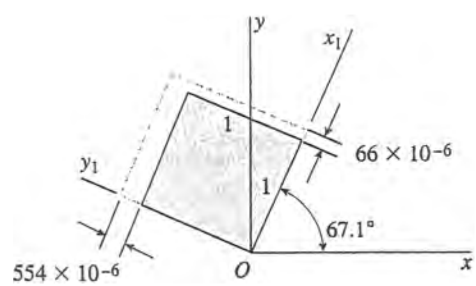
Problem 7.7-7 The strains for an element of material in *plane strain* (see figure) are as follows: $\epsilon_x = 480 \times 10^{-6}$, $\epsilon_y = 140 \times 10^{-6}$, and $\gamma_{xy} = -350 \times 10^{-6}$.

Determine the principal strains and maximum shear strains, and show these strains on sketches of properly oriented elements.

Solution 7.7-7 Element in plane strain

$$\varepsilon_x = 480 \times 10^{-6} \qquad \varepsilon_y = 140 \times 10^{-6}$$

$$\gamma_{xy} = -350 \times 10^{-6}$$



PRINCIPAL STRAINS

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 310 \times 10^{-6} \pm 244 \times 10^{-6}$$

$$\varepsilon_1 = 554 \times 10^{-6} \qquad \varepsilon_2 = 66 \times 10^{-6}$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = -1.0294$$

$$2\theta_p = -45.8° \quad \text{and} \quad 134.2°$$

$$\theta_p = -22.9° \quad \text{and} \quad 67.1°$$

For $\theta_p = -22.9°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$= 554 \times 10^{-6}$$

$$\therefore\ \theta_{p_1} = -22.9° \qquad \varepsilon_1 = 554 \times 10^{-6} \qquad \leftarrow$$

$$\theta_{p_2} = 67.1° \qquad \varepsilon_2 = 66 \times 10^{-6} \qquad \leftarrow$$

MAXIMUM SHEAR STRAINS

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 244 \times 10^{-6}$$

$$\gamma_{max} = 488 \times 10^{-6}$$

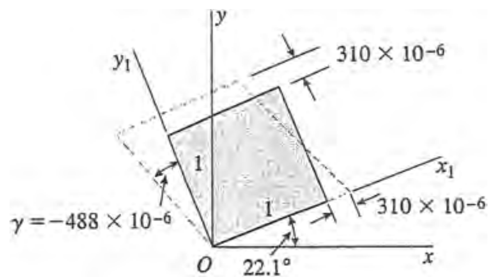
$$\theta_{s_1} = \theta_{p_1} - 45° = -67.9° \text{ or } 112.1°$$

$$\gamma_{max} = 488 \times 10^{-6} \qquad \leftarrow$$

$$\theta_{s_2} = \theta_{s_1} + 90° = 22.1°$$

$$\gamma_{min} = -488 \times 10^{-6} \qquad \leftarrow$$

$$\varepsilon_{aver} = \frac{\varepsilon_x + \varepsilon_y}{2} = 310 \times 10^{-6}$$



Problem 7.7-8 Solve the preceding problem for the following strains: $\epsilon_x = 120 \times 10^{-6}$, $\epsilon_y = -450 \times 10^{-6}$, and $\gamma_{xy} = -360 \times 10^{-6}$.

Solution 7.7-8 Element in plane strain

$\varepsilon_x = 120 \times 10^{-6} \qquad \varepsilon_y = -450 \times 10^{-6}$

$\gamma_{xy} = -360 \times 10^{-6}$

Principal strains

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= -165 \times 10^{-6} \pm 377 \times 10^{-6}$$

$$\varepsilon_1 = 172 \times 10^{-6} \qquad \varepsilon_2 = -502 \times 10^{-6}$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = -0.6316$$

$$2\theta_p = 327.7° \quad \text{and} \quad 147.7°$$

$$\theta_p = 163.9° \quad \text{and} \quad 73.9°$$

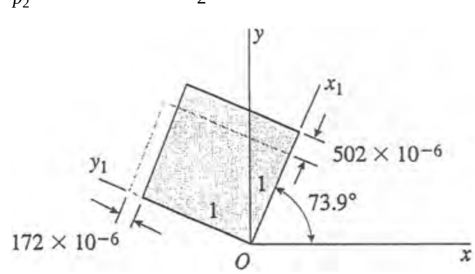
For $\theta_p = 163.9°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$= 172 \times 10^{-6}$$

$$\therefore \theta_{p_1} = 163.9° \qquad \varepsilon_1 = 172 \times 10^{-6} \quad \leftarrow$$

$$\theta_{p_2} = 73.9° \qquad \varepsilon_2 = -502 \times 10^{-6} \quad \leftarrow$$



Maximum shear strains

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 337 \times 10^{-6}$$

$$\gamma_{max} = 674 \times 10^{-6}$$

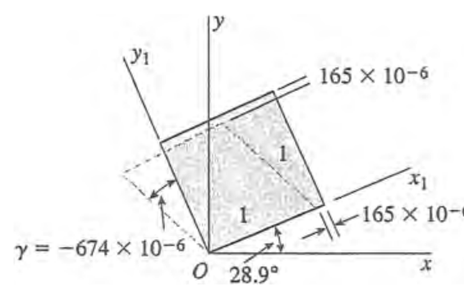
$$\theta_{s_1} = \theta_{p_1} - 45° = 118.9°$$

$$\gamma_{max} = 674 \times 10^{-6} \quad \leftarrow$$

$$\theta_{s_2} = \theta_{s_1} - 90° = 28.9°$$

$$\gamma_{min} = -674 \times 10^{-6} \quad \leftarrow$$

$$\varepsilon_{aver} = \frac{\varepsilon_x + \varepsilon_y}{2} = -165 \times 10^{-6}$$



Problem 7.7-9 A element of material in *plane strain* (see figure) is subjected to strains $\epsilon_x = 480 \times 10^{-6}$, $\epsilon_y = 70 \times 10^{-6}$, and $\gamma_{xy} = 420 \times 10^{-6}$.

Determine the following quantities: (a) the strains for an element oriented at an angle $\theta = 75°$, (b) the principal strains, and (c) the maximum shear strains. Show the results on sketches of properly oriented element.

Solution 7.7-9 Element in plane strain

$$\varepsilon_x = 480 \times 10^{-6} \qquad \varepsilon_y = 70 \times 10^{-6}$$

$$\gamma_{xy} = 420 \times 10^{-6}$$

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

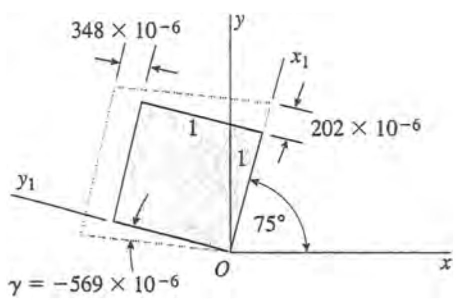
$$\frac{\gamma_{x_1y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$\varepsilon_{y_1} = \varepsilon_x + \varepsilon_y - \varepsilon_{x_1}$$

For $\theta = 75°$:

$$\varepsilon_{x_1} = 202 \times 10^{-6} \qquad \gamma_{x_1y_1} = -569 \times 10^{-6}$$

$$\varepsilon_{y_1} = 348 \times 10^{-6}$$



PRINCIPAL STRAINS

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 275 \times 10^{-6} \pm 293 \times 10^{-6}$$

$$\varepsilon_1 = 568 \times 10^{-6} \qquad \varepsilon_2 = -18 \times 10^{-6}$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = 1.0244$$

$$2\theta_p = 45.69° \quad \text{and} \quad 225.69°$$

$$\theta_p = 22.85° \quad \text{and} \quad 112.85°$$

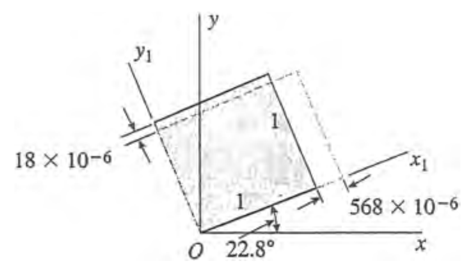
For $\theta_p = 22.85°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$= 568 \times 10^{-6}$$

$$\therefore \theta_{p_1} = 22.8° \qquad \varepsilon_1 = 568 \times 10^{-6} \qquad \leftarrow$$

$$\theta_{p_2} = 112.8° \qquad \varepsilon_2 = -18 \times 10^{-6} \qquad \leftarrow$$



MAXIMUM SHEAR STRAINS

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = 293 \times 10^{-6}$$

$$\gamma_{max} = 587 \times 10^{-6}$$

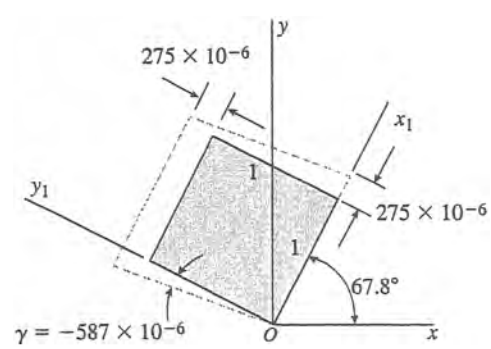
$$\theta_{s_1} = \theta_{p_1} - 45° = -22.2° \text{ or } 157.8°$$

$$\gamma_{max} = 587 \times 10^{-6} \qquad \leftarrow$$

$$\theta_{s_2} = \theta_{s_1} + 90° = 67.8°$$

$$\gamma_{min} = -587 \times 10^{-6} \qquad \leftarrow$$

$$\varepsilon_{aver} = \frac{\varepsilon_x + \varepsilon_y}{2} = 275 \times 10^{-6}$$



Problem 7.7-10 Solve the preceding problem for the following data: $\epsilon_x = -1120 \times 10^{-6}$, $\epsilon_y = -430 \times 10^{-6}$, $\gamma_{xy} = 780 \times 10^{-6}$, and $\theta = 45°$.

Solution 7.7-10 Element in plane strain

$$\varepsilon_x = -1120 \times 10^{-6} \qquad \varepsilon_y = -430 \times 10^{-6}$$

$$\gamma_{xy} = 780 \times 10^{-6}$$

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

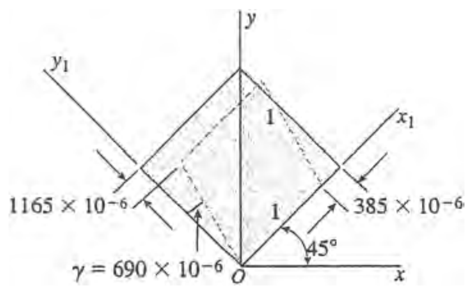
$$\frac{\gamma_{x_1y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$\varepsilon_{y_1} = \varepsilon_x + \varepsilon_y - \varepsilon_{x_1}$$

For $\theta = 45°$:

$$\varepsilon_{x_1} = -385 \times 10^{-6} \qquad \gamma_{x_1y_1} = 690 \times 10^{-6}$$

$$\varepsilon_{y_1} = -1165 \times 10^{-6}$$



Principal strains

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= -775 \times 10^{-6} \pm 521 \times 10^{-6}$$

$$\varepsilon_1 = -254 \times 10^{-6} \qquad \varepsilon_2 = -1296 \times 10^{-6}$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = -1.1304$$

$$2\theta_p = 131.5° \quad \text{and} \quad 311.5°$$

$$\theta_p = 65.7° \quad \text{and} \quad 155.7°$$

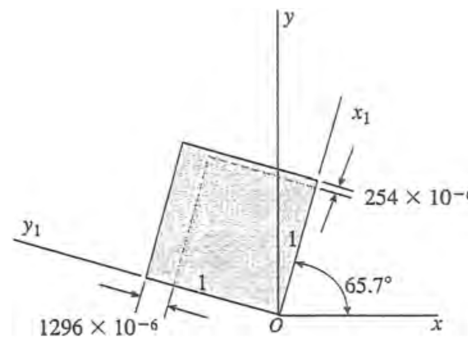
For $\theta_p = 65.7°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$= -254 \times 10^{-6}$$

$$\therefore\ \theta_{p_1} = 65.7° \qquad \varepsilon_1 = -254 \times 10^{-6} \quad \longleftarrow$$

$$\theta_{p_2} = 155.7° \qquad \varepsilon_2 = -1296 \times 10^{-6} \quad \longleftarrow$$



Maximum shear strains

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 521 \times 10^{-6}$$

$$\gamma_{max} = 1041 \times 10^{-6}$$

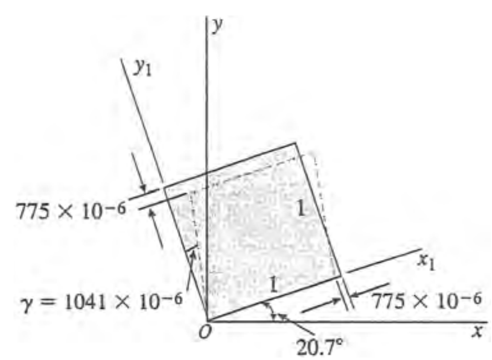
$$\theta_{s_1} = \theta_{p_1} - 45° = 20.7°$$

$$\gamma_{max} = 1041 \times 10^{-6} \quad \longleftarrow$$

$$\theta_{s_2} = \theta_{s_1} + 90° = 110.7°$$

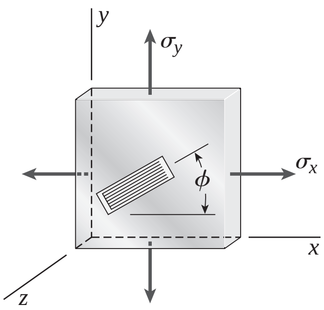
$$\gamma_{min} = -1041 \times 10^{-6} \quad \longleftarrow$$

$$\varepsilon_{aver} = \frac{\varepsilon_x + \varepsilon_y}{2} = -775 \times 10^{-6}$$



Problem 7.7-11 A steel plate with modulus of elasticity $E = 30 \times 10^6$ psi and Poisson's ratio $\nu = 0.30$ is loaded in *biaxial stress* by normal stresses σ_x and σ_y (see figure). A strain gage is bonded to the plate at an angle $\phi = 30°$.

If the stress σ_x is 18,000 psi and the strain measured by the gage is $\epsilon = 407 \times 10^{-6}$, what is the maximum in-plane shear stress $(\tau_{max})_{xy}$ and shear strain $(\gamma_{max})_{xy}$? What is the maximum shear strain $(\gamma_{max})_{xz}$ in the xz plane? What is the maximum shear strain $(\gamma_{max})_{yz}$ in the yz plane?



Probs. 7.7-11 and 7.7-12

Solution 7.7-11 Steel plate in biaxial stress

$\sigma_x = 18{,}000$ psi $\quad \gamma_{xy} = 0 \quad \sigma_y = ?$

$E = 30 \times 10^6$ psi $\quad \nu = 0.30$

Strain gage: $\phi = 30°$ $\quad \varepsilon = 407 \times 10^{-6}$

UNITS: All stresses in psi.

STRAIN IN BIAXIAL STRESS (EQS. 7-39)

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = \frac{1}{30 \times 10^6}(18{,}000 - 0.3\sigma_y) \qquad (1)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = \frac{1}{30 \times 10^6}(\sigma_y - 5400) \qquad (2)$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) = -\frac{0.3}{30 \times 10^6}(18{,}000 + \sigma_y) \qquad (3)$$

STRAINS AT ANGLE $\phi = 30°$ (EQ. 7-71a)

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$407 \times 10^{-6} = \left(\frac{1}{2}\right)\left(\frac{1}{30 \times 10^6}\right)(12{,}600 + 0.7\sigma_y) + \left(\frac{1}{2}\right)\left(\frac{1}{30 \times 10^6}\right)(23{,}400 - 1.3\sigma_y)\cos 60°$$

Solve for σ_y: $\quad \sigma_y = 2400$ psi $\qquad (4)$

MAXIMUM IN-PLANE SHEAR STRESS

$$(\tau_{max})_{xy} = \frac{\sigma_x - \sigma_y}{2} = 7800 \text{ psi} \quad \longleftarrow$$

STRAINS FROM EQS. (1), (2), AND (3)

$\varepsilon_x = 576 \times 10^{-6} \qquad \varepsilon_y = -100 \times 10^{-6}$

$\varepsilon_z = -204 \times 10^{-6}$

MAXIMUM SHEAR STRAINS (EQ. 7-75)

$$xy \text{ plane: } \frac{(\gamma_{max})_{xy}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\gamma_{xy} = 0 \quad (\gamma_{max})_{xy} = 676 \times 10^{-6} \quad \longleftarrow$$

$$xz \text{ plane: } \frac{(\gamma_{max})_{xz}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_z}{2}\right)^2 + \left(\frac{\gamma_{xz}}{2}\right)^2}$$

$$\gamma_{xz} = 0 \quad (\gamma_{max})_{xz} = 780 \times 10^{-6} \quad \longleftarrow$$

$$yz \text{ plane: } \frac{(\gamma_{max})_{yz}}{2} = \sqrt{\left(\frac{\varepsilon_y - \varepsilon_z}{2}\right)^2 + \left(\frac{\gamma_{yz}}{2}\right)^2}$$

$$\gamma_{yz} = 0 \quad (\gamma_{max})_{yz} = 104 \times 10^{-6} \quad \longleftarrow$$

Problem 7.7-12 Solve the preceding problem if the plate is made of aluminum with $E = 72$ GPa and $\nu = 1/3$, the stress σ_x is 86.4 MPa, the angle ϕ is 21°, and the strain ϵ is 946×10^{-6}.

Solution 7.7-12 Aluminum plate in biaxial stress

$\sigma_x = 86.4$ MPa $\quad \gamma_{xy} = 0 \quad \sigma_y = ?$

$E = 72$ GPa $\quad \nu = 1/3$

Strain gage: $\phi = 21°$ $\quad \varepsilon = 946 \times 10^{-6}$

UNITS: All stresses in MPa.

STRAIN IN BIAXIAL STRESS (EQS. 7-39)

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = \frac{1}{72{,}000}\left(86.4 - \frac{1}{3}\sigma_y\right) \qquad (1)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = \frac{1}{72{,}000}(\sigma_y - 28.8) \qquad (2)$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) = -\frac{1/3}{72{,}000}(86.4 + \sigma_y) \qquad (3)$$

STRAINS AT ANGLE $\phi = 21°$ (EQ. 7-71a)

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$946 \times 10^{-6} = \left(\frac{1}{2}\right)\left(\frac{1}{72{,}000}\right)\left(57.6 + \frac{2}{3}\sigma_y\right)$$

$$+ \left(\frac{1}{2}\right)\left(\frac{1}{72{,}000}\right)\left(115.2 - \frac{4}{3}\sigma_y\right)\cos 42°$$

Solve for σ_y: $\quad \sigma_y = 21.55$ MPa $\qquad (4)$

MAXIMUM IN-PLANE SHEAR STRESS

$$(\tau_{max})_{xy} = \frac{\sigma_x - \sigma_y}{2} = 32.4 \text{ MPa} \quad \leftarrow$$

STRAINS FROM EQS. (1), (2), AND (3)

$\varepsilon_x = 1100 \times 10^{-6} \quad \varepsilon_y = -101 \times 10^{-6}$

$\varepsilon_z = -500 \times 10^{-6}$

MAXIMUM SHEAR STRAINS (EQ. 7-75)

xy plane: $$\frac{(\gamma_{max})_{xy}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$\gamma_{xy} = 0 \quad (\gamma_{max})_{xy} = 1200 \times 10^{-6} \quad \leftarrow$

xz plane: $$\frac{(\gamma_{max})_{xz}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_z}{2}\right)^2 + \left(\frac{\gamma_{xz}}{2}\right)^2}$$

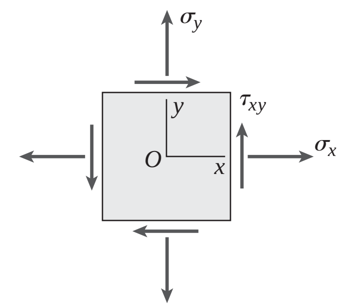
$\gamma_{xz} = 0 \quad (\gamma_{max})_{xz} = 1600 \times 10^{-6} \quad \leftarrow$

yz plane: $$\frac{(\gamma_{max})_{yz}}{2} = \sqrt{\left(\frac{\varepsilon_y - \varepsilon_z}{2}\right)^2 + \left(\frac{\gamma_{yz}}{2}\right)^2}$$

$\gamma_{yz} = 0 \quad (\gamma_{max})_{yz} = 399 \times 10^{-6} \quad \leftarrow$

Problem 7.7-13 An element in *plane stress* is subjected to stresses $\sigma_x = -8400$ psi, $\sigma_y = 1100$ psi, and $\tau_{xy} = -1700$ psi (see figure). The material is aluminum with modulus of elasticity $E = 10{,}000$ ksi and Poisson's ratio $\nu = 0.33$.

Determine the following quantities: (a) the strains for an element oriented at an angle $\theta = 30°$, (b) the principal strains, and (c) the maximum shear strains. Show the results on sketches of properly oriented elements.



Probs. 7.7-13 and 7.7-14

Solution 7.7-13 Element in plane strain

$\sigma_x = -8400$ psi $\quad \sigma_y = 1100$ psi

$\tau_{xy} = -1700$ psi $\quad E = 10{,}000$ ksi $\quad \nu = 0.33$

HOOKE'S LAW (EQS. 7-34 AND 7-35)

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = -876.3 \times 10^{-6}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = 387.2 \times 10^{-6}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2\tau_{xy}(1 + \nu)}{E} = -452.2 \times 10^{-6}$$

FOR $\theta = 30°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

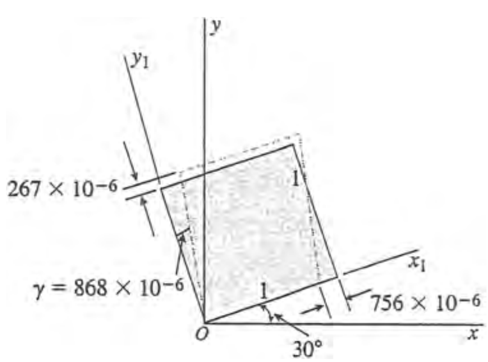
$$= -756 \times 10^{-6}$$

$$\frac{\gamma_{x_1y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$= 434 \times 10^{-6}$$

$$\gamma_{x_1y_1} = 868 \times 10^{-6}$$

$$\varepsilon_{y_1} = \varepsilon_x + \varepsilon_y - \varepsilon_{x_1} = 267 \times 10^{-6}$$



PRINCIPAL STRAINS

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= -245 \times 10^{-6} \pm 671 \times 10^{-6}$$

$$\varepsilon_1 = 426 \times 10^{-6} \quad \varepsilon_2 = -961 \times 10^{-6}$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = 0.3579$$

$$2\theta_p = 19.7° \text{ and } 199.7°$$

$$\theta_p = 9.8° \text{ and } 99.8°$$

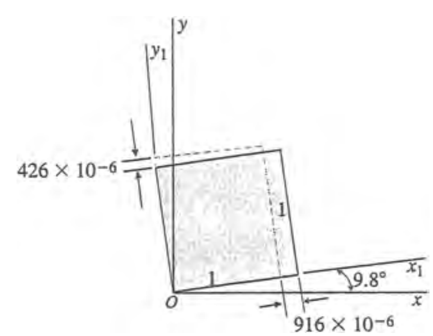
FOR $\theta_p = 9.8°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$= -916 \times 10^{-6}$$

$$\therefore\ \theta_{p_1} = 99.8° \quad \varepsilon_1 = 426 \times 10^{-6} \quad \leftarrow$$

$$\theta_{p_2} = 9.8° \quad \varepsilon_2 = -916 \times 10^{-6} \quad \leftarrow$$



MAXIMUM SHEAR STRAINS

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 671 \times 10^{-6}$$

$$\gamma_{max} = 1342 \times 10^{-6}$$

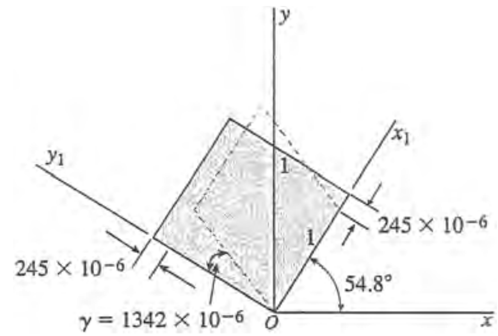
$$\theta_{s_1} = \theta_{p_1} - 45° = 54.8°$$

$$\gamma_{max} = 1342 \times 10^{-6} \quad \longleftarrow$$

$$\theta_{s_2} = \theta_{s_1} + 90° = 144.8°$$

$$\gamma_{min} = -1342 \times 10^{-6} \quad \longleftarrow$$

$$\varepsilon_{aver} = \frac{\varepsilon_x + \varepsilon_y}{2} = -245 \times 10^{-6}$$



Problem 7.7-14 Solve the preceding problem for the following data: $\sigma_x = -150$ MPa, $\sigma_y = -210$ MPa, $\tau_{xy} = -16$ MPa, and $\theta = 50°$. The material is brass with $E = 100$ GPa and $\nu = 0.34$.

Solution 7.7-14 Element in plane strain

$$\sigma_x = -150 \text{ MPa} \quad \sigma_y = -210 \text{ MPa}$$

$$\tau_{xy} = -16 \text{ MPa} \quad E = 100 \text{ GPa} \quad \nu = 0.34$$

HOOKE'S LAW (EQS. 7-34 AND 7-35)

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = -786 \times 10^{-6}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = -1590 \times 10^{-6}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2\tau_{xy}(1 + \nu)}{E} = -429 \times 10^{-6}$$

FOR $\theta = 50°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

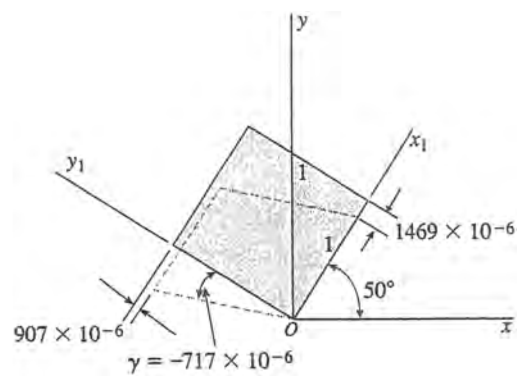
$$= -1469 \times 10^{-6}$$

$$\frac{\gamma_{x_1y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$= -358.5 \times 10^{-6}$$

$$\gamma_{x_1y_1} = -717 \times 10^{-6}$$

$$\varepsilon_{y_1} = \varepsilon_x + \varepsilon_y - \varepsilon_{x_1} = -907 \times 10^{-6}$$



PRINCIPAL STRAINS

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= -1188 \times 10^{-6} \pm 456 \times 10^{-6}$$

$$\varepsilon_1 = -732 \times 10^{-6} \quad \varepsilon_2 = -1644 \times 10^{-6}$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = -0.5333$$

$$2\theta_p = 151.9° \text{ and } 331.9°$$

$$\theta_p = 76.0° \text{ and } 166.0°$$

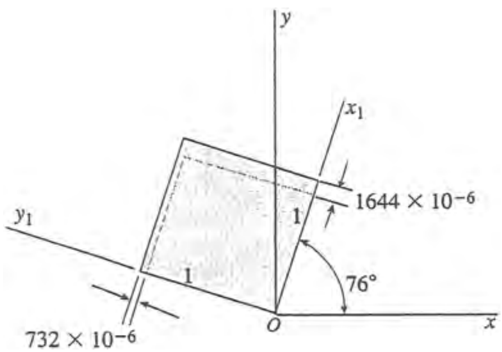
FOR $\theta_p = 76.0°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$= -1644 \times 10^{-6}$$

$$\therefore\ \theta_{p_1} = 166.0° \quad \varepsilon_1 = -732 \times 10^{-6} \quad \leftarrow$$

$$\theta_{p_2} = 76.0° \quad \varepsilon_2 = -1644 \times 10^{-6} \quad \leftarrow$$



MAXIMUM SHEAR STRAINS

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 456 \times 10^{-6}$$

$$\gamma_{max} = 911 \times 10^{-6}$$

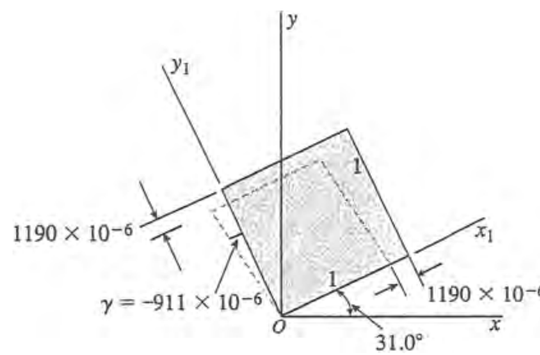
$$\theta_{s_1} = \theta_{p_1} - 45° = 121.0°$$

$$\gamma_{max} = 911 \times 10^{-6} \quad \leftarrow$$

$$\theta_{s_2} = \theta_{s_1} - 90° = 31.0°$$

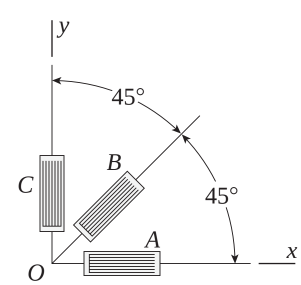
$$\gamma_{min} = -911 \times 10^{-6} \quad \leftarrow$$

$$\varepsilon_{aver} = \frac{\varepsilon_x + \varepsilon_y}{2} = -1190 \times 10^{-6}$$



Problem 7.7-15 During a test of an airplane wing, the strain gage readings from a 45° rosette (see figure) are as follows: gage *A*, 520×10^{-6}; gage *B*, 360×10^{-6}; and gage *C*, -80×10^{-6}.

Determine the principal strains and maximum shear strains, and show them on sketches of properly oriented elements.



Probs. 7.7-15 and 7.7-16

Solution 7.7-15 45° strain rosette

$\varepsilon_A = 520 \times 10^{-6} \quad \varepsilon_B = 360 \times 10^{-6}$

$\varepsilon_C = -80 \times 10^{-6}$

FROM EQS. (7-77) AND (7-78) OF EXAMPLE 7-8:

$\varepsilon_x = \varepsilon_A = 520 \times 10^{-6} \quad \varepsilon_y = \varepsilon_C = -80 \times 10^{-6}$

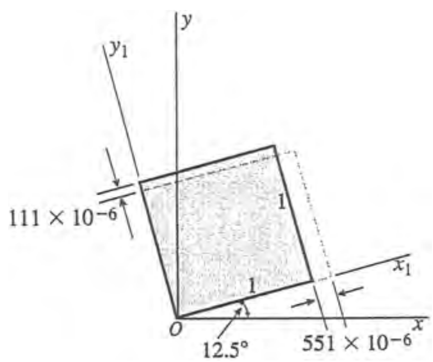
$\gamma_{xy} = 2\varepsilon_B - \varepsilon_A - \varepsilon_C = 280 \times 10^{-6}$

PRINCIPAL STRAINS

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 220 \times 10^{-6} \pm 331 \times 10^{-6}$$

$$\varepsilon_1 = 551 \times 10^{-6} \qquad \varepsilon_2 = -111 \times 10^{-6}$$



$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = 0.4667$$

$2\theta_p = 25.0°$ and $205.0°$

$\theta_p = 12.5°$ and $102.5°$

For $\theta_p = 12.5°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$= 551 \times 10^{-6}$$

$\therefore\ \theta_{p_1} = 12.5° \quad \varepsilon_1 = 551 \times 10^{-6} \quad \leftarrow$

$\theta_{p_2} = 102.5° \quad \varepsilon_2 = -111 \times 10^{-6} \quad \leftarrow$

MAXIMUM SHEAR STRAINS

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 331 \times 10^{-6}$$

$\gamma_{max} = 662 \times 10^{-6}$

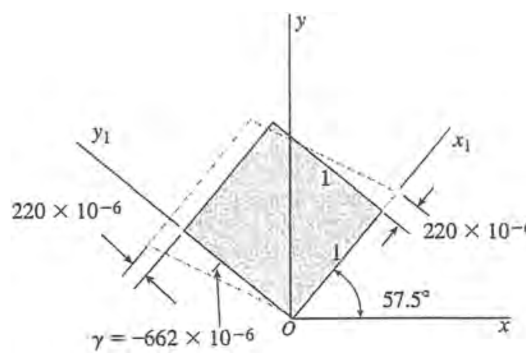
$\theta_{s_1} = \theta_{p_1} - 45° = -32.5°$ or $147.5°$

$\gamma_{max} = 662 \times 10^{-6} \quad \leftarrow$

$\theta_{s_2} = \theta_{s_1} + 90° = 57.5°$

$\gamma_{min} = -662 \times 10^{-6} \quad \leftarrow$

$$\varepsilon_{aver} = \frac{\varepsilon_x + \varepsilon_y}{2} = 220 \times 10^{-6}$$



Problem 7.7-16 A 45°strain rosette (see figure) mounted on the surface of an automobile frame gives the following readings: gage A, 310×10^{-6}; gage B, 180×10^{-6}; and gage C, -160×10^{-6}.

Determine the principal strains and maximum shear strains, and show them on sketches of properly oriented elements.

Solution 7.7-16 45° strain rosette

$\varepsilon_A = 310 \times 10^{-6} \quad \varepsilon_B = 180 \times 10^{-6}$

$\varepsilon_C = -160 \times 10^{-6}$

FROM EQS. (7-77) AND (7-78) OF EXAMPLE 7-8:

$\varepsilon_x = \varepsilon_A = 310 \times 10^{-6} \quad \varepsilon_y = \varepsilon_C = -160 \times 10^{-6}$

$\gamma_{xy} = 2\varepsilon_B - \varepsilon_A - \varepsilon_C = 210 \times 10^{-6}$

PRINCIPAL STRAINS

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 75 \times 10^{-6} \pm 257 \times 10^{-6}$$

$$\varepsilon_1 = 332 \times 10^{-6} \quad \varepsilon_2 = -182 \times 10^{-6}$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = 0.4468$$

$2\theta_p = 24.1°$ and $204.1°$

$\theta_p = 12.0°$ and $102.0°$

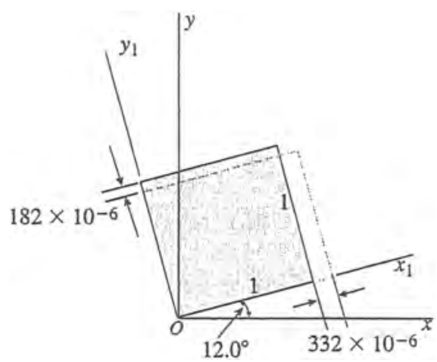
FOR $\theta_p = 12.0°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$= 332 \times 10^{-6}$$

$\therefore\ \theta_{p1} = 12.0° \quad \varepsilon_1 = 332 \times 10^{-6} \quad \leftarrow$

$\theta_{p2} = 102.0° \quad \varepsilon_2 = -182 \times 10^{-6} \quad \leftarrow$



MAXIMUM SHEAR STRAINS

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 257 \times 10^{-6}$$

$\gamma_{max} = 515 \times 10^{-6}$

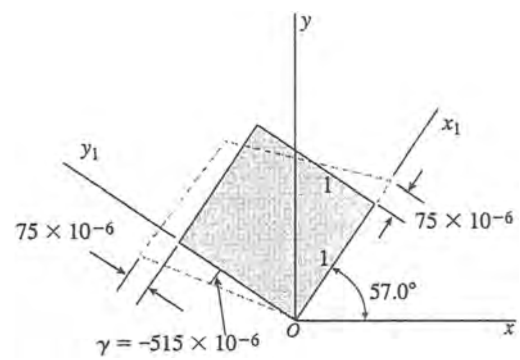
$\theta_{s_1} = \theta_{p_1} - 45° = -33.0°$ or $147.0°$

$\gamma_{max} = 515 \times 10^{-6} \quad \leftarrow$

$\theta_{s_2} = \theta_{s_1} + 90° = 57.0°$

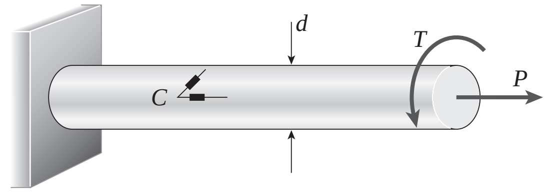
$\gamma_{min} = -515 \times 10^{-6} \quad \leftarrow$

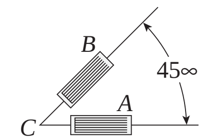
$$\varepsilon_{aver} = \frac{\varepsilon_x + \varepsilon_y}{2} = 75 \times 10^{-6}$$



Problem 7.7-17 A solid circular bar of diameter $d = 1.5$ in. is subjected to an axial force P and a torque T (see figure). Strain gages A and B mounted on the surface of the bar give reading $\epsilon_a = 100 \times 10^{-6}$ and $\epsilon_b = -55 \times 10^{-6}$. The bar is made of steel having $E = 30 \times 10^6$ psi and $\nu = 0.29$.

(a) Determine the axial force P and the torque T.
(b) Determine the maximum shear strain γ_{max} and the maximum shear stress τ_{max} in the bar.





Solution 7.7-17 Circular bar (plane stress)

Bar is subjected to a torque T and an axial force P.

$E = 30 \times 10^6$ psi $\quad \nu = 0.29$

Diameter $d = 1.5$ in.

STRAIN GAGES

At $\theta = 0°$: $\quad \varepsilon_A = \varepsilon_x = 100 \times 10^{-6}$

At $\theta = 45°$: $\quad \varepsilon_B = -55 \times 10^{-6}$

ELEMENT IN PLANE STRESS

$$\sigma_x = \frac{P}{A} = \frac{4P}{\pi d^2} \qquad \sigma_y = 0 \qquad \tau_{xy} = -\frac{16T}{\pi d^3}$$

$$\varepsilon_x = 100 \times 10^{-6} \qquad \varepsilon_y = -\nu\varepsilon_x = -29 \times 10^{-6}$$

AXIAL FORCE P

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{4P}{\pi d^2 E} \qquad P = \frac{\pi d^2 E \varepsilon_x}{4} = 5300 \text{ lb} \quad \leftarrow$$

SHEAR STRAIN

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2\tau_{xy}(1+\nu)}{E} = -\frac{32T(1+\nu)}{\pi d^3 E}$$

$$= -(0.1298 \times 10^{-6})T \quad (T = \text{lb-in.})$$

STRAIN AT $\theta = 45°$

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta \qquad (1)$$

$$\varepsilon_{x_1} = \varepsilon_B = -55 \times 10^{-6} \qquad 2\theta = 90°$$

Substitute numerical values into Eq. (1):

$$-55 \times 10^{-6} = 35.5 \times 10^{-6} - (0.0649 \times 10^{-6})T$$

Solve for T: $\quad T = 1390$ lb-in $\quad \leftarrow$

MAXIMUM SHEAR STRAIN AND MAXIMUM SHEAR STRESS

$$\gamma_{xy} = -(0.1298 \times 10^{-6})T = -180.4 \times 10^{-6} \text{ rad}$$

Eq. (7-75): $$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

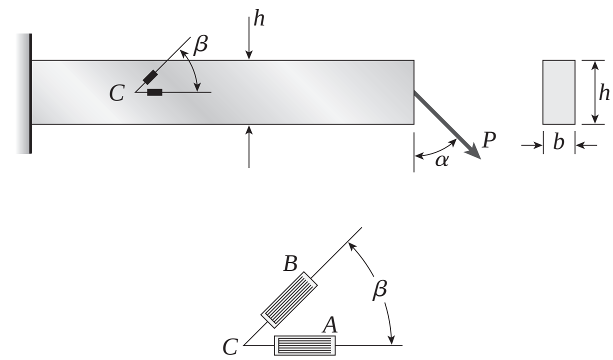
$$= 111 \times 10^{-6} \text{ rad}$$

$$\gamma_{max} = 222 \times 10^{-6} \text{ rad} \quad \leftarrow$$

$$\tau_{max} = G\gamma_{max} = 2580 \text{ psi} \quad \leftarrow$$

Problem 7.7-18 A cantilever beam of rectangular cross section (width $b = 25$ mm, height $h = 100$ mm) is loaded by a force P that acts at the midheight of the beam and is inclined at an angle α to the vertical (see figure). Two strain gages are placed at point C, which also is at the midheight of the beam. Gage A measures the strain in the horizontal direction and gage B measures the strain at an angle $\beta = 60°$ to the horizontal. The measured strains are $\epsilon_a = 125 \times 10^{-6}$ and $\epsilon_b = -375 \times 10^{-6}$.

Determine the force P and the angle α, assuming the material is steel with $E = 200$ GPa and $\nu = 1/3$.



Probs. 7.7-18 and 7.7-19

Solution 7.7-18 Cantilever beam (plane stress)

Beam loaded by a force P acting at an angle α.

$E = 200$ GPa $\quad \nu = 1/3 \quad b = 25$ mm

$h = 100$ mm

Axial force $F = P \sin \alpha$

Shear force $V = P \cos \alpha$

(At the neutral axis, the bending moment produces no stresses.)

STRAIN GAGES

At $\theta = 0°$: $\quad \varepsilon_A = \varepsilon_x = 125 \times 10^{-6}$

At $\theta = 60°$: $\quad \varepsilon_B = -375 \times 10^{-6}$

ELEMENT IN PLANE STRESS

$$\sigma_x = \frac{F}{A} = \frac{P \sin \alpha}{bh} \qquad \sigma_y = 0$$

$$\tau_{xy} = -\frac{3V}{2A} = -\frac{3P \cos \alpha}{2bh}$$

$$\varepsilon_x = 125 \times 10^{-6} \qquad \varepsilon_y = -\nu\varepsilon_x = -41.67 \times 10^{-6}$$

HOOKE'S LAW

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{P \sin \alpha}{bhE}$$

$$P \sin \alpha = bhE\varepsilon_x = 62{,}500 \text{ N} \qquad (1)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = -\frac{3P \cos \alpha}{2bhG} = -\frac{3(1 + \nu) P \cos \alpha}{bhE}$$

$$= -(8.0 \times 10^{-9}) P \cos \alpha \qquad (2)$$

FOR $\theta = 60°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta \qquad (3)$$

$$\varepsilon_{x_1} = \varepsilon_B = -375 \times 10^{-6} \qquad 2\theta = 120°$$

Substitute into Eq. (3):

$$-375 \times 10^{-6} = 41.67 \times 10^{-6} - 41.67 \times 10^{-6}$$
$$- (3.464 \times 10^{-9}) P \cos \alpha$$

or $P \cos \alpha = 108{,}260$ N $\qquad (4)$

SOLVE EQS. (1) AND (4):

$\tan \alpha = 0.5773 \qquad \alpha = 30° \quad \leftarrow$

$P = 125$ kN $\quad \leftarrow$

Problem 7.7-19 Solve the preceding problem if the cross-sectional dimensions are $b = 1.0$ in. and $h = 3.0$ in., the gage angle is $\beta = 75°$, the measure strains are $\epsilon_a = 171 \times 10^{-6}$ and $\epsilon_b = -266 \times 10^{-6}$, and the material is a magnesium alloy with modulus $E = 6.0 \times 10^6$ psi and Poisson's ratio $\nu = 0.35$.

Solution 7.7-19 Cantilever beam (plane stress)

Beam loaded by a force P acting at an angle α.

$E = 6.0 \times 10^6$ psi $\qquad \nu = 0.35 \qquad b = 1.0$ in.

$h = 3.0$ in.

Axial force $F = P \sin \alpha$ Shear foce $V = P \cos \alpha$
(At the neutral axis, the bending moment produces no stresses.)

STRAIN GAGES

At $\theta = 0°$: $\qquad \varepsilon_A = \varepsilon_x = 171 \times 10^{-6}$

At $\theta = 75°$: $\qquad \varepsilon_B = -266 \times 10^{-6}$

ELEMENT IN PLANE STRESS

$$\sigma_x = \frac{F}{A} = \frac{P \sin \alpha}{bh} \qquad \sigma_y = 0$$

$$\tau_{xy} = -\frac{3V}{2A} = -\frac{3P \cos \alpha}{2bh}$$

$$\varepsilon_x = 171 \times 10^{-6} \qquad \varepsilon_y = -\nu\varepsilon_x = -59.85 \times 10^{-6}$$

HOOKE'S LAW

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{P \sin \alpha}{bhE}$$

$$P \sin \alpha = bhE\varepsilon_x = 3078 \text{ lb} \qquad (1)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = -\frac{3P \cos \alpha}{2bhG} = -\frac{3(1 + \nu)P \cos \alpha}{bhE}$$

$$= -(225.0 \times 10^{-9})P \cos \alpha \qquad (2)$$

FOR $\theta = 75°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta \qquad (3)$$

$$\varepsilon_{x_1} = \varepsilon_B = -266 \times 10^{-6} \qquad 2\theta = 150°$$

Substitute into Eq. (3):

$$-266 \times 10^{-6} = 55.575 \times 10^{-6} - 99.961 \times 10^{-6} - (56.25 \times 10^{-9})P \cos \alpha$$

$$\text{or } P \cos \alpha = 3939.8 \text{ lb} \qquad (4)$$

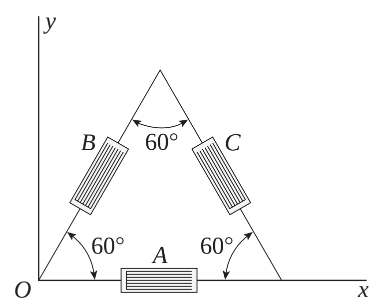
SOLVE EQS. (1) AND (4):

$\tan \alpha = 0.7813 \qquad \alpha = 38° \qquad \leftarrow$

$P = 5000$ lb $\qquad \leftarrow$

Problem 7.7-20 A 60° strain rosette, or *delta rosette*, consists of three electrical-resistance strain gages arranged as shown in the figure. Gage A measures the normal strain ϵ_a in the direction of the x axis. Gages B and C measure the strains ϵ_b and ϵ_c in the inclined directions shown.

Obtain the equations for the strains ϵ_x, ϵ_y, and γ_{xy} associated with the xy axis.



Solution 7.7-20 Delta rosette (60° strain rosette)

STRAIN GAGES

Gage A at $\theta = 0°$ $\qquad$ Strain $= \varepsilon_A$

Gage B at $\theta = 60°$ $\qquad$ Strain $= \varepsilon_B$

Gage C at $\theta = 120°$ $\qquad$ Strain $= \varepsilon_C$

FOR $\theta = 0°$: $\qquad \varepsilon_x = \varepsilon_A \qquad \leftarrow$

FOR $\theta = 60°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

$$\varepsilon_B = \frac{\varepsilon_A + \varepsilon_y}{2} + \frac{\varepsilon_A - \varepsilon_y}{2} (\cos 120°) + \frac{\gamma_{xy}}{2} (\sin 120°)$$

$$\varepsilon_B = \frac{\varepsilon_A}{4} + \frac{3\varepsilon_y}{4} + \frac{\gamma_{xy}\sqrt{3}}{4} \qquad (1)$$

FOR $\theta = 120°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\varepsilon_C = \frac{\varepsilon_A + \varepsilon_y}{2} + \frac{\varepsilon_A - \varepsilon_y}{2}(\cos 240°)\frac{\gamma_{xy}}{2}(\sin 240°)$$

$$\varepsilon_C = \frac{\varepsilon_A}{4} + \frac{3\varepsilon_y}{4} - \frac{\gamma_{xy}\sqrt{3}}{4} \qquad (2)$$

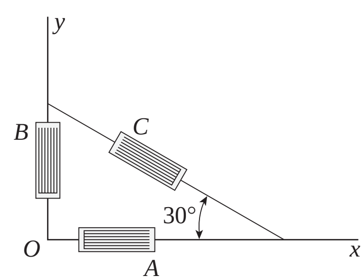
SOLVE EQS. (1) AND (2):

$$\varepsilon_y = \frac{1}{3}(2\varepsilon_B + 2\varepsilon_C - \varepsilon_A) \quad \leftarrow$$

$$\gamma_{xy} = \frac{2}{\sqrt{3}}(\varepsilon_B - \varepsilon_C) \quad \leftarrow$$

Problem 7.7-21 On the surface of a structural component in a space vehicle, the strainsare monitored by means of three strain gages arranged as shown in the figure. During a certain maneuver, the following strains were recorded: $\epsilon_a = 1100 \times 10^{-6}$, $\epsilon_b = 200 \times 10^{-6}$, and $\epsilon_c = 200 \times 10^{-6}$.

Determine the principal strains and principal stresses in the material, which is a magnesium alloy for which $E = 6000$ ksi and $\nu = 0.35$. (Show the principal strains and principal stresses on sketches of properly oriented element.)



Solution 7.7-21 **30-60-90° strain rosette**

Magnesium alloy: $E = 6000$ ksi $\quad \nu = 0.35$

STRAIN GAGES

Gage A at $\theta = 0°$ $\quad \epsilon_A = 1100 \times 10^{-6}$

Gage B at $\theta = 90°$ $\quad \varepsilon_B = 200 \times 10^{-6}$

Gage C at $\theta = 150°$ $\quad \varepsilon_C = 200 \times 10^{-6}$

FOR $\theta = 0°$: $\quad \varepsilon_x = \varepsilon_A = 1100 \times 10^{-6}$

FOR $\theta = 90°$: $\quad \varepsilon_y = \varepsilon_B = 200 \times 10^{-6}$

FOR $\theta = 150°$:

$$\varepsilon_{x_1} = \varepsilon_C = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$200 \times 10^{-6} = 650 \times 10^{-6} + 225 \times 10^{-6} - 0.43301\gamma_{xy}$$

Solve for γ_{xy}: $\gamma_{xy} = 1558.9 \times 10^{-6}$

PRINCIPAL STRAINS

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 650 \times 10^{-6} \pm 900 \times 10^{-6}$$

$$\varepsilon_1 = 1550 \times 10^{-6} \qquad \varepsilon_2 = -250 \times 10^{-6}$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = \sqrt{3} = 1.7321$$

$$2\theta_p = 60° \qquad \theta_p = 30°$$

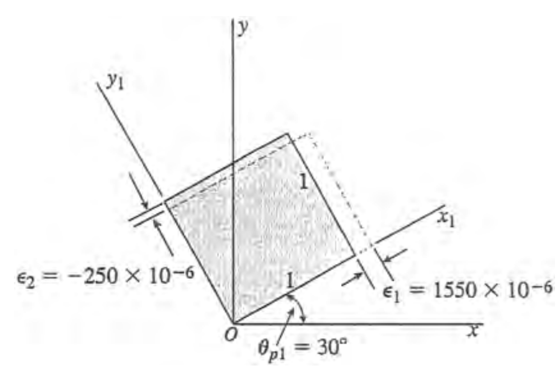
FOR $\theta_p = 30°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$= 1550 \times 10^{-6}$$

$$\therefore \theta_{p1} = 30° \qquad \varepsilon_1 = 1550 \times 10^{-6} \quad \leftarrow$$

$$\theta_{p2} = 120° \qquad \varepsilon_2 = -250 \times 10^{-6} \quad \leftarrow$$

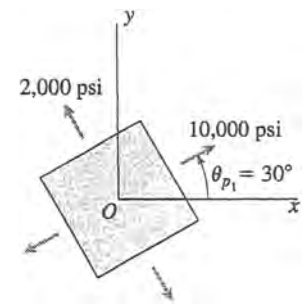


PRINCIPAL STRESSES (see Eqs. 7-36)

$$\sigma_1 = \frac{E}{1 - \nu^2}(\varepsilon_1 + \nu\varepsilon_2) \qquad \sigma_2 = \frac{E}{1 - \nu^2}(\varepsilon_2 + \nu\varepsilon_1)$$

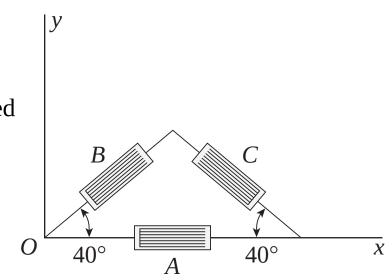
Substitute numerical values:

$\sigma_1 = 10{,}000$ psi $\qquad \sigma_2 = 2{,}000$ psi $\quad \leftarrow$



Problem 7.7-22 The strains on the surface of an experimental device made of pure aluminum ($E = 70$ Gpa, $\nu = 0.33$) and tested in a space shuttle were measured by means of strain gages. The gages were oriented as shown in the figure, and the measured strains were $\epsilon_a = 1100 \times 10^{-6}$, $\epsilon_b = 1496 \times 10^{-6}$, and $\epsilon_c = -39.44 \times 10 \times ^{-6}$.

What is the stress σ_x in the x direction?



Solution 7.7-22 40-40-100° strain rosette

Pure aluminum: $E = 70$ GPa $\qquad \nu = 0.33$

STRAIN GAGES

Gage A at $\theta = 0°$ $\qquad \varepsilon_A = 1100 \times 10^{-6}$

Gage B at $\theta = 40°$ $\qquad \varepsilon_B = 1496 \times 10^{-6}$

Gage C at $\theta = 140°$ $\qquad \varepsilon_C = -39.44 \times 10^{-6}$

FOR $\theta = 0°$: $\qquad \varepsilon_x = \varepsilon_A = 1100 \times 10^{-6}$

FOR $\theta = 40°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

Substitute $\varepsilon_{x_1} = \varepsilon_B = 1496 \times 10^{-6}$ and $\varepsilon_x = 1100 \times 10^{-6}$; then simplify and rearrange:

$$0.41318\varepsilon_y + 0.49240\gamma_{xy} = 850.49 \times 10^{-6} \qquad (1)$$

FOR $\theta = 140°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

Substitute $\varepsilon_{x_1} = \varepsilon_c = -39.44 \times 10^{-6}$ and $\varepsilon_x = 1100 \times 10^{-6}$; then simplify and rearrange:

$$0.41318\varepsilon_y - 0.49240\gamma_{xy} = -684.95 \times 10^{-6} \qquad (2)$$

SOLVE EQS. (1) AND (2):

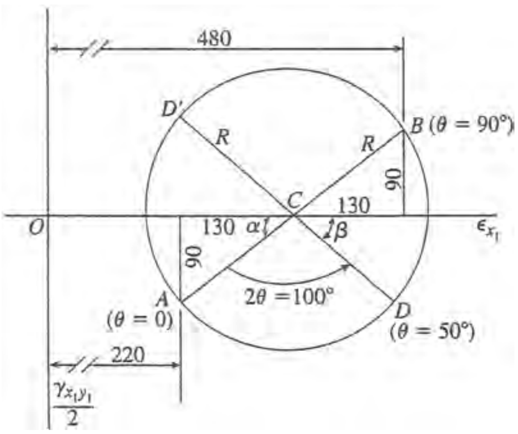
$\varepsilon_y = 200.3 \times 10^{-6} \qquad \gamma_{xy} = 1559.2 \times 10^{-6}$

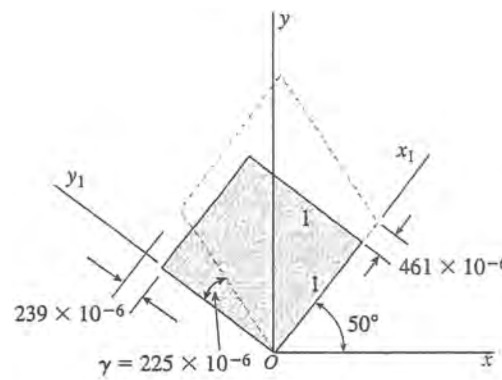
HOOKE'S LAW

$$\sigma_x = \frac{E}{1 - \nu^2}(\varepsilon_x + \nu\varepsilon_y) = 91.6 \text{ MPa} \qquad \leftarrow$$

Problem 7.7-23 Solve Problem 7.7-5 by using Mohr's circle for plane strain.

Solution 7.7-23 Element in plane strain





$\varepsilon_x = 220 \times 10^{-6} \qquad \varepsilon_y = 480 \times 10^{-6}$

$\gamma_{xy} = 180 \times 10^{-6} \qquad \dfrac{\gamma_{xy}}{2} = 90 \times 10^{-6} \qquad \theta = 50°$

$R = \sqrt{(130 \times 10^{-6})^2 + (90 \times 10^{-6})^2}$

$\quad = 158.11 \times 10^{-6}$

$\alpha = \arctan \dfrac{90}{130} = 34.70°$

$\beta = 180° - \alpha - 2\theta = 45.30°$

Point C: $\varepsilon_{x_1} = 350 \times 10^{-6}$

Point D ($\theta = 50°$):

$\varepsilon_{x_1} = 350 \times 10^{-6} + R \cos \beta = 461 \times 10^{-6}$

$\dfrac{\gamma_{x_1y_1}}{2} = R \sin \beta = 112.4 \times 10^{-6}$

$\gamma_{x_1y_1} = 225 \times 10^{-6}$

Point D′ ($\theta = 140°$):

$\varepsilon_{x_1} = 350 \times 10^{-6} - R \cos \beta = 239 \times 10^{-6}$

$\dfrac{\gamma_{x_1y_1}}{2} = -R \sin \beta = -112.4 \times 10^{-6}$

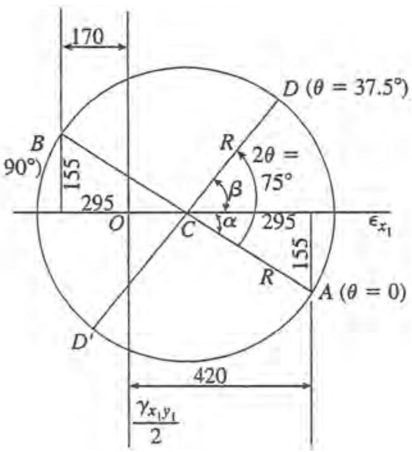
$\gamma_{x_1y_1} = -225 \times 10^{-6}$

Problem 7.7-24 Solve Problem 7.7-6 by using Mohr's circle for plane strain.

Solution 7.7-24 Element in plane strain

$\varepsilon_x = 420 \times 10^{-6}$ $\quad\varepsilon_y = -170 \times 10^{-6}$

$\gamma_{xy} = 310 \times 10^{-6}$ $\quad\dfrac{\gamma_{xy}}{2} = 155 \times 10^{-6}$ $\quad\theta = 37.5°$



$$R = \sqrt{(295 \times 10^{-6})^2 + (155 \times 10^{-6})^2}$$
$$= 333.24 \times 10^{-6}$$

$$\alpha = \arctan\frac{155}{295} = 27.72°$$

$$\beta = 2\theta - \alpha = 47.28°$$

Point C: $\varepsilon_{x_1} = 125 \times 10^{-6}$

Point D ($\theta = 37.5°$):

$$\varepsilon_{x_1} = 125 \times 10^{-6} + R\cos\beta = 351 \times 10^{-6}$$

$$\frac{\gamma_{x_1y_1}}{2} = -R\sin\beta = -244.8 \times 10^{-6}$$

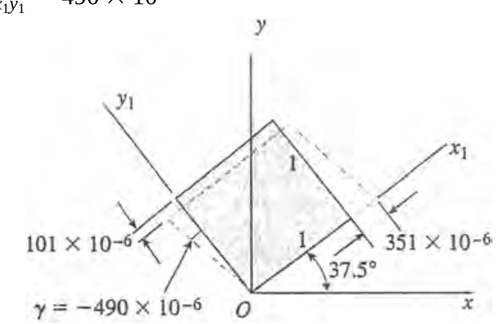
$$\gamma_{x_1y_1} = -490 \times 10^{-6}$$

Point D' ($\theta = 127.5°$):

$$\varepsilon_{x_1} = 125 \times 10^{-6} - R\cos\beta = -101 \times 10^{-6}$$

$$\frac{\gamma_{x_1y_1}}{2} = R\sin\beta = 244.8 \times 10^{-6}$$

$$\gamma_{x_1y_1} = 490 \times 10^{-6}$$

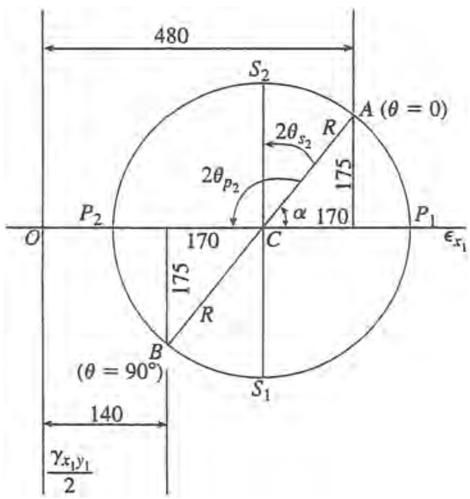


Problem 7.7-25 Solve Problem 7.7-7 by using Mohr's circle for plane strain.

Solution 7.7-25 Element in plane strain

$\varepsilon_x = 480 \times 10^{-6} \qquad \varepsilon_y = 140 \times 10^{-6}$

$\gamma_{xy} = -350 \times 10^{-6} \qquad \dfrac{\gamma_{xy}}{2} = -175 \times 10^{-6}$



$$R = \sqrt{(175 \times 10^{-6})^2 + (170 \times 10^{-6})^2}$$
$$= 243.98 \times 10^{-6}$$

$$\alpha = \arctan \frac{175}{170} = 45.83°$$

POINT C: $\varepsilon_{x_1} = 310 \times 10^{-6}$

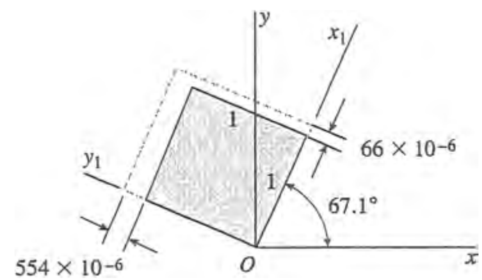
PRINCIPAL STRAINS

$2\theta_{p_2} = 180° - \alpha = 134.2° \qquad \theta_{p_2} = 67.1°$

$2\theta_{p_1} = 2\theta_{p_2} + 180° = 314.2° \qquad \theta_{p_1} = 157.1°$

Point P_1: $\varepsilon_1 = 310 \times 10^{-6} + R = 554 \times 10^{-6}$

Point P_2: $\varepsilon_2 = 310 \times 10^{-6} - R = 66 \times 10^{-6}$



MAXIMUM SHEAR STRAINS

$2\theta_{s_2} = 90° - \alpha = 44.17° \qquad \theta_{s_2} = 22.1°$

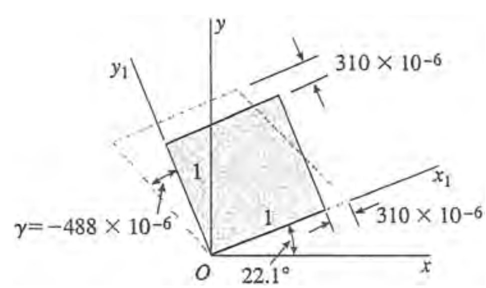
$2\theta_{s_1} = 2\theta_{s_2} + 180° = 224.17° \qquad \theta_{s_1} = 112.1°$

Point S_1: $\varepsilon_{aver} = 310 \times 10^{-6}$

$\gamma_{max} = 2R = 488 \times 10^{-6}$

Point S_2: $\varepsilon_{aver} = 310 \times 10^{-6}$

$\gamma_{min} = -488 \times 10^{-6}$

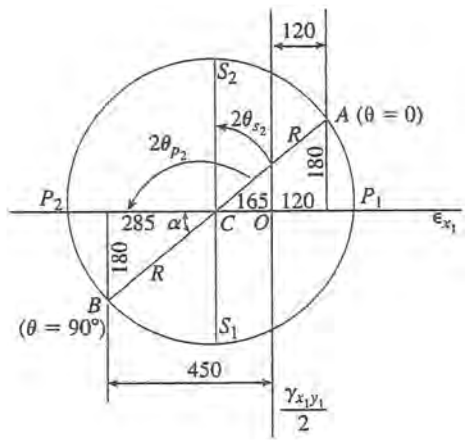


Problem 7.7-26 Solve Problem 7.7-8 by using Mohr's circle for plane strain.

Solution 7.7-26 Element in plane strain

$\varepsilon_x = 120 \times 10^{-6} \qquad \varepsilon_y = -450 \times 10^{-6}$

$\gamma_{xy} = -360 \times 10^{-6} \qquad \dfrac{\gamma_{xy}}{2} = -180 \times 10^{-6}$



$$R = \sqrt{(285 \times 10^{-6})^2 + (180 \times 10^{-6})^2}$$
$$= 337.08 \times 10^{-6}$$

$$\alpha = \arctan \frac{180}{285} = 32.28°$$

Point C: $\varepsilon_{x_1} = -165 \times 10^{-6}$

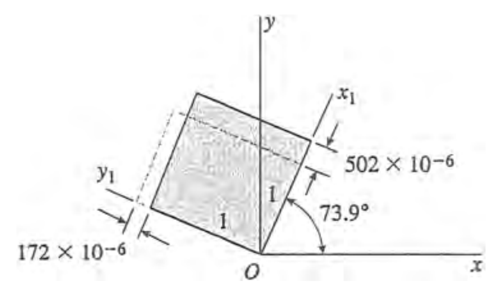
PRINCIPAL STRAINS

$2\theta_{p_2} = 180° - \alpha = 147.72° \qquad \theta_{p_2} = 73.9°$

$2\theta_{p_1} = 2\theta_{p_2} + 180° = 327.72° \qquad \theta_{p_1} = 163.9°$

Point P_1: $\varepsilon_1 = R - 165 \times 10^{-6} = 172 \times 10^{-6}$

Point P_2: $\varepsilon_2 = -165 \times 10^{-6} - R = -502 \times 10^{-6}$



MAXIMUM SHEAR STRAINS

$2\theta_{s_2} = 90° - \alpha = 57.72° \qquad \theta_{s_2} = 28.9°$

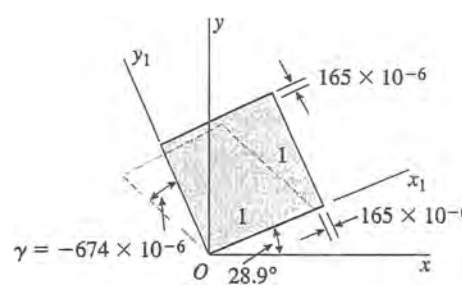
$2\theta_{s_1} = 2\theta_{s_2} + 180° = 237.72° \qquad \theta_{s_1} = 118.9°$

Point S_1: $\varepsilon_{aver} = -165 \times 10^{-6}$

$\gamma_{max} = 2R = 674 \times 10^{-6}$

Point S_2: $\varepsilon_{aver} = -165 \times 10^{-6}$

$\gamma_{min} = -674 \times 10^{-6}$

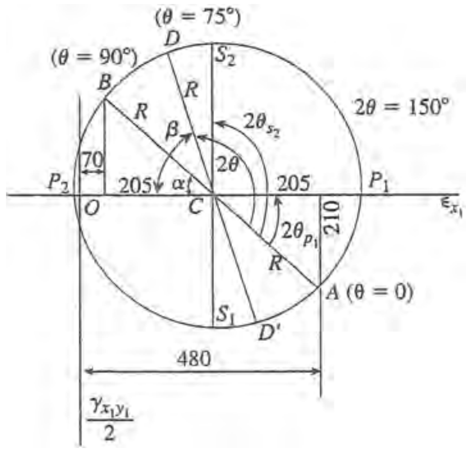


Problem 7.7-27 Solve Problem 7.7-9 by using Mohr's circle for plane strain.

Solution 7.7-27 Element in plane strain

$\varepsilon_x = 480 \times 10^{-6}$ $\quad \varepsilon_y = 70 \times 10^{-6}$

$\gamma_{xy} = 420 \times 10^{-6}$ $\quad \dfrac{\gamma_{xy}}{2} = 210 \times 10^{-6}$ $\quad \theta = 75°$



$R = \sqrt{(205 \times 10^{-6})^2 + (210 \times 10^{-6})^2}$

$= 293.47 \times 10^{-6}$

$\alpha = \arctan \dfrac{210}{205} = 45.69°$

$\beta = \alpha + 180° - 2\theta = 75.69°$

Point C: $\varepsilon_{x_1} = 275 \times 10^{-6}$

Point D ($\theta = 75°$):

$\varepsilon_{x_1} = 275 \times 10^{-6} - R\cos\beta = 202 \times 10^{-6}$

$\dfrac{\gamma_{x_1y_1}}{2} = -R\sin\beta = -284.36 \times 10^{-6}$

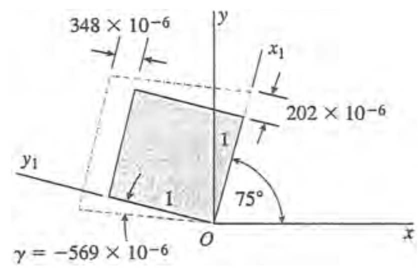
$\gamma_{x_1y_1} = -569 \times 10^{-6}$

Point D' ($\theta = 165°$):

$\varepsilon_{x_1} = 275 \times 10^{-6} + R\cos\beta = 348 \times 10^{-6}$

$\dfrac{\gamma_{x_1y_1}}{2} = R\sin\beta = 284.36 \times 10^{-6}$

$\gamma_{x_1y_1} = 569 \times 10^{-6}$



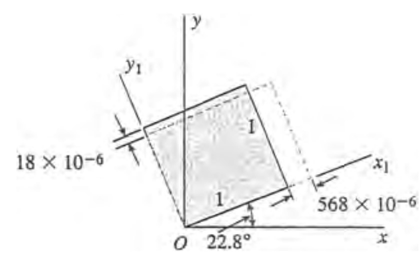
Principal strains

$2\theta_{p_1} = \alpha = 45.69°$ $\quad \theta_{p_1} = 22.8°$

$2\theta_{p_2} = 2\theta_{p_1} + 180° = 225.69°$ $\quad \theta_{p_2} = 112.8°$

Point P_1: $\varepsilon_1 = 275 \times 10^{-6} + R = 568 \times 10^{-6}$

Point P_2: $\varepsilon_2 = 275 \times 10^{-6} - R = -18 \times 10^{-6}$



Maximum shear strains

$2\theta_{s_2} = 90° + \alpha = 135.69°$ $\quad \theta_{s_2} = 67.8°$

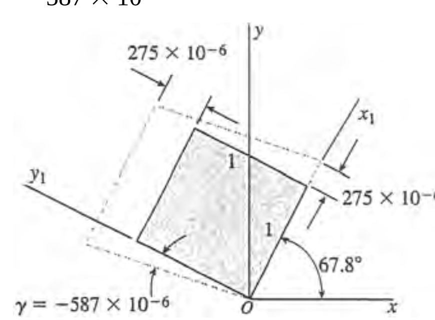
$2\theta_{s_1} = 2\theta_{s_2} + 180° = 315.69°$ $\quad \theta_{s_1} = 157.8°$

Point S_1: $\varepsilon_{aver} = 275 \times 10^{-6}$

$\gamma_{max} = 2R = 587 \times 10^{-6}$

Point S_2: $\varepsilon_{aver} = 275 \times 10^{-6}$

$\gamma_{min} = -587 \times 10^{-6}$

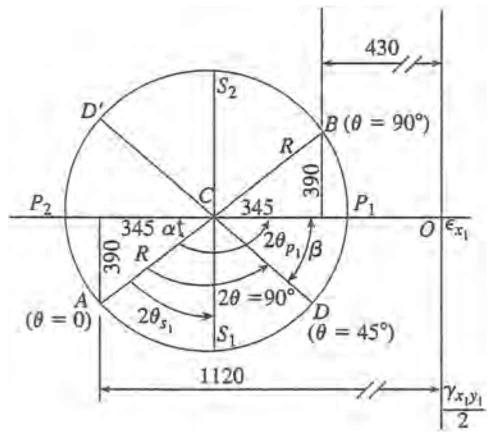


Problem 7.7-28 Solve Problem 7.7-10 by using Mohr's circle for plane strain.

Solution 7.7-28 Element in plane strain

$\varepsilon_x = -1120 \times 10^{-6} \qquad \varepsilon_y = -430 \times 10^{-6}$

$\gamma_{xy} = 780 \times 10^{-6} \qquad \dfrac{\gamma_{xy}}{2} = 390 \times 10^{-6} \qquad \theta = 45°$



$R = \sqrt{(345 \times 10^{-6})^2 + (390 \times 10^{-6})^2}$

$= 520.70 \times 10^{-6}$

$\alpha = \arctan \dfrac{390}{345} = 48.50°$

$\beta = 180° - \alpha - 2\theta = 41.50°$

Point C: $\varepsilon_{x_1} = -775 \times 10^{-6}$

Point D: ($\theta = 45°$):

$\varepsilon_{x_1} = -775 \times 10^{-6} + R\cos\beta = -385 \times 10^{-6}$

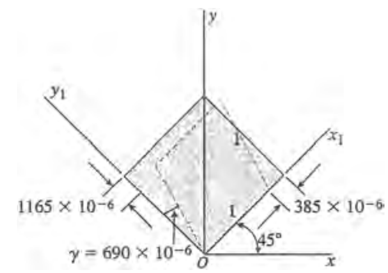
$\dfrac{\gamma_{x_1y_1}}{2} = R\sin\beta = 345 \times 10^{-6} \qquad \gamma_{x_1y_1} = 690 \times 10^{-6}$

Point D′: ($\theta = 135°$)

$\varepsilon_{x_1} = -775 \times 10^{-6} - R\cos\beta = -1165 \times 10^{-6}$

$\dfrac{\gamma_{x_1y_1}}{2} = -R\sin\beta = -345 \times 10^{-6}$

$\gamma_{x_1y_1} = -690 \times 10^{-6}$



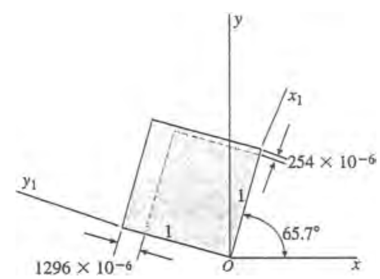
Principal strains

$2\theta_{p_1} = 180° - \alpha = 131.50° \qquad \theta_{p_1} = 65.7°$

$2\theta_{p_2} = 2\theta_{p_1} + 180° = 311.50° \qquad \theta_{p_2} = 155.7°$

Point P_1: $\varepsilon_1 = -775 \times 10^{-6} + R = -254 \times 10^{-6}$

Point P_2: $\varepsilon_2 = -775 \times 10^{-6} - R = -1296 \times 10^{-6}$



Maximum shear strains

$2\theta_{s_1} = 90° - \alpha = 41.50° \qquad \theta_{s_1} = 20.7°$

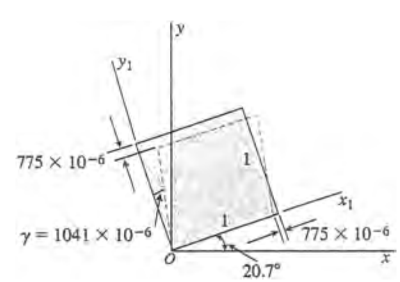
$2\theta_{s_2} = 2\theta_{s_1} + 180° = 221.50° \qquad \theta_{s_2} = 110.7°$

Point S_1: $\varepsilon_{aver} = -775 \times 10^{-6}$

$\gamma_{max} = 2R = 1041 \times 10^{-6}$

Point S_2: $\varepsilon_{aver} = -775 \times 10^{-6}$

$\gamma_{min} = -1041 \times 10^{-6}$



8

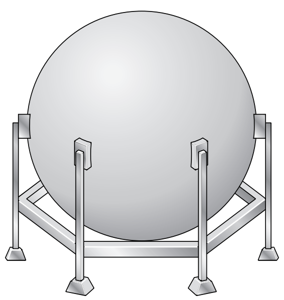
Applications of Plane Stress (Pressure Vessels, Beams, and Combined Loadings)

Spherical Pressure Vessels

When solving the problems for Section 8.2, assume that the given radius or diameter is an inside dimension and that all internal pressures are gage pressures.

Problem 8.2-1 A large spherical tank (see figure) contains gas at a pressure of 450 psi. The tank is 42 ft in diameter and is constructed of high-strength steel having a yield stress in tension of 80 ksi.

Determine the required thickness (to the nearest 1/4 inch) of the wall of the tank if a factor of safety of 3.5 with respect to yielding is required.



Probs. 8.2-1 and 8.2-2

Solution 8.2-1

Radius: $r = \frac{1}{2} 42 \times 12 \qquad r = 252 \text{ in.}$

Internal Pressure: $p = 450$ psi

Yield stress: $\sigma_Y = 80$ ksi (steel)

Factor of safety: $n = 3.5$

MINIMUM WALL THICKNESS $t_{\min}$

From Eq. (8-1): $\sigma_{\max} = \frac{pr}{2t}$ or $\frac{\sigma_Y}{n} = \frac{pr}{2t}$

$t = \frac{prn}{2\sigma_Y} \qquad t = 2.481 \text{ in.}$

to nearest 1/4 inch, $t_{\min} = 2.5$ in. $\leftarrow$

Problem 8.2-2 Solve the preceding problem if the internal pressure is 3.75 MPa, the diameter is 19 m, the yield stress is 570 MPa, and the factor of safety is 3.0.

Determine the required thickness to the nearest millimeter.

Solution 8.2-2

Radius: $r = \frac{1}{2}(19 \text{ m}) \qquad r = 9.5 \times 10^3 \text{ mm}$

Internal Pressure: $p = 3.75$ MPa

Yield stress: $\sigma_Y = 570$ MPa

Factor of safety: $n = 3$

MINIMUM WALL THICKNESS t_{min}

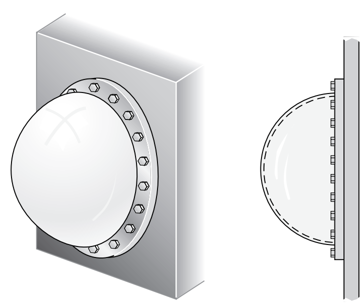
From Eq. (8-1): $\sigma_{max} = \dfrac{pr}{2t}$ or $\dfrac{\sigma_Y}{n} = \dfrac{pr}{2t}$

$t = \dfrac{prn}{2\sigma_Y} \qquad t = 93.8 \text{ mm}$

Use the next higher millimeter $t_{min} = 94 \text{ mm}$ ←

Problem 8.2-3 A hemispherical window (or *viewport*) in a decompression chamber (see figure) is subjected to an internal air pressure of 80 psi. The port is attached to the wall of the chamber by 18 bolts.

Find the tensile force F in each bolt and the tensile stress σ in the viewport if the radius of the hemisphere is 7.0 in. and its thickness is 1.0 in.



Solution 8.2-3 Hemispherical viewport

FREE-BODY DIAGRAM

Radius: $r = 7.0$ in.

Internal pressure: $p = 80$ psi

Wall thickness: $t = 1.0$ in.

18 bolts

T = total tensile force in 18 bolts

$\sum F_{HORIZ} = T - pA = 0 \qquad T = pA = p(\pi r^2)$

F = force in one bolt

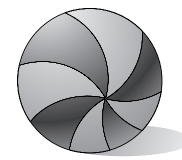
$F = \dfrac{T}{18} = \dfrac{1}{18}(\pi p r^2) = 684 \text{ lb}$ ←

TENSILE STRESS IN VIEWPORT (EQ. 8-1)

$\sigma = \dfrac{pr}{2t} = 280 \text{ psi}$ ←

Problem 8.2-4 A rubber ball (see figure) is inflated to a pressure of 60 kPa. At that pressure the diameter of the ball is 230 mm and the wall thickness is 1.2 mm. The rubber has modulus of elasticity $E = 3.5$ MPa and Poisson's ratio $\nu = 0.45$.

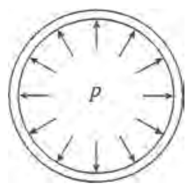
Determine the maximum stress and strain in the ball.



Prob. 8.2-4, 8.2-5

Solution 8.2-4 Rubber ball

CROSS-SECTION



Radius: $r = (230 \text{ mm})/2 = 115 \text{ mm}$

Internal pressure: $p = 60$ kPa

Wall thickness: $t = 1.2$ mm

Modulus of elasticity: $E = 3.5$ MPa (rubber)

Poisson's ratio: $\nu = 0.45$ (rubber)

MAXIMUM STRESS (EQ. 8-1)

$$\sigma_{\max} = \frac{pr}{2t} = \frac{(60 \text{ kPa})(115 \text{ mm})}{2(1.2 \text{ mm})}$$

$$= 2.88 \text{ MPa} \quad \leftarrow$$

MAXIMUM STRAIN (EQ. 8-4)

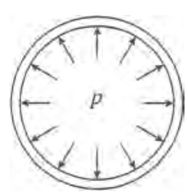
$$\varepsilon_{\max} = \frac{pr}{2tE}(1 - \nu) = \frac{(60 \text{ kPa})(115 \text{ mm})}{2(1.2 \text{ mm})(3.5 \text{ MPa})}(0.55)$$

$$= 0.452 \quad \leftarrow$$

Problem 8.2-5 Solve the preceding problem if the pressure is 9.0 psi, the diameter is 9.0 in., the wall thickness is 0.05 in., the modulus of elasticity is 500 psi, and Poisson's ratio is 0.45.

Solution 8.2-5 Rubber ball

CROSS-SECTION



Radius: $r = \frac{1}{2}(9.0 \text{ in.}) = 4.5 \text{ in.}$

Internal pressure: $p = 9.0$ psi

Wall thickness: $t = 0.05$ in.

Modulus of elasticity: $E = 500$ psi (rubber)

Poisson's ratio: $\nu = 0.45$ (rubber)

MAXIMUM STRESS (EQ. 8-1)

$$\sigma_{\max} = \frac{pr}{2t} = \frac{(9.0 \text{ psi})(4.5 \text{ in.})}{2(0.05 \text{ in.})} = 405 \text{ psi} \quad \leftarrow$$

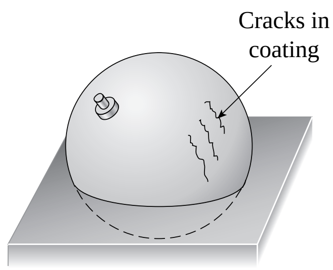
MAXIMUM STRAIN (EQ. 8-4)

$$\varepsilon_{\max} = \frac{pr}{2tE}(1 - \nu) = \frac{(9.0 \text{ psi})(4.5 \text{ in.})}{2(0.05 \text{ in.})(500 \text{ psi})}(0.55)$$

$$= 0.446 \quad \leftarrow$$

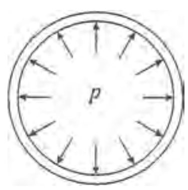
Problem 8.2-6 A spherical steel pressure vessel (diameter 480 mm, thickness 8.0 mm) is coated with brittle lacquer that cracks when the strain reaches 150×10^{-6} (see figure).

What internal pressure p will cause the lacquer to develop cracks? (Assume $E = 205$ GPa and $v = 0.30$.)



Solution 8.2-6 Spherical vessel with brittle coating

CROSS-SECTION



$r = 240$ mm $\qquad E = 205$ GPa (steel)

$t = 8.0$ mm $\qquad v = 0.30$

Cracks occur when $\varepsilon_{\max} = 150 \times 10^{-6}$

From Eq. (8-4): $\varepsilon_{\max} = \dfrac{pr}{2tE}(1 - v)$

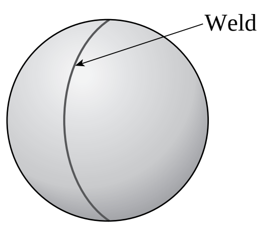
$$\therefore P = \frac{2tE\,\varepsilon_{\max}}{r(1-v)}$$

$$P = \frac{2(8.0 \text{ mm})(205 \text{ GPa})(150 \times 10^{-6})}{(240 \text{ mm})(0.70)}$$

$$= 2.93 \text{ MPa} \qquad \leftarrow$$

Problem 8.2-7 A spherical tank of diameter 48 in. and wall thickness 1.75 in. contains compressed air at a pressure of 2200 psi. The tank is constructed of two hemispheres joined by a welded seam (see figure).

(a) What is the tensile load f (lb per in. of length of weld) carried by the weld?
(b) What is the maximum shear stress $\tau_{\max}$ in the wall of the tank?
(c) What is the maximum normal strain ε in the wall? (For steel, assume $E = 30 \times 10^6$ psi and $v = 0.29$.)



Probs. 8.2-7 and 8.2-8

Solution 8.2-7

$r = 24$ in. $\qquad E = 30 \times 10^6$ psi

$t = 1.75$ in. $\qquad v = 0.29$ (steel)

(a) TENSILE LOAD CARRIED BY WELD

T = Total load $\qquad f$ = load per inch

$T = pA = p\pi r^2 \qquad c$ = Circumference of tank $= 2\pi r$

$$f = \frac{T}{c} = \frac{p(\pi r^2)}{2\pi r} = \frac{pr}{2} = \frac{(2200 \text{ psi})(24 \text{ in.})}{2}$$

$$= 26.4 \text{ k/in.} \qquad \leftarrow$$

(b) MAXIMUM SHEAR STRESS IN WALL (EQ. 8-3)

$$\tau_{\max} = \frac{pr}{4t} = \frac{(2200 \text{ psi})(24 \text{ in.})}{4(1.75 \text{ in.})} = 7543 \text{ psi}$$

(c) MAXIMUM NORMAL STRAIN IN WALL (EQ. 8-4)

$$\varepsilon_{\max} = \frac{pr(1 - v)}{2tE} = \frac{(2200 \text{ psi})(24 \text{ in.})(0.71)}{2(1.75 \text{ in.})(3010^6 \text{ psi})}$$

$$= 3.57 \times 10^{-4} \qquad \leftarrow$$

Problem 8.2-8 Solve the preceding problem for the following data: diameter 1.0 m, thickness 48 mm, pressure 22 MPa, modulus 210 GPa, and Poisson's ratio 0.29.

Solution 8.2-8

$r = 0.5$ m $\quad E = 210$ GPa

$t = 48$ mm $\quad v = 0.29$ (steel)

(a) Tensile load carried by weld

T = Total load $\quad f$ = load per inch

$T = pA = p\pi r^2 \quad c$ = Circumference of tank $= 2\pi r$

$$f = \frac{T}{c} = \frac{p(\pi r^2)}{2\pi r} = \frac{pr}{2} = \frac{(22 \text{ MPa})(0.5 \text{ m})}{2}$$

$$= 5.5 \text{ MN/m} \quad \leftarrow$$

(b) Maximum shear stress in wall (eq. 8-3)

$$\tau_{max} = \frac{pr}{4t} = \frac{(22 \text{ MPa})(0.5 \text{ m})}{4(48 \text{ mm})}$$

$$= 57.3 \text{ MPa} \quad \leftarrow$$

(c) Maximum normal strain in wall (eq. 8-4)

$$\varepsilon_{max} = \frac{pr(1 - v)}{2tE} = \frac{(22 \text{ MPa})(0.5 \text{ m})0.71}{2(48 \text{ mm})(210 \text{ GPa})}$$

$$= 3.87 \times 10^{-4} \quad \leftarrow$$

Problem 8.2-9 A spherical stainless-steel tank having a diameter of 22 in. is used to store propane gas at a pressure of 2450 psi. The properties of the steel are as follows: yield stress in tension, 140,000 psi; yield stress in shear, 65,000 psi; modulus of elasticity, 30×10^6 psi; and Poisson's ratio, 0.28. The desired factor of safety with respect to yielding is 2.8. Also, the normal strain must not exceed 1100×10^{-6}.

Determine the minimum permissible thickness t_{min} of the tank.

Solution 8.2-9

$r = 11$ in. $\quad E = 30 \times 10^6$ psi

$p = 2450$ psi $\quad v = 0.28$ (steel)

$\sigma_Y = 140000$ psi $\quad n = 2.8$

$\tau_Y = 65000$ psi $\quad \varepsilon_{max} = 1100 \times 10^{-6}$

Mimimum wall thickness t

(1) Tension (eq. 8-1) $\quad \sigma_{max} = \dfrac{pr}{2t_1}$

$$t_1 = \frac{pr}{2\sigma_{max}} = \frac{pr}{2\left(\dfrac{\sigma_Y}{n}\right)}$$

$$= \frac{(2450 \text{ psi})(11 \text{ in.})}{2\dfrac{140000 \text{ psi}}{2.8}} = 0.269 \text{ in.}$$

(2) Shear (eq. 8-3) $\quad \tau_{max} = \dfrac{pr}{4t_2}$

$$t_2 = \frac{pr}{4\dfrac{\tau_Y}{n}} = \frac{(2450 \text{ psi})(11 \text{ in.})}{4\dfrac{65000 \text{ psi}}{2.8}} = 0.29 \text{ in.}$$

(3) Strain (eq. 8-4) $\quad \varepsilon_{max} = \dfrac{pr}{2t_3E}(1 - v)$

$$t_3 = \frac{pr}{2\varepsilon_{max}E}(1 - v)$$

$$= \frac{(2450 \text{ psi})(11 \text{ in.})}{2(1100 \times 10^{-6})(30 \times 10^6 \text{ psi})}0.72$$

$$= 0.294 \text{ in.}$$

$t_3 > t_2 > t_1 \quad$ Thus, $t_{min} = 0.294$ in. $\quad \leftarrow$

Problem 8.2-10 Solve the preceding problem if the diameter is 500 mm, the pressure is 18 MPa, the yield stress in tension is 975 MPa, the yield stress in shear is 460 MPa, the factor of safety is 2.5, the modulus of elasticity is 200 GPa, Poisson's ratio is 0.28, and the normal strain must not exceed 1210×10^{-6}.

Solution 8.2-10

$r = 250$ mm $\qquad E = 200$ GPa

$p = 18$ MPa $\qquad v = 0.28$ (steel)

$\sigma_Y = 975$ MPa $\qquad n = 2.5$

$\tau_Y = 460$ MPa $\qquad \varepsilon_{max} = 1210 \times 10^{-6}$

MINIMUM WALL THICKNESS t

(1) TENSION (EQ. 8-1) $\quad \sigma_{max} = \dfrac{pr}{2t_1}$

$$t_1 = \frac{pr}{2\sigma_{max}} = \frac{pr}{2\left(\dfrac{\sigma_Y}{n}\right)}$$

$$= \frac{(18\text{ MPa})(250\text{ mm})}{2\dfrac{975\text{ MPa}}{2.5}} = 5.769\text{ mm}$$

(2) SHEAR (EQ. 8-3) $\quad \tau_{max} = \dfrac{pr}{4t_2}$

$$t_2 = \frac{pr}{4\dfrac{\tau_Y}{n}} = \frac{(18\text{ MPa})(250\text{ mm})}{4\dfrac{460\text{ MPa}}{2.5}} = 6.114\text{ mm}$$

(3) STRAIN (EQ. 8-4) $\quad \varepsilon_{max} = \dfrac{pr}{2t_3E}(1 - v)$

$$t_3 = \frac{pr}{2\varepsilon_{max}E}(1 - v)$$

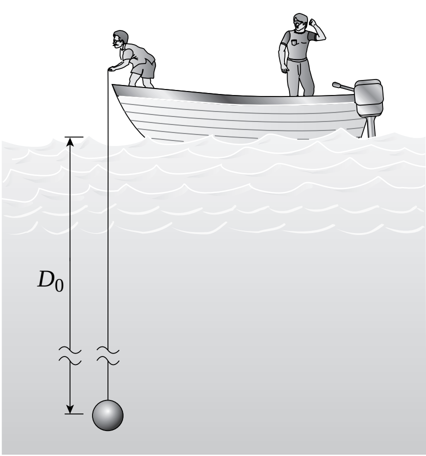
$$= \frac{(18\text{ MPa})(250\text{ mm})}{2(1210 \times 10^{-6})(200\text{ GPa})}0.72$$

$$= 6.694\text{ mm}$$

$t_3 > t_2 > t_1 \qquad$ Thus, $t_{min} = 6.69$ mm $\quad \longleftarrow$

Problem 8.2-11 A hollow pressurized sphere having radius $r = 4.8$ in. and wall thickness $t = 0.4$ in. is lowered into a lake (see figure). The compressed air in the tank is at a pressure of 24 psi (gage pressure when the tank is out of the water).

At what depth D_0 will the wall of the tank be subjected to a compressive stress of 90 psi?



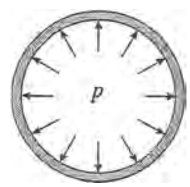
Solution 8.2-11 Pressurized sphere under water

CROSS-SECTION

$r = 4.8$ in. $\quad p_1 = 24$ psi

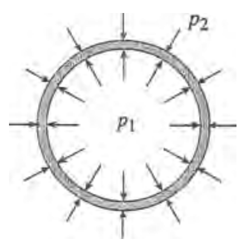
$t = 0.4$ in. $\quad \gamma =$ density of water $= 62.4$ lb/ft^3

(1) IN AIR: $p_1 = 24$ psi



(1) IN AIR

(2) UNDER WATER: $p_1 = 24$ psi



(2) UNDER WATER

D_0 = depth of water (in.)

$$p_2 = \gamma D_0 = \left(\frac{62.4 \text{ lb/ft}^3}{1728 \text{ in.}^3/\text{ ft}^3}\right) D_0 = 0.036111\, D_0 \text{ (psi)}$$

Compressive stress in tank wall equals 90 psi. (Note: σ is positive in tension.)

$$\sigma = \frac{pr}{2t} = \frac{(p_1 - p_2)r}{2t} \qquad \sigma = -90 \text{ psi}$$

$$-90 \text{ psi} = \frac{(24 \text{ psi} - 0.03611\, D_0)(4.8 \text{ in.})}{2(0.4 \text{ in.})}$$

$$= 144 - 0.21667\, D_0$$

$$\text{Solve for } D_0\text{: } D_0 = \frac{234}{0.21667}$$

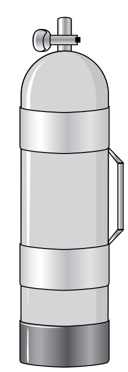
$$= 1080 \text{ in.} = 90 \text{ ft} \qquad \leftarrow$$

Cylindrical Pressure Vessels

When solving the problems for Section 8.3, assume that the given radius or diameter is an inside dimension and that all internal pressures are gage pressures.

Problem 8.3-1 A scuba tank (see figure) is being designed for an internal pressure of 1600 psi with a factor of safety of 2.0 with respect to yielding. The yield stress of the steel is 35,000 psi in tension and 16,000 psi in shear.

If the diameter of the tank is 7.0 in., what is the minimum required wall thickness?



Solution 8.3-1 Scuba tank

Cylindrical pressure vessel

$p = 1600\ \text{psi} \qquad n = 2.0 \qquad d = 7.0\ \text{in.}$

$r = 3.5\ \text{in.} \qquad \sigma_Y = 35{,}000\ \text{psi} \qquad \tau_Y = 16{,}000\ \text{psi}$

$$\sigma_{\text{allow}} = \frac{\sigma_Y}{n} = 17{,}500\ \text{psi} \qquad \tau_{\text{allow}} = \frac{\tau_Y}{n} = 8{,}000\ \text{psi}$$

Find required wall thickness t.

(1) BASED ON TENSION (EQ. 8-5) $\sigma_{\max} = \dfrac{pr}{t}$

$$t_1 = \frac{pr}{\sigma_{\text{allow}}} = \frac{(1600\ \text{psi})(3.5\ \text{in.})}{17{,}500\ \text{psi}} = 0.320\ \text{in.}$$

(2) BASED ON SHEAR (EQ. 8-10) $\tau_{\max} = \dfrac{pr}{2t}$

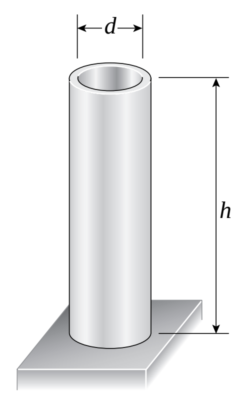
$$t_2 = \frac{pr}{2\tau_{\text{allow}}} = \frac{(1600\ \text{psi})(3.5\ \text{in.})}{2(8{,}000\ \text{psi})} = 0.350\ \text{in.}$$

Shear governs since $t_2 > t_1$

$\therefore t_{\min} = 0.350\ \text{in.} \quad \longleftarrow$

Problem 8.3-2 A tall standpipe with an open top (see figure) has diameter $d = 2.2$ m and wall thickness $t = 20$ mm.

(a) What height h of water will produce a circumferential stress of 12 MPa in the wall of the standpipe?
(b) What is the axial stress in the wall of the tank due to the water pressure?



Solution 8.3-2

$d = 2.2\ \text{m} \qquad r = 1.1\ \text{m} \qquad t = 20\ \text{mm}$

weight density of water $\gamma = 9.81\ \text{kN/m}^3$.

height of water h

water pressure $p = \gamma h$

(a) HEIGHT OF WATER

$$\sigma_1 = \frac{pr}{t} = 12\ \text{MPa} = \frac{0.00981h(1.1\ \text{m})}{20\ \text{mm}}$$

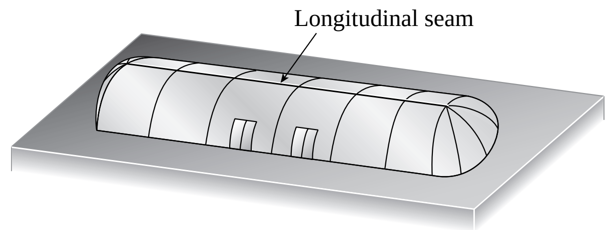
$$h = \frac{12(20)}{0.00981(1.1)} = 22.2\ \text{m}$$

(b) AXIAL STRESS IN THE WALL DUE TO WATER PRESSURE ALONE

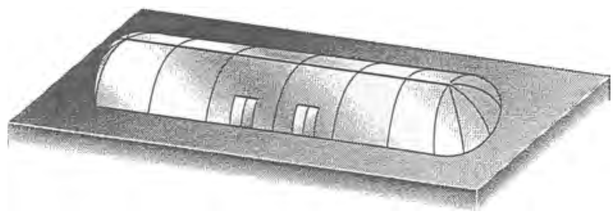
Because the top of the tank is open, the internal pressure of the water produces no axial (longitudinal) stresses in the wall of the tank. Axial stress equals *zero*. $\longleftarrow$

Problem 8.3-3 An inflatable structure used by a traveling circus has the shape of a half-circular cylinder with closed ends (see figure). The fabric and plastic structure is inflated by a small blower and has a radius of 40 ft when fully inflated. A longitudinal seam runs the entire length of the "ridge" of the structure.

If the longitudinal seam along the ridge tears open when it is subjected to a tensile load of 540 pounds per inch of seam, what is the factor of safety n against tearing when the internal pressure is 0.5 psi and the structure is fully inflated?



Solution 8.3-3 Inflatable structure



Half-circular cylinder
$r = 40$ ft $= 480$ in.
Internal pressure $p = 0.5$ psi

T = tensile force per unit length of longitudinal seam
Seam tears when $T = T_{max} = 540$ lb/in.
Find factor of safety against tearing.

CIRCUMFERENTIAL STRESS (EQ. 8-5)

$\sigma_1 = \dfrac{pr}{t}$ where t = thickness of fabric

Actual value of T due to internal pressure $= \sigma_1 t$

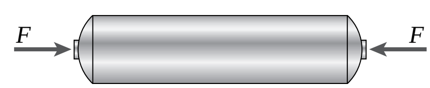
$\therefore T = \sigma_1 t = pr = (0.5 \text{ psi})(480 \text{ in.}) = 240$ lb/in.

FACTOR OF SAFETY

$$n = \frac{T_{max}}{T} = \frac{540 \text{ lb/in.}}{240 \text{ lb/in.}} = 2.25 \quad \leftarrow$$

Problem 8.3-4 A thin-walled cylindrical pressure vessel of radius r is subjected simultaneously to internal gas pressure p and a compressive force F acting at the ends (see figure).

What should be the magnitude of the force F in order to produce pure shear in the wall of the cylinder?



Solution 8.3-4 Cylindrical pressure vessel

r = Radius

p = Internal pressure

STRESSES (SEE EQ. 8-5 AND 8-6):

$$\sigma_1 = \frac{pr}{t}$$

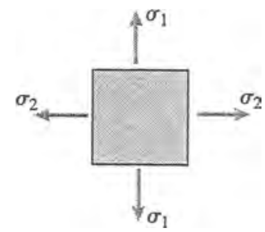
$$\sigma_2 = \frac{pr}{2t} - \frac{F}{A} = \frac{pr}{2t} - \frac{F}{2\pi rt}$$

FOR PURE SHEAR, the stresses σ_1 and σ_2 must be equal in magnitude and opposite in sign (see, e.g., Fig. 7-11 in Section 7.3).

$\therefore \sigma_1 = -\sigma_2$

OR $\quad \dfrac{pr}{t} = -\left(\dfrac{pr}{2t} - \dfrac{F}{2\pi rt}\right)$

Solve for F: $F = 3\pi pr^2 \quad \leftarrow$



Problem 8.3-5 A strain gage is installed in the longitudinal direction on the surface of an aluminum beverage can (see figure). The radius-to-thickness ratio of the can is 200. When the lid of the can is popped open, the strain changes by $\epsilon_0 = 170 \times 10^{-6}$.

What was the internal pressure p in the can? (Assume $E = 10 \times 10^6$ psi and $v = 0.33$.)

Solution 8.3-5 Aluminum can

$\dfrac{r}{t} = 200 \qquad E = 10 \times 10^6 \text{ psi} \qquad v = 0.33$

ε_0 = change in strain when pressure is released

$= 170 \times 10^{-6}$

Find internal pressure p.

STRAIN IN LONGITUDINAL DIRECTION (EQ. 8-11a)

$$\varepsilon_2 = \frac{pr}{2tE}(1 - 2v) \quad \text{or} \quad p = \frac{2tE\varepsilon_2}{r(1 - 2v)}$$

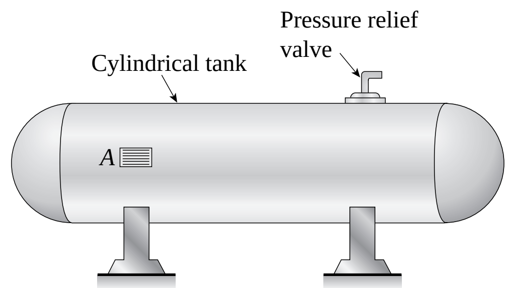
$$\varepsilon_2 = \varepsilon_0 \qquad \therefore \qquad p = \frac{2tE\varepsilon_0}{(r)(1 - 2v)} = \frac{2E\varepsilon_0}{(r/t)\,(1 - 2v)}$$

Substitute numerical values:

$$p = \frac{2(10 \times 10^6 \text{ psi})(170 \times 10^{-6})}{(200)(1 - 0.66)} = 50 \text{ psi} \qquad \leftarrow$$

Problem 8.3-6 A circular cylindrical steel tank (see figure) contains a volatile fuel under pressure. A strain gage at point A records the longitudinal strain in the tank and transmits this information to a control room. The ultimate shear stress in the wall of the tank is 84 MPa, and a factor of safety of 2.5 is required.

At what value of the strain should the operators take action to reduce the pressure in the tank? (Data for the steel are as follows: modulus of elasticity $E = 205$ GPa and Poisson's ratio $\nu = 0.30$.)



Solution 8.3-6

$\tau_{ULT} = 84$ MPa $\quad E = 205$ GPa $\quad \nu = 0.3$

$n = 2.5 \quad \tau_{max} = \dfrac{\tau_{ULT}}{n} \quad \tau_{max} = 33.6$ MPa

Find maximum allowable strain reading at the gage

$$\sigma_1 = \frac{pr}{t} \qquad \sigma_2 = \frac{pr}{2t}$$

From Eq. (8-10)

$$\tau_{max} = \frac{\sigma_1}{2} = \frac{pr}{2t} \qquad P_{max} = \frac{2t\tau_{max}}{r}$$

From Eq. (8-11a)

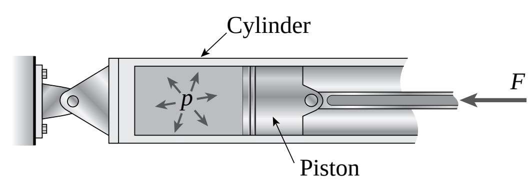
$$\varepsilon_2 = \frac{pr}{2tE}(1 - 2\nu)$$

$$\varepsilon_{2max} = \frac{p_{max}r}{2tE}(1 - 2\nu) = \frac{\tau_{max}}{E}(1 - 2\nu)$$

$$\varepsilon_{max} = \frac{\tau_{max}}{E}(1 - 2\nu) \qquad \varepsilon_{max} = 6.56 \times 10^{-5}$$

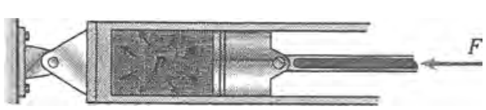
Problem 8.3-7 A cylinder filled with oil is under pressure from a piston, as shown in the figure. The diameter d of the piston is 1.80 in. and the compressive force F is 3500 lb. The maximum allowable shear stress τ_{allow} in the wall of the cylinder is 5500 psi.

What is the minimum permissible thickness t_{min} of the cylinder wall? (See the figure on the next page.)



Probs. 8.3-7 and 8.3-8

Solution 8.3-7 Cylinder with internal pressure



$d = 1.80$ in. $\quad r = 0.90$ in.

$F = 3500$ lb $\quad \tau_{allow} = 5500$ psi

Find minimum thickness t_{min}.

Pressure in cylinder: $p = \dfrac{F}{A} = \dfrac{F}{\pi r^2}$

Maximum shear stress (Eq. 8-10):

$$\tau_{max} = \frac{pr}{2t} = \frac{F}{2\pi rt}$$

Minimum thickness:

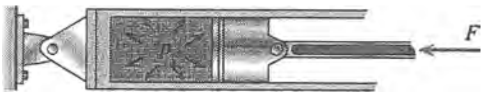
$$t_{min} = \frac{F}{2\pi r\tau_{allow}}$$

Substitute numerical values:

$$t_{min} = \frac{3500 \text{ lb}}{2\pi(0.90 \text{ in.})(5500 \text{ psi})} = 0.113 \text{ in.} \qquad \leftarrow$$

Problem 8.3-8 Solve the preceding problem if $d = 90$ mm, $F = 42$ kN, and $\tau_{allow} = 40$ MPa.

Solution 8.3-8 Cylinder with internal pressure



$d = 90$ mm $\quad r = 45$ mm

$F = 42.0$ kN $\quad \tau_{allow} = 40$ MPa

Find minimum thickness t_{min}.

Pressure in cylinder: $p = \dfrac{F}{A} = \dfrac{F}{\pi r^2}$

Maximum shear stress (Eq. 8-10):

$$\tau_{max} = \frac{pr}{2t} = \frac{F}{2\pi rt}$$

Minimum thickness:

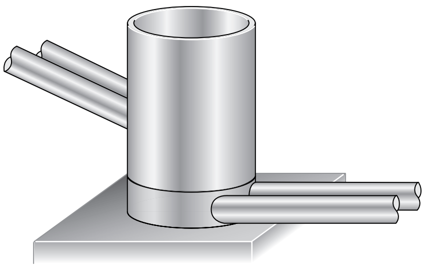
$$t_{min} = \frac{F}{2\pi r \tau_{allow}}$$

Substitute numerical values:

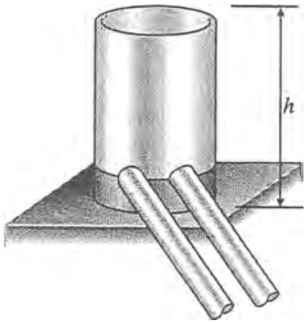
$$t_{min} = \frac{42.0 \text{ kN}}{2\pi(45 \text{ mm})(40 \text{ MPa})} = 3.71 \text{ mm} \quad \leftarrow$$

Problem 8.3-9 A standpipe in a water-supply system (see figure) is 12 ft in diameter and 6 inches thick. Two horizontal pipes carry water out of the standpipe; each is 2 ft in diameter and 1 inch thick. When the system is shut down and water fills the pipes but is not moving, the hoop stress at the bottom of the standpipe is 130 psi.

(a) What is the height h of the water in the standpipe?
(b) If the bottoms of the pipes are at the same elevation as the bottom of the standpipe, what is the hoop stress in the pipes?



Solution 8.3-9 Vertical standpipe



$d = 12$ ft $= 144$ in. $\quad r = 72$ in. $\quad t = 6$ in.

$$\gamma = 62.4 \text{ lb/ft}^3 = \frac{62.4}{1728} \text{ lb/in.}^3$$

σ_1 = hoop stress at bottom of standpipe = 130 psi

(a) Find height h of water in the standpipe

p = pressure at bottom of standpipe = γh

From Eq. (8-5): $\sigma_1 = \dfrac{pr}{t} = \dfrac{\gamma hr}{t} \quad$ or $\quad h = \dfrac{\sigma_1 t}{\gamma r}$

Substitute numerical values:

$$h = \frac{(130 \text{ psi})(6 \text{ in.})}{\left(\frac{62.4}{1728} \text{ lb/in.}^3\right)(72 \text{ in.})} = 300 \text{ in.}$$

$$= 25 \text{ ft} \quad \leftarrow$$

HORIZONTAL PIPES

$d_1 = 2 \text{ ft} = 24 \text{ in.} \qquad r_1 = 12 \text{ in.} \qquad t_1 = 1.0 \text{ in.}$

(b) FIND HOOP STRESS σ_1 IN THE PIPES

Since the pipes are 2 ft in diameter, the depth of water to the center of the pipes is about 24 ft.

$h_1 \approx 24 \text{ ft} = 288 \text{ in.} \qquad p_1 = \gamma h_1$

$$\sigma_1 = \frac{p_1 r_1}{t_1} = \frac{\gamma h_1 r_1}{t_1}$$

$$= \frac{\left(\dfrac{62.4}{1728} \text{ lb/in.}^3\right)(288 \text{ in.})(12 \text{ in.})}{1.0 \text{ in.}}$$

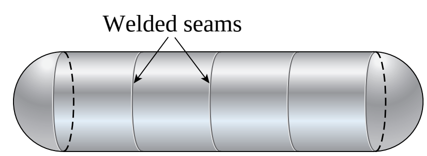
$$= 125 \text{ psi}$$

Based on the average pressure in the pipes:

$\sigma_1 \approx 125 \text{ psi} \qquad \leftarrow$

Problem 8.3-10 A cylindrical tank with hemispherical heads is constructed of steel sections that are welded circumferentially (see figure). The tank diameter is 1.25 m, the wall thickness is 22 mm, and the internal pressure is 1750 kPa.

(a) Determine the maximum tensile stress σ_h in the heads of the tank.
(b) Determine the maximum tensile stress σ_c in the cylindrical part of the tank.
(c) Determine the tensile stress σ_w acting perpendicular to the welded joints.
(d) Determine the maximum shear stress τ_h in the heads of the tank.
(e) Determine the maximum shear stress τ_c in the cylindrical part of the tank.



Probs. 8.3-10 and 8.3-11

Solution 8.3-10

$d = 1.25 \text{ m} \qquad r = \dfrac{d}{2} \qquad t = 22 \text{ mm} \qquad p = 1750 \text{ kPa}$

(a) MAXIMUM TENSILE STRESS IN HEMISPHERES (EQ. 8-1)

$$\sigma_h = \frac{pr}{2t} \qquad \sigma_h = 24.9 \text{ MPa} \qquad \leftarrow$$

(b) MAXIMUM STRESS IN CYLINDER (EQ. 8-5)

$$\sigma_c = \frac{pr}{t} \qquad \sigma_c = 49.7 \text{ MPa} \qquad \leftarrow$$

(c) TENSILE STRESS IN WELDS (EQ. 8-6)

$$\sigma_w = \frac{pr}{2t} \qquad \sigma_w = 24.9 \text{ MPa} \qquad \leftarrow$$

(d) MAXIMUM SHEAR STRESS IN HEMISPHERES (EQ. 8-3)

$$\tau_h = \frac{pr}{4t} \qquad \tau_h = 12.43 \text{ MPa} \qquad \leftarrow$$

(e) MAXIMUM SHEAR STRESS IN CYLINDER (EQ. 8-10)

$$\tau_c = \frac{pr}{2t} \qquad \tau_c = 24.9 \text{ MPa} \qquad \leftarrow$$

Problem 8.3-11 A cylindrical tank with diameter $d = 18$ in. is subjected to internal gas pressure $p = 450$ psi. The tank is constructed of steel sections that are welded circumferentially (see figure). The heads of the tank are hemispherical. The allowable tensile and shear stresses are 8200 psi and 3000 psi, respectively. Also, the allowable tensile stress perpendicular to a weld is 6250 psi.

Determine the minimum required thickness t_{min} of (a) the cylindrical part of the tank and (b) the hemispherical heads.

Solution 8.3-11

$d = 18$ in. $\quad r = \dfrac{d}{2} \quad p = 450$ psi

$\sigma_{allow} = 8200$ psi (tension)

$\tau_{allow} = 3000$ psi (shear)

WELD $\quad \sigma_a = 6250$ psi (tension)

(a) FIND MINIMUM THICKNESS OF CYLINDER

TENSION $\quad \sigma_{max} = \dfrac{pr}{t}$

$$t_{min} = \frac{pr}{\sigma_{allow}} \qquad t_{min} = 0.494 \text{ in.}$$

SHEAR $\quad \tau_{max} = \dfrac{pr}{2t}$

$$t_{min} = \frac{pr}{2\tau_{allow}} \qquad t_{min} = 0.675 \text{ in.}$$

WELD $\quad \sigma = \dfrac{pr}{2t}$

$$t_{min} = \frac{pr}{2\sigma_a} \qquad t_{min} = 0.324 \text{ in.}$$

$$t_{min} = 0.675 \text{ in.} \quad \leftarrow$$

(b) FIND MINIMUM THICKNESS OF HEMISPHERES

TENSION $\quad \sigma_{max} = \dfrac{pr}{2t}$

$$t_{min} = \frac{pr}{2\sigma_{allow}} \qquad t_{min} = 0.247 \text{ in.}$$

SHEAR $\quad \tau_{max} = \dfrac{pr}{4t}$

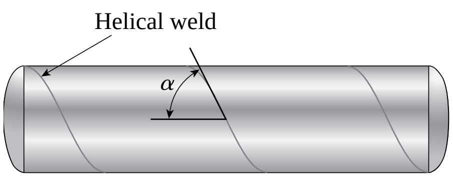
$$t_{min} = \frac{pr}{4\tau_{allow}} \qquad t_{min} = 0.338 \text{ in.}$$

$$t_{min} = 0.338 \text{ in.} \quad \leftarrow$$

Problem *8.3-12 A pressurized steel tank is constructed with a helical weld that makes an angle $\alpha = 55°$ with the longitudinal axis (see figure). The tank has radius $r = 0.6$ m, wall thickness $t = 18$ mm, and internal pressure $p = 2.8$ MPa. Also, the steel has modulus of elasticity $E = 200$ GPa and Poisson's ratio $v = 0.30$.

Determine the following quantities for the cylindrical part of the tank.

(a) The circumferential and longitudinal stresses.
(b) The maximum in-plane and out-of-plane shear stresses.
(c) The circumferential and longitudinal strains.
(d) The normal and shear stresses acting on planes parallel and perpendicular to the weld (show these stresses on a properly oriented stress element).



Probs. 8.3-12 and 8.3-13

Solution 8.3-12

$\alpha = 55$ deg $\quad r = 0.6$ m $\quad t = 18$ mm

$p = 2.8$ MPa $\quad E = 200$ GPa $\quad v = 0.3$

(a) CIRCUMFERENTIAL STRESS

$$\sigma_1 = \frac{pr}{t} \qquad \sigma_1 = 93.3 \text{ MPa} \quad \leftarrow$$

LONGITUDIAL STRESS

$$\sigma_2 = \frac{pr}{2t} \qquad \sigma_2 = 46.7 \text{ MPa} \quad \leftarrow$$

(b) IN-PLANE SHEAR STRESS

$$\tau_1 = \frac{\sigma_1 - \sigma_2}{2} \qquad \tau_1 = 23.3 \text{ MPa} \quad \leftarrow$$

OUT-OF-PLANE SHEAR STRESS

$$\tau_2 = \frac{\sigma_1}{2} \qquad \tau_2 = 46.7 \text{ MPa} \quad \leftarrow$$

(c) Circumferential strain

$$\varepsilon_1 = \frac{\sigma_1}{2E}(2 - v) \qquad \varepsilon_1 = 3.97 \times 10^{-4} \quad \leftarrow$$

Longitudinal strain

$$\varepsilon_2 = \frac{\sigma_2}{E}(1 - 2v) \qquad \varepsilon_2 = 9.33 \times 10^{-5} \quad \leftarrow$$

(d) Stress on weld

$\theta = 90\text{ deg} - \alpha \qquad \theta = 35\text{ deg}$

$\sigma_x = \sigma_2 \qquad \sigma_x = 46.667\text{ MPa} \qquad \sigma_y = \sigma_1$

$\sigma_y = 93.333\text{ MPa} \qquad \tau_{xy} = 0$

For $\theta = 35$ deg

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$\sigma_{x1} = 62.0\text{ MPa} \quad \leftarrow$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

$\tau_{x1y1} = 21.9\text{ MPa} \quad \leftarrow$

$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1} \qquad \sigma_{y1} = 78.0\text{ MPa} \quad \leftarrow$

Problem *8.3-13 Solve the preceding problem for a welded tank with $\alpha = 62°$, $r = 19$ in., $t = 0.65$ in., $p = 240$ psi, $E = 30 \times 10^6$ psi, and $v = 0.30$.

Solution 8.3-13

$\alpha = 62\text{ deg} \qquad r = 19\text{ in.} \qquad t = 0.65\text{ in.}$

$p = 240\text{ psi} \qquad E = 30 \times 10^6\text{ psi} \qquad v = 0.3$

(a) Circumferential stress

$$\sigma_1 = \frac{pr}{t} \qquad \sigma_1 = 7015\text{ psi} \quad \leftarrow$$

Longitudial stress

$$\sigma_2 = \frac{pr}{2t} \qquad \sigma_2 = 3508\text{ psi} \quad \leftarrow$$

(b) In-plane shear stress

$$\tau_1 = \frac{\sigma_1 - \sigma_2}{2} \qquad \tau_1 = 1754\text{ psi} \quad \leftarrow$$

Out-of-plane shear stress

$$\tau_2 = \frac{\sigma_1}{2} \qquad \tau_2 = 3508\text{ psi} \quad \leftarrow$$

(c) Circumferential strain

$$\varepsilon_1 = \frac{\sigma_1}{2E}(2 - v) \qquad \varepsilon_1 = 1.988 \times 10^{-4} \quad \leftarrow$$

Longitudinal strain

$$\varepsilon_2 = \frac{\sigma_2}{E}(1 - 2v) \qquad \varepsilon_2 = 4.68 \times 10^{-5} \quad \leftarrow$$

(d) Stress on weld

$\theta = 90\text{ deg} - \alpha \qquad \theta = 28\text{ deg}$

$\sigma_x = \sigma_2 \qquad \sigma_x = 3.508 \times 10^3\text{ psi} \qquad \sigma_y = \sigma_1$

$\sigma_y = 7.015 \times 10^3\text{ psi} \qquad \tau_{xy} = 0$

For $\theta = 28$ deg

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$\sigma_{x1} = 4281\text{ psi} \quad \leftarrow$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

$\tau_{x1y1} = 1454\text{ psi} \quad \leftarrow$

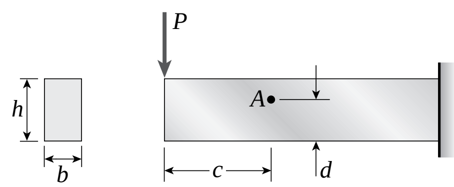
$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1} \qquad \sigma_{y1} = 6242\text{ psi} \quad \leftarrow$

Maximum Stresses in Beams

When solving the problems for Section 8.4, consider only the in-plane stresses and disregard the weights of the beams

Problem 8.4-1 A cantilever beam of rectangular cross section is subjected to a concentrated load $P = 17$ k acting at the free end (see figure). The beam has width $b = 3$ in. and height $h = 12$ in. Point A is located at distance $c = 2.5$ ft from the free end and distance $d = 9$ in. from the bottom of the beam.

Calculate the principal stresses σ_1 and σ_2 and the maximum shear stress τ_{max} at point A. Show these stresses on sketches of properly oriented elements.



Probs. 8.4-1 and 8.4-2

Solution 8.4-1

$P = 17$ k $\quad c = 2.5$ ft $\quad b = 3$ in. $\quad d = 9$ in.

$h = 12$ in. $\quad v = 0.3$

STRESS AT POINT A

$$I = \frac{bh^3}{12} \qquad I = 432 \text{ in.}^4$$

$$M = -Pc \qquad M = -5.1 \times 10^5 \text{ lb-in.}$$

$$V = P \qquad V = 1.7 \times 10^4 \text{ lb}$$

$$y_A = -\frac{h}{2} + d \qquad y_A = 3 \text{ in.}$$

$$\sigma_x = -\frac{My_A}{I} \qquad \sigma_x = 3.542 \times 10^3 \text{ psi}$$

$$Q = bd\left(\frac{h}{2} - \frac{d}{2}\right) \qquad Q = 40.5 \text{ in.}^3$$

$$\tau = \frac{VQ}{Ib} \qquad \tau = 531.25 \text{ psi} \qquad \tau_{xy} = \tau$$

$$\sigma_x = 3.542 \times 10^3 \text{ psi} \qquad \sigma_y = 0 \qquad \tau_{xy} = 531.25 \text{ psi}$$

PRINCIPAL STRESSES

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = 0.3$$

$$\theta_p = \frac{1}{2}\,\mathrm{atan}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right) \qquad \theta_p = 8.35 \text{ deg}$$

For $\theta_1 = \theta_p \qquad \theta_1 = 8.35$ deg

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_1) + \tau_{xy}\sin(2\theta_1)$$

$$\sigma_{x1} = 60.306 \text{ Mpa}$$

For $\theta_2 = 90 \text{ deg} + \theta_p \qquad \theta_2 = 98.35$ deg

$$\sigma_{x2} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_2) + \tau_{xy}\sin(2\theta_2)$$

$$\sigma_{x2} = -77.971 \text{ psi}$$

Therefore

$$\sigma_1 = \sigma_{x2} \qquad \theta_{p1} = \theta_2 \qquad \sigma_2 = \sigma_{x1} \qquad \theta_{p2} = \theta_1$$

$$\sigma_1 = -78.0 \text{ psi} \quad \longleftarrow \qquad \theta_{p1} = 98.4 \text{ deg} \quad \longleftarrow$$

$$\sigma_2 = 3620 \text{ psi} \quad \longleftarrow \qquad \theta_{p2} = 8.35 \text{ deg} \quad \longleftarrow$$

MAXIMUM SHEAR STRESSES

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{max} = 1849 \text{ psi} \quad \longleftarrow$$

$$\theta_{s1} = \theta_{p1} - 45 \text{ deg} \qquad \theta_{s1} = 53.4 \text{ deg} \quad \longleftarrow$$

$$\theta_{s2} = \theta_{s1} + 90 \text{ deg} \qquad \theta_{s2} = 143.4 \text{ deg} \quad \longleftarrow$$

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{avg} = 1771 \text{ psi} \quad \longleftarrow$$

Problem 8.4-2 Solve the preceding problem for the following data: $P = 130$ kN, $b = 80$ mm, $h = 260$ mm, $c = 0.6$ m, and $d = 220$ mm.

Solution 8.4-2

$P = 130\text{ kN} \qquad c = 0.6\text{ m} \qquad b = 80\text{ mm}$

$d = 220\text{ mm} \qquad h = 260\text{ mm} \qquad v = 0.3$

STRESS AT POINT A

$$I = \frac{bh^3}{12} \qquad I = 1.172 \times 10^8\text{ mm}^4$$

$$M = -Pc \qquad M = -7.8 \times 10^4\text{ N}\cdot\text{m}$$

$$V = P \qquad V = 1.3 \times 10^5\text{ N}$$

$$y_A = -\frac{h}{2} + d \qquad y_A = 90\text{ mm}$$

$$\sigma_x = -\frac{My_A}{I} \qquad \sigma_x = 59.911\text{ MPa}$$

$$Q = bd\left(\frac{h}{2} - \frac{d}{2}\right) \qquad Q = 3.52 \times 10^5\text{ mm}^3$$

$$\tau = \frac{VQ}{Ib} \qquad \tau = 4.882\text{ MPa} \qquad \tau_{xy} = \tau$$

$$\sigma_x = 59.911\text{ MPa} \qquad \sigma_y = 0 \qquad \tau_{xy} = 4.882\text{ MPa}$$

PRINCIPAL STRESSES

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = 0.163$$

$$\theta_p = \frac{1}{2}\,\text{atan}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right) \qquad \theta_p = 4.628\text{ deg}$$

For $\theta_1 = \theta_p \qquad \theta_1 = 4.628$ deg

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_1) + \tau_{xy}\sin(2\theta_1)$$

$$\sigma_{x1} = 60.306\text{ MPa}$$

For $\theta_2 = 90\text{ deg} + \theta_p \qquad \theta_2 = 94.628$ deg

$$\sigma_{x2} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_2) + \tau_{xy}\sin(2\theta_2)$$

$$\sigma_{x2} = -0.395\text{ MPa}$$

Therefore

$$\sigma_1 = \sigma_{x1} \qquad \theta_{p1} = \theta_1 \qquad \sigma_2 = \sigma_{x2} \qquad \theta_{p2} = \theta_2$$

$$\sigma_1 = 60.3\text{ MPa} \quad \longleftarrow \qquad \theta_{p1} = 4.63\text{ deg} \quad \longleftarrow$$

$$\sigma_2 = -0.395\text{ MPa} \quad \longleftarrow \qquad \theta_{p2} = 94.6\text{ deg} \quad \longleftarrow$$

MAXIMUM SHEAR STRESSES

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{max} = 30.4\text{ MPa} \quad \longleftarrow$$

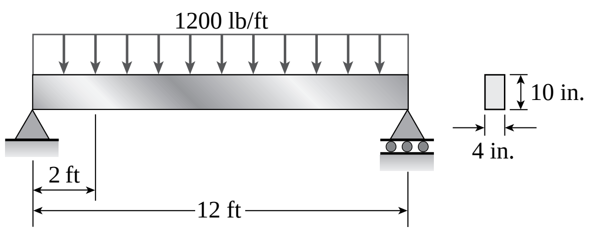
$$\theta_{s1} = \theta_{p1} - 45\text{ deg} \qquad \theta_{s1} = -40.4\text{ deg} \quad \longleftarrow$$

$$\theta_{s2} = \theta_{s1} + 90\text{ deg} \qquad \theta_{s2} = 49.6\text{ deg} \quad \longleftarrow$$

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} \qquad \sigma_{avg} = 30.0\text{ deg} \quad \longleftarrow$$

Problem 8.4-3 A simple beam of rectangular cross section (width 4 in., height 10 in.) carries a uniform load of 1200 lb/ft on a span of 12 ft (see figure).

Find the principal stresses σ_1 and σ_2 and the maximum shear stress τ_{max} at a cross section 2 ft from the left-hand support at each of the following locations: (a) the neutral axis, (b) 2 in. above the neutral axis and (c) the top of the beam. (Disregard the direct compressive stresses produced by the uniform load bearing against the top of the beam.)



Solution 8.4-3

$b = 4$ in. $\quad h = 10$ in. $\quad A = bh$

$I = \dfrac{bh^3}{12} \quad I = 333.333\text{ in.}^4$

$q = 1200$ lb/ft $\quad c = 2$ ft $\quad L = 12$ ft

$R_A = \dfrac{qL}{2} \quad R_A = 7.2 \times 10^3$ lb

$M = R_A c - q\dfrac{c^2}{2} \quad M = 1.44 \times 10^5$ lb · in.

$V = R_A - qc \quad V = 4.8 \times 10^3$ lb

(a) Neutral axis

$\sigma_x = 0 \quad \sigma_y = 0 \quad \tau_{xy} = -\dfrac{3V}{2A}$

$\tau_{xy} = -180$ psi

Pure shear: $\sigma_1 = -\tau_{xy} \quad \sigma_2 = -\sigma_1 \quad \tau_{max} = \sigma_1$

$\sigma_1 = 180$ psi $\quad \sigma_2 = -180$ psi

$\tau_{max} = 180$ psi $\quad \leftarrow$

(b) 2 in. above the neutral axis

$y = 2$ in. $\quad d = 3$ in.

$\sigma_x = -\dfrac{My}{I} \quad \sigma_x = -864$ psi $\quad \sigma_y = 0$

$Q = bd\left(\dfrac{h}{2} - \dfrac{d}{2}\right) \quad Q = 42\text{ in.}^3$

$\tau_{xy} = -\dfrac{VQ}{Ib} \quad \tau_{xy} = -151.2$ psi

$$\sigma_1 = \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

$\sigma_1 = 25.7$ psi $\quad \leftarrow$

$$\sigma_2 = \frac{\sigma_x}{2} - \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

$\sigma_2 = -890$ psi $\quad \leftarrow$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

$\tau_{max} = 458$ psi $\quad \leftarrow$

(c) Top of the beam

$$\sigma_x = -\frac{M\left(\frac{h}{2}\right)}{I} \qquad \sigma_x = -2.16 \times 10^3 \text{ psi}$$

$\sigma_y = 0 \quad \tau_{xy} = 0$

Uniaxial stress: $\sigma_1 = \sigma_y \quad \sigma_1 = 0$ psi

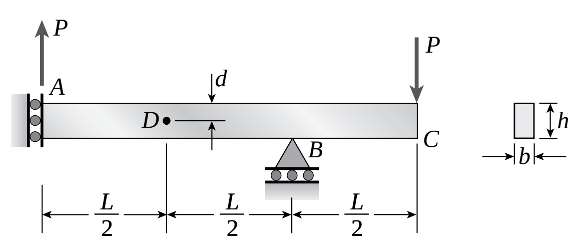
$\sigma_2 = \sigma_x \quad \sigma_2 = -2.16 \times 10^3$ psi

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

$\tau_{max} = 1080$ psi $\quad \leftarrow$

Problem 8.4-4 An overhanging beam ABC with a guided support at A is of rectangular cross section and supports concentrated loads P both at A and at the free end C (see figure). The span length from A to B is L, and the length of the overhang is $L/2$. The cross section has width b and height h. Point D is located midway between the supports at a distance d from the top face of the beam.

Knowing that the maximum tensile stress (principal stress) at point D is $\sigma_1 = 35$ MPa. determine the magnitude of the load P. Data for the beam are as follows: $L = 1.75$ m, $b = 50$ mm, $h = 200$ mm, and $d = 40$ mm.



Probs. 8.4-4 and 8.4-5

Solution 8.4-4

$L = 1.75$ m $\quad b = 50$ mm $\quad h = 200$ mm

$d = 40$ mm

Maximum principal stress at point D:

$$R_B = 0 \qquad M_A = PL + P\frac{L}{2} \qquad M_A = \frac{3}{2}PL$$

$$M_D = -\frac{3}{2}PL + P\frac{L}{2} = -PL \qquad V_D = P$$

STRESS AT POINT D

$$I = \frac{bh^3}{12} \qquad I = 3.333 \times 10^7 \text{ mm}^4$$

$$y = \frac{h}{2} - d \qquad y = 60 \text{ mm}$$

$$\sigma_x = -\frac{My}{I} = \frac{-(-PL)y}{I} = (3150P)\ \text{N/m}^2 \qquad \sigma_y = 0$$

$$Q = bd\left(\frac{h}{2} - \frac{d}{2}\right) \qquad Q = 1.6 \times 10^5 \text{ mm}^3$$

$$\tau_{xy} = \frac{VQ}{Ib} = (96P)\ \text{N/m}^2$$

PRINCIPAL STRESSES

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2} = (3.153 \times 10^3)P$$

With $\sigma_1 = 35$ MPa $\qquad P = \dfrac{\sigma_1}{3.153 \times 10^3}$

$P = 11.10$ kN $\quad \leftarrow$

Problem 8.4-5 Solve the preceding problem if the stress and dimensions are as follows: $\sigma_1 = 2450$ psi, $L = 80$ in., $b = 2.5$ in., $h = 10$ in., and $d = 2.5$ in.

Solution 8.4-5

$L = 80$ in. $\quad b = 2.5$ in. $\quad h = 10$ in. $\quad d = 2.5$ in.

Maximum principal stress at point D:

$$R_B = 0 \qquad M_A = PL + P\frac{L}{2} \qquad M_A = \frac{3}{2}PL$$

$$M_D = -\frac{3}{2}PL + P\frac{L}{2} = -PL \qquad V_D = P$$

STRESS AT POINT D

$$I = \frac{bh^3}{12} \qquad I = 208.333 \text{ in.}^4$$

$$y = \frac{h}{2} - d \qquad y = 2.5 \text{ in.}$$

$$\sigma_x = -\frac{My}{I} = \frac{-(-PL)y}{I} = (0.96P)\ \frac{\text{lb}}{\text{in.}^2}$$

$$\sigma_y = 0$$

$$Q = bd\left(\frac{h}{2} - \frac{d}{2}\right) \qquad Q = 23.438 \text{ in.}^3$$

$$\tau_{xy} = \frac{VQ}{Ib} = (0.045P)\ \text{lb/in.}^2$$

PRINCIPAL STRESSES

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

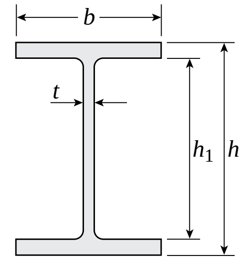
$$= \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2} = (0.962P)\frac{\text{1b}}{\text{in.}^2}$$

With $\sigma_1 = 2450$ psi $\qquad P = \dfrac{\sigma_1}{0.962}$

$P = 2.55$ k $\quad \leftarrow$

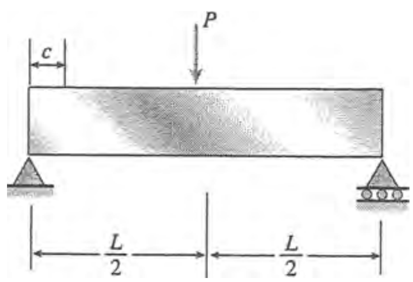
Problem 8.4-6 A beam of wide-flange cross section (see figure) has the following dimensions: $b = 120$ mm, $t = 10$ mm, $h = 300$ mm, and $h_1 = 260$ mm. The beam is simply supported with span length $L = 3.0$ m. A concentrated load $P = 120$ kN acts at the midpoint of the span.

At a cross section located 1.0 m from the left-hand support, determine the principal stresses σ_1 and σ_2 and the maximum shear stress τ_{max} at each of the following locations: (a) the top of the beam, (b) the top of the web, and (c) the neutral axis.



Probs. 8.4-6 and 8.4-7

Solution 8.4-6 Simply supported beam



$P = 120$ kN $\quad L = 3.0$ m $\quad c = 1.0$ m

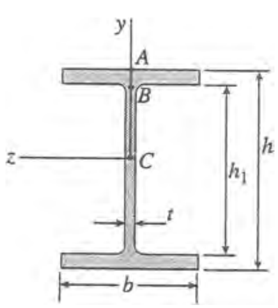
$$M = \frac{Pc}{2} = 60 \text{ kN}\cdot\text{m} \qquad V = \frac{P}{2} = 60 \text{ kN}$$

$b = 120$ mm $\quad t = 10$ mm

$h = 300$ mm $\quad h_1 = 260$ mm

$$I = \frac{bh^3}{12} - \frac{(b-t)h_1^3}{12} = 108.89 \times 10^6 \text{ mm}^4$$

(a) TOP OF THE BEAM (POINT A)



$$\sigma_x = \frac{Mc}{I} = -\frac{(60 \text{ kN}\cdot\text{m})(150 \text{ mm})}{108.89 \times 10^6 \text{ mm}^4}$$

$$= -82.652 \text{ MPa}$$

$\sigma_y = 0 \quad \tau_{xy} = 0$

Uniaxial stress: $\sigma_1 = 0$

$$\left.\begin{aligned} \sigma_2 &= -82.7 \text{ MPa} \\ \tau_{max} &= 41.3 \text{ MPa} \end{aligned}\right] \quad \leftarrow$$

(b) TOP OF THE WEB (POINT B)

$$\sigma_x = -\frac{My}{I} = -\frac{(60 \text{ kN}\cdot\text{ m})(130 \text{ mm})}{108.89 \times 10^6 \text{ mm}^4}$$

$$= -71.63 \text{ MPa}$$

$\sigma_y = 0$

$$Q = (b)\left(\frac{h - h_1}{2}\right)\left(\frac{h + h_1}{4}\right) = 336 \times 10^3 \text{ mm}^3$$

$$\tau_{xy} = -\frac{VQ}{It} = -\frac{(60 \text{ kN})(336 \times 10^3 \text{ mm}^3)}{(108.89 \times 10^6 \text{ mm}^4)(10 \text{ mm})}$$

$$= -18.51 \text{ MPa}$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= -35.82 \pm 40.32 \text{ MPa}$$

$$\sigma_1 = 4.5 \text{ MPa}, \sigma_2 = -76.1 \text{ MPa} \quad \leftarrow$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 40.3 \text{ MPa} \quad \leftarrow$$

(c) NEUTRAL AXIS (POINT C)

$\sigma_x = 0 \quad \sigma_y = 0$

$$Q = b\left(\frac{h}{2}\right)\left(\frac{h}{4}\right) - (b - t)\left(\frac{h_1}{2}\right)\left(\frac{h_1}{4}\right)$$

$$= 420.5 \times 10^3 \text{ mm}^3$$

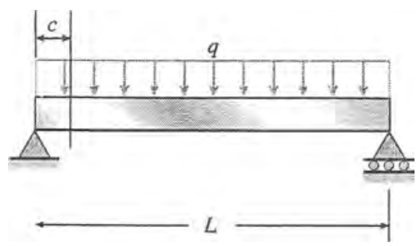
$$\tau_{xy} = -\frac{VQ}{It} = -\frac{(60 \text{ kN})(420.5 \times 10^3 \text{ mm}^3)}{(108.89 \times 10^6 \text{ mm}^4)(10 \text{ mm})}$$
$$= -23.17 \text{ MPa}$$

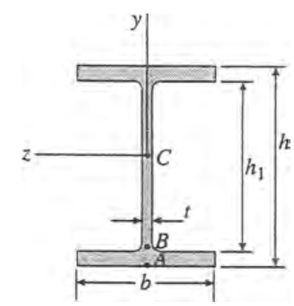
Pure shear: $\sigma_1 = 23.2$ MPa, $\sigma_2 = -23.2$ MPa, $\tau_{max} = 23.2$ MPa ←

Problem 8.4-7 A beam of wide-flange cross section (see figure) has the following dimensions: $b = 5$ in., $t = 0.5$ in., $h = 12$ in., and $h_1 = 10.5$ in. The beam is simply supported with span length $L = 10$ ft and supports a uniform load $q = 6$ k/ft.

Calculate the principal stresses σ_1 and σ_2 and the maximum shear stress τ_{max} at a cross section located 3 ft from the left-hand support at each of the following locations: (a) the bottom of the beam, (b) the bottom of the web, and (c) the neutral axis.

Solution 8.4-7 Simply supported beam





$q = 6.0$ k/ft $\quad L = 10$ ft $= 120$ in.

$$c = 3 \text{ ft} = 36 \text{ in.} \qquad M = \frac{qLc}{2} - \frac{qc^2}{2} = 756{,}000 \text{ lb-in.}$$

$$V = \frac{qL}{2} - qc = 12{,}000 \text{ lb}$$

$b = 5.0$ in. $\quad t = 0.5$ in. $\quad h = 12$ in. $\quad h_1 = 10.5$ in.

$$I = \frac{bh^3}{12} - \frac{(b - t)h_1^3}{12} = 285.89 \text{ in.}^4$$

(a) Bottom of the beam (point A)

$$\sigma_x = -\frac{Mc}{I} = \frac{(756{,}000 \text{ lb-in.})(-6.0 \text{ in.})}{285.89 \text{ in.}^4}$$
$$= 15{,}866 \text{ psi}$$

$\sigma_y = 0 \quad \tau_{xy} = 0$

Uniaxial stress: $\sigma_1 = 15{,}870$ psi, $\sigma_2 = 0 \quad \tau_{max} = 7930$ psi ←

(b) Bottom of the web (point B)

$$\sigma_x = -\frac{My}{I} = -\frac{(756{,}000 \text{ lb-in.})(-5.25 \text{ in.})}{285.89 \text{ in.}^4}$$
$$= 13{,}883 \text{ psi}$$

$$\sigma_y = 0 \quad Q = b\left(\frac{h - h_1}{2}\right)\left(\frac{h + h_1}{4}\right) = 21.094 \text{ in.}^3$$

$$\tau_{xy} = -\frac{VQ}{It} = -\frac{(12{,}000 \text{ lb})(21.094 \text{ in.}^3)}{(285.89 \text{ in.}^4)(0.5 \text{ in.})}$$
$$= -1771 \text{ psi}$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$
$$= 6941.5 \pm 7163.9 \text{ psi}$$

$\sigma_1 = 14{,}100 \text{ psi}, \sigma_2 = -220 \text{ psi} \quad \leftarrow$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 7160 \text{ psi} \quad \leftarrow$$

(c) Neutral axis (point C)

$\sigma_x = 0 \qquad \sigma_y = 0$

$$Q = b\left(\frac{h}{2}\right)\left(\frac{h}{4}\right) - (b - t)\left(\frac{h_1}{2}\right)\left(\frac{h_1}{4}\right)$$

$$= 27.984 \text{ in.}^3$$

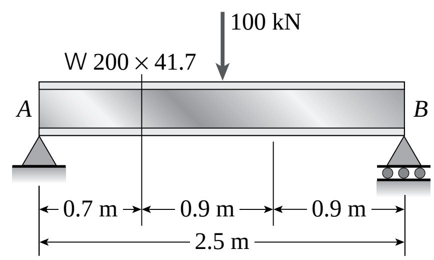
$$\tau_{xy} = -\frac{VQ}{It} = -\frac{(12{,}000 \text{ lb})(27.984 \text{ in.}^3)}{(285.89 \text{ in.}^4)(0.5 \text{ in.})}$$

$$= -2349 \text{ psi}$$

$$\left.\begin{aligned} \text{Pure shear: } \sigma_1 &= 2350 \text{ psi}, \\ \sigma_2 &= -2350 \text{ psi}, \\ \tau_{max} &= 2350 \text{ psi} \end{aligned}\right] \quad \leftarrow$$

Problem 8.4-8 A W 200 × 41.7 wide-flange beam (see Table E-1(b). Appendix E) is simply supported with a span length of 2.5 m (see figure). The beam supports a concentrated load of 100 kN at 0.9 m from support B.

At a cross section located 0.7 m from the left-hand support, determine the principal stresses σ_1 and σ_2 and the maximum shear stress τ_{max} at each of the following locations: (a) the top of the beam, (b) the top of the web, and (c) the neutral axis.



Solution 8.4-8

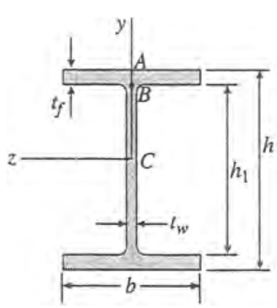
$$R_B = 100 \text{ kN}\left(\frac{0.7 + 0.9}{2.5}\right) \qquad R_B = 64 \text{ kN} \quad \text{(upward)}$$

$R_A = 100 \text{ kN} - R_B \qquad R_A = 36 \text{ kN} \quad \text{(upward)}$

At the point D

$M = R_A(0.7 \text{ m}) \qquad M = 25.2 \text{ kN} \cdot \text{m}$

$V = R_A \qquad V = 36 \text{ kN}$



W200 × 41.7

$I = 40.8 \times 10^6 \text{ mm}^4$

$b = 166 \text{ mm}$

$t_w = 7.24 \text{ mm}$

$t_f = 11.8 \text{ mm}$

$h = 205 \text{ mm}$

$h_1 = h - 2t_f$

$h_1 = 181.4 \text{ mm}$

(a) Top of the beam (point A)

$$\sigma_x = -\frac{M\left(\frac{h}{2}\right)}{I} \qquad \sigma_x = -63.309 \text{ MPa}$$

$\sigma_y = 0 \qquad \tau_{xy} = 0$

Uniaxial stress: $\sigma_1 = \sigma_y \qquad \sigma_2 = \sigma_x \qquad \tau_{max} = \left|\frac{\sigma_x}{2}\right|$

$\sigma_1 = 0 \quad \leftarrow \qquad \sigma_2 = -63.3 \text{ MPa} \quad \leftarrow$

$\tau_{max} = 31.7 \text{ MPa} \quad \leftarrow$

(b) Top of the web (point B)

$$\sigma_x = -\frac{M\left(\frac{h_1}{2}\right)}{I} \qquad \sigma_x = -56.021 \text{ MPa}$$

$\sigma_y = 0$

$$Q = b\left(\frac{h - h_1}{2}\right)\left(\frac{h + h_1}{4}\right)$$

$Q = 1.892 \times 10^5 \text{ mm}^3$

$$\tau_{xy} = -\frac{VQ}{It_w} \qquad \tau_{xy} = -23.061 \text{ MPa}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_1 = 8.27 \text{ MPa} \quad \leftarrow$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = -64.3 \text{ MPa} \quad \leftarrow$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{max} = 36.3 \text{ MPa} \quad \leftarrow$$

(c) Neutral axis (point C)

$$\sigma_x = 0 \qquad \sigma_y = 0$$

$$Q = b\left(\frac{h}{2}\right)\left(\frac{h}{4}\right) - (b - t_w)\left(\frac{h_1}{2}\right)\left(\frac{h_1}{4}\right)$$

$$Q = 2.19 \times 10^5 \text{ mm}^3$$

$$\tau_{xy} = -\frac{VQ}{It_w} \qquad \tau_{xy} = -26.69 \text{ MPa}$$

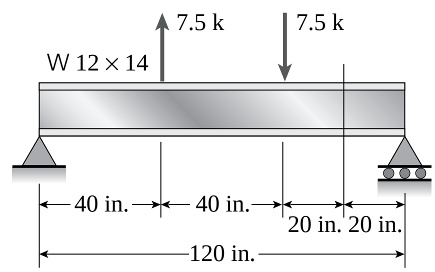
Pure shear: $\sigma_1 = |\tau_{xy}| \qquad \sigma_1 = 26.7 \text{ Mpa} \quad \leftarrow$

$$\sigma_2 = -\sigma_1 \qquad \sigma_2 = -26.7 \text{ Mpa} \quad \leftarrow$$

$$\tau_{max} = \tau_{xy} \qquad \tau_{max} = -26.7 \text{ Mpa} \quad \leftarrow$$

Problem 8.4-9 A W 12 × 14 wide-flange beam (see Table E-1(a), Appendix E) is simply supported with a span length of 120 in. (see figure). The beam supports two anti-symmetrically placed concentrated loads of 7.5 k each.

At a cross section located 20 in. from the right-hand support, determine the principal stresses σ_1 and σ_2 and the maximum shear stress τ_{max} at each of the following locations: (a) the top of the beam, (b) the top of the web, and (c) the neutral axis.



Solution 8.4-9

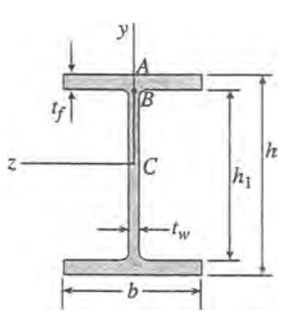
$$R_B = \frac{7.5 \cdot 80 - 7.5 \cdot 40}{120} \text{ k} \qquad R_B = 2.5 \text{ k} \quad \text{(upward)}$$

$$R_A = -R_B \qquad R_A = -2.5 \text{ k} \quad \text{(downward)}$$

At Section C-C

$$M = R_B \cdot 20 \text{ in.} \qquad M = 50 \text{ k} \cdot \text{in.}$$

$$V = -R_B \qquad V = -2.5 \text{ k}$$



W12 × 14

$I = 88.6 \text{ in.}^4$

$b = 3.970 \text{ in.}$

$t_w = 0.200 \text{ in.}$

$t_f = 0.225 \text{ in.}$

$h = 11.91 \text{ in.}$

$h_1 = h - 2t_f$

$h_1 = 11.46 \text{ in.}$

(a) Top of the beam (point A)

$$\sigma_x = -\frac{M\left(\frac{h}{2}\right)}{I} \qquad \sigma_x = -3.361 \times 10^3 \text{ psi}$$

$$\sigma_y = 0 \qquad \tau_{xy} = 0$$

Uniaxial stress: $\sigma_1 = \sigma_y \qquad \sigma_2 = \sigma_x \qquad \tau_{max} = \left|\frac{\sigma_x}{2}\right|$

$$\sigma_1 = 0 \quad \leftarrow \qquad \sigma_2 = -3361 \text{ psi} \quad \leftarrow$$

$$\tau_{max} = 1680 \text{ psi} \quad \leftarrow$$

(b) Top of the web (point B)

$$\sigma_x = -\frac{M\left(\frac{h_1}{2}\right)}{I} \qquad \sigma_x = -3.234 \times 10^3 \text{ psi}$$

$$\sigma_y = 0$$

$$Q = b\left(\frac{h - h_1}{2}\right)\left(\frac{h + h_1}{4}\right) \qquad Q = 5.219 \text{ in.}^3$$

$$\tau_{xy} = -\frac{VQ}{It_w} \qquad \tau_{xy} = 736.289 \text{ psi}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_1 = 159.8 \text{ psi} \quad \leftarrow$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = -3393 \text{ psi} \quad \leftarrow$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{max} = 1777 \text{ psi} \quad \leftarrow$$

(c) Neutral axis (point C)

$$\sigma_x = 0 \qquad \sigma_y = 0$$

$$Q = b\left(\frac{h}{2}\right)\left(\frac{h}{4}\right) - (b - t_w)\left(\frac{h_1}{2}\right)\left(\frac{h_1}{4}\right)$$

$$Q = 8.502 \text{ in}^3$$

$$\tau_{xy} = -\frac{VQ}{It_w} \qquad \tau_{xy} = 1.2 \times 10^3 \text{ psi}$$

Pure shear:

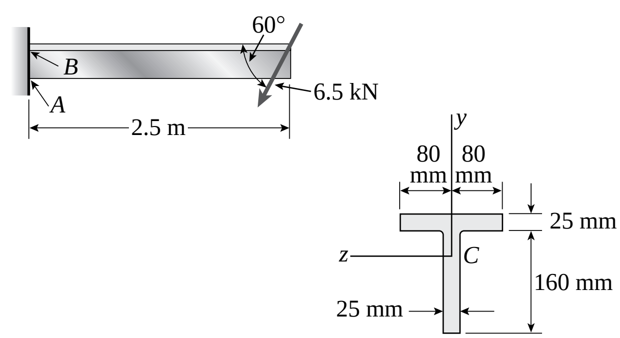
$$\sigma_1 = |\tau_{xy}| \qquad \sigma_1 = 1200 \text{ psi} \quad \leftarrow$$

$$\sigma_2 = -\sigma_1 \qquad \sigma_2 = -1200 \text{ psi} \quad \leftarrow$$

$$\tau_{max} = \tau_{xy} \qquad \tau_{max} = 1200 \text{ psi} \quad \leftarrow$$

Problem *8.4-10 A cantilever beam of T-section is loaded by an inclined force of magnitude 6.5 kN (see figure). The line of action of the force is inclined at an angle of 60° to the horizontal and intersects the top of the beam at the end cross section. The beam is 2.5 m long and the cross section has the dimensions shown.

Determine the principal stresses σ_1 and σ_2 and the maximum shear stress τ_{max} at points A and B in the web of the beam near the support.



Solution 8.4-10

$P = 6.5$ kN $\qquad L = 2.5$ m $\qquad A = 2(160 \text{ mm})(25 \text{ mm})$

$A = 8 \times 10^3 \text{ mm}^2 \qquad b = 160$ mm $\qquad t = 25$ mm

Location of centroid C $\qquad$ From Eq. (12-7b) in Chapter 12:

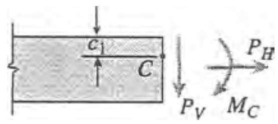
$$c_2 = \frac{\Sigma(y_i A_i)}{A} \qquad c_2 = \frac{(160 \text{ mm})(25 \text{ mm})\left(160 + \frac{25}{2}\right)\text{mm} + (160 \text{ mm})(25 \text{ mm})(80 \text{ mm})}{A}$$

$$c_2 = 126.25 \text{ mm} \qquad c_1 = 185 \text{ mm} - c_2 \qquad c_1 = 58.75 \text{ mm}$$

Moment of intertia

$$I_Z = \frac{1}{3}tc_2^3 + \frac{1}{3}bc_1^3 - \frac{1}{3}(b - t)(c_1 - t)^3 \qquad I_Z = 2.585 \times 10^7 \text{ mm}^4$$

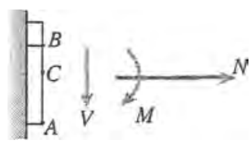
Equivalent loads at free end of beam



$P_H = -P \cos(60 \text{ deg})$ $\quad P_H = -3.25 \text{ kN}$ $\quad P_v = P \sin(60 \text{ deg})$

$P_v = 5.629 \text{ kN}$ $\quad M_c = P_H\, c_1$ $\quad M_c = -190.938 \text{ N} \cdot \text{m}$

Stress resultants at fixed end of beam



$N_0 = P_H$ $\quad N_0 = -3.25 \text{ kN}$

$V = P_v$ $\quad V = 5.629 \text{ kN}$

$M = -M_c - P_v L$ $\quad M = -1.388 \times 10^4 \text{ N} \cdot \text{m}$

Stress at point A (bottom of web)

$$\sigma_x = \frac{N_0}{A} + \frac{Mc_1}{I_Z} \qquad \sigma_x = -31.951 \text{ MPa} \qquad \sigma_y = 0 \qquad \tau_{xy} = 0$$

Uniaxial stress: $\sigma_1 = \sigma_y \qquad \sigma_2 = \sigma_x \qquad \tau_{max} = \left|\frac{\sigma_x}{2}\right|$

$\sigma_1 = 0 \quad \leftarrow \qquad \sigma_2 = -32.0 \text{ MPa} \quad \leftarrow \qquad \tau_{max} = 15.98 \text{ MPa} \quad \leftarrow$

Stress at point B (top of web)

$$\sigma_x = \frac{N_0}{A} - \frac{M(c_1 - t)}{I_z} \qquad \sigma_x = 17.715 \text{ MPa} \qquad \sigma_y = 0$$

$$Q = bt\left(c_1 - \frac{t}{2}\right) \qquad Q = 1.85 \times 10^5 \text{ mm}^3$$

$$\tau_{xy} = -\frac{VQ}{I_z t} \qquad \tau_{xy} = -1.611 \text{ MPa}$$

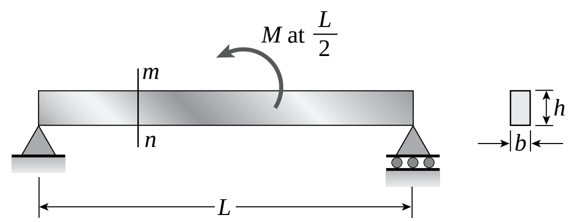
$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \sigma_1 = 17.86 \text{ MPa} \quad \leftarrow$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \sigma_2 = -0.145 \text{ MPa} \quad \leftarrow$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} \qquad \tau_{max} = 9.00 \text{ MPa} \quad \leftarrow$$

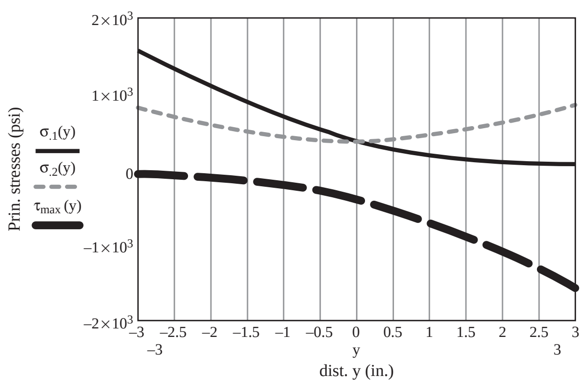
Problem *8.4-11 A simple beam of rectangular cross section has span length $L = 62$ in. and supports a concentrated moment $M = 560$ k-in at midspan (see figure). The height of the beam is $h = 6$ in. and the width is $b = 2.5$ in.

Plot graphs of the principal stresses σ_1 and σ_2 and the maximum shear stress τ_{max}, showing how they vary over the height of the beam at cross section mn, which is located 24 in. from the left-hand support.



Probs. 8.4-11 and 8.4-12

Solution 8.4-11



$M = 560\ \text{k}\cdot\text{in.} \qquad L = 62\ \text{in.} \qquad c = 24\ \text{in.}$

$b = 2.5\ \text{in.} \qquad h = 6\ \text{in.}$

$$I = \frac{bh^3}{12} \qquad I = 45\ \text{in}^4$$

$$R_A = \frac{M}{L} \qquad R_A = 9.032\ \text{k} \quad \text{(upward)}$$

$$R_B = -R_A \qquad R_B = -9.032\ \text{k} \quad \text{(downward)}$$

At section *m-n*

$$M = R_A c \qquad M = 216.774\ \text{k}\cdot\text{in.}$$

$$V = R_A \qquad V = 9.032\ \text{k}$$

$$\sigma_x(y) = -\frac{My}{I} \qquad \sigma_y = 0$$

$$Q(y) = b\left(\frac{h}{2} - y\right)\left(\frac{1}{2}\right)\left(\frac{h}{2} + y\right)$$

$$\tau_{xy}(y) = \frac{VQ(y)}{Ib}$$

PRINCIPAL STRESSES

$$\sigma_1(y) = \frac{\sigma_x(y)}{2} + \sqrt{\left(\frac{\sigma_x(y)}{2}\right)^2 + \tau_{xy}(y)^2}$$

$$\sigma_2(y) = \frac{\sigma_x(y)}{2} - \sqrt{\left(\frac{\sigma_x(y)}{2}\right)^2 + \tau_{xy}(y)^2}$$

$$\tau_{max}(y) = \sqrt{\left(\frac{\sigma_x(y)}{2}\right)^2 + \tau_{xy}(y)^2}$$

$\sigma_1(3\ \text{in.}) = 0$ psi

$\sigma_2(3\ \text{in.}) = -14{,}452$ psi

$\tau_{max}(3\ \text{in.}) = 7226$ psi

$\sigma_1(0) = 903$ psi $\qquad \sigma_2(0) = -903$ psi

$\tau_{max}(0) = 903$ psi

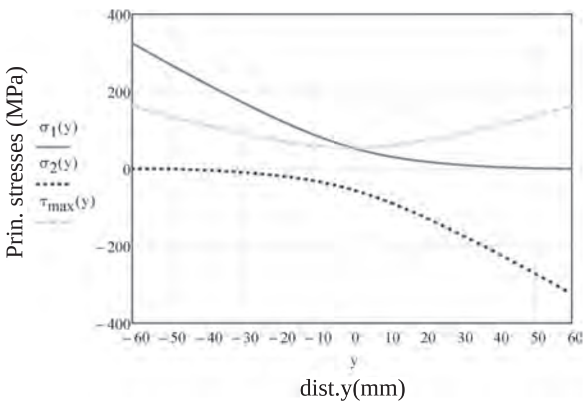
$\sigma_1(-3\ \text{in.}) = 14{,}452$ psi

$\sigma_2(-3\ \text{in.}) = 0$ psi

$\tau_{max}(-3\ \text{in.}) = 7226$ psi

Problem *8.4-12 Solve the preceding problem for a cross section mn located 0.18 m from the support if $L = 0.75$ m, $M = 65\ \text{kN}\cdot\text{m}$, $h = 120$ mm, and $b = 20$ mm.

Solution 8.4-12



$M = 65\ \text{kN} \cdot \text{m} \qquad L = 0.75\ \text{m} \qquad c = 0.18\ \text{m}$

$b = 20\ \text{mm} \qquad h = 120\ \text{mm}$

$I = \dfrac{bh^3}{12} \qquad I = 2.88 \times 10^6\ \text{mm}^4$

$R_A = \dfrac{M}{L} \qquad R_A = 86.667\ \text{kN} \quad \text{(upward)}$

$R_B = -R_A \qquad R_B = -86.667\ \text{kN} \quad \text{(downward)}$

At section *m-n*

$M = R_A\, c \qquad M = 15.6\ \text{kN} \cdot \text{m}$

$V = R_A \qquad V = 86.667\ \text{kN}$

$$\sigma_x(y) = \frac{My}{I} \qquad \sigma_y = 0$$

$$Q(y) = b\left(\frac{h}{2} - y\right)\left(\frac{1}{2}\right)\left(\frac{h}{2} + y\right)$$

$$\tau_{xy}(y) = \frac{VQ(y)}{Ib}$$

PRINCIPAL STRESSES

$$\sigma_1(y) = \frac{\sigma_x(y)}{2} + \sqrt{\left(\frac{\sigma_x(y)}{2}\right)^2 + \tau_{xy}(y)^2}$$

$$\sigma_2(y) = \frac{\sigma_x(y)}{2} - \sqrt{\left(\frac{\sigma_x(y)}{2}\right)^2 + \tau_{xy}(y)^2}$$

$$\tau_{max}(y) = \sqrt{\left(\frac{\sigma_x(y)}{2}\right)^2 + \tau_{xy}(y)^2}$$

$\sigma_1(60\ \text{mm}) = 0\ \text{MPa} \qquad \sigma_2(60\ \text{mm}) = -325\ \text{MPa}$

$\tau_{max}(60\ \text{mm}) = 162.5\ \text{MPa}$

$\sigma_1(0) = 54.2\ \text{MPa} \qquad \sigma_2(0) = -54.2\ \text{MPa}$

$\tau_{max}(0) = 54.2\ \text{MPa}$

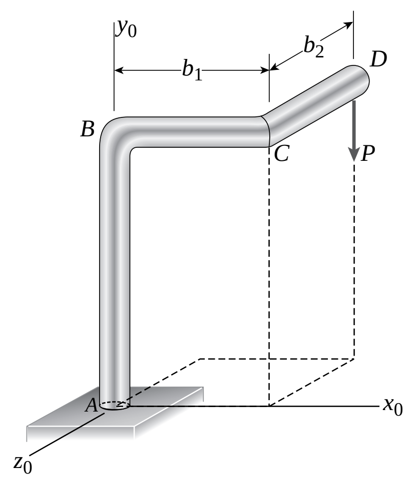
$\sigma_1(-60\ \text{mm}) = 325\ \text{MPa} \qquad \sigma_2(-60\ \text{mm}) = 0\ \text{MPa}$

$\tau_{max}(-60\ \text{mm}) = 162.5\ \text{MPa}$

Combined Loadings

The problems for Section 8.5 are to be solved assuming that the structures behave linearly elastically and that the stresses caused by two or more loads may be superimposed to obtain the resultant stresses acting at a point. Consider both in-plane and out-of-plane shear stresses unless otherwise specified.

Problem 8.5-1 A bracket *ABCD* having a hollow circular cross section consists of a vertical arm *AB*, a horizontal arm *BC* parallel to the x_0 axis, and a horizontal arm *CD* parallel to the z_0 axis (see figure). The arms *BC* and *CD* have lengths $b_1 = 3.6$ ft and $b_2 = 2.2$ ft, respectively. The outer and inner diameters of the bracket are $d_2 = 7.5$ in. and $d_1 = 6.8$ in. A vertical load $P = 1400$ lb acts at point *D*. Determine the maximum tensile, compressive, and shear stresses in the vertical arm.



Solution 8.5-1

$b_1 = 3.6\ \text{ft} \qquad b_2 = 2.2\ \text{ft} \qquad P = 1400\ \text{lb}$

$d_2 = 7.5\ \text{in.} \qquad d_1 = 6.8\ \text{in.}$

$$A = \frac{\pi}{4}\left(d_2^2 - d_1^2\right) \qquad A = 7.862\ \text{in.}^2$$

$$I = \frac{\pi}{64}\left(d_2^4 - d_1^4\right) \qquad I = 50.36\ \text{in.}^4$$

Vertical arm AB

$$M = P\sqrt{b_1^2 + b_2^2} \qquad M = 7.088 \times 10^4\ \text{lb} \cdot \text{in.}$$

Maximum tensile stress

$$\sigma_t = -\frac{P}{A} + \frac{M\left(\frac{d_2}{2}\right)}{I} \qquad \sigma_t = 5100\ \text{psi} \quad \longleftarrow$$

Maximum compressive stress

$$\sigma_c = -\frac{P}{A} - \frac{M\left(\frac{d_2}{2}\right)}{I}$$

$$\sigma_c = -5456\ \text{psi} \quad \longleftarrow$$

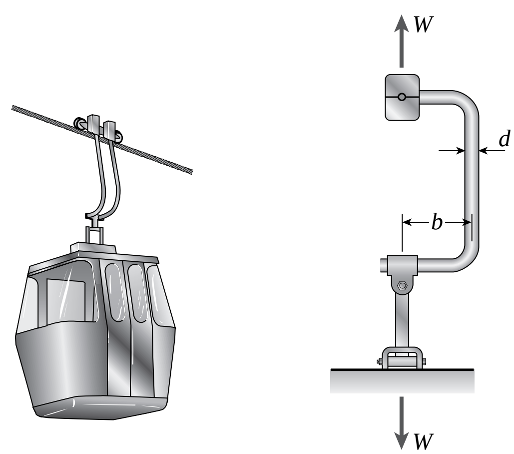
Maximum shear stress

Uniaxial stress

$$\tau_{max} = |\sigma_c| \qquad \tau_{max} = 5456\ \text{psi} \quad \longleftarrow$$

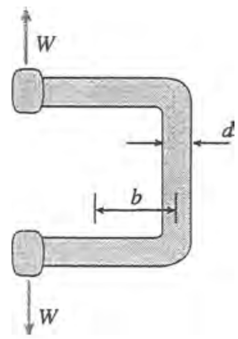
Problem 8.5-2 A gondola on a ski lift is supported by two bent arms, as shown in the figure. Each arm is offset by the distance $b = 180$ mm from the line of action of the weight force W. The allowable stresses in the arms are 100 MPa in tension and 50 MPa in shear.

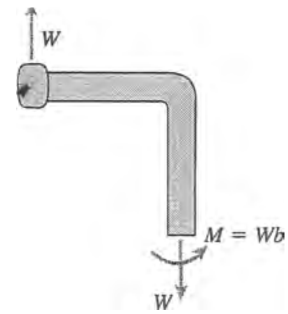
If the loaded gondola weighs 12 kN, what is the minimum diameter d of the arms?



(a) (b)

Solution 8.5-2 Gondola on a ski lift





$b = 180 \text{ mm} \qquad W = \dfrac{12 \text{ kN}}{2} = 6 \text{ kN}$

$\sigma_{allow} = 100 \text{ MPa (tension)} \qquad \tau_{allow} = 50 \text{ MPa}$

Find d_{min}

$$A = \frac{\pi d^2}{4} \quad S = \frac{\pi d^3}{32}$$

Maximum tensile stress

$$\sigma_t = \frac{W}{A} + \frac{M}{S} = \frac{4W}{\pi d^2} + \frac{32\,Wb}{\pi d^3}$$

$$\text{or } \left(\frac{\pi \sigma_t}{4W}\right) d^3 - d - 8b = 0$$

Substitute numerical values:

$$\frac{\pi \sigma_t}{4W} = \frac{\pi \sigma_{allow}}{4W} = \frac{\pi(100 \text{ Mpa})}{4(6 \text{ kN})} = 13{,}089.97 \frac{1}{m^2}$$

$8b = 1.44$ m

$13{,}090\, d^3 - d - 1.44 = 0 \qquad (d = \text{meters})$

Solve numerically: $d = 0.04845$ m

$\therefore d_{min} = 48.4 \text{ mm} \qquad \leftarrow$

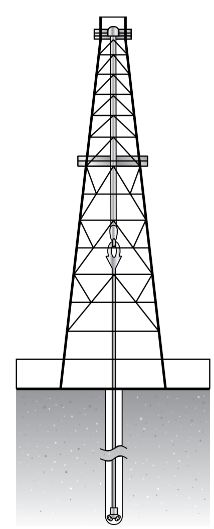
Maximum shear stress

$$\tau_{max} = \frac{\sigma_t}{2} \text{ (uniaxial stress)}$$

Since τ_{allow} is one-half of σ_{allow}, the minimum diameter for shear is the same as for tension.

Problem 8.5-3 The hollow drill pipe for an oil well (see figure) is 6.2 in. in outer diameter and 0.75 in. in thickness. Just above the bit, the compressive force in the pipe (due to the weight of the pipe) is 62 k and the torque (due to drilling) is 185 k-in.

Determine the maximum tensile, compressive, and shear stresses in the drill pipe.



Solution 8.5-3

P = compressive force

T = Torque

d_2 = outer diameter

d_1 = inner diameter

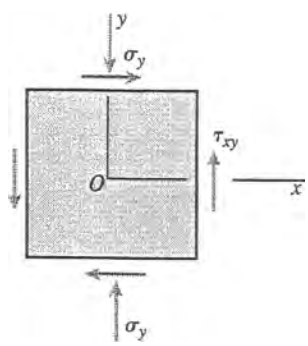
$P = 62\text{ k}$ $\quad T = 185\text{ k}\cdot\text{in.}$ $\quad d_2 = 6.2\text{ in.}$

$t = 0.75\text{ in.}$ $\quad d_1 = d_2 - 2t$ $\quad d_1 = 4.7\text{ in.}$

$$A = \frac{\pi}{4}\left(d_2^2 - d_1^2\right) \qquad A = 12.841\text{ in.}^2$$

$$I_p = \frac{\pi}{32}\left(d_2^4 - d_1^4\right) \qquad I_p = 97.16\text{ in.}^4$$

STRESSES AT THE OUTER SURFACE



$$\sigma_y = -\frac{P}{A} \qquad \sigma_y = -4828\text{ psi}$$

$\sigma_x = 0$

$$\tau_{xy} = \frac{T\left(\dfrac{d_2}{2}\right)}{I_p} \qquad \tau_{xy} = 5903\text{ psi}$$

PRINCIPAL STRESSES

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy^2}}$$

$\sigma_1 = 3963\text{ psi}$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy^2}}$$

$\sigma_2 = -8791\text{ psi}$

MAXIMUM TENSILE STRESS $\quad \sigma_t = \sigma_1$

$\sigma_t = 3963\text{ psi}$ $\quad \leftarrow$

MAXIMUM COMPRESSIVE STRESS $\quad \sigma_c = \sigma_2$

$\sigma_c = -8791\text{ psi}$ $\quad \leftarrow$

MAXIMUM IN-PLANESHEAR STRESS

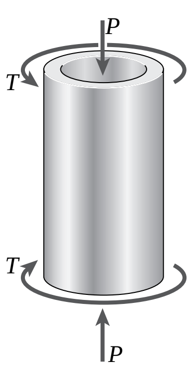
$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right) + \tau_{xy^2}}$$

$\tau_{\max} = 6377\text{ psi}$ $\quad \leftarrow$

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

Problem 8.5-4 A segment of a generator shaft is subjected to a torque T and an axial force P, as shown in the figure. The shaft is hollow (outer diameter $d_2 = 300$ mm and inner diameter $d_1 = 250$ mm) and delivers 1800 kW at 4.0 Hz.

If the compressive force $P = 540$ kN, what are the maximum tensile, compressive, and shear stresses in the shaft?



Probs. 8.5-4 and 8.5-5

Solution 8.5-4

P = Compressive_force

P_0 = Power $\quad f$ = frequency

$$T = \text{torgue} = \frac{P_0}{2\pi f}$$

d_2 = outer diameter

d_1 = inner diameter

$f = 4.0$ Hz

$P = 540$ kN $\quad P_0 = 1800$ kW

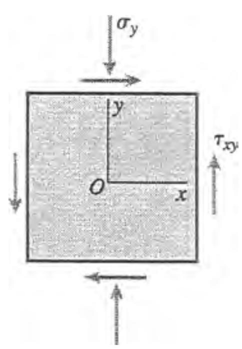
$$T = \frac{P_0}{2\pi f} \qquad T = 7.162 \times 10^4\ \text{N} \cdot \text{m}$$

$d_2 = 300$ mm $\quad d_1 = 250$ mm

$$A = \frac{\pi}{4}\left(d_2^2 - d_1^2\right) \qquad I_p = \frac{\pi}{32}\left(d_2^4 - d_1^4\right)$$

$A = 2.16 \times 10^4\ \text{mm}^2 \quad I_p = 4.117 \times 10^8\ \text{mm}^4$

Stresses at the outer surface



$$\sigma_y = -\frac{P}{A} \qquad \sigma_y = -25.002\ \text{MPa}$$

$\sigma_x = 0$

$$\tau_{xy} = \frac{T\left(\frac{d_2}{2}\right)}{I_p} \qquad \tau_{xy} = 26.093\ \text{MPa}$$

Principal stresses

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$\sigma_1 = 16.432$ MPa

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$\sigma_2 = -41.434$ MPa

Maximum tensile stress $\quad \sigma_t = \sigma_1$

$\sigma_t = 16.43$ MPa $\quad \leftarrow$

Maximum compressive stress $\quad \sigma_c = \sigma_2$

$\sigma_c = -41.4$ MPa $\quad \leftarrow$

Maximum in-planeshear stress

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$\tau_{\max} = 28.9$ MPa $\quad \leftarrow$

Note: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

Problem 8.5-5 A segment of a generator shaft of hollow circular cross section is subjected to a torque $T = 240$ k-in. (see figure). The outer and inner diameters of the shaft are 8.0 in. and 6.25 in., respectively.

What is the maximum permissible compressive load P that can be applied to the shaft if the allowable in-plane shear stress is $\tau_{\text{allow}} = 6250$ psi?

Solution 8.5-5

P = compressive force $\qquad T$ = Torque

d_2 = outer diameter $\qquad d_1$ = inner diameter

$T = 240\ \text{k} \cdot \text{in.}$

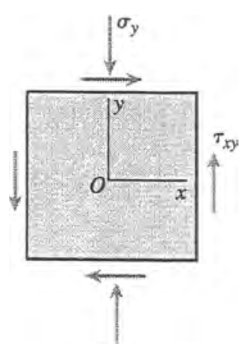
$d_2 = 8\ \text{in.} \qquad d_1 = 6.25\ \text{in.}$

$$A = \frac{\pi}{4}\left(d_2^2 - d_1^2\right) \qquad A = 19.586\ \text{in.}^2$$

$$\tau_{\text{allow}} = 6250\ \text{psi} \qquad I_p = \frac{\pi}{32}\left(d_2^4 - d_1^4\right)$$

$I_p = 252.321\ \text{in.}^4$

STRESSES AT THE OUTER SURFACE



$$\sigma_y = -\frac{P}{A}$$

$$\sigma_x = 0$$

$$\tau_{xy} = \frac{T\left(\dfrac{d_2}{2}\right)}{I_p} \qquad \tau_{xy} = 3805\ \text{psi}$$

MAXIMUM IN-PLANESHEAR STRESS $\qquad \tau_{\max} = \tau_{\text{allow}}$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

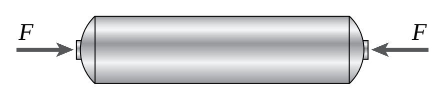
$$\sigma_y = \sqrt{(\tau_{\max}^2 - \tau_{xy}^2)\,4} \qquad \sigma_y = 9917\ \text{psi}$$

$$P = \sigma_y A \qquad P = 194.2\ \text{k} \qquad \longleftarrow$$

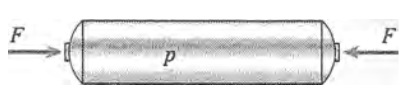
NOTE: The maximum in-plane shear is larger than the maximum out-of-plane shear stress.

Problem 8.5-6 A cylindrical tank subjected to internal pressure p is simultaneously compressed by an axial force $F = 72$ kN (see figure). The cylinder has diameter $d = 100$ mm and wall thickness $t = 4$ mm.

Calculate the maximum allowable internal pressure $p_{\max}$ based upon an allowable shear stress in the wall of the tank of 60 MPa.



Solution 8.5-6 Cylindrical tank with compressive force



$F = 72$ kN

p = internal pressure

$d = 100\ \text{mm} \qquad t = 4\ \text{mm} \qquad \tau_{\text{allow}} = 60\ \text{MPa}$

CIRCUMFERENTIAL STRESS (TENSION)

$$\sigma_1 = \frac{pr}{t} = \frac{p(50\ \text{mm})}{4\ \text{mm}} = 12.5\,p$$

Units: σ_1 = MPa $\qquad p$ = MPa

LONGITUDINAL STRESS (TENSION)

$$\sigma_2 = \frac{pr}{2t} - \frac{F}{A} = \frac{pr}{2t} - \frac{F}{2\pi rt}$$

$$= 6.25p - \frac{72{,}000\ N}{2\pi(50\ \text{mm})(4\ \text{mm})}$$

$$= 6.25p - 57.296\ \text{Mpa}$$

Units: σ_2 = MPa $\quad p$ = MPa

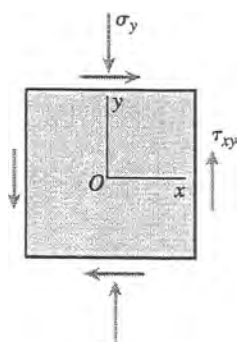
BIAXIAL STRESS

IN-PLANE SHEAR STRESS (CASE 1)

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = 3.125\,p + 28.648\ \text{Mpa}$$

60 MPa = 3.125 p + 28.648 MPa

Solving, p_1 = 10.03 MPa



OUT-OF-PLANE SHEAR STRESSES

Case 2: $\tau_{max} = \dfrac{\sigma_1}{2} = 6.25\,p$; 60 MPa = 6.25 p

Solving, p_2 = 9.60 MPa

Case 3: $\tau_{max} = \dfrac{\sigma_2}{2} = 3.125\,p - 28.648$ MPa

60 MPa = 3.125 p − 28.648 MPa

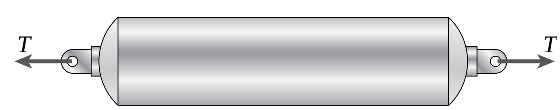
Solving, p_3 = 28.37 MPa

CASE 2, OUT-OF-PLANE SHEAR STRESS GOVERNS

p_{max} = 9.60 MPa $\quad\longleftarrow$

Problem 8.5-7 A cylindrical tank having diameter d = 2.5 in. is subjected to internal gas pressure p = 600 psi and an external tensile load T = 1000 lb (see figure).

Determine the minimum thickness t of the wall of the tank based upon an allowable shear stress of 3000 psi.



Solution 8.5-7 Cylindrical tank with tensile load

T = 1000 lb $\quad t$ = thickness

p = 600 psi

d = 2.5 in. $\quad \tau_{allow}$ = 3000 psi

CIRCUMFERENTIAL STRESS (TENSION)

$$\sigma_1 = \frac{pr}{t} = \frac{(600\ \text{psi})(1.25\ \text{in.})}{t} = \frac{750}{t}$$

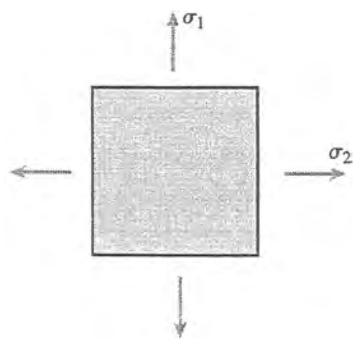
Units: σ_1 = psi $\quad t$ = inches $\quad \sigma_2$ = psi

LONGITUDINAL STRESS (TENSION)

$$\sigma_2 = \frac{pr}{2t} + \frac{T}{A} = \frac{pr}{2t} + \frac{T}{2\pi rt}$$

$$= \frac{375}{t} + \frac{1000 \text{ lb}}{2\pi(1.25 \text{ in.})t} = \frac{375}{t} + \frac{127.32}{t} + \frac{502.32}{t}$$

BIAXIAL STRESS



IN-PLANE SHEAR STRESS (CASE 1)

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{247.68}{2t} = \frac{123.84}{t}$$

$$3000 \text{ psi} = \frac{123.84}{t}$$

Solving, $t_1 = 0.0413$ in.

OUT-OF-PLANE SHEAR STRESSES

Case 2: $\tau_{max} = \dfrac{\sigma_1}{2} = \dfrac{375}{t}$; $3000 = \dfrac{375}{t}$

Solving, $t_2 = 0.125$ in.

Case 3: $\tau_{max} = \dfrac{\sigma_2}{2} = \dfrac{251.16}{t}$; $3000 = \dfrac{251.16}{t}$

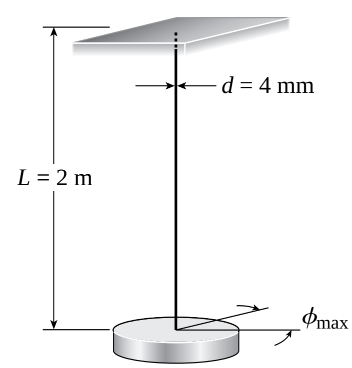
Solving, $t_3 = 0.0837$ in.

CASE 2, OUT-OF-PLANE SHEAR STRESS GOVERNS

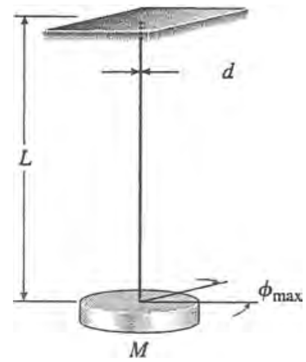
$t_{min} = 0.125$ in. $\longleftarrow$

Problem 8.5-8 The torsional pendulum shown in the figure consists of a horizontal circular disk of mass $M = 60$ kg suspended by a vertical steel wire ($G = 80$ GPa) of length $L = 2$ m and diameter $d = 4$ mm.

Calculate the maximum permissible angle of rotation ϕ_{max} of the disk (that is, the maximum amplitude of torsional vibrations) so that the stresses in the wire do not exceed 100 MPa in tension or 50 MPa in shear.



Solution 8.5-8 Torsional pendulum



$L = 2.0$ m $\quad d = 4.0$ mm

$M = 60$ kg $\quad G = 80$ GPa

$\sigma_{allow} = 100$ MPa $\quad \tau_{allow} = 50$ MPa

$$A = \frac{\pi d^2}{4} = 12.5664 \text{ mm}^2$$

$$W = Mg = (60 \text{ kg})(9.81 \text{ m/s}^2) = 588.6 \text{ N}$$

TORQUE: $T = \dfrac{GI_p\phi_{max}}{L}$ (EQ. 3-15)

SHEAR STRESS: $\tau = \dfrac{Tr}{I_p}$ (EQ. 3-11)

$$\tau = \left(\frac{GI_p\phi_{max}}{L}\right)\left(\frac{r}{I_P}\right) = \frac{Gr\phi_{max}}{L} = (80 \times 10^6 \text{ Pa})\phi_{max}$$

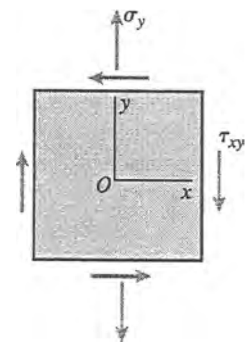
$\tau = 80\ \phi_{max}$ $\quad$ Units: τ = MPa $\quad \phi_{max}$ = radians

TENSILE STRESS $\quad \sigma_x = \dfrac{W}{A} = 46.839$ MPa

$\sigma_x = 0$

$\sigma_y = \sigma_t = 46.839$ MPa

$\tau_{xy} = -80\ \phi_{max}$ (MPa)



PRINCIPAL STRESSES

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_{1,2} = 23.420 \pm \sqrt{(23.420)^2 + 6400\phi_{max}^2} \quad \text{(MPa)}$$

Note that σ_1 is positive and σ_2 is negative. Therefore, the maximum in-plane shear stress is greater than the maximum out-of-plane shear stress.

MAXIMUM ANGLE OF ROTATION BASED ON TENSILE STRESS

σ_1 = maximum tensile stress $\quad \sigma_{allow} = 100$ MPa

$\therefore\ 100 \text{ MPa} = 23.420 \pm \sqrt{(23.420)^2 + 6400\phi_{max}^2}$

$(100 - 23.420)^2 = (23.420)^2 + 6400\phi_{max}^2$

$5316 = 6400\phi_{max}^2 \qquad \phi_{max} = 0.9114 \text{ rad} = 52.2°$

MAXIMUM ANGLE OF ROTATION BASED ON IN-PLANE SHEAR STRESS

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{(23.420)^2 + 6400\phi_{max}^2}$$

$\tau_{allow} = 50$ MPa $\qquad 50 = \sqrt{(23.420)^2 + 6400\phi_{max}^2}$

$(50)^2 = (23.420)^2 + 6400\phi_{max}^2$

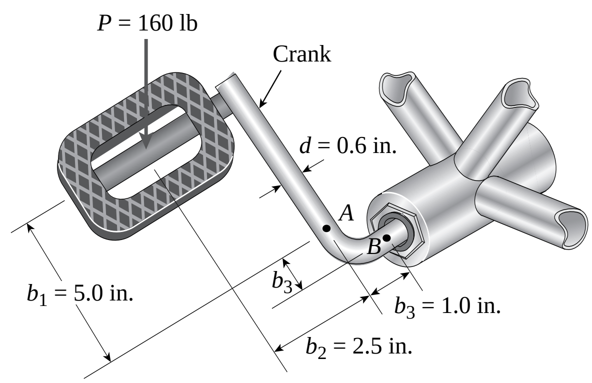
Solving, $\phi_{max} = 0.5522 \text{ rad} = 31.6°$

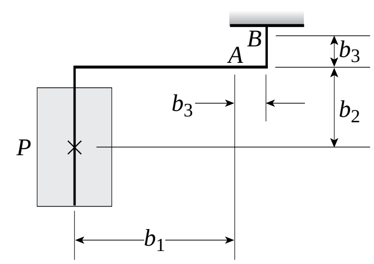
SHEAR STRESS GOVERNS

$\phi_{max} = 0.552 \text{ rad} = 31.6° \qquad \leftarrow$

Problem 8.5-9 Determine the maximum tensile, compressive, and shear stresses at points A and B on the bicycle pedal crank shown in the figure.

The pedal and crank are in a horizontal plane and points A and B are located on the top of the crank. The load $P = 160$ lb acts in the vertical direction and the distances (in the horizontal plane) between the line of action of the load and points A and B are $b_1 = 5.0$ in., $b_2 = 2.5$ in. and $b_3 = 1.0$ in. Assume that the crank has a solid circular cross section with diameter $d = 0.6$ in.





Top view

Solution 8.5-9

$P = 160$ lb $\quad d = 0.6$ in.

$b_1 = 5.0$ in. $\quad b_2 = 2.5$ in.

$b_3 = 1.0$ in. $\quad S = \dfrac{\pi d^3}{32}$

STRESS RESULTANTS on cross section at **point A**:

Torque: $T_A = Pb_2$

Moment: $M_A = Pb_1$

Shear Force: $V_A = P$

STRESS RESULTANTS at **point B**:

$T_B = P(b_1 + b_3)$

$M_B = P(b_2 + b_3)$

$V_B = P$

STRESS AT POINT A:

$$\tau = \frac{16\,T_A}{\pi d^3} \qquad \tau = 9.431 \times 10^3 \text{ psi}$$

$$\sigma = \frac{M_A}{S} \qquad \sigma = 3.773 \times 10^4 \text{ psi}$$

(The shear force V produces no shear stresses at point A)

PRINCIPAL STRESSES AND MAXIMUM SHEAR STRESS

$\sigma_x = 0 \qquad \sigma_y = \sigma \qquad \tau_{xy} = -\tau$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$\sigma_1 = 3.995 \times 10^4$ psi

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$\sigma_2 = -2.226 \times 10^3$ psi

MAXIMUM TENSILE STRESS

$\sigma_t = \sigma_1 \qquad \sigma_t = 39{,}950 \text{ psi} \qquad \longleftarrow$

MAXIMUM COMPRESSIVE STRESS

$\sigma_c = \sigma_2 \qquad \sigma_c = -2226 \text{ psi} \qquad \longleftarrow$

MAXIMUM IN-PLANE SHEAR STRESS

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$\tau_{\max} = 21{,}090 \text{ psi} \qquad \longleftarrow$

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

STRESS AT POINT B

$$\tau = \frac{16T_B}{\pi d^3} \qquad \tau = 2.264 \times 10^4 \text{ psi}$$

$$\sigma = \frac{M_B}{S} \qquad \sigma = 2.641 \times 10^4 \text{ psi}$$

(The shear force V produces no shear stresses at point A)

PRINCIPAL STRESSES AND MIXIMUM SHEAR STRESS

$\sigma_x = 0 \qquad \sigma_y = \sigma \qquad \tau_{xy} = -\tau$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$\sigma_1 = 3.941 \times 10^4 \text{ psi}$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$\sigma_2 = -1.3 \times 10^4$

MAXIMUM TENSILE STRESS

$\sigma_t = \sigma_1 \qquad \sigma_t = 39{,}410 \text{ psi} \qquad \longleftarrow$

MAXIMUM COMPRESSIVE STRESS

$\sigma_c = \sigma_2 \qquad \sigma_c = -13{,}000 \text{ psi} \qquad \longleftarrow$

MAXIMUM IN-PLANE SHEAR STRESS

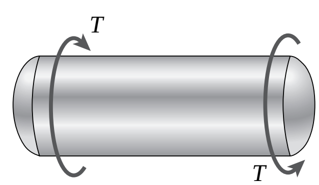
$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$\tau_{\max} = 26{,}210 \text{ psi} \qquad \longleftarrow$

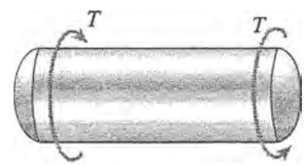
NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

Problem 8.5-10 A cylindrical pressure vessel having radius $r = 300$ mm and wall thickness $t = 15$ mm is subjected to internal pressure $p = 2.5$ MPa. In addition, a torque $T = 120$ kN·m acts at each end of the cylinder (see figure).

(a) Determine the maximum tensile stress $\sigma_{\max}$ and the maximum in-plane shear stress $\tau_{\max}$ in the wall of the cylinder.
(b) If the allowable in-plane shear stress is 30 MPa, what is the maximum allowable torque T?



Solution 8.5-10 Cylindrical pressure vessel



$T = 120\ \text{kN} \cdot \text{m} \qquad r = 300\ \text{mm}$

$t = 15\ \text{mm} \qquad P = 2.5\ \text{MPa}$

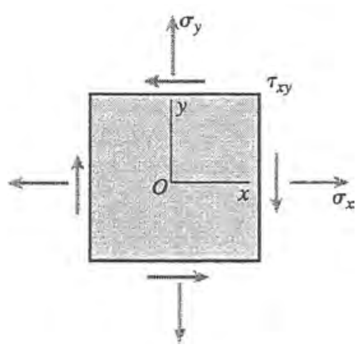
STRESSES IN THE WALL OF THE VESSEL

$$\sigma_x = \frac{pr}{2t} = 25\ \text{MPa} \qquad \sigma_y = \frac{pr}{t} = 50\ \text{MPa}$$

$$\tau_{xy} = -\frac{Tr}{Ip} \quad (\text{EQ. } 3\text{-}11)$$

$$I_p = 2\pi r^3 t \quad (\text{EQ. } 3\text{-}18)$$

$$\tau_{xy} = -\frac{T}{2\pi r^2 t} = -14.147\ \text{MPa}$$



(a) PRINCIPAL STRESSES

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= 37.5 \pm 18.878\ \text{MPa}$$

$\sigma_1 = 56.4\ \text{MPa} \qquad \sigma_2 = 18.6\ \text{MPa}$

$\therefore \sigma_{max} = 56.4\ \text{MPa} \quad \leftarrow$

MAXIMUM IN-PLANE SHEAR STRESS

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 18.9\ \text{MPa} \quad \leftarrow$$

(b) MAXIMUM ALLOWABLE TORQUE T

$\tau_{allow} = 30$ MPa (in-plane shear stress)

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad (1)$$

$$\sigma_x = \frac{pr}{2t} = 25\ \text{MPa} \quad \sigma_y = \frac{pr}{t} = 50\ \text{MPa}$$

$$\tau_{xy} = -\frac{T}{2\pi r^2 t} = -117.893 \times 10^{-6}\, T$$

Units: $\tau_{xy} = \text{MPa} \qquad T = \text{N} \cdot \text{m}$

Substitute into Eq. (1):

$$\tau_{max} = \tau_{allow} = 30\ \text{MPa}$$
$$= \sqrt{(-12.5\ \text{MPa})^2 + (-117.893 \times 10^{-6}\, T)^2}$$

Square both sides, rearrange, and solve for T:

$$(30)^2 = (12.5)^2 + (117.893 \times 10^{-6})^2\, T^2$$

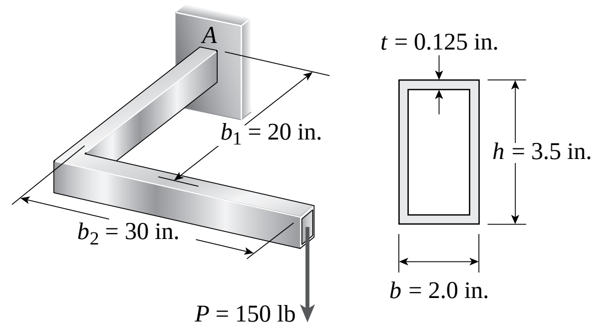
$$T^2 = \frac{743.750}{13{,}899 \times 10^{-12}} = 53{,}512 \times 10^6\, (\text{N} \cdot \text{m})^2$$

$$T = 231.3 \times 10^3\ \text{N} \cdot \text{m}$$

$$T_{max} = 231\ \text{kN} \cdot \text{m} \quad \leftarrow$$

Problem 8.5-11 An L-shaped bracket lying in a horizontal plane supports a load $P = 150$ lb (see figure). The bracket has a hollow rectangular cross section with thickness $t = 0.125$ in. and outer dimensions $b = 2.0$ in. and $h = 3.5$ in. The centerline lengths of the arms are $b_1 = 20$ in. and $b_2 = 30$ in.

Considering only the load P, calculate the maximum tensile stress σ_t, maximum compressive stress σ_c, and maximum shear stress τ_{max} at point A, which is located on the top of the bracket at the support.

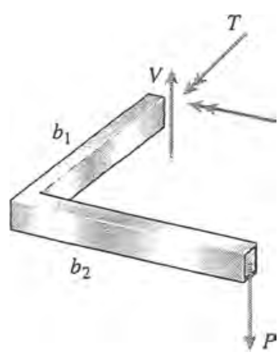


Solution 8.5-11 L-shaped bracket

$P = 150$ lb $\quad b_1 = 20$ in. $\quad b_2 = 30$ in.

$t = 0.125$ in. $\quad h = 3.5$ in. $\quad b = 2.0$ in.

FREE-BODY DIAGRAM OF BRACKET



STRESS RESULTANTS AT THE SUPPORT

Torque: $T = Pb_2 = (150 \text{ lb})(30 \text{ in.}) = 4500 \text{ lb-in.}$

Moment: $M = Pb_1 = (150 \text{ lb})(20 \text{ in.}) = 3000 \text{ lb-in.}$

Shear force: $V = P = 150$ lb

PROPERTIES OF THE CROSS SECTION

For torsion:

$A_m = (b - t)(h - t) = (1.875 \text{ in.})(3.375 \text{ in.}) = 6.3281 \text{ in.}^2$

For bending: $c = \dfrac{h}{2} = 1.75$ in.

$$I = \frac{1}{12}(bh^3) - \frac{1}{12}(b - 2t)(h - 2t)^3$$

$$= \frac{1}{12}(2.0 \text{ in.})(3.5 \text{ in.})^3 - \frac{1}{12}(1.75 \text{ in.})(3.25 \text{ in.})^3$$

$$= 2.1396 \text{ in.}^4$$

STRESSES AT POINT A ON THE TOP OF THE BRACKET

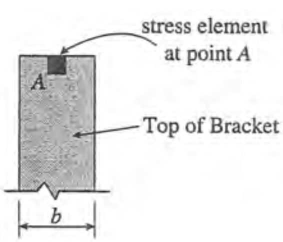
$$\tau = \frac{T}{2tA_m} = \frac{4500 \text{ lb-in.}}{2(0.125 \text{ in.})(6.3281 \text{ in.}^2)} = 2844 \text{ psi}$$

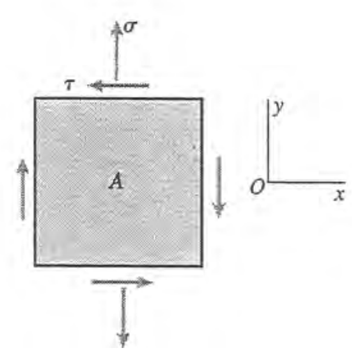
$$\sigma = \frac{Mc}{I} = \frac{(3000 \text{ lb-in.})(1.75 \text{ in.})}{2.1396 \text{ in.}^4} = 2454 \text{ psi}$$

(The shear force V produces no stresses at point A.)

STRESS ELEMENT AT POINT A

(This view is looking downward at the top of the bracket.)





$\sigma_x = 0 \quad \sigma_y = \sigma = 2454$ psi

$\tau_{xy} = -\tau = -2844$ psi

PRINCIPAL STRESSES AND MAXIMUM SHEAR STRESS

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= 1227 \text{ psi} \pm 3097 \text{ psi}$$

$$\sigma_1 = 4324 \text{ psi} \qquad \sigma_2 = -1870 \text{ psi}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 3097 \text{ psi}$$

MAXIMUM TENSILE STRESS:

$$\sigma_t = 4320 \text{ psi} \qquad \leftarrow$$

MAXIMUM COMPRESSIVE STRESS:

$$\sigma_c = -1870 \text{ psi} \qquad \leftarrow$$

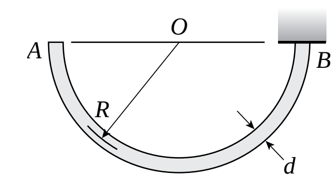
MAXIMUM SHEAR STRESS:

$$\tau_{max} = 3100 \text{ psi} \qquad \leftarrow$$

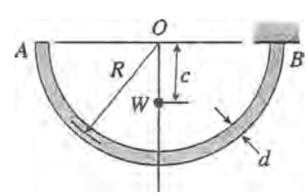
NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear stress is larger than the maximum out-of-plane shear stress.

Problem 8.5-12 A semicircular bar AB lying in a horizontal plane is supported at B (see figure). The bar has centerline radius R and weight q per unit of length (total weight of the bar equals πqR). The cross section of the bar is circular with diameter d.

Obtain formulas for the maximum tensile stress σ_t, maximum compressive stress σ_c, and maximum in-plane shear stress τ_{max} at the top of the bar at the support due to the weight of the bar.



Solution 8.5-12 Semicircular bar



d = diameter of bar $\qquad R$ = radius of bar

q = weight of bar per unit length

W = weight of bar = πqR

Weight of bar acts at the center of gravity

From Case 23, Appendix D, with $\beta = \pi/2$, we get

$$\bar{y} = \frac{2R}{\pi} \qquad \therefore\ c = \frac{2R}{\pi}$$

Bending moment at B: $M_B = Wc = 2qR^2$

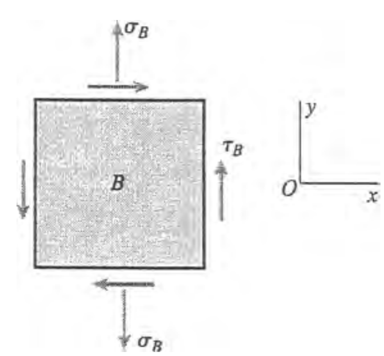
Torque at B: $T_B = WR = \pi qR^2$

(Shear force at B produces no shear stress at the top of the bar.)

STRESSES AT THE TOP OF THE BAR AT B

$$\sigma_B = \frac{M_B(d/2)}{I} = \frac{(2qR^2)(d/2)}{\pi d^4/64} = \frac{64qR^2}{\pi d^3}$$

$$\tau_B = \frac{T_B(d/2)}{I_P} = \frac{(\pi qR^2)(d/2)}{\pi d^4/32} = \frac{16qR^2}{d^3}$$



STRESS ELEMENT AT THE TOP OF THE BAR AT B

$$\sigma_x = 0$$

$$\sigma_y = \sigma_B$$

$$\tau_{xy} = \tau_B$$

PRINCIPAL STRESSES:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{\sigma_B}{2} \pm \sqrt{\left(-\frac{\sigma_B}{2}\right)^2 + \tau_B^2}$$

$$= \frac{32qR^2}{\pi d^3} \pm \sqrt{\left(\frac{32qR^2}{\pi d^3}\right)^2 + \left(\frac{16qR^2}{d^3}\right)^2}$$

$$= \frac{16qR^2}{\pi d^3}(2 \pm \sqrt{4 + \pi^2})$$

MAXIMUM TENSILE STRESS

$$\sigma_t = \sigma_1 = \frac{16qR^2}{\pi d^3}(2 + \sqrt{4 + \pi^2})$$

$$= 29.15\frac{qR^2}{d^3} \qquad \leftarrow$$

MAXIMUM COMPRESSIVE STRESS

$$\sigma_c = \sigma_2 = \frac{16qR^2}{\pi d^3}(2 - \sqrt{4 + \pi^2})$$

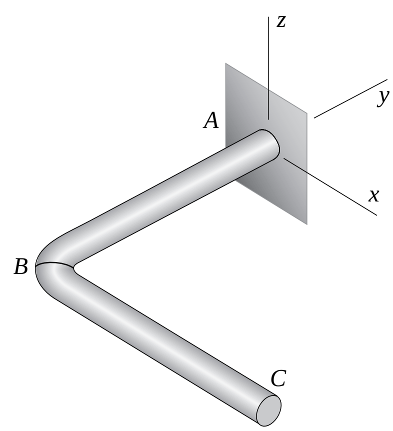
$$= -8.78\frac{qR^2}{d^3} \qquad \leftarrow$$

MAXIMUM IN-PLANE SHEAR STRESS (EQ. 7-26)

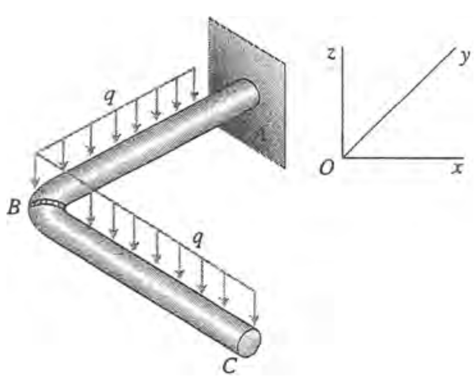
$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{16qR^2}{\pi d^3}\sqrt{4 + \pi^2} \qquad \leftarrow$$

Problem 8.5-13 An arm *ABC* lying in a horizontal plane and supported at *A* (see figure) is made of two identical solid steel bars *AB* and *BC* welded together at a right angle. Each bar is 20 in. long.

Knowing that the maximum tensile stress (principal stress) at the top of the bar at support *A* due solely to the weights of the bars is 932 psi, determine the diameter *d* of the bars.



Solution 8.5-13 Horizontal arm *ABC*



L = length of *AB* and *BC*

d = diameter of *AB* and *BC*

A = cross-sectional area

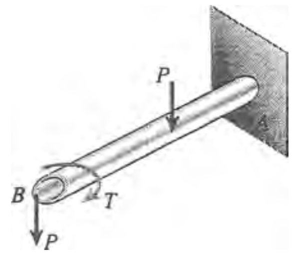
$= \pi d^2/4$

γ = weight density of steel

q = weight per unit length of bars

$$= \gamma A = \pi \gamma d^2/4 \qquad (1)$$

RESULTANT FORCES ACTING ON AB



P = weight of AB and BC

$$P = qL = \pi\gamma L d^2/4 \qquad (2)$$

T = torque due to weight of BC

$$T = (qL)\left(\frac{L}{2}\right) = \frac{qL^2}{2} = \frac{\pi\gamma L^2 d^2}{8} \qquad (3)$$

M_A = bending moment at A

$$M_A = PL + PL/2 = 3PL/2 = 3\pi\gamma L^2 d^2/8 \qquad (4)$$

STRESSES AT THE TOP OF THE BAR AT A

σ_A = normal stress due to M_A

$$\sigma_A = \frac{M(d/2)}{I} = \frac{M(d/2)}{\pi d^4/64} = \frac{32M}{\pi d^3} = \frac{12\gamma L^2}{d} \qquad (5)$$

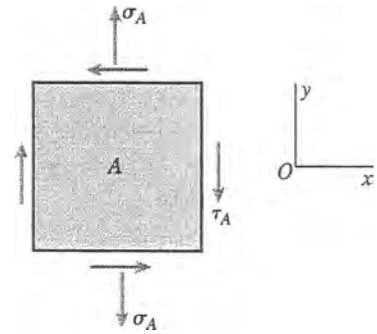
τ_A = shear stress due to torque T

$$\tau_A = \frac{T(d/2)}{I_p} = \frac{T(d/2)}{\pi d^4/32} = \frac{16T}{\pi d^3} = \frac{2\gamma L^2}{d} \qquad (6)$$

STRESS ELEMENT ON TOP OF THE BAR AT A

σ_1 = principal tensile stress (maximum tensile stress)

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad (7)$$



$$\sigma_x = 0 \qquad \sigma_y = \sigma_A \qquad \tau_{xy} = -\tau_A \qquad (8)$$

Substitute (8) into (7):

$$\sigma_1 = \frac{\sigma_A}{2} + \sqrt{\left(\frac{\sigma_A}{2}\right)^2 + \tau_A^2} \qquad (9)$$

Substitute from (5) and (6) and simplify:

$$\sigma_1 = \frac{\gamma L^2}{d}(6 + \sqrt{40}) = \frac{2\gamma L^2}{d}(3 + \sqrt{10}) \qquad (10)$$

SOLVE FOR d

$$d = \frac{2\gamma L^2}{\sigma_1}(3 + \sqrt{10}) \quad \longleftarrow \qquad (11)$$

SUBSTITUTE NUMERICAL VALUES INTO EQ. (11):

$\gamma = 490\ \text{lb/ft}^3 = 0.28356\ \text{lb/in.}^3$

$L = 20$ in. $\qquad \sigma_1 = 932$ psi

$d = 1.50$ in. $\quad \longleftarrow$

Problem 8.5-14 A pressurized cylindrical tank with flat ends is loaded by torques T and tensile forces P (see figure). The tank has radius $r = 50$ mm and wall thickness $t = 3$ mm. The internal pressure $p = 3.5$ MPa and the torque $T = 450$ N·m.

What is the maximum permissible value of the forces P if the allowable tensile stress in the wall of the cylinder is 72 MPa?

Solution 8.5-14 Cylindrical tank

$r = 50\text{ mm} \qquad t = 3.0\text{ mm} \qquad p = 3.5\text{ MPa}$

$T = 450\text{ N}\cdot\text{m} \qquad \sigma_{allow} = 72\text{ MPa}$

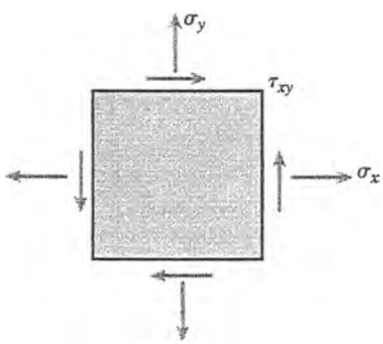
CROSS SECTION

$A = 2\pi rt = 2\pi(50\text{ mm})(3.0\text{ mm}) = 942.48\text{ mm}^2$

$I_P = 2\pi r^3 t = 2\pi(50\text{ mm})^3(3.0\text{ mm})$

$= 2.3562 \times 10^6\text{ mm}^4$

STRESSES IN THE WALL OF THE TANK



$$\sigma_x = \frac{pr}{2t} + \frac{P}{A}$$

$$= \frac{(3.5\text{ MPa})(50\text{ mm})}{2(3.0\text{ mm})} + \frac{P}{942.48\text{ mm}^2}$$

$$= 29.167\text{ MPa} + 1.0610 \times 10^{-3}P$$

Units: σ_x = MPa, P = newtons

$$\sigma_y = \frac{pr}{t} = 58.333\text{ MPa}$$

$$\tau_{xy} = -\frac{Tr}{I_P} = -\frac{(450\text{ N}\cdot\text{m})(50\text{ mm})}{2.3562 \times 10^6\text{ mm}^4}$$

$$= -9.5493\text{ MPa}$$

MAXIMUM TENSILE STRESS

$$\sigma_{max} = \sigma_{allow} = 72\text{ MPa}$$

$$= \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$72 = 43.750 + (530.52 \times 10^{-6})P$$

$$+ \sqrt{[-14.583 + (530.52 \times 10^{-6})P]^2 + (-9.5493)^2}$$

or

$$28.250 - 0.00053052P$$

$$= \sqrt{(-14.583 + 0.00053052\,P)^2 + 91.189}$$

Square both sides and simplify:

$$494.21 = 0.014501\,P$$

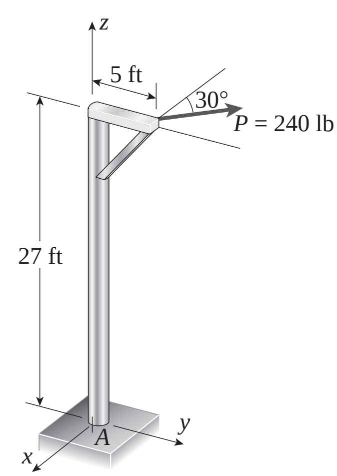
SOLVE FOR P $\qquad P = 34{,}080\text{ N}$ $\qquad$ OR

$P_{max} = 34.1\text{ kN}$ $\quad \longleftarrow$

Problem 8.5-15 A post having a hollow circular cross section supports a horizontal load $P = 240$ lb acting at the end of an arm that is 5 ft long (see figure). The height of the post is 27 ft, and its section modulus is $S = 15\text{ in.}^3$ Assume that outer radius of the post, $r_2 = 4.5$ in., and inner radius $r_1 = 4.243$ in.

(a) Calculate the maximum tensile stress σ_{max} and maximum in-plane shear stress τ_{max} at point A on the outer surface of the post along the x-axis due to the load P. Load P acts in a horizontal plane at an angle of 30° from a line which is parallel to the $(-x)$ axis.

(b) If the maximum tensile stress and maximum in-plane shear stress at point A are limited to 16,000 psi and 6000 psi, respectively, what is the largest permissible value of the load P?



Solution 8.5-15

$P = 240$ lb

$b = 5$ ft length of arm

$h = 27$ ft height of post

$r_2 = 4.5$ in. $r_1 = 4.243$ in.

$t = r_2 - r_1$ $t = 0.257$ in.

$A = \pi(r_2^2 - r_1^2)$ $A = 7.059$ in.^{2}

$I = \frac{\pi}{4}(r_2^4 - r_1^4)$ $I = 67.507$ in.^{4}

$I_p = 2I$ $I_p = 135.014$ in.^{4}

$Q = \frac{2}{3}(r_2^3 - r_1^3)$ $Q = 9.825$ in.^{3}

REACTIONS AT THE SUPPORT

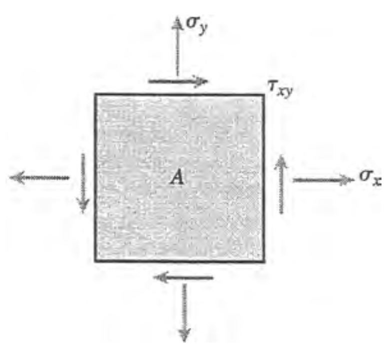
$M = P\cos(30\text{deg})h$ $M = 6.734 \times 10^4$ lb · in

$T = -P\cos(30\text{deg})b$ $T = -1.247 \times 10^4$ lb · in

$V_x = P\cos(30\text{deg})$ $V_x = 207.846$ lb

$V_y = -P\sin(30\text{deg})$ $V_y = -120$ lb

STRESSES AT POINT A



$\sigma_x = 0$

$\sigma_y = \frac{Mr_2}{I}$ $\sigma_y = 4.489 \times 10^3$ psi

$\tau = \frac{Tr_2}{I_p} + \frac{V_y Q}{I2t}$ $\tau = -449.628$ psi

(The shear force V_x produces no stress at point A)

(a) MAXIMUM TENSILE STRESS AND MAXIMUM SHEAR STRESS

$\sigma_x = 0$ $\sigma_y = 4.489 \times 10^3$ psi $\tau_{xy} = \tau$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$\sigma_1 = 4.534 \times 10^3$ psi

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$\sigma_2 = -44.593$ psi

MAXIMUM TENSILE STRESS

$\sigma_{max} = \sigma_1$ $\sigma_{max} = 4534$ psi $\leftarrow$

MAXIMUM SHEAR STRESS

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$\tau_{max} = 2289$ psi $\leftarrow$

(b) ALLOWABLE LOAD P

$\sigma_{allow} = 16000$ psi $\tau_{allow} = 6000$ psi

The stresses at point A are proportional to the load P.

Based on tensile stress:

$\frac{P_{allow}}{P} = \frac{\sigma_{allow}}{\sigma_{max}}$ $P_{allow} = \frac{\sigma_{allow} P}{\sigma_{max}}$

$P_{allow} = 847.01$ lb

Based on shear stress:

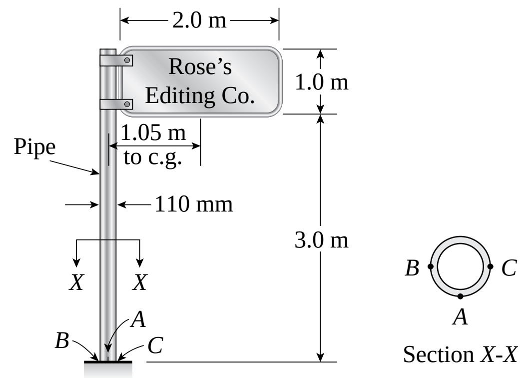
$\frac{P_{allow}}{P} = \frac{\tau_{allow}}{\tau_{max}}$ $P_{allow} = \frac{\tau_{allow} P}{\tau_{max}}$

$P_{allow} = 629.07$ lb

$P_{allow} = 629$ lb $\leftarrow$

Problem 8.5-16 A sign is supported by a pipe (see figure) having outer diameter 110 mm and inner diameter 90 mm. The dimensions of the sign are 2.0 m × 1.0 m, and its lower edge is 3.0 m above the base. Note that the center of gravity of the sign is 1.05 m from the axis of the pipe. The wind pressure against the sign is 1.5 kPa.

Determine the maximum in-plane shear stresses due to the wind pressure on the sign at points *A*, *B*, and *C*, located on the outer surface at the base of the pipe.



Solution 8.5-16

PIPE: $d_2 = 110$ mm $\quad d_1 = 90$ mm $\quad t = 10$ mm

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) \qquad I = 3.966 \times 10^6 \text{ mm}^4$$

$$I_p = 2I \qquad I_p = 7.933 \times 10^6 \text{ mm}^4$$

$$Q = \frac{1}{12}(d_2^3 - d_1^3)$$

$$Q = 5.017 \times 10^4 \text{ mm}^3$$

SIGN: $A = 2\text{m}^2$ Area

$h = \left(3 + \frac{1}{2}\right)$ m Height from the base to the center of gravity of the sign

$b = 1.05$ m horizontal distance from the center of gravity of the sign to the axis of the pipe

WIND PRESSURE $p_w = 1.5$ kPa

$$P = p_w A \qquad P = 3 \text{ kN}$$

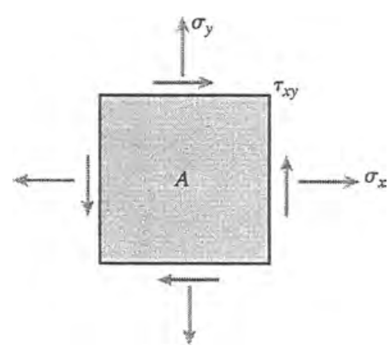
STRESS RESULTANTS AT THE BASE

$M = Ph \qquad M = 10.5$ kN·m

$T = Pb \qquad T = 3.15$ kN·m

$V = P \qquad V = 3$ kN

MAXIMUM SHEAR STRESS AT POINT *A*



$$\sigma_x = 0$$

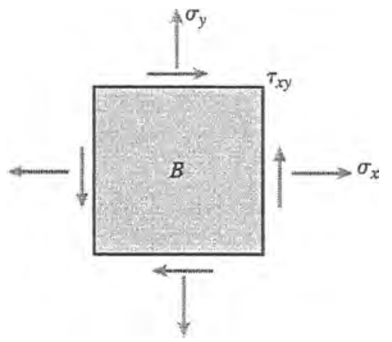
$$\sigma_y = \frac{Md_2}{2I} \qquad \sigma_y = 145.603 \text{ MPa}$$

$$\tau_{xy} = \frac{Td_2}{2I_p} \qquad \tau_{xy} = 21.84 \text{ MPa}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{max} = 76.0 \text{ MPa} \quad \leftarrow$$

MAXIMUM SHEAR STRESS AT POINT B



$\sigma_x = 0$

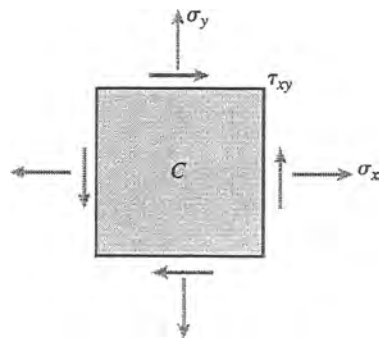
$\sigma_y = 0$

$$\tau_{xy} = \frac{Td_2}{2I_p} - \frac{VQ}{I(2t)} \qquad \tau_{xy} = 19.943 \text{ MPa}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Pure shear $\quad \tau_{max} = 19.94 \text{ MPa} \quad \leftarrow$

MAXIMUM SHEAR STRESS AT POINT C



$\sigma_x = 0$

$\sigma_y = 0$

$$\tau_{xy} = \frac{Td_2}{2I_p} + \frac{VQ}{I(2t)} \qquad \tau_{xy} = 23.738 \text{ MPa}$$

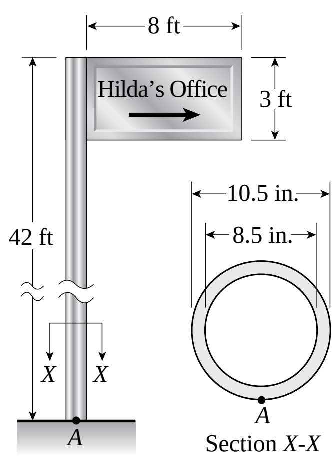
$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Pure shear $\quad \tau_{max} = 23.7 \text{ MPa} \quad \leftarrow$

Problem 8.5-17 A sign is supported by a pole of hollow circular cross section, as shown in the figure. The outer and inner diameters of the pole are 10.5 in. and 8.5 in., respectively. The pole is 42 ft high and weighs 4.0 k. The sign has dimensions 8 ft × 3 ft and weighs 500 lb. Note that its center of gravity is 53.25 in. from the axis of the pole. The wind pressure against the sign is 35 lb/ft^2.

(a) Determine the stresses acting on a stress element at point A, which is on the outer surface of the pole at the "front" of the pole, that is, the part of the pole nearest to the viewer.

(b) Determine the maximum tensile, compressive, and shear stresses at point A.



Solution 8.5-17

PIPE: $d_2 = 10.5$ in. $\quad d_1 = 8.5$ in. $\quad c = \dfrac{d_2}{2}$

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) \qquad A = 29.845 \text{ in.}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) \qquad I = 340.421 \text{ in.}^4$$

$$I_p = 2I \qquad I_p = 680.842 \text{ in.}^4$$

$$Q = \frac{1}{12}(d_2^3 - d_1^3) \qquad Q = 45.292 \text{ in.}^3$$

$$W_1 = 4000 \text{ lb}$$

SIGN: $A_s = (8)(3)$ ft^2 $\quad$ Area $\quad A_s = 24$ ft^2

Height from the base to the center of gravity of the sign

$$h = \left(42 - \frac{3}{2}\right) \text{ ft}$$

Horizontal distance from the center of gravity of the sign to the axis of the pipe

$$b = \left((4)12 + \frac{10.5}{2}\right) \text{ in.}$$

$$b = 53.25 \text{ in.}$$

$$W_2 = 500 \text{ lb}$$

WIND PRESSURE $\quad p_w = 35$ lb/ft^2

$$P = p_w A_s \qquad P = 840 \text{ lb}$$

STRESS RESULTANTS AT THE BASE

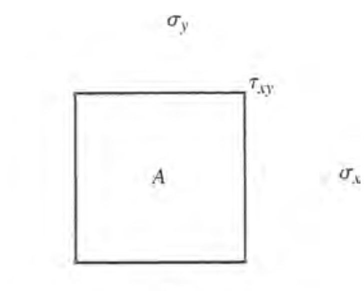
$$M = Ph \qquad M = 4.082 \times 10^5 \text{ lb} \cdot \text{in.}$$

$$T = Pb \qquad T = 4.473 \times 10^4 \text{ lb} \cdot \text{in.}$$

$$V = P \qquad V = 840 \text{ lb}$$

$$N_z = W_1 + W_2 \qquad N_z = 4.5 \times 10^3 \text{ lb}$$

(a) STRESSES AT POINT A



$$\sigma_x = 0 \quad \leftarrow$$

$$\sigma_y = -\frac{N_z}{A} + \frac{Md_2}{2I} \qquad \sigma_y = 6145 \text{ psi} \quad \leftarrow$$

$$\sigma_{xy} = \frac{Td_2}{2I_p} \qquad \tau_{xy} = 345 \text{ psi} \quad \leftarrow$$

(b) MAXIMUM STRESSES AT POINT A

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Max. tensile stress $\quad \sigma_1 = 6164$ psi $\quad \leftarrow$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Max. compressive stress

$$\sigma_2 = -19.30 \text{ psi} \quad \leftarrow$$

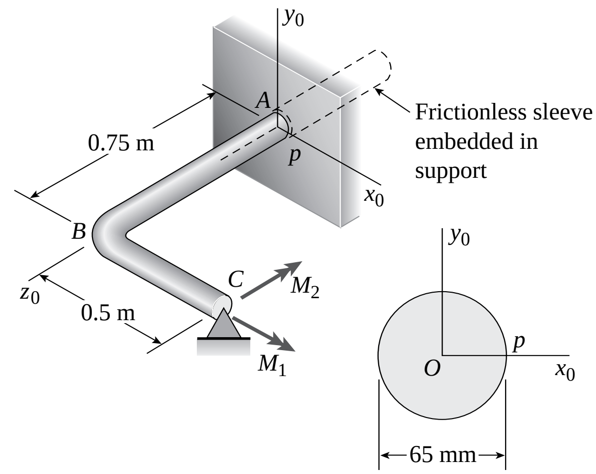
$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Max. shear stress $\quad \tau_{max} = 3092$ psi $\quad \leftarrow$

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

Problem 8.5-18 A horizontal bracket ABC consists of two perpendicular arms AB of length 0.5 m, and BC of length of 0.75 m. The bracket has a solid circular cross section with diameter equal to 65 mm. The bracket is inserted in a frictionless sleeve at A (which is slightly larger in diameter) so is free to rotate about the z_0 axis at A, and is supported by a pin at C. Moments are applied at point C as follows: $M_1 = 1.5\ \text{kN}\cdot\text{m}$ in the x-direction and $M_2 = 1.0\ \text{kN}\cdot\text{m}$ acts in the $(-z)$ direction.

Considering only the moments M_1 and M_2, calculate the maximum tensile stress σ_t, the maximum compressive stress σ_c, and the maximum in-plane shear stress τ_{max} at point p, which is located at support A on the side of the bracket at midheight.



Cross section at A

Solution 8.5-18

$d = 65$ mm

$b_1 = 0.5$ m length of arm BC

$b_2 = 0.75$ m length of arm AB

$M_1 = 1.5\ \text{kN}\cdot\text{m}$

$M_2 = 1.0\ \text{kN}\cdot\text{m}$

PROPERTIES OF THE CROSS SECTION

$d = 65\ \text{mm} \qquad r = \dfrac{d}{2}$

$A = \dfrac{\pi}{4}d^2 \qquad A = 3.318 \times 10^3\ \text{mm}^2$

$I = \dfrac{\pi}{64}d^4 \qquad I = 8.762 \times 10^5\ \text{mm}^4$

$I_p = 2I \qquad I_p = 1.752 \times 10^6\ \text{mm}^4$

$Q = \dfrac{2}{3}r^3 \qquad Q = 2.289 \times 10^4\ \text{mm}^3$

STRESS RESULTANTS AT SUPPORT A

$N_z = 0$ Axial force

$M_y = 0$

$M_x = -M_2\dfrac{b_2}{b_1} + M_1 \qquad M_x = 0$

$T = 0$ Torsional frictionless sleeve at support A (M_Z)

$V_y = -\dfrac{M_2}{b_1}$

$\sigma_x = 0$

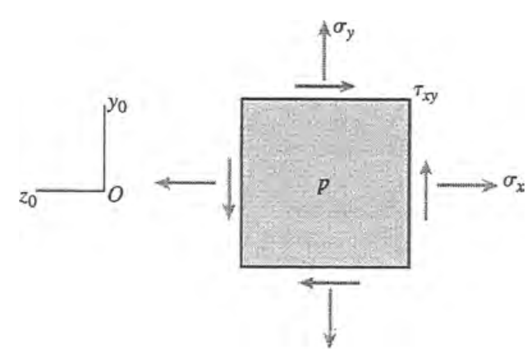
$\sigma_y = 0$

$\tau_{xy} = \dfrac{V_y Q}{Id} \qquad \tau_{xy} = -0.804\ \text{MPa}$

$\tau_{max} = \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

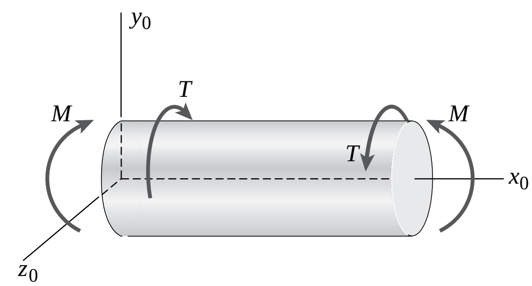
Pure Shear $\tau_{max} = 0.804\ \text{MPa}$ ←

STRESSES AT POINT p ON THE SIDE OF THE BRACKET



Problem 8.5-19 A cylindrical pressure vessel with flat ends is subjected to a torque T and a bending moment M (see figure). The outer radius is 12.0 in. and the wall thickness is 1.0 in. The loads are as follows: $T = 800$ k-in., $M = 1000$ k-in., and the internal pressure $p = 900$ psi.

Determine the maximum tensile stress σ_t, maximum compressive stress σ_c, and maximum shear stress τ_{max} in the wall of the cylinder.



Solution 8.5-19 Cylindrical pressure vessal

Internal pressure: $p = 900$ psi

Bending moment: $M = 1000$ k-in.

Torque: $T = 800$ k-in.

Outer radius: $r_2 = 12$ in.

Wall thickness: $t = 1.0$ in.

Mean radius: $r = r_2 - t/2 = 11.5$ in.

Outer diameter: $d_2 = 24$ in.

Inner diameter: $d_1 = 22$ in.

MOMENT OF INERTIA

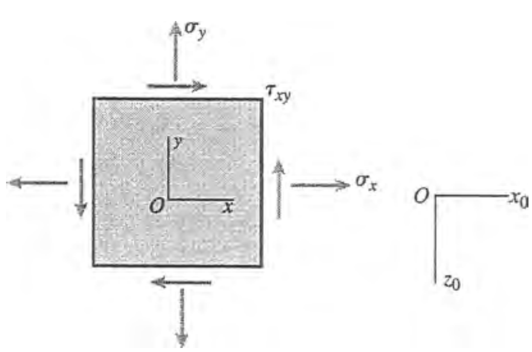
$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 4787.0 \text{ in.}^4$$

$$I_p = 2I = 9574.0 \text{ in.}^4$$

NOTE: Since the stresses due to T and p are the same everywhere in the cylinder, the maximum stresses occur at the top and bottom of the cylinder where the bending stresses are the largest.

PART (a). TOP OF THE CYLINDER

Stress element on the top of the cylinder as seen from above.



$$\sigma_x = \frac{pr}{2t} - \frac{Mr_2}{I} = 5175.0 \text{ psi} - 2506.8 \text{ psi}$$
$$= 2668.2 \text{ psi}$$

$$\sigma_y = \frac{pr}{t} = 10{,}350 \text{ psi}$$

$$\tau_{xy} = \frac{Tr_2}{I_p} = -1002.7 \text{ psi}$$

PRINCIPAL STRESSES

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$
$$= 6509.1 \text{ psi} \pm 3969.6 \text{ psi}$$
$$\sigma_1 = 10{,}479 \text{ psi} \qquad \sigma_2 = 2540 \text{ psi}$$

MAXIMUM SHEAR STRESSES

In-plane: $\tau = 3970$ psi

Out-of-plane:

$$\tau = \frac{\sigma_1}{2} \quad or \quad \frac{\sigma_2}{2} \qquad \tau = \frac{\sigma_1}{2} = 5240 \text{ psi}$$

$$\therefore \tau_{max} = 5240 \text{ psi}$$

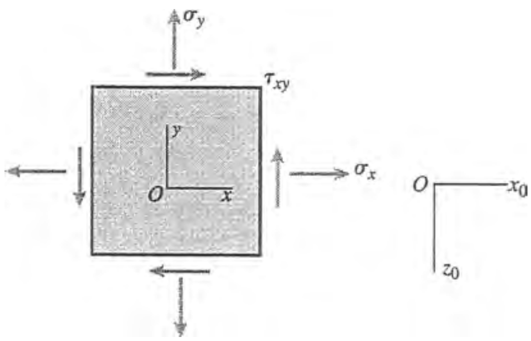
MAXIMUM STRESSES FOR THE TOP OF THE CYLINDER

$\sigma_t = 10{,}480$ psi $\qquad \sigma_c = 0$ (No compressive stresses)

$\tau_{max} = 5240$ psi

PART (b). BOTTOM OF THE CYLINDER

Stress element on the bottom of the cylinder as seen from below.



$$\sigma_x = \frac{pr}{2t} + \frac{Mr_2}{I} = 5175.0 \text{ psi} + 2506.8 \text{ psi}$$

$$= 7681.8 \text{ psi}$$

$$\sigma_y = \frac{pr}{t} = 10{,}350 \text{ psi}$$

$$\tau_{xy} = -\frac{Tr_2}{I_p} = -1002.7 \text{ psi}$$

PRINCIPAL STRESSES

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= 9015.9 \text{ psi} \pm 1668.9 \text{ psi}$$

$$\sigma_1 = 10{,}685 \text{ psi} \qquad \sigma_2 = 7347 \text{ psi}$$

MAXIMUM SHEAR STRESSES

In-plane: $\tau = 1669$ psi

Out-of-plane:

$$\tau = \frac{\sigma_1}{2} \quad or \quad \frac{\sigma_2}{2} \qquad \tau = \frac{\sigma_1}{2} = 5340 \text{ psi}$$

$$\therefore \tau_{max} = 5340 \text{ psi}$$

MAXIMUM STRESSES FOR THE BOTTOM OF THE CYLINDER

$\sigma_t = 10{,}680$ psi $\quad \sigma_c = 0$ (No compressive stresses)

$\tau_{max} = 5340$ psi

PART (c). ENTIRE CYLINDER

The largest stresses are at the bottom of the cylinder.

$\sigma_t = 10{,}680$ psi $\quad \leftarrow$

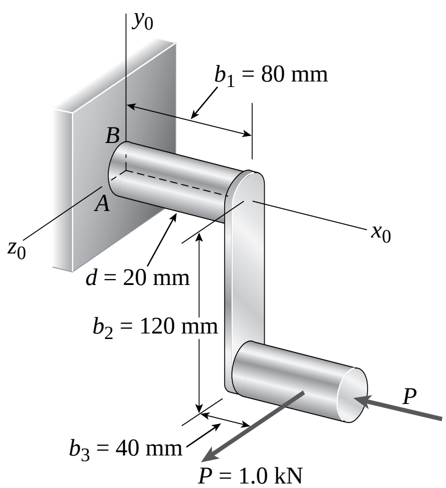
$\sigma_c = 0$ (No compressive stresses) $\quad \leftarrow$

$\tau_{max} = 5340$ psi $\quad \leftarrow$

Problem 8.5-20 For purposes of analysis, a segment of the crankshaft in a vehicle is represented as shown in the figure. Two loads P act as shown, one parallel to $(-x_0)$ and another parallel to z_0; each load P equals 1.0 kN. The crankshaft dimensions are $b_1 = 80$ mm, $b_2 = 120$ mm, and $b_3 = 40$ mm. The diameter of the upper shaft is $d = 20$ mm.

(a) Determine the maximum tensile, compressive, and shear stresses at point A, which is located on the surface of the upper shaft at the z_0 axis.

(b) Determine the maximum tensile, compressive, and shear stresses at point B, which is located on the surface of the shaft at the y_0 axis.



Solution 8.5-20

$P = 1.0\text{ kN}$

$b_1 = 80\text{ mm}$

$b_2 = 120\text{ mm}$

$b_3 = 40\text{ mm}$

PROPERTIES OF THE CROSS SECTION

$d = 20\text{ mm} \qquad r = \dfrac{d}{2}$

$A = \dfrac{\pi}{4}d^2 \qquad A = 314.159\text{ mm}^2$

$I = \dfrac{\pi}{64}d^4 \qquad I = 7.854 \times 10^3\text{ mm}^4$

$I_p = 2I \qquad I_p = 1.571 \times 10^4\text{ mm}^4$

$Q = \dfrac{2}{3}r^3 \quad Q = 666.667\text{ mm}^3$

STRESS RESULTANTS AT THE SUPPORT

$V_x = P$ (Axial force in X-dir.)

$V_y = 0$ (Shear force in Y-dir.)

$V_z = P$ (Shear force in Z-dir.)

$M_x = Pb_2$ (Torsional Moment)

$M_x = 120\text{ kN}\cdot\text{mm}$

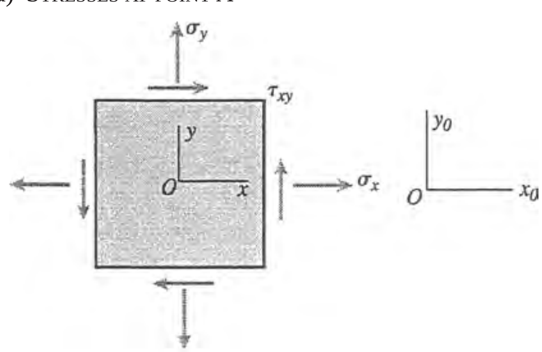
$M_y = P(b_1 + b_3)$ (Bending Moment)

$M_y = 120\text{ kN}\cdot\text{mm}$

$M_z = Pb_2$ (Bending Moment)

$M_z = 120\text{ kN}\cdot\text{mm}$

(a) STRESSES AT POINT A



$$\sigma_x = -\frac{V_x}{A} - \frac{M_y}{I}r$$

$\sigma_x = -155.972\text{ MPa}$ (compressive)

$\sigma_y = 0$

$\tau_{xy} = \dfrac{M_x d}{2I_p} \qquad \tau_{xy} = 76.394\text{ MPa}$

$$\sigma_1 = \frac{\sigma_x - \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

MAX. TENSILE STRESS $\quad \sigma_1 = 31.2\text{ MPa} \quad \leftarrow$

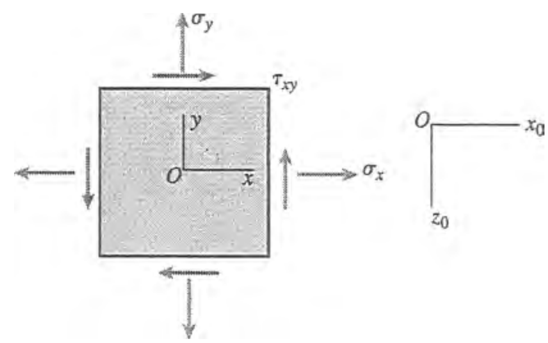
MAX. COMPRESSIVE STRESS

$\sigma_2 = -187.2\text{ MPa} \quad \leftarrow$

MAX. SHEAR STRESS $\quad \tau_{max} = 109.2\text{ MPa} \quad \leftarrow$

(b) STRESSES AT POINT B

$$\sigma_x = -\frac{V_x}{A} + \frac{M_z}{I}r$$



$\sigma_x = 149.606\text{ MPa}$ (tensile)

$\sigma_y = 0$

$\tau_{xy} = \dfrac{M_x d}{2I_p} + \dfrac{V_z Q}{Id} \qquad \tau_{xy} = 80.639\text{ MPa}$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

MAX. TENSILE STRESS $\sigma_1 = 184.8$ MPa ←

MAX. COMPRESSIVE STRESS

$\sigma_2 = -35.2$ MPa ←

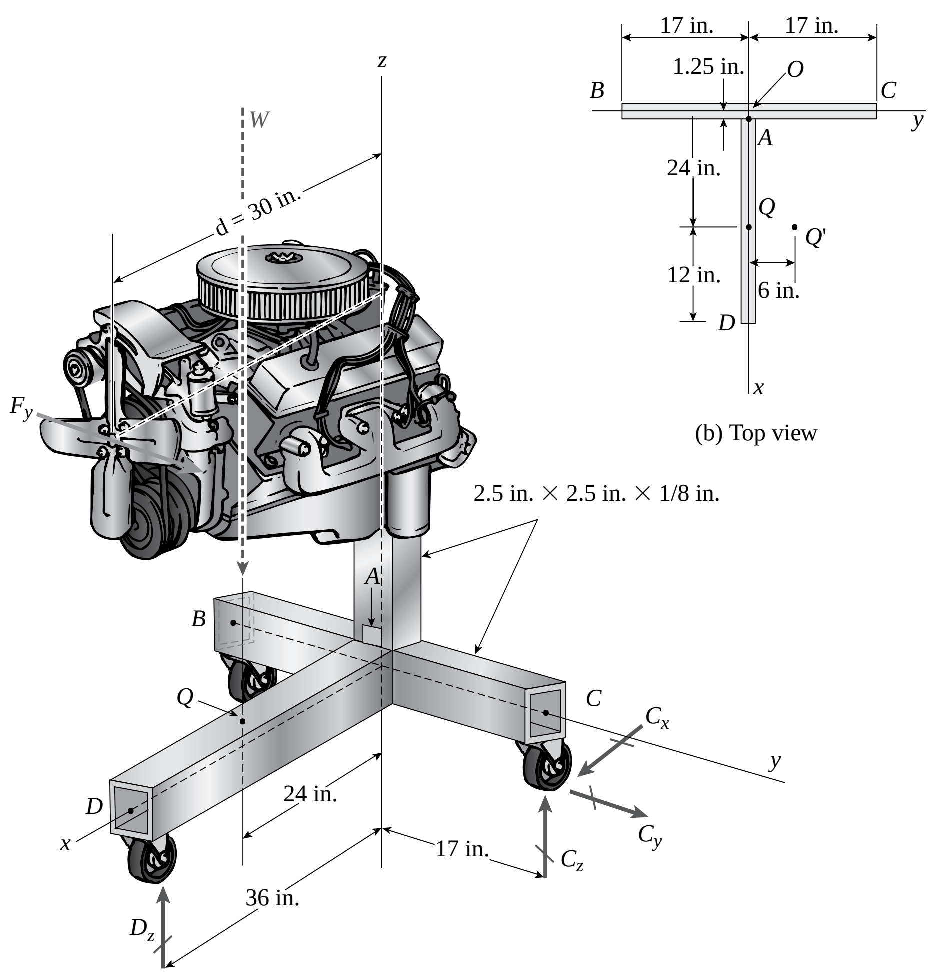
MAX. SHEAR STRESS $\tau_{max} = 110.0$ MPa ←

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

Problem 8.5-21 A moveable steel stand supports an automobile engine weighing $W = 750$ lb as shown in figure part (a). The stand is constructed of 2.5 in. × 2.5 in. × 1/8 in. thick steel tubing. Once in position the stand is restrained by pin supports at B and C. Of interest are stresses at point A at the base of the vertical post; point A has coordinates ($x = 1.25$, $y = 0$, $z = 1.25$) (inches). Neglect the weight of the stand.

(a) Initially, the engine weight acts in the ($-z$) direction through point Q which has coordinates (24, 0, 1.25); find the maximum tensile, compressive, and shear stresses at point A.

(b) Repeat (a) assuming now that, during repair, the engine is rotated about its own longitudinal axis (which is parallel to the x axis) so that W acts through Q′ (with coordinates (24, 6, 1.25)) and force $F_y = 200$ lb is applied parallel to the y axis at distance $d = 30$ in.



(a)

(b) Top view

Solution 8.5-21

$W = 750 \text{ lb} \qquad F_y = 200 \text{ lb}$

$b = 2.5 \text{ in.} \qquad t = \frac{1}{8} \text{ in.} \qquad d = 30 \text{ in.} \qquad x_1 = 24 \text{ in.}$

$A = b^2 - (b - 2t)^2 \qquad A = 1 \text{ in.}^2 \qquad A_m = (b - t)^2$

$A_m = 6 \text{ in.}^2 \qquad y_1 = 6 \text{ in.} \qquad c = \frac{b}{2}.$

$I = \frac{1}{12}[b^4 - (b - 2t)^4] \qquad I = 1 \text{ in.}^4$

(a) ENGINE WEIGHT ACTS THROUGH X-AXIS (POINT Q)

$M_x = 0 \qquad M_y = Wx_1$

SHEAR & NORMAL STRESSES AT A

$$\sigma = \frac{-W}{A} - \frac{M_y c}{I} \qquad \sigma = -20730 \text{ psi} \qquad \tau = 0$$

PRINCIPAL STRESSES & MAX SHEAR STRESS

$\sigma_x = 0 \qquad \sigma_y = \sigma \qquad \tau_{xy} = \tau$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$\sigma_1 = 0 \quad \leftarrow$

$\sigma_2 = -20730 \text{ psi} \quad \leftarrow$

$\tau_{max} = 10365 \text{ psi} \quad \leftarrow$

(b) ENGINE WEIGHT ACTS THROUGH POINT Q' & FORCE FY ACTS IN Y-DIR

$M_y = 18000 \text{ in.-lb} \qquad M_x = Wy_1$

SHEAR & NORMAL STRESSES AT A

$$\sigma = \frac{-W}{A} - \frac{M_y c}{I} \qquad \sigma = -20730 \text{ psi}$$

same since element A lies on NA for bending about x-axis

SHEAR STRESS DEPENDS ON TRANSVERSE SHEAR DUE TO FY & TORSION DUE TO MZ (MAX TRANSVERSE SHEAR IN WEB-SEE PROB. #5.10-11)

$b_1 = b - 2t \qquad b_1 = 2 \text{ in.}$

$Q = \frac{1}{8}(b^3 - b_1^3) \qquad Q = 1 \text{ in.}^3$

$$\tau_t = \frac{F_y Q}{I(2t)} \qquad \tau_t = 378 \text{ psi}$$

SHEAR DUE TO TORSIONAL MOMENT MZ USING APPROX. THEORY (EQU. 3-65)

$$M_z = F_y d \qquad M_z = 6000 \text{ in.-lb} \qquad \tau_T = \frac{M_z}{2tA_m}$$

$\tau_T = 4255 \text{ psi}$

$\tau = \tau_t + \tau_T \qquad \tau = 4633 \text{ psi}$

PRINCIPAL STRESSES & MAX SHEAR STRESS

$\sigma_x = 0 \qquad \sigma_y = \sigma \qquad \tau_{xy} = \tau$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

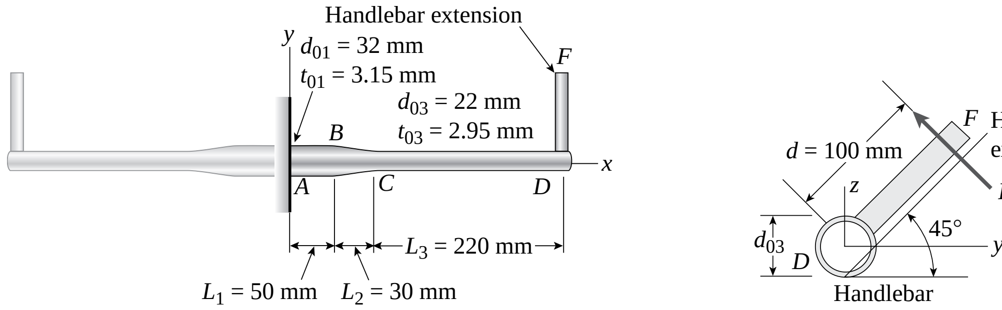
$\sigma_1 = 988 \text{ psi} \quad \leftarrow$

$\sigma_2 = -21719 \text{ psi} \quad \leftarrow$

$\tau_{max} = 11354 \text{ psi} \quad \leftarrow$

Problem 8.5-22 A mountain bike rider going uphill applies force $P = 65$ N to each end of the handlebars $ABCD$, made of aluminum alloy 7075-T6, by pulling on the handlebar extenders (DF on right handlebar segment). Consider the right half of the handlebar assembly only (assume the bars are fixed at the fork at A). Segments AB and CD are prismatic with lengths L_1 and L_3 and with outer diameters and thicknesses d_{01}, t_{01} and d_{03}, t_{03}, respectively, as shown. Segment BC of length L_2, however, is tapered and, outer diameter and thickness vary linearly between dimensions at B and C. Consider shear, torsion, and bending effects only for segment AD; assume DF is rigid.

Find maximum tensile, compressive, and shear stresses adjacent to support A. Show where each maximum stress value occurs.



(a) (b) Section D–F

Solution 8.5-22

$P = 65$ N

$L_1 = 50$ mm

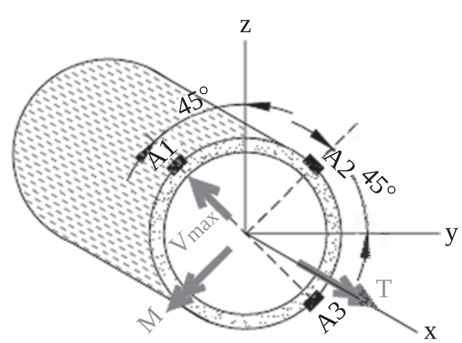
$L_2 = 30$ mm

$L_3 = 220$ mm

$d_{01} = 32$ mm

$d_{03} = 22$ mm

$d = 100$ mm



Section at point A

PROPERTIES OF THE CROSS SECTION AT POINT A

$r = \dfrac{d_{01}}{2}$ $\qquad t_{01} = 3.15$ mm

$A = \dfrac{\pi}{4}\left[d_{01}^2 - (d_{01} - t_{01})^2\right]$ $\qquad A = 150.543\text{ mm}^2$

$I = \dfrac{\pi}{64}\left[d_{01}^4 - (d_{01} - t_{01})^4\right]$ $\qquad I = 1.747 \times 10^4\text{ mm}^4$

$I_p = 2I$ $\qquad I_p = 3.493 \times 10^4\text{ mm}^4$

$Q = \dfrac{1}{12}\left[d_{01}^3 - (d_{01} - t_{01})^3\right]$ $\qquad Q = 729.625\text{ mm}^3$

STRESS RESULTANTS AT POINT $A1$

$N_x = 0$ (Axial force in X-dir.)

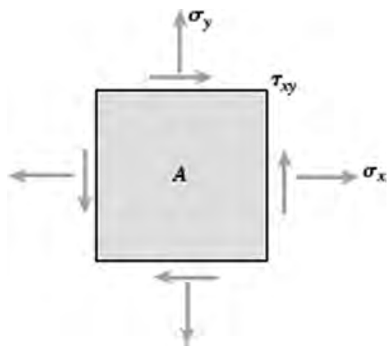
$V_{max.} = P$ (Max. Shear force)

$V_{max.} = 0.065$ kN

$T = Pd$ (Torsional Moment) $\qquad T = 6.5\text{ kN}\cdot\text{mm}$

$M = P(L_1 + L_2 + L_3)$ (Bending Moment)

$M = 19.5\text{ kN}\cdot\text{mm}$



$$\sigma_x = -\frac{M}{I} r$$

$\sigma_x = -17.863$ MPa (compressive stress)

$\sigma_y = 0$

$$\tau_{xy} = \frac{T\,d_{01}}{2I_p} \qquad \tau_{xy} = 2.977 \text{ MPa}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

MAX. TENSILE STRESS $\sigma_1 = 0.483$ MPa ←

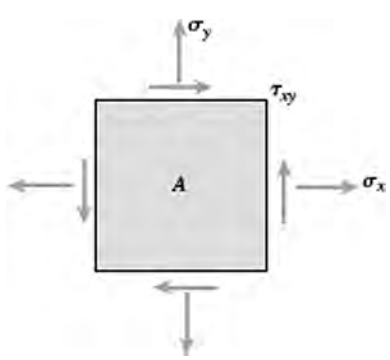
MAX. COMPRESSIVE STRESS

$\sigma_2 = -18.35$ MPa ←

MAX. SHEAR STRESS $\tau_{\max} = 9.42$ MPa ←

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

STRESS RESULTANTS AT POINT $A2$



$\sigma_x = 0$

$\sigma_y = 0$

$$\tau_{xy} = \frac{Td_{01}}{2I_p} + \frac{V_{\max.}Q}{I2\,t_{01}} \qquad \tau_{xy} = 3.408 \text{ MPa}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

MAX. TENSILE STRESS $\sigma_1 = 3.41$ MPa ←

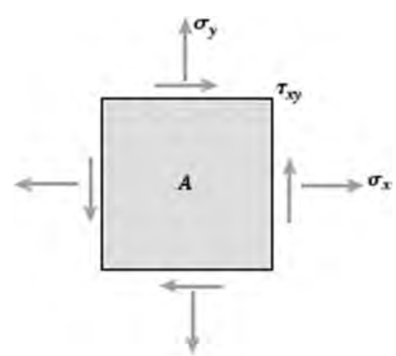
MAX. COMPRESSIVE STRESS

$\sigma_2 = -3.41$ MPa ←

MAX. SHEAR STRESS $\tau_{\max} = 3.41$ MPa ←

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

STRESS RESULTANTS AT POINT $A3$



$$\sigma_x = \frac{M}{I} r \qquad \sigma_x = 17.863 \text{ MPa (tensile stress)}$$

$\sigma_y = 0$

$$\tau_{xy} = \frac{Td_{01}}{2\,I_p} \qquad \tau_{xy} = 2.977 \text{ MPa}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

MAX. TENSILE STRESS $\sigma_1 = 18.35$ MPa ←

MAX. COMPRESSIVE STRESS

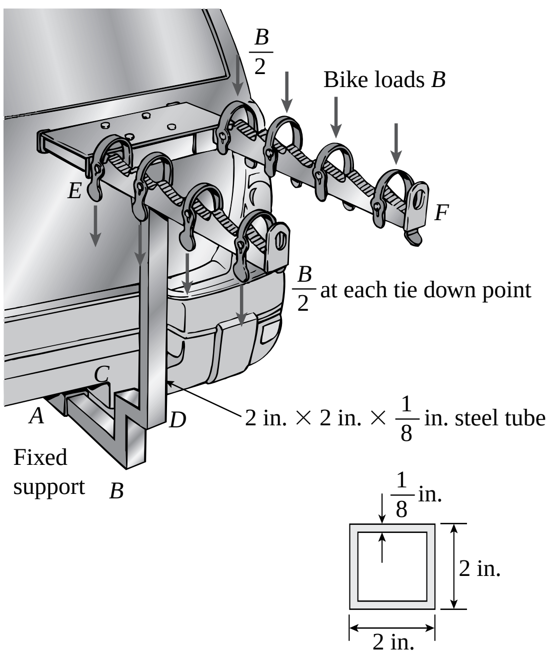
$\sigma_2 = -0.483$ MPa ←

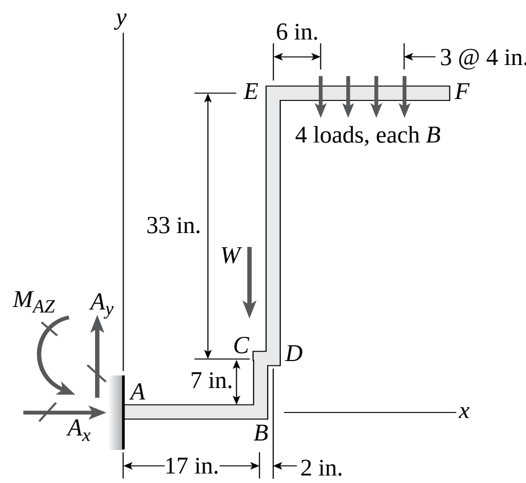
MAX. SHEAR STRESS $\tau_{max} = 9.42$ MPa ←

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

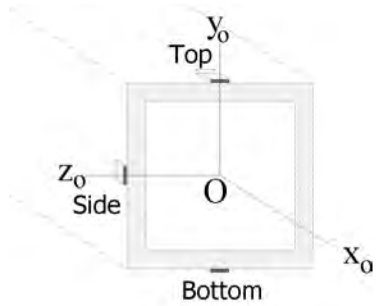
Problem 8.5-23 Determine the maximum tensile, compressive, and shear stresses acting on the cross section of the tube at point *A* of the hitch bicycle rack shown in the figure.

The rack is made up of 2 in. × 2 in. steel tubing which is 1/8 in. thick. Assume that the weight of each of four bicycles is distributed evenly between the two support arms so that the rack can be represented as a cantilever beam (*ABCDEF*) in the *x*-*y* plane. The overall weight of the rack alone is $W = 60$ lb. directed through *C*, and the weight of each bicycle is $B = 30$ lb.





Solution 8.5-23



Cross section at point A

Cross section $t = 0.125$ in. Thickness

$d_2 = 2$ in. Outer width

$d_1 = d_2 - 2t$ Inner width

$A = d_2^2 - d_2^1$ $A = 0.938$ in.^{2}

$I = \dfrac{d_2^4}{12} - \dfrac{d_1^4}{12}$ $I = 0.552$ in.^{4}

$$Q = (d_2 - 2t)\,t\left(\frac{d_2}{2} - \frac{t}{2}\right) + 2\,\frac{d_2}{2}\;t\,\frac{d_2}{4}$$

$Q = 0.33$ in.^{3}

The distance between point A and the center of load W

$b_1 = 17$ in.

The distance between the point A and the center of a bike load B

$$b_2 = \left(17 + 2 + 6 + \frac{3.4}{2}\right) \text{ in.}$$

$W = 60$ lb $B = 30$ lb

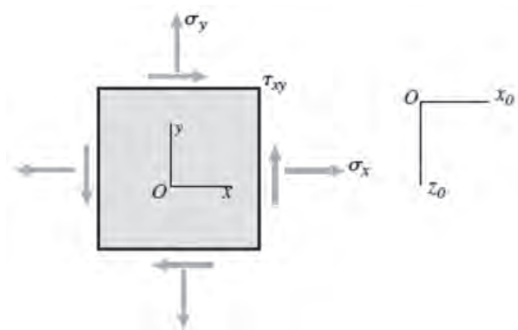
REACTIONS AT SUPPORT A

$M_{Az} = Wb_1 + 4Bb_2$ $M_{Az} = 4.74 \times 10^3$ lb · in.

$A_y = W + 4B$ $A_y = 180$ lb

$A_x = 0$

TOP OF THE CROSS SECTION (AT POINT A)



$$\sigma_x = \frac{M_{Az}\,d_2}{2I}$$

$\sigma_x = 8.591 \times 10^3$ psi (tensile stress)

$\sigma_y = 0$

$\tau_{xy} = 0$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

MAX. TENSILE STRESS

$\sigma_1 = 8591$ psi $\leftarrow$

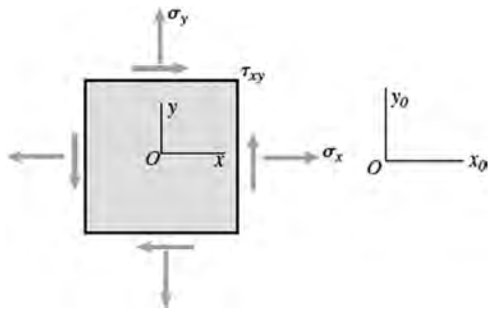
MAX. COMPRESSIVE STRESS $\sigma_2 = 0$ $\leftarrow$

MAX. SHEAR STRESS

$\tau_{\max} = 4295$ psi $\leftarrow$

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

SIDE OF THE CROSS SECTION (AT POINT A)



$\sigma_x = 0$

$\sigma_y = 0$

$\tau_{xy} = \dfrac{A_y Q}{I 2 t} \qquad \tau_{xy} = 430.726 \text{ psi}$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

MAX. TENSILE STRESS $\quad \sigma_1 = 431 \text{ psi} \quad \leftarrow$

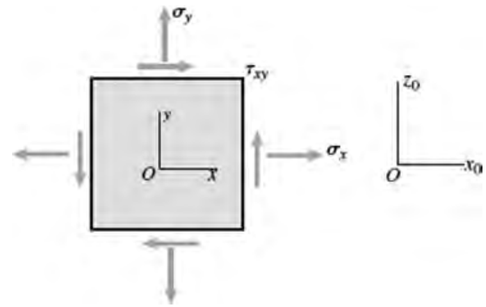
MAX. COMPRESSIVE STRESS

$\sigma_2 = -431 \text{ psi} \quad \leftarrow$

MAX. SHEAR STRESS $\quad \tau_{\max} = 431 \text{ psi} \quad \leftarrow$

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

BOTTOM OF THE CROSS SECTION (AT POINT A)



$$\sigma_x = -\frac{M_A z d_2}{2I}$$

$\sigma_x = -8.591 \times 10^3 \text{ psi}$ (compressive stress)

$\sigma_y = 0$

$\tau_{xy} = 0$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

MAX. TENSILE STRESS $\quad \sigma_1 = 0 \quad \leftarrow$

MAX. COMPRESSIVE STRESS

$\sigma_2 = -8591 \text{ psi} \quad \leftarrow$

MAX. SHEAR STRESS

$\tau_{\max} = 4295 \text{ psi} \quad \leftarrow$

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

9

Deflections of Beams

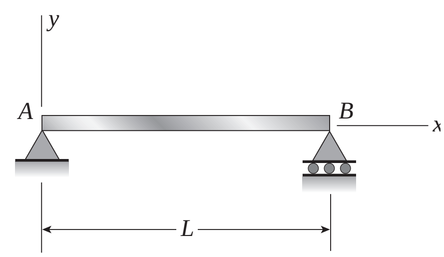
Differential Equations of the Deflection Curve

The beams described in the problems for Section 9.2 have constant flexural rigidity EI.

Problem 9.2-1 The deflection curve for a simple beam *AB* (see figure) is given by the following equation:

$$v = -\frac{q_0 x}{360LEI}(7L^4 - 10L^2x^2 + 3x^4)$$

Describe the load acting on the beam.



Probs. 9.2-1 and 9.2-2

Solution 9.2-1 Simple beam

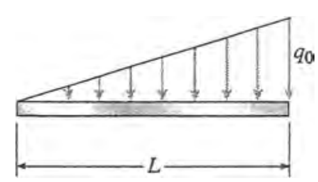
$$v = -\frac{q_0 x}{360LEI}(7L^4 - 10L^2x^2 + 3x^4)$$

Take four consecutive derivatives and obtain:

$$v'''' = -\frac{q_0 x}{LEI}$$

From Eq. (9-12c): $q = -EIv'''' = \dfrac{q_0 x}{L}$ $\leftarrow$

The load is a downward triangular load of maximum intensity q_0. $\leftarrow$



Problem 9.2-2 The deflection curve for a simple beam AB (see figure) is given by the following equation:

$$v = -\frac{q_0 L^4}{\pi^4 EI} \sin \frac{\pi x}{L}$$

(a) Describe the load acting on the beam.
(b) Determine the reactions R_A and R_B at the supports.
(c) Determine the maximum bending moment M_{max}.

Solution 9.2-2 Simple beam

$$v = -\frac{q_0 L^4}{\pi^4 EI} \sin \frac{\pi x}{L}$$

$$v' = -\frac{q_0 L^3}{\pi^3 EI} \cos \frac{\pi x}{L}$$

$$v'' = \frac{q_0 L^2}{\pi^2 EI} \sin \frac{\pi x}{L}$$

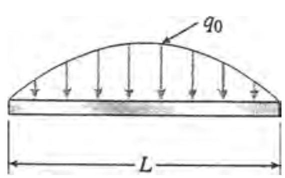
$$v''' = \frac{q_0 L}{\pi EI} \cos \frac{\pi x}{L}$$

$$v'''' = -\frac{q_0}{EI} \sin \frac{\pi x}{L}$$

(a) LOAD (EQ. 9-12c)

$$q = -EIv'''' = q_0 \sin \frac{\pi x}{L} \quad \leftarrow$$

The load has the shape of a sine curve, acts downward, and has maximum intensity q_0. $\leftarrow$



(b) REACTIONS (EQ. 9-12b)

$$V = EIv''' = \frac{q_0 L}{\pi} \cos \frac{\pi x}{L}$$

$$\text{At } x = 0\text{: } V = R_A = \frac{q_0 L}{\pi} \quad \leftarrow$$

$$\text{At } x = L\text{: } V = -R_B = -\frac{q_0 L}{\pi};$$

$$R_B = \frac{q_0 L}{\pi} \quad \leftarrow$$

(c) MAXIMUM BENDING MOMENT (EQ. 9-12a)

$$M = EIv'' = \frac{q_0 L^2}{\pi^2} \sin \frac{\pi x}{L}$$

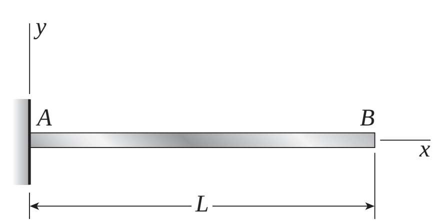
For maximum moment, $x = \frac{L}{2}$;

$$M_{max} = \frac{q_0 L^2}{\pi^2} \quad \leftarrow$$

Problem 9.2-3 The deflection curve for a cantilever beam AB (see figure) is given by the following equation:

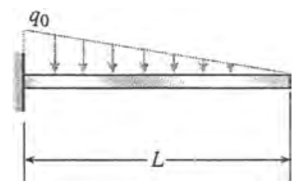
$$v = -\frac{q_0 x^2}{120LEI}(10L^3 - 10L^2 x + 5Lx^2 - x^3)$$

Describe the load acting on the beam.



Probs. 9.2-3 and 9.2.-4

Solution 9.2-3 Cantilever beam



$$\nu = -\frac{q_0 x^2}{120LEI}(10L^3 - 10L^2x + 5Lx^2 - x^3)$$

Take four consecutive derivatives and obtain:

$$\nu'''' = -\frac{q_0}{LEI}(L - x)$$

From Eq. (9-12c):

$$q = -EI\nu'''' = q_0\left(1 - \frac{x}{L}\right) \quad \leftarrow$$

The load is a downward triangular load of maximum intensity q_0. $\leftarrow$

Problem 9.2-4 The deflection curve for a cantilever beam *AB* (see figure) is given by the following equation:

$$\nu = -\frac{q_0 x^2}{360L^2EI}(45L^4 - 40L^3x + 15L^2x^2 - x^4)$$

(a) Describe the load acting on the beam.
(b) Determine the reactions R_A and M_A at the support.

Solution 9.2-4 Cantilever beam

$$\nu = -\frac{q_0 x^2}{360L^2EI}(45L^4 - 40L^3x + 15L^2x^2 - x^4)$$

$$\nu'' = -\frac{q_0}{60L^2EI}(15L^4x - 20L^3x^2 + 10L^2x^2 - x^5)$$

$$\nu'' = -\frac{q_0}{12L^2EI}(3L^4 - 8L^3x + 6L^2x^2 - x^4)$$

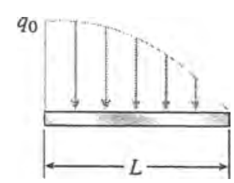
$$\nu''' = -\frac{q_0}{3L^2EI}(-2L^3 + 3L^2x - x^3)$$

$$\nu'''' = -\frac{q_0}{L^2EI}(L^2 - x^2)$$

(a) LOAD (EQ. 9-12c)

$$q = -EI\nu'''' = q_0\left(1 - \frac{x^2}{L^2}\right) \quad \leftarrow$$

The load is a downward parabolic load of maximum intensity q_0. $\leftarrow$



(b) REACTIONS R_A AND M_A (EQ. 9-12b AND EQ. 9-12a)

$$V = EI\nu''' = -\frac{q_0}{3L^2}(-2L^3 + 3L^2x - x^3)$$

$$\text{At } x = 0\text{: } V = R_A = \frac{2q_0L}{3} \quad \leftarrow$$

$$M = EI\nu'' = -\frac{q_0}{12L^2}(3L^4 - 8L^3x + 6L^2x^2 - x^4)$$

$$\text{At } x = 0\text{: } M = M_A = -\frac{q_0L^2}{4} \quad \leftarrow$$

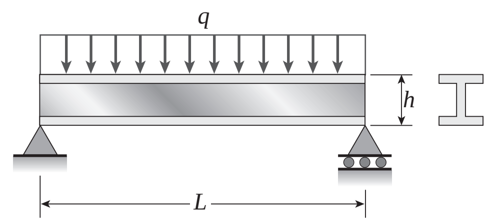
NOTE: Reaction R_A is positive upward.
Reaction M_A is positive clockwise (minus means M_A is counterclockwise).

Deflection Formulas

Problems 9.3-1 through 9.3-7 require the calculation of deflections using the formulas derived in Examples 9-1, 9-2, and 9-3. All beams have constant flexural rigidity EI.

Problem 9.3-1 A wide-flange beam (W 12 × 35) supports a uniform load on a simple span of length $L = 14$ ft (see figure).

Calculate the maximum deflection δ_{max} at the midpoint and the angles of rotation θ at the supports if $q = 1.8$ k/ft and $E = 30 \times 10^6$ psi. Use the formulas of Example 9-1.



Probs. 9.3-1 through 9.3-3

Solution 9.3-1 Simple beam (uniform load)

W 12 × 35 $\quad L = 14 \text{ ft} = 168 \text{ in.}$

$q = 1.8 \text{ k/ft} = 150 \text{ lb/in.} \qquad E = 30 \times 10^6 \text{ psi}$

$I = 285 \text{ in.}^4$

MAXIMUM DEFLECTION (EQ. 9-18)

$$\delta_{max} = \frac{5\,qL^4}{384\,EI} = \frac{5(150 \text{ lb/in.})(168 \text{ in.})^4}{384(30 \times 10^6 \text{ psi})(285 \text{ in.}^4)}$$
$$= 0.182 \text{ in.} \quad \leftarrow$$

ANGLE OF ROTATION AT THE SUPPORTS (EQS. 9-19 AND 9-20)

$$\theta = \theta_A = \theta_B = \frac{qL^3}{24\,EI}$$
$$= \frac{(150 \text{ lb/in.})(168 \text{ in.})^3}{24(30 \times 10^6 \text{ psi})(285 \text{ in.}^4)}$$
$$= 0.003466 \text{ rad} = 0.199° \quad \leftarrow$$

Problem 9.3-2 A uniformly loaded steel wide-flange beam with simple supports (see figure) has a downward deflection of 10 mm at the midpoint and angles of rotation equal to 0.01 radians at the ends.

Calculate the height h of the beam if the maximum bending stress is 90 MPa and the modulus of elasticity is 200 GPa. (*Hint*: Use the formulas of Example 9-1.)

Solution 9.3-2 Simple beam (uniform load)

$\delta = \delta_{max} = 10 \text{ mm} \qquad \theta = \theta_A = \theta_B = 0.01 \text{ rad}$

$\sigma = \sigma_{max} = 90 \text{ MPa} \qquad E = 200 \text{ GPa}$

Calculate the height h of the beam.

$$\text{Eq. (9-18): } \delta = \delta_{max} = \frac{5qL^4}{384EI} \text{ or } q = \frac{384EI\delta}{5L^4} \tag{1}$$

$$\text{Eq. (9-19): } \theta = \theta_A = \frac{qL^3}{24EI} \text{ or } q = \frac{24EI\theta}{L^3} \tag{2}$$

$$\text{Equate (1) and (2) and solve for } L\text{: } L = \frac{16\delta}{5\theta} \tag{3}$$

$$\text{Flexure formula: } \sigma = \frac{Mc}{I} = \frac{Mh}{2I}$$

Maximum bending moment:

$$M = \frac{qL^2}{8} \qquad \therefore\ \sigma = \frac{qL^2h}{16I} \tag{4}$$

$$\text{Solve Eq. (4) for } h\text{: } h = \frac{16I\sigma}{qL^2} \tag{5}$$

Substitute for q from (2) and for L from (3):

$$h = \frac{32\sigma\delta}{15E\theta^2} \quad \leftarrow$$

Substitute numerical values:

$$h = \frac{32(90 \text{ MPa})(10 \text{ mm})}{15(200 \text{ GPa})(0.01 \text{ rad})^2} = 96 \text{ mm} \quad \leftarrow$$

Problem 9.3-3 What is the span length L of a uniformly loaded simple beam of wide-flange cross section (see figure) if the maximum bending stress is 12,000 psi, the maximum deflection is 0.1 in., the height of the beam is 12 in., and the modulus of elasticity is 30×10^6 psi? (Use the formulas of Example 9-1.)

Solution 9.3-3 Simple beam (uniform load)

$\sigma = \sigma_{max} = 12{,}000$ psi $\qquad \delta = \delta_{max} = 0.1$ in.

$h = 12$ in. $\qquad E = 30 \times 10^6$ psi

Calculate the span length L.

Eq. (9-18): $\delta = \delta_{max} = \dfrac{5qL^4}{384EI}$ or $q = \dfrac{384EI\delta}{5L^4}$ (1)

Flexure formula: $\sigma = \dfrac{Mc}{I} = \dfrac{Mh}{2I}$

Maximum bending moment:

$M = \dfrac{qL^2}{8} \qquad \therefore \sigma = \dfrac{qL^2h}{16I}$ (2)

Solve Eq. (2) for q: $q = \dfrac{16I\sigma}{L^2h}$ (3)

Equate (1) and (2) and solve for L:

$L^2 = \dfrac{24\,Eh\delta}{5\sigma} \qquad L = \sqrt{\dfrac{24\,Eh\delta}{5\sigma}}$ ←

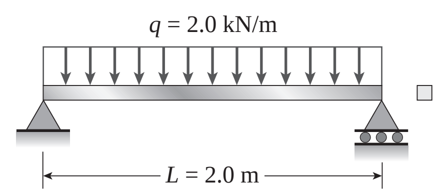
Substitute numerical values:

$$L^2 = \frac{24(30 \times 10^6 \text{ psi})(12 \text{ in.})(0.1 \text{ in.})}{5(12{,}000 \text{ psi})} = 14{,}400 \text{ in.}^2$$

$L = 120$ in. $= 10$ ft ←

Problem 9.3-4 Calculate the maximum deflection δ_{max} of a uniformly loaded simple beam (see figure) if the span length $L = 2.0$ m, the intensity of the uniform load $q = 2.0$ kN/m, and the maximum bending stress $\sigma = 60$ MPa.

The cross section of the beam is square, and the material is aluminum having modulus of elasticity $E = 70$ GPa. (Use the formulas of Example 9-1.)



Solution 9.3-4 Simple beam (uniform load)

$L = 2.0$ m $\qquad q = 2.0$ kN/m

$\sigma = \sigma_{max} = 60$ MPa $\qquad E = 70$ GPa

Cross section (square; b = width)

$I = \dfrac{b^4}{12} \qquad S = \dfrac{b^3}{6}$

Maximum deflection (Eq. 9-18): $\delta = \dfrac{5qL^4}{384\,EI}$ (1)

Substitute for I: $\delta = \dfrac{5qL^4}{32\,Eb^4}$ (2)

Flexure formula with $M = \dfrac{qL^2}{8}$: $\sigma = \dfrac{M}{S} = \dfrac{qL^2}{8S}$

Substitute for S: $\sigma = \dfrac{3qL^2}{4b^3}$ (3)

Solve for b^3: $b^3 = \dfrac{3qL^2}{4\sigma}$ (4)

Substitute b into Eq. (2): $\delta_{max} = \dfrac{5L\sigma}{24E}\left(\dfrac{4L\sigma}{3q}\right)^{1/3}$ ←

(The term in parentheses is nondimensional.)

Substitute numerical values:

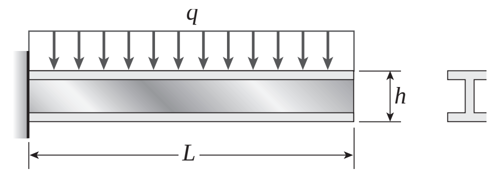
$$\frac{5L\sigma}{24E} = \frac{5(2.0 \text{ m})(60 \text{ MPa})}{24(70 \text{ GPa})} = \frac{1}{2800} \text{ m} = \frac{1}{2.8} \text{ mm}$$

$$\left(\frac{4\,L\sigma}{3q}\right)^{1/3} = \left[\frac{4(2.0 \text{ m})(60 \text{ MPa})}{3(2000 \text{ N/m})}\right]^{1/3} = 10(80)^{1/3}$$

$\delta_{max} = \dfrac{10(80)^{1/3}}{2.8}$ mm $= 15.4$ mm ←

Problem 9.3-5 A cantilever beam with a uniform load (see figure) has a height h equal to 1/8 of the length L. The beam is a steel wideflange section with $E = 28 \times 10^6$ psi and an allowable bending stress of 17,500 psi in both tension and compression.

Calculate the ratio δ/L of the deflection at the free end to the length, assuming that the beam carries the maximum allowable load. (Use the formulas of Example 9-2.)



Solution 9.3-5 Cantilever beam (uniform load)

$$\frac{h}{L} = \frac{1}{8} \qquad E = 28 \times 10^6 \text{ psi} \qquad \sigma = 17{,}500 \text{ psi}$$

Calculate the ratio δ/L.

Maximum deflection (Eq. 9-26): $\delta_{max} = \dfrac{qL^4}{8EI}$ (1)

$$\therefore \frac{\delta}{L} = \frac{qL^3}{8EI} \qquad (2)$$

Flexure formula with $M = \dfrac{qL^2}{2}$:

$$\sigma = \frac{Mc}{I} = \left(\frac{qL^2}{2}\right)\left(\frac{h}{2I}\right) = \frac{qL^2h}{4I}$$

Solve for q:

$$q = \frac{4I\sigma}{L^2h} \qquad (3)$$

Substitute q from (3) into (2):

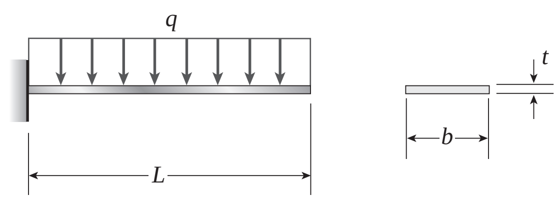
$$\frac{\delta}{L} = \frac{\sigma}{2E}\left(\frac{L}{h}\right) \quad \longleftarrow$$

Substitute numerical values:

$$\frac{\delta}{L} = \frac{17{,}500 \text{ psi}}{2(28 \times 10^6 \text{ psi})}(8) = \frac{1}{400} \quad \longleftarrow$$

Problem 9.3-6 A gold-alloy microbeam attached to a silicon wafer behaves like a cantilever beam subjected to a uniform load (see figure). The beam has length $L = 27.5$ μm and rectangular cross section of width $b = 4.0$ μm and thickness $t = 0.88$ μm. The total load on the beam is 17.2 μN.

If the deflection at the end of the beam is 2.46 μm, what is the modulus of elasticity E_g of the gold alloy? (Use the formulas of Example 9-2.)



Solution 9.3-6 Gold-alloy microbeam

Cantilever beam with a uniform load.

$$L = 27.5 \text{ μm} \qquad b = 4.0 \text{ μm} \qquad t = 0.88 \text{ μm}$$

$$qL = 17.2 \text{ μN} \qquad \delta_{max} = 2.46 \text{ μm}$$

Determine E_g.

Eq. (9-26): $\delta = \dfrac{qL^4}{8E_gI}$ or $E_g = \dfrac{qL^4}{8I\delta_{max}}$

$$I = \frac{bt^3}{12} \qquad E_g = \frac{3qL^4}{2bt^3\delta_{max}} \quad \longleftarrow$$

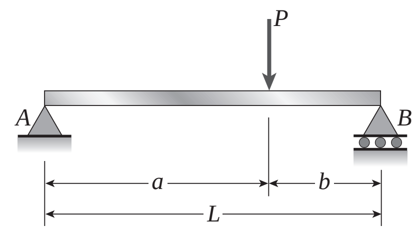
Substitute numerical values:

$$E_g = \frac{3(17.2 \text{ mN})(27.5 \text{ mm})^3}{2(4.0 \text{ mm})(0.88 \text{ mm})^3(2.46 \text{ mm})}$$

$$= 80.02 \times 10^9 \text{ N/m}^2 \quad \text{or} \quad E_g = 80.0 \text{ GPa} \quad \longleftarrow$$

Problem 9.3-7 Obtain a formula for the ratio δ_C/δ_{max} of the deflection at the midpoint to the maximum deflection for a simple beam supporting a concentrated load P (see figure).

From the formula, plot a graph of δ_C/δ_{max} versus the ratio a/L that defines the position of the load ($0.5 < a/L < 1$). What conclusion do you draw from the graph? (Use the formulas of Example 9-3.)



Solution 9.3-7 Simple beam (concentrated load)

Eq. (9-35): $\delta_C = \dfrac{Pb(3L^2 - 4b^2)}{48EI} \quad (a \geq b)$

Eq. (9-34): $\delta_{max} = \dfrac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}LEI} \quad (a \geq b)$

$$\frac{\delta_c}{\delta_{max}} = \frac{(3\sqrt{3}L)(3L^2 - 4b^2)}{16(L^2 - b^2)^{3/2}} \quad (a \geq b)$$

Replace the distance b by the distance a by substituting $L - a$ for b:

$$\frac{\delta_c}{\delta_{max}} = \frac{(3\sqrt{3}L)(-L^2 + 8aL - 4a^2)}{16(2aL - a^2)^{3/2}}$$

Divide numerator and denominator by L^2:

$$\frac{\delta_c}{\delta_{max}} = \frac{(3\sqrt{3}L)\left(-1 + 8\frac{a}{L} - 4\frac{a^2}{L^2}\right)}{16L\left(2\frac{a}{L} - \frac{a^2}{L^2}\right)^{3/2}}$$

$$\frac{\delta_c}{\delta_{max}} = \frac{(3\sqrt{3}L)\left(-1 + 8\frac{a}{L} - 4\frac{a^2}{L^2}\right)}{16\left(2\frac{a}{L} - \frac{a^2}{L^2}\right)^{3/2}} \quad \leftarrow$$

ALTERNATIVE FORM OF THE RATIO

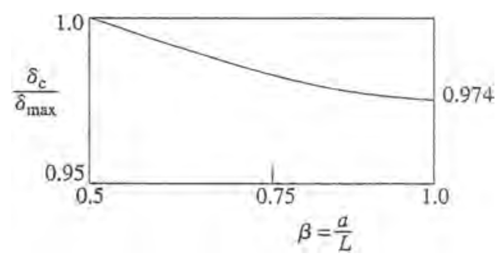
Let $\beta = \dfrac{a}{L}$

$$\frac{\delta_c}{\delta_{max}} = \frac{(3\sqrt{3})(-1 + 8\beta - 4\beta^2)}{16(2\beta - \beta^2)^{3/2}} \quad \leftarrow$$

GRAPH OF δ_c/δ_{max} VERSUS $\beta = a/L$

Because $a \geq b$, the ratio β versus from 0.5 to 1.0.

β	$\dfrac{\delta_c}{\delta_{max}}$
0.5	1.0
0.6	0.996
0.7	0.988
0.8	0.981
0.9	0.976
1.0	0.974

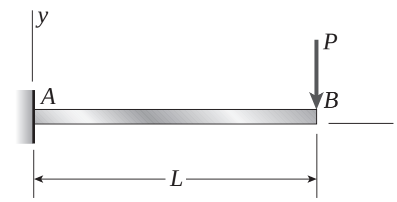


NOTE: The deflection δ_c at the midpoint of the beam is almost as large as the maximum deflection δ_{max}. The greatest difference is only 2.6% and occurs when the load reaches the end of the beam ($\beta = 1$).

Deflections by Integration of the Bending-Moment Equation

Problems 9.3-8 through 9.3-16 are to be solved by integrating the second-order differential equation of the deflection curve (the bending-moment equation). The origin of coordinates is at the left-hand end of each beam, and all beams have constant flexural rigidity EI.

Problem 9.3-8 Derive the equation of the deflection curve for a cantilever beam *AB* supporting a load *P* at the free end (see figure). Also, determine the deflection δ_B and angle of rotation θ_B at the free end. (*Note*: Use the second-order differential equation of the deflection curve.)



Solution 9.3-8 Cantilever beam (concentrated load)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -P(L - x)$$

$$EIv' = -PLx + \frac{Px^2}{2} + C_1$$

B.C. $v'(0) = 0 \qquad \therefore C_1 = 0$

$$EIv = -\frac{PLx^2}{2} + \frac{Px^3}{6} + C_2$$

B.C. $v(0) = 0 \qquad \therefore C_2 = 0$

$$v = -\frac{Px^2}{6EI}(3L - x) \quad \leftarrow$$

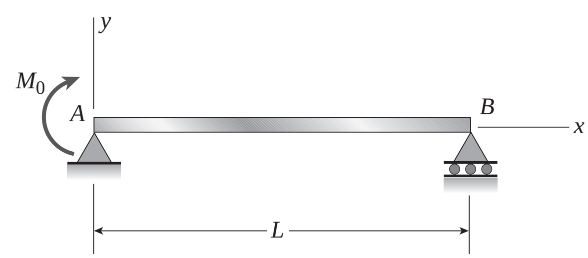
$$v' = -\frac{Px}{2EI}(2L - x)$$

$$\delta_B = -v(L) = \frac{PL^3}{3EI} \quad \leftarrow$$

$$\theta_B = -v'(L) = \frac{PL^2}{2EI} \quad \leftarrow$$

(These results agree with Case 4, Table G-1.)

Problem 9.3-9 Derive the equation of the deflection curve for a simple beam *AB* loaded by a couple M_0 at the left-hand support (see figure). Also, determine the maximum deflection δ_{max}. (*Note*: Use the second-order differential equation of the deflection curve.)



Solution 9.3-9 Simple beam (couple M_0)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = M_0\left(1 - \frac{x}{L}\right)$$

$$EIv' = M_0\left(x - \frac{x^2}{2L}\right) + C_1$$

$$EIv = M_0\left(\frac{x^2}{2} - \frac{x^3}{6L}\right) + C_1x + C_2$$

B.C. $v(0) = 0 \qquad \therefore C_2 = 0$

B.C. $v(L) = 0 \qquad \therefore C_1 = -\frac{M_0L}{3}$

$$v = -\frac{M_0x}{6LEI}(2L^2 - 3Lx + x^2) \quad \leftarrow$$

MAXIMUM DEFLECTION

$$v' = -\frac{M_0}{6LEI}(2L^2 - 6Lx + 3x^2)$$

Set $v' = 0$ and solve for x:

$$x_1 = L\left(1 - \frac{\sqrt{3}}{3}\right) \quad \leftarrow$$

Substitute x_1 into the equation for v:

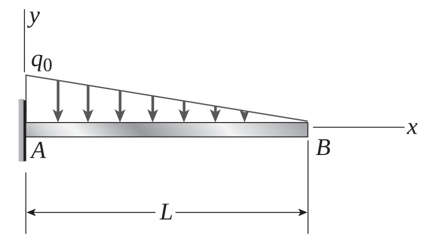
$$\delta_{\max} = -(v)_{x=x_1}$$

$$= \frac{M_0 L^2}{9\sqrt{3}EI} \quad \leftarrow$$

(These results agree with Case 7, Table G-2.)

Problem 9.3-10 A cantilever beam AB supporting a triangularly distributed load of maximum intensity q_0 is shown in the figure.

Derive the equation of the deflection curve and then obtain formulas for the deflection δ_B and angle of rotation θ_B at the free end. (*Note*: Use the second-order differential equation of the deflection curve.)



Solution 9.3-10 Cantilever beam (triangular load)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -\frac{q_0}{6L}(L - x)^3$$

$$EIv' = \frac{q_0}{24L}(L - x)^4 + C_1$$

B.C. $v'(0) = 0 \qquad \therefore C_1 = -\frac{q_0 L^3}{24}$

$$EIv = -\frac{q_0}{120L}(L - x)^5 - \frac{q_0 L^3 x}{24} + C_2$$

B.C. $v(0) = 0 \qquad \therefore C_2 = \frac{q_0 L^4}{120}$

$$v = -\frac{q_0 x^2}{120LEI}(10L^3 - 10L^2x + 5Lx^2 - x^3) \quad \leftarrow$$

$$v' = -\frac{q_0 x}{24LEI}(4L^3 - 6L^2x + 4Lx^2 - x^3)$$

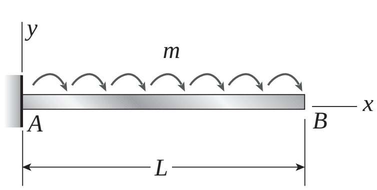
$$\delta_B = -v(L) = \frac{q_0 L^4}{30EI} \quad \leftarrow$$

$$\theta_B = -v'(L) = \frac{q_0 L^3}{24EI} \quad \leftarrow$$

(These results agree with Case 8, Table G-1.)

Problem 9.3-11 A cantilever beam AB is acted upon by a uniformly distributed moment (bending moment, not torque) of intensity m per unit distance along the axis of the beam (see figure).

Derive the equation of the deflection curve and then obtain formulas for the deflection δ_B and angle of rotation θ_B at the free end. (*Note*: Use the second-order differential equation of the deflection curve.)



Solution 9.3-11 Cantilever beam (distributed moment)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -m(L - x)$$

$$EIv' = -m\left(Lx - \frac{x^2}{2}\right) + C_1$$

B.C. $v'(0) = 0 \qquad \therefore C_1 = 0$

$$EIv = -m\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_2$$

B.C. $v(0) = 0 \qquad \therefore C_2 = 0$

$$v = -\frac{mx^2}{6\,EI}(3L - x) \qquad \leftarrow$$

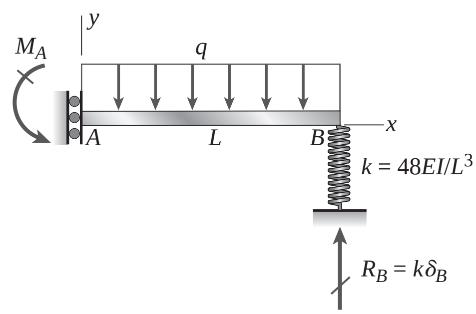
$$v' = -\frac{mx}{2EI}(2L - x)$$

$$\delta_B = -v(L) = \frac{mL^3}{3\,EI} \qquad \leftarrow$$

$$\theta_B = -v'(L) = \frac{mL^2}{2EI} \qquad \leftarrow$$

Problem 9.3-12 The beam shown in the figure has a guided support at A and a spring support at B. The guided support permits vertical movement but no rotation.

Derive the equation of the deflection curve and determine the deflection δ_B at end B due to the uniform load of intensity q. (*Note*: Use the second-order differential equation of the deflection curve.)



Solution 9.3-12

BENDING-MOMENT EQUATION

$$EIv'' = M(x) = \frac{qL^2}{2} - \frac{qx^2}{2}$$

$$EIv' = \frac{qL^2x}{2} - \frac{qx^3}{24} + C_1$$

$$EIv' = \frac{qL^2x^2}{2} - \frac{qx^4}{24} + C_1x + C_2$$

B.C. $v'(0) = 0 \qquad C_1 = 0$

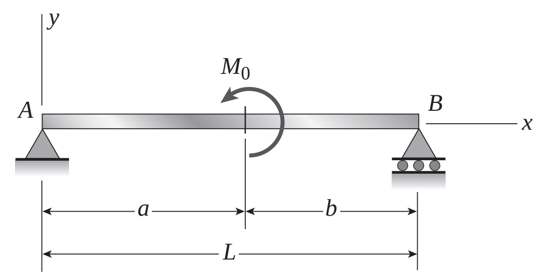
B.C. $v(L) = \dfrac{qL}{k} = -\dfrac{qL^4}{48EI} \qquad C_2 = -\dfrac{11qL^4}{48}$

$$v(x) = -\frac{q}{48EI}(2x^4 - 12x^2L^2 + 11L^4) \qquad \leftarrow$$

$$\delta_B = -v(L) = \frac{qL^4}{48EI} \qquad \leftarrow$$

Note that $R_B = k\delta_B = qL$ which agrees with $\sum F_{\text{vert}} = 0$

Problem 9.3-13 Derive the equations of the deflection curve for a simple beam *AB* loaded by a couple M_0 acting at distance *a* from the left-hand support (see figure). Also, determine the deflection δ_0 at the point where the load is applied. (*Note*: Use the second-order differential equation of the deflection curve.)



Solution 9.3-13 Simple beam (couple M_0)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = \frac{M_0 x}{L} \quad (0 \le x \le a)$$

$$EIv' = \frac{M_0 x^2}{2L} + C_1 \quad (0 \le x \le a)$$

$$EIv'' = M = -\frac{M_0}{L}(L - x) \quad (a \le x \le L)$$

$$EIv' = -\frac{M_0}{L}\left(Lx - \frac{x^2}{2}\right) + C_2 \quad (a \le x \le L)$$

B.C. 1 $(v')_{\text{Left}} = (v')_{\text{Right}}$ at $x = a$

$$\therefore C_2 = C_1 + M_0 a$$

$$EIv = \frac{M_0 x^3}{6L} + C_1 x + C_3 \quad (0 \le x \le a)$$

B.C. 2 $v(0) = 0$ $\therefore C_3 = 0$

$$EIv = -\frac{M_0 x^2}{2} + \frac{M_0 x^3}{6L} + C_1 x + M_0 a x + C_4 \quad (a \le x \le L)$$

B.C. 3 $v(L) = 0$ $\therefore C_4 = -M_0 L\left(a - \frac{L}{3}\right) - C_1 L$

B.C. 4 $(v)_{\text{Left}} = (v)_{\text{Right}}$ at $x = a$

$$\therefore C_4 = -\frac{M_0 a^2}{2}$$

$$C_1 = \frac{M_0}{6L}(2L^2 - 6aL + 3a^2)$$

$$v = -\frac{M_0 x}{6\,LEI}(6aL - 3a^2 - 2L^2 - x^2) \quad (0 \le x \le a) \quad \leftarrow$$

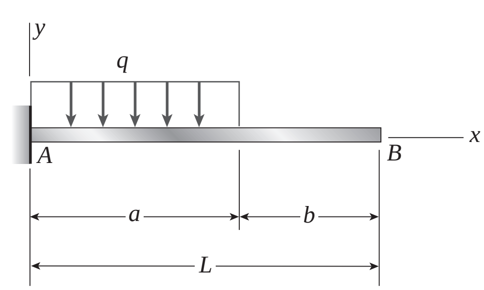
$$v = -\frac{M_0}{6\,LEI}(3a^2L - 3a^2x - 2L^2x + 3Lx^2 - x^3) \quad (a \le x \le L) \quad \leftarrow$$

$$\delta_0 = -v(a) = \frac{M_0 a(L - a)(2a - L)}{3LEI}$$

$$= \frac{M_0 ab(2a - L)}{3LEI} \quad \leftarrow$$

NOTE: δ_0 is positive downward. The preceding results agree with Case 9, Table G-2.

Problem 9.3-14 Derive the equations of the deflection curve for a cantilever beam *AB* carrying a uniform load of intensity *q* over part of the span (see figure). Also, determine the deflection δ_B at the end of the beam. (*Note:* Use the second-order differential equation of the deflection curve.)



Solution 9.3-14 Cantilever beam (partial uniform load)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -\frac{q}{2}(a - x)^2 = -\frac{q}{2}(a^2 - 2ax + x^2)$$

$$(0 \le x \le a)$$

$$EIv' = -\frac{q}{2}\left(a^2x - ax^2 + \frac{x^3}{3}\right) + C_1 \quad (0 \le x \le a)$$

B.C. 1 $v'(0) = 0 \qquad \therefore C_1 = 0$

$$EIv'' = M = 0 \quad (a \le x \le L)$$

$$EIv' = C_2 \qquad (a \le x \le L)$$

B.C. 2 $(v')_{\text{Left}} = (v')_{\text{Right}}$ at $x = a \qquad \therefore C_2 = -\frac{qa^3}{6}$

$$EIv = -\frac{q}{2}\left(\frac{a^2x^2}{2} - \frac{ax^3}{3} + \frac{x^4}{12}\right) + C_3 \quad (0 \le x \le a)$$

B.C. 3 $v(0) = 0 \qquad \therefore C_3 = 0$

$$EIv = C_2x + C_4 = -\frac{qa^3x}{6} + C_4 \quad (a \le x \le L)$$

B.C. 4 $(v)_{\text{Left}} = (v)_{\text{Right}}$ at $x = a \qquad \therefore C_4 = \frac{qa^4}{24}$

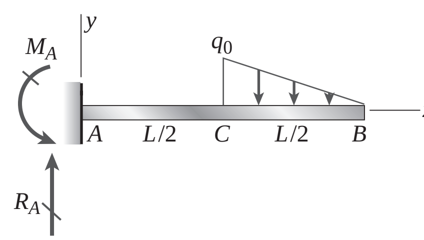
$$v = -\frac{qx^2}{24EI}(6a^2 - 4ax + x^2) \quad (0 \le x \le a) \qquad \leftarrow$$

$$v = -\frac{qa^3}{24EI}(4x - a) \quad (a \le x \le L) \qquad \leftarrow$$

$$\delta_B = -v(L) = \frac{qa^3}{24EI}(4L - a) \qquad \leftarrow$$

(These results agree with Case 2, Table G-1.)

Problem 9.3-15 Derive the equations of the deflection curve for a cantilever beam *AB* supporting a distributed load of peak intensity q_0 acting over one-half of the length (see figure). Also, obtain formulas for the deflections δ_B and δ_C at points *B* and *C*, respectively. (*Note*: Use the second-order differential equation of the deflection curve.)



Solution 9.3-15

BENDING-MOMENT EQUATION

For $0 \le x \le \frac{L}{2}$

$$EIv'' = M(x) = \frac{q_0Lx}{4} - \frac{q_0L^2}{6}$$

$$EIv' = \frac{q_0Lx^2}{8} - \frac{q_0L^2x}{6} + C_1$$

$$EIv = \frac{q_0Lx^3}{24} - \frac{q_0L^2x^2}{12} + C_1x + C_2$$

B.C. $v'(0) = 0 \qquad C_1 = 0$

B.C. $v(0) = 0 \qquad C_2 = 0$

$$v'\left(\frac{L}{2}\right) = -\frac{5q_0L^3}{96EI}$$

$$v(x) = \frac{q_0L}{24EI}(x^3 - 2Lx^2) \qquad \leftarrow$$

$$\delta_C = -v\left(\frac{L}{2}\right) = \frac{q_0L^4}{64EI} \qquad \leftarrow$$

For $\frac{L}{2} \le x \le L$

$$EIv'' = M(x) = \frac{q_0Lx}{4} - \frac{q_0L^2}{6} - \frac{q_0}{L}(L - x)$$

$$\left(x - \frac{L}{2}\right)^2 - \frac{1}{2}\left[q_0 - \frac{2q_0}{L}\right.$$

$$\left.(L - x)\right]\left(x - \frac{L}{2}\right)^2\frac{2}{3}$$

$$EIv'' = M(x) = \frac{-q_0}{3L}(-3L^2x + L^3 + 3Lx^2 - x^3)$$

$$EIv' = -\frac{q_0}{3L}\left(\frac{-3}{2}L^2x^2 + L^3x + Lx^3 - \frac{x^4}{4}\right) + C_3$$

$$EIv = -\frac{q_0}{3L}\left(\frac{-1}{2}L^2x^3 + \frac{1}{2}L^3x^2 + \frac{1}{4}Lx^4 - \frac{1}{20}x^5\right) + C_3x + C_4$$

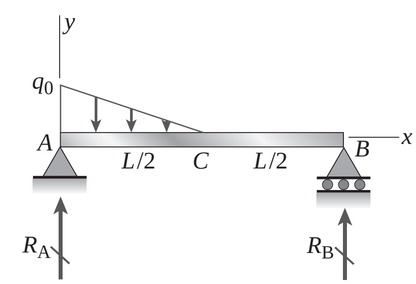
B.C. $v'\left(\frac{L}{2}\right) = -\frac{5q_0L^3}{96EI}$ $\qquad C_3 = \frac{5}{192}q_0L^3$

B.C. $v\left(\frac{L}{2}\right) = -\frac{q_0L^4}{64EI}$ $\qquad C_4 = \frac{-1}{320}q_0L^4$

$$v(x) = \frac{-q_0}{960LEI}(-160L^2x^3 + 160L^3x^2 + 80Lx^4 - 16x^5 - 25L^4x + 3L^5) \quad \leftarrow$$

$$\delta_B = -v(L) = \frac{7q_0L^4}{160EI} \quad \leftarrow$$

Problem 9.3-16 Derive the equations of the deflection curve for a simple beam AB with a distributed load of peak intensity q_0 acting over the left-hand half of the span (see figure). Also, determine the deflection δ_C at the midpoint of the beam. (*Note*: Use the second-order differential equation of the deflection curve.)



Solution 9.3-16

BENDING-MOMENT EQUATION

For $0 \le x \le \frac{L}{2}$

$$EIv'' = M(x) = \frac{5q_0Lx}{24} - \frac{2q_0}{L}\left(\frac{L}{2} - x\right)\left(\frac{x^2}{2}\right) - \frac{1}{2}\left[q_0 - \frac{2q_0}{L}\left(\frac{L}{2} - x\right)\right]x\frac{2}{3}x$$

$$EIv'' = \frac{q_0}{24L}(5L^2x - 12x^2L + 8x^3)$$

$$EIv' = \frac{q_0}{24L}\left(\frac{5L^2x^2}{2} - 4x^3L + 2x^4\right) + C_1 \qquad (1)$$

$$EIv = \frac{q_0}{24L}\left(\frac{5L^2x^3}{6} - x^4L + \frac{2x^5}{5}\right) + C_1x + C_2$$

B.C. $v(0) = 0 \qquad C_2 = 0$

$$EIv = \frac{q_0}{24L}\left(\frac{5L^2x^3}{6} - x^4L + \frac{2x^5}{5}\right) + C_1x \qquad (2)$$

For $\frac{L}{2} \le x \le L$

$$EIv'' = M(x) = \frac{5q_0Lx}{24} - \frac{1}{2}q_0\frac{L}{2}\left(x - \frac{L}{6}\right)$$

$$EIv'' = \frac{Lq_0}{24}(-x + L)$$

$$EIv' = \frac{Lq_0}{24}\left(\frac{-x^2}{2} + Lx\right) + C_3 \qquad (3)$$

$$EIv = \frac{Lq_0}{24}\left(\frac{-x^3}{6} + \frac{Lx^2}{2}\right) + C_3x + C_4 \qquad (4)$$

B.C. $v(L) = 0 \qquad 0 = \frac{q_0L^4}{72} + C_3L + C_4 \qquad (5)$

B.C. $v'_L\left(\frac{L}{2}\right) = v'_R\left(\frac{L}{2}\right)$

$$\frac{1}{96}q_0L^3 + C_1 = \frac{1}{64}q_0L^3 + C_3 \qquad (6)$$

B.C. $v_L\left(\frac{L}{2}\right) = v_R\left(\frac{L}{2}\right)$

$$\frac{13}{5760}q_0L^4 + \frac{1}{2}C_1L = \frac{5}{1152}q_0L^4 + \frac{1}{2}C_3L + C_4 \qquad (7)$$

From (5)–(7)

$$C_1 = \frac{-53}{5760}q_0L^3 \qquad C_3 = \frac{-83}{5760}q_0L^3$$

$$C_4 = \frac{1}{1920}q_0L^4$$

For $0 \le x \le \frac{L}{2}$

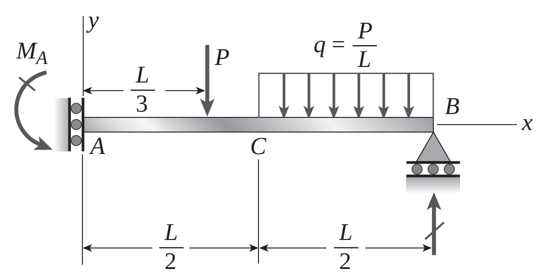
$$v(x) = \frac{q_0x}{5760LEI}(200x^2L^2 - 240x^3L + 96x^4 - 53L^4) \quad \leftarrow$$

For $\frac{L}{2} \le x \le L$

$$v(x) = \frac{-Lq_0}{5760EI}(40x^3 - 120Lx^2 + 83L^2x - 3L^3) \quad \leftarrow$$

$$\delta_C = -v\left(\frac{L}{2}\right) = \frac{3q_0L^4}{1280EI} \quad \leftarrow$$

Problem 9.3-17 The beam shown in the figure has a guided support at A and a roller support at B. The guided support permits vertical movement but no rotation. Derive the equation of the deflection curve and determine the deflection δ_A at end A and also δ_C at point C due to the uniform load of intensity $q = P/L$ applied over segment CB and load P at $x = L/3$. (*Note*: Use the second-order differential equation of the deflection curve.)



Solution 9.3-17

BENDING-MOMENT EQUATION

For $0 \le x \le \frac{L}{3}$ $\qquad EIv'' = M(x) = \frac{19}{24}PL$

$$EIv' = \frac{19}{24}PLx + C_1$$

$$EIv = \frac{19}{48}PLx^2 + C_1x + C_2$$

B.C. $v'(0) = 0 \qquad C_1 = 0 \qquad EIv' = \frac{19}{24}PLx \qquad EIv = \frac{19}{48}PLx^2 + C_2$

For $\frac{L}{3} \le x \le \frac{L}{2}$ $\qquad EIv'' = M(x) = \frac{19}{24}PL - P\left(x - \frac{L}{3}\right)$

$$EIv'' = M(x) = \frac{19}{24}PL - Px + \frac{PL}{3}$$

$$EIv' = \frac{19}{24}PLx - \frac{Px^2}{2} + \frac{PLx}{3} + C_3$$

$$EIv = \frac{19}{48}PLx^2 - \frac{Px^3}{6} + \frac{PLx^2}{6} + C_3x + C_4$$

For $\frac{L}{2} \le x \le L$ $\qquad EIv'' = M(x) = \frac{19}{24}PL - Px + \frac{PL}{3} - \frac{P}{L}\left(x - \frac{L}{2}\right)^2\frac{1}{2}$

$$EIv'' = M(x) = \frac{19}{24}PL - Px + \frac{PL}{3} - \frac{Px^2}{2L} + \frac{Px}{2} - \frac{PL}{8}$$

$$EIv' = \frac{19}{24}PLx - \frac{Px^2}{2} + \frac{PLx}{3} - \frac{Px^3}{6L} + \frac{Px^2}{4} - \frac{PLx}{8} + C_5$$

$$EIv = \frac{19}{48}PLx^2 - \frac{Px^3}{6} + \frac{PLx^2}{6} - \frac{Px^4}{24L} + \frac{Px^3}{12} - \frac{PLx^2}{16} + C_5x + C_6$$

B.C. $v(L) = 0 \qquad 0 = \frac{19}{48}PLL^2 - \frac{PL^3}{6} + \frac{PLL^2}{6} - \frac{PL^4}{24L} + \frac{PL^3}{12} - \frac{PLL^2}{16} + C_5L + C_6$ (1)

B.C. $v'_L\left(\frac{L}{3}\right) = v'_R\left(\frac{L}{3}\right) \qquad 0 = -\frac{P\left(\frac{L}{3}\right)^2}{2} + \frac{PL\left(\frac{L}{3}\right)}{3} + C_3$ (2)

B.C. $v_L\left(\frac{L}{3}\right) = v_R\left(\frac{L}{3}\right) \qquad C_2 = -\frac{P\left(\frac{L}{3}\right)^3}{6} + \frac{PL\left(\frac{L}{3}\right)^2}{6} + C_3\left(\frac{L}{3}\right) + C_4$ (3)

B.C. $v'_L\left(\frac{L}{2}\right) = v'_R\left(\frac{L}{2}\right) \qquad C_3 = -\frac{P\left(\frac{L}{2}\right)^3}{6L} + \frac{P\left(\frac{L}{2}\right)^2}{4} - \frac{PL\left(\frac{L}{2}\right)}{8} + C_5$ (4)

B.C. $v_L(a) = v_R(a) \qquad C_3\frac{L}{2} + C_4 = -\frac{P\left(\frac{L}{2}\right)^4}{24L} + \frac{P\left(\frac{L}{2}\right)^3}{12} - \frac{PL\left(\frac{L}{2}\right)^2}{16} + C_5\left(\frac{L}{2}\right) + C_6$ (5)

From (1)–(5)

$C_2 = \frac{-3565}{10368}PL^3 \qquad C_3 = \frac{-1}{18}PL^2 \qquad C_4 = \frac{-389}{1152}PL^3 \qquad C_5 = \frac{-5}{144}PL^2 \qquad C_6 = \frac{-49}{144}PL^3$

For $0 \le x \le \frac{L}{3} \qquad v(x) = \frac{-PL}{10368EI}(-4104x^2 + 3565L^2) \quad \leftarrow$

For $\frac{L}{3} \le x \le \frac{L}{2} \qquad v(x) = \frac{-P}{1152EI}(-648Lx^2 + 192x^3 + 64L^2x + 389L^3) \quad \leftarrow$

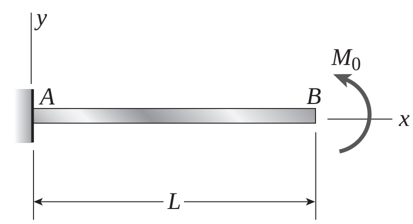
For $\frac{L}{2} \le x \le L \qquad v(x) = \frac{-P}{144EIL}(-72L^2x^2 + 12x^3L + 6x^4 + 5L^3x + 49L^4) \quad \leftarrow$

$\delta_A = -v(0) = \frac{3565PL^3}{10368EI} \quad \leftarrow \qquad \delta_C = -v\left(\frac{L}{3}\right) = \frac{3109PL^3}{10368EI} \quad \leftarrow$

Deflections by Integration of the Shear Force and Load Equations

The beams described in the problems for Section 9.4 have constant flexural rigidity EI. Also, the origin of coordinates is at the left-hand end of each beam.

Problem 9.4-1 Derive the equation of the deflection curve for a cantilever beam AB when a couple M_0 acts counterclockwise at the free end (see figure). Also, determine the deflection δ_B and slope θ_B at the free end. Use the third-order differential equation of the deflection curve (the shear-force equation).



Solution 9.4-1 Cantilever beam (couple M_0)

SHEAR-FORCE EQUATION (EQ. 9-12b).

$$EIv''' = V = 0$$

$$EIv'' = C_1$$

B.C. 1 $M = M_0$ $\quad EIv'' = M = M_0 = C_1$

$$EIv' = C_1 x + C_2 = M_0 x + C_2$$

B.C. 2 $v'(0) = 0 \quad \therefore C_2 = 0$

$$EIv = \frac{M_0 x^2}{2} + C_3$$

B.C. 3 $v(0) = 0 \quad \therefore C_3 = 0$

$$v = \frac{M_0 x^2}{2EI} \quad \leftarrow$$

$$v' = \frac{M_0 x}{EI}$$

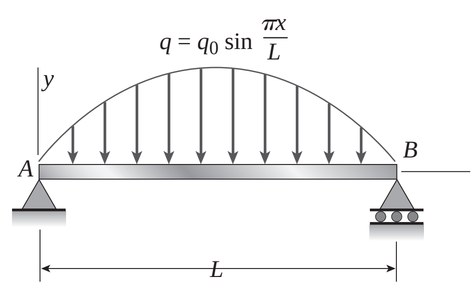
$$\delta_B = v(L) = \frac{M_0 L^2}{2EI} \text{ (upward)} \quad \leftarrow$$

$$\theta_B = v'(L) = \frac{M_0 L}{EI} \text{ (counterclockwise)} \quad \leftarrow$$

(These results agree with Case 6, Table G-1.)

Problem 9.4-2 A simple beam AB is subjected to a distributed load of intensity $q = q_0 \sin \pi x/L$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the deflection δ_{max} at the midpoint of the beam. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-2 Simple beam (sine load)

LOAD EQUATION (EQ. 9-12c).

$$EIv'''' = -q = -q_0 \sin \frac{\pi x}{L}$$

$$EIv''' = q_0\left(\frac{L}{\pi}\right) \cos \frac{\pi x}{L} + C_1$$

$$EIv'' = q_0\left(\frac{L}{\pi}\right)^2 \sin \frac{\pi x}{L} + C_1 x + C_2$$

B.C. 1 $EIv'' = M \quad EIv''(0) = 0 \quad \therefore C_2 = 0$

B.C. 2 $EIv''(L) = 0 \quad \therefore C_1 = 0$

$$EIv' = -q_0\left(\frac{L}{\pi}\right)^3 \cos \frac{\pi x}{L} + C_3$$

$$EIv = -q_0\left(\frac{L}{\pi}\right)^4 \sin \frac{\pi x}{L} + C_3 x + C_4$$

B.C. 3 $v(0) = 0 \qquad \therefore C_4 = 0$

B.C. 4 $v(L) = 0 \qquad \therefore C_3 = 0$

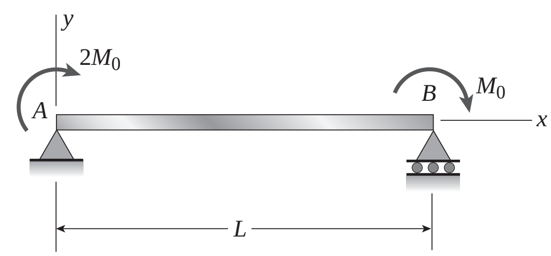
$$v = -\frac{q_0 L^4}{\pi^4 EI} \sin \frac{\pi x}{L} \quad \leftarrow$$

$$\delta_{max} = -v\left(\frac{L}{2}\right) = \frac{q_0 L^4}{\pi^4 EI} \quad \leftarrow$$

(These results agree with Case 13, Table G-2.)

Problem 9.4-3 The simple beam AB shown in the figure has moments $2M_0$ and M_0 acting at the ends.

Derive the equation of the deflection curve, and then determine the maximum deflection δ_{max}. Use the third-order differential equation of the deflection curve (the shear-force equation).



Solution 9.4-3 Simple beam with two couples

Reaction at support A: $R_A = \frac{3M_0}{L}$ (downward)

Shear force in beam: $V = -R_A = -\frac{3M_0}{L}$

SHEAR-FORCE EQUATION (EQ. 9-12b)

$$EIv''' = V = -\frac{3M_0}{L}$$

$$EIv'' = -\frac{3M_0 x}{L} + C_1$$

B.C. 1 $EIv'' = M \qquad EIv''(0) = 2M_0 \qquad \therefore C_1 = 2M_0$

$$EIv' = -\frac{3M_0 x^2}{2L} + 2M_0 x + C_2$$

$$EIv = -\frac{M_0 x^3}{2L} + M_0 x^2 + C_2 x + C_3$$

B.C. 2 $v(0) = 0 \qquad \therefore C_3 = 0$

B.C. 3 $v(L) = 0 \qquad \therefore C_2 = -\frac{M_0 L}{2}$

$$v = -\frac{M_0 x}{2LEI}(L^2 - 2Lx + x^2)$$

$$= -\frac{M_0 x}{2LEI}(L - x)^2 \quad \leftarrow$$

$$v' = -\frac{M_0}{2LEI}(L - x)(L - 3x)$$

MAXIMUM DEFLECTION

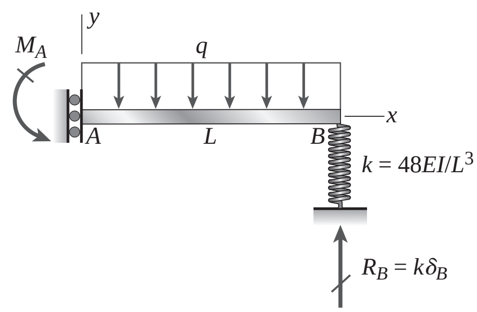
Set $v' = 0$ and solve for x:

$$x_1 = L \text{ and } x_2 = \frac{L}{3}$$

Maximum deflection occurs at $x_2 = \frac{L}{3}$.

$$\delta_{max} = -v\left(\frac{L}{3}\right) = \frac{2M_0 L^2}{27EI} \quad \text{(downward)} \quad \leftarrow$$

Problem 9.4-4 A beam with a uniform load has a guided support at one end and spring support at the other. The spring has stiffness $k = 48EI/L^3$. Derive the equation of the deflection curve by starting with the third-order differential equation (the shear-force equation). Also, determine the angle of rotation θ_B at support B.



Solution 9.4-4

SHEAR-FORCE EQUATION

$$EIv''' = V = -qx$$

$$EIv'' = -\frac{qx^2}{2} + C_1$$

B.C. $v''(L) = M(L) = 0 \qquad C_1 = \frac{qL^2}{2}$

$$EIv'' = \frac{qL^2}{2} - \frac{qx^2}{2}$$

$$EIv' = \frac{qL^2x}{2} - \frac{qx^3}{6} + C_2$$

$$EIv = \frac{qL^2x^2}{4} - \frac{qx^4}{24} + C_2x + C_3$$

B.C. $v'(0) = 0 \qquad C_2 = 0$

B.C. $v(L) = \frac{qL}{k} = -\frac{qL^4}{48EI}$

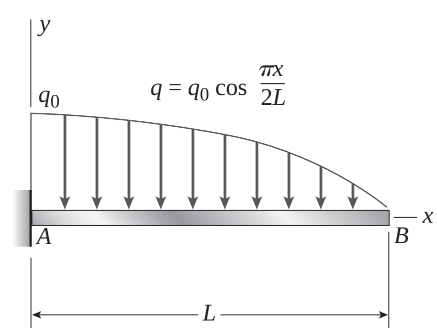
$$C_3 = -\frac{11qL^4}{48}$$

$$v(x) = -\frac{q}{48EI}(2x^4 - 12x^2L^2 + 11L^4) \quad \leftarrow$$

$$\theta_B = -v'(L) = -\frac{qL^3}{3EI} \quad \text{(Counterclockwise)} \quad \leftarrow$$

Problem 9.4-5 The distributed load acting on a cantilever beam AB has an intensity q given by the expression $q_0 \cos \pi x/2L$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the deflection δ_B at the free end. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-5 Cantilever beam (cosine load)

LOAD EQUATION (EQ. 9-12c)

$$EIv'''' = -q = -q_0 \cos \frac{\pi x}{2L}$$

$$EIv''' = -q_0\left(\frac{2L}{\pi}\right) \sin \frac{\pi x}{2L} + C_1$$

B.C. 1 $EIv''' = V$ $\quad EIv'''(L) = 0 \quad \therefore C_1 = \frac{2q_0 L}{\pi}$

$$EIv'' = q_0\left(\frac{2L}{\pi}\right)^2 \cos \frac{\pi x}{2L} + \frac{2q_0 Lx}{\pi} + C_2$$

B.C. 2 $EIv'' = M$ $\quad EIv''(L) = 0 \quad \therefore C_2 = -\frac{2q_0 L^2}{\pi}$

$$EIv' = q_0\left(\frac{2L}{\pi}\right)^3 \sin \frac{\pi x}{2L} + \frac{q_0 Lx^2}{\pi} - \frac{2q_0 L^2 x}{\pi} + C_3$$

B.C. 3 $v'(0) = 0 \quad \therefore C_3 = 0$

$$EIv = -q_0\left(\frac{2L}{\pi}\right)^4 \cos \frac{\pi x}{2L} + \frac{q_0 Lx^3}{3\pi} - \frac{q_0 L^2 x^2}{\pi} + C_4$$

B.C. 4 $v(0) = 0 \quad \therefore C_4 = \frac{16 q_0 L^4}{\pi^4}$

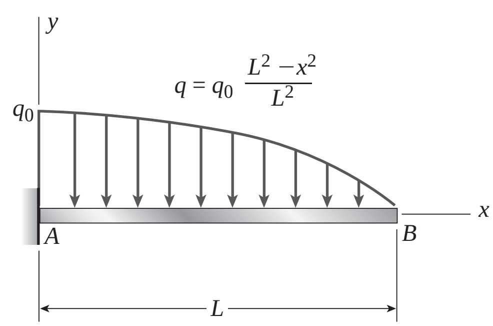
$$v = -\frac{q_0 L}{3\pi^4 EI}\left(48L^3 \cos \frac{\pi x}{2L} - 48L^3 + 3\pi^3 L x^2 - \pi^3 x^3\right) \quad \leftarrow$$

$$\delta_B = -v(L) = \frac{2q_0 L^4}{3\pi^4 EI}(\pi^3 - 24) \quad \leftarrow$$

(These results agree with Case 10, Table G-1.)

Problem 9.4-6 A cantilever beam AB is subjected to a parabolically varying load of intensity $q = q_0(L^2 - x^2)/L^2$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the deflection δ_B and angle of rotation θ_B at the free end. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-6 Cantilever beam (parabolic load)

LOAD EQUATION (EQ. 9-12c)

$$EIv'''' = -q = -\frac{q_0}{L^2}(L^2 - x^2)$$

$$EIv''' = -\frac{q_0}{L^2}\left(L^2 x - \frac{x^3}{3}\right) + C_1$$

B.C. 1 $EIv''' = V$ $\quad EIv'''(L) = 0 \quad \therefore C_1 = \frac{2q_0 L}{3}$

$$EIv'' = -\frac{q_0}{L^2}\left(\frac{L^2 x^2}{2} - \frac{x^4}{12}\right) + \frac{2q_0 L}{3}x + C_2$$

B.C. 2 $EIv'' = M$ $\quad EIv''(L) = 0 \quad \therefore C_2 = -\frac{q_0 L^2}{4}$

$$EIv' = -\frac{q_0}{L^2}\left(\frac{L^2 x^3}{6} - \frac{x^5}{60}\right) + \frac{q_0 L x^2}{3} - \frac{q_0 L^2 x}{4} + C_3$$

B.C. 3 $v'(0) = 0 \quad \therefore C_3 = 0$

$$EIv = -\frac{q_0}{L^2}\left(\frac{L^2 x^4}{24} - \frac{x^6}{360}\right) + \frac{q_0 L x^3}{9} - \frac{q_0 L^2 x^2}{8} + C_4$$

B.C. 4 $v(0) = 0 \qquad \therefore C_4 = 0$

$$v = -\frac{q_0 x^2}{360\,L^2 EI}(45L^4 - 40L^3 x + 15L^2 x^2 - x^4) \quad \longleftarrow$$

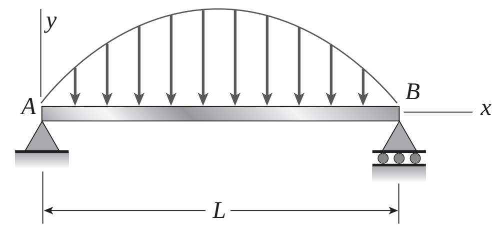
$$\delta_B = -v(L) = \frac{19 q_0 L^4}{360\,EI} \quad \longleftarrow$$

$$v' = -\frac{q_0 x}{60L^2 EI}(15L^4 - 20L^3 x + 10L^2 x^2 - x^4)$$

$$\theta_B = -v'(L) = \frac{q_0 L^3}{15EI} \quad \longleftarrow$$

Problem 9.4-7 A beam on simple supports is subjected to a parabolically distributed load of intensity $q = 4q_0 x(L - x)/L^2$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the maximum deflection δ_{max}. Use the fourthorder differential equation of the deflection curve (the load equation).



Solution 9.4-7 Simple beam (parabolic load)

LOAD EQUATION (EQ. 9-12c)

$$EIv'''' = -q = -\frac{4q_0 x}{L^2}(L - x) = -\frac{4q_0}{L^2}(Lx - x^2)$$

$$EIv''' = -\frac{2q_0}{3L^2}(3Lx^2 - 2x^3) + C_1$$

$$EIv'' = -\frac{q_0}{3L^2}(2Lx^3 - x^4) + C_1 x + C_2$$

B.C. 1 $EIv'' = M \qquad EIv''(0) = 0 \qquad \therefore C_2 = 0$

B.C. 2 $EIv''(L) = 0 \qquad \therefore C_1 = \frac{q_0 L}{3}$

$$EIv' = -\frac{q_0}{30L^2}(-5L^3 x^2 + 5Lx^4 - 2x^5) + C_3$$

B.C. 3 (Symmetry) $v'\left(\frac{L}{2}\right) = 0 \qquad \therefore C_3 = -\frac{q_o L^3}{30}$

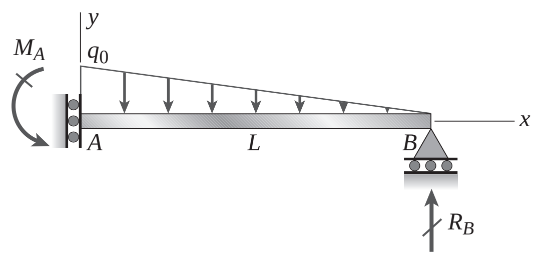
$$EIv = -\frac{q_0}{30L^2}\left(L^5 x - \frac{5L^3 x^3}{3} + Lx^5 - \frac{x^6}{3}\right) + C_4$$

B.C. 4 $v(0) = 0 \qquad \therefore C_4 = 0$

$$v = -\frac{q_0 x}{90L^2 EI}(3L^5 - 5L^3 x^2 + 3Lx^4 - x^5) \quad \longleftarrow$$

$$\delta_{max} = -v\left(\frac{L}{2}\right) = \frac{61 q_0 L^4}{5760\,EI} \quad \longleftarrow$$

Problem 9.4-8 Derive the equation of the deflection curve for beam AB, with guided support at A and roller at B, carrying a triangularly distributed load of maximum intensity q_0 (see figure). Also, determine the maximum deflection δ_{max} of the beam. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-8

LOAD EQUATION

$$EIv'''' = -q = -q_0 + \frac{q_0 x}{L}$$

$$EIv''' = -q_0 x + \frac{q_0 x^2}{2L} + C_1$$

B.C. $v'''(0) = V(0) = 0 \qquad C_1 = 0$

$$EIv''' = -q_0 x + \frac{q_0 x^2}{2L}$$

$$EIv'' = -\frac{q_0 x^2}{2} + \frac{q_0 x^3}{6L} + C_2$$

B.C. $v''(L) = M(L) = 0 \qquad C_2 = \frac{q_0 L^2}{3}$

$$EIv'' = -\frac{q_0 x^2}{2} + \frac{q_0 x^3}{6L} + \frac{q_0 L^2}{3}$$

$$EIv' = -\frac{q_0 x^3}{6} + \frac{q_0 x^4}{24L} + \frac{q_0 L^2 x}{3} + C_3$$

$$EIv = -\frac{q_0 x^4}{24} + \frac{q_0 x^5}{120L} + \frac{q_0 L^2 x^2}{6} + C_3 x + C_4$$

B.C. $v'(0) = 0 \qquad C_3 = 0$

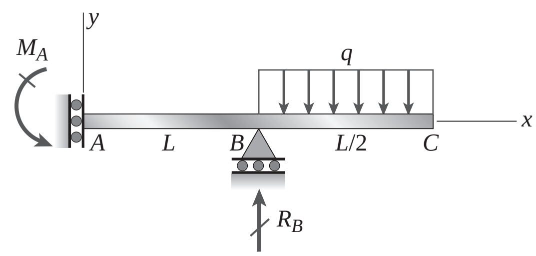
B.C. $v(L) = 0 \qquad C_4 = -\frac{2q_0 L^4}{15}$

$$v(x) = \frac{q_0}{120EIL}(-5x^4 L + x^5 + 20L^3 x^2 - 16L^5) \quad \leftarrow$$

MAXIMUM DEFLECTION

$$\delta_{max} = -v(0) = \frac{2q_0 L^4}{15EI} \quad \leftarrow$$

Problem 9.4-9 Derive the equations of the deflection curve for beam *ABC*, with guided support at *A* and roller support at *B*, supporting a uniform load of intensity *q* acting on the over-hang portion of the beam (see figure). Also, determine deflection δ_C and angle of rotation θ_C. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-9

LOAD EQUATION

$EIv'''' = -q = 0 \quad (0 \le x \le L)$

$EIv''' = C_1 \quad (0 \le x \le L)$

$EIv'' = C_1 x + C_2 \quad (0 \le x \le L)$

B.C. $v'''(0) = V(0) = 0 \qquad C_1 = 0$

B.C. $v''(0) = M(0) = -\frac{qL^2}{8} \qquad C_2 = -\frac{qL^2}{8}$

$$EIv'' = -\frac{qL^2}{8}$$

$$EIv' = -\frac{qL^2 x}{8} + C_3$$

B.C. $v'(0) = 0 \qquad C_3 = 0$

$$EIv = -\frac{qL^2 x^2}{16} + C_4$$

B.C. $v(L) = 0 \qquad C_4 = \frac{qL^4}{16}$

$$v(x) = -\frac{qL^2}{16EI}(x^2 - L^2) \quad (0 \le x \le L) \quad \leftarrow$$

LOAD EQUATION

$$EIv'''' = -q \quad \left(L \le x \le \frac{3L}{2}\right)$$

$$EIv''' = -qx + C_5 \quad \left(L \le x \le \frac{3L}{2}\right)$$

$$EIv'' = \frac{-qx^2}{2} + C_5 x + C_6 \quad \left(L \le x \le \frac{3L}{2}\right)$$

$$\text{B.C. } v'''\left(\frac{3L}{2}\right) = V\left(\frac{3L}{2}\right) = 0 \qquad C_5 = \frac{3qL}{2}$$

$$\text{B.C. } v''\left(\frac{3L}{2}\right) = M\left(\frac{3L}{2}\right) = 0 \qquad C_6 = \frac{9qL^2}{8}$$

$$EIv'' = \frac{-qx^2}{2} + \frac{3qLx}{2} - \frac{9qL^2}{8}$$

$$EIv' = \frac{-qx^3}{6} + \frac{3qLx^2}{4} - \frac{9qL^2x}{8} + C_7$$

$$\text{B.C. } v'_L(L) = v'_R(L)$$

$$-\frac{qL^3}{8} = \frac{-qL^3}{6} + \frac{3qL^3}{4} - \frac{9qL^3}{8} + C_7$$

$$C_7 = \frac{5}{12}qL^3$$

$$EIv' = \frac{-qx^3}{6} + \frac{3qLx^2}{4} - \frac{9qL^2x}{8} + \frac{5}{12}qL^3$$

$$EIv = \frac{-qx^4}{24} + \frac{3qLx^3}{12} - \frac{9qL^2x^2}{16} + \frac{5}{12}qL^3x + C_8$$

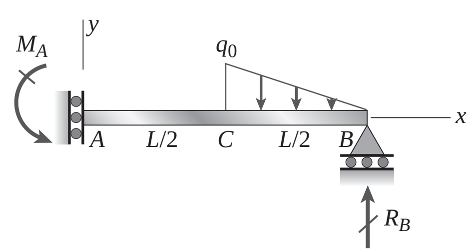
$$\text{B.C. } v(L) = 0 \qquad C_8 = \frac{-1}{16}qL^4$$

$$v(x) = \frac{-q}{48EI}(-20xL^3 + 27L^2x^2 - 12Lx^3 + 2x^4 + 3L^4) \quad \left(L \le x \le \frac{3L}{2}\right) \quad \leftarrow$$

$$\delta_C = -v\left(\frac{3L}{2}\right) = \frac{9qL^4}{128EI} \quad \leftarrow$$

$$\theta_C = -v'\left(\frac{3L}{2}\right) = \frac{7qL^3}{48EI} \quad \text{(Clockwise)} \quad \leftarrow$$

Problem 9.4-10 Derive the equations of the deflection curve for beam *AB*, with guided support at *A* and roller support at *B*, supporting a distributed load of maximum intensity q_0 acting on the right-hand half of the beam (see figure). Also, determine deflection δ_A, angle of rotation θ_B, and deflection δ_C at the midpoint. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-10

Load equation

$$EIv'''' = -q = 0 \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$EIv''' = C_1 \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$EIv'' = C_1x + C_2 \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$\text{B.C. } v'''(0) = V(0) = 0 \qquad C_1 = 0$$

$$\text{B.C. } v''(0) = M(0) = \frac{q_0L^2}{12} \qquad C_2 = \frac{q_0L^2}{12}$$

$$EIv'' = \frac{q_0L^2}{12} \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$EIv' = \frac{q_0L^2x}{12} + C_3 \quad \left(0 \le x \le \frac{L}{2}\right)$$

B.C. $\nu'(0) = 0 \qquad C_3 = 0$

$$EI\nu = \frac{q_0 L^2 x^2}{24} + C_4 \quad \left(0 \le x \le \frac{L}{2}\right)$$

LOAD EQUATION

$$EI\nu'''' = -q_0 + \frac{2q_0}{L}\left(x - \frac{L}{2}\right) \quad \left(\frac{L}{2} \le x \le L\right)$$

$$EI\nu'''' = -2q_0 + \frac{2q_0 x}{L}$$

$$EI\nu''' = -2q_0 x + \frac{q_0 x^2}{L} + C_5 \quad \left(\frac{L}{2} \le x \le L\right)$$

$$EI\nu'' = -q_0 x^2 + \frac{q_0 x^3}{3L} + C_5 x + C_6$$

$$\left(\frac{L}{2} \le x \le L\right)$$

B.C. $\nu'''\left(\frac{L}{2}\right) = V\left(\frac{L}{2}\right) = 0 \qquad C_5 = \frac{3q_0 L}{4}$

B.C. $\nu''\left(\frac{L}{2}\right) = M\left(\frac{L}{2}\right) = \frac{q_0 L^2}{12} \qquad C_6 = \frac{q_0 L^2}{12}$

$$EI\nu'' = -q_0 x^2 + \frac{q_0 x^3}{3L} + \frac{3q_0 L x}{4} - \frac{q_0 L^2}{12}$$

$$EI\nu' = -\frac{q_0 x^3}{3} + \frac{q_0 x^4}{12L} + \frac{3q_0 L x^2}{8} - \frac{q_0 L^2 x}{12} + C_7$$

B.C. $\nu'_L\left(\frac{L}{2}\right) = \nu'_R\left(\frac{L}{2}\right) \qquad C_7 = \frac{5}{192} q_0 L^3$

$$EI\nu' = -\frac{q_0 x^3}{3} + \frac{q_0 x^4}{12L} + \frac{3q_0 L x^2}{8} + \frac{q_0 L^2 x}{12} + \frac{5q_0 L^3}{192}$$

$$EI\nu = -\frac{q_0 x^4}{12} + \frac{q_0 x^5}{60L} + \frac{q_0 L x^3}{8} - \frac{q_0 L^2 x^2}{24} + \frac{5q_0 L^3 x}{192} + C_8$$

B.C. $\nu(L) = 0 \qquad C_8 = \frac{-41}{960} q_0 L^4$

$$EI\nu = -\frac{q_0 x^4}{12} + \frac{q_0 x^5}{60L} + \frac{q_0 L x^3}{8} - \frac{q_0 L^2 x^2}{24} + \frac{5q_0 L^3 x}{192} - \frac{41}{960} q_0 L^4$$

$$\left(\frac{L}{2} \le x \le L\right) \quad \longleftarrow$$

B.C. $\nu_L\left(\frac{L}{2}\right) = \nu_R\left(\frac{L}{2}\right)$

$$\frac{q_0 L^2\left(\frac{L}{2}\right)^2}{24} + C_4 = -\frac{q_0\left(\frac{L}{2}\right)^4}{12} + \frac{q_0\left(\frac{L}{2}\right)^5}{60L} + \frac{q_0 L\left(\frac{L}{2}\right)^3}{8} - \frac{q_0 L^2\left(\frac{L}{2}\right)^2}{24} + \frac{5q_0 L^3 \frac{L}{2}}{192} - \frac{41}{960} q_0 L^4$$

$$C_4 = \frac{-19}{480} q_0 L^4$$

$$EI\nu = \frac{q_0 L^2 x^2}{24} - \frac{19}{480} q_0 L^4 \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$\nu(x) = -\frac{q_0 L^2}{480EI}(-20x^2 + 19L^2)$$

$$(0 \le x \le L) \quad \longleftarrow$$

$$\nu(x) = -\frac{q_0}{960LEI}(80x^4 L - 16x^5 - 120L^2 x^3 + 40L^3 x^2 - 25L^4 x + 41L^5$$

$$\left(\frac{L}{2} \le x \le L\right) \quad \longleftarrow$$

$$\delta_A = -\nu(0) = \frac{19}{480EI} q_0 L^4 \quad \longleftarrow$$

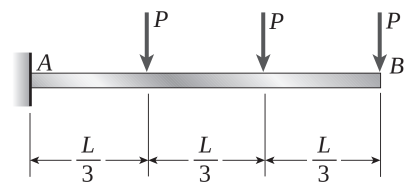
$$\theta_B = -\nu'(L) = -\frac{13}{192EI} q_0 L^3 \quad \longleftarrow$$

$$\delta_C = -\nu\left(\frac{L}{2}\right) = \frac{7}{240EI} q_0 L^4 \quad \longleftarrow$$

Method of Superposition

The problems for Section 9.5 are to be solved by the method of superposition. All beams have constant flexural rigidity EI.

Problem 9.5-1 A cantilever beam AB carries three equally spaced concentrated loads, as shown in the figure. Obtain formulas for the angle of rotation θ_B and deflection δ_B at the free end of the beam.



Solution 9.5-1 Cantilever beam with 3 loads

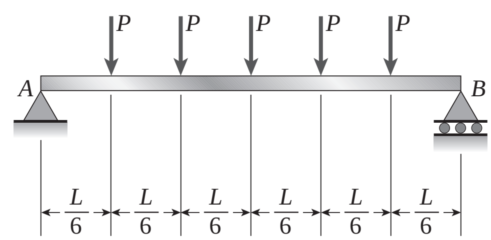
Table G-1, Cases 4 and 5

$$\theta_B = \frac{P\left(\frac{L}{3}\right)^2}{2EI} + \frac{P\left(\frac{2L}{3}\right)^2}{2EI} + \frac{PL^2}{2EI} = \frac{7PL^2}{9EI} \quad \leftarrow$$

$$\delta_B = \frac{P\left(\frac{L}{3}\right)^2}{6EI}\left(3L - \frac{L}{3}\right) + \frac{P\left(\frac{2L}{3}\right)^2}{6EI}\left(3L - \frac{2L}{3}\right) + \frac{PL^3}{3EI} = \frac{5PL^3}{9EI} \quad \leftarrow$$

Problem 9.5-2 A simple beam AB supports five equally spaced loads P (see figure).

(a) Determine the deflection δ_1 at the midpoint of the beam.
(b) If the same total load ($5P$) is distributed as a uniform load on the beam, what is the deflection δ_2 at the midpoint?
(c) Calculate the ratio of δ_1 to δ_2.



Solution 9.5-2 Simple beam with 5 loads

(a) Table G-2, Cases 4 and 6

$$\delta_1 = \frac{P\left(\frac{L}{6}\right)}{24\,EI}\left[3L^2 - 4\left(\frac{L}{6}\right)^2\right] + \frac{P\left(\frac{L}{3}\right)}{24EI}\left[3L^2 - 4\left(\frac{L}{3}\right)^2\right] + \frac{PL^3}{48EI}$$

$$= \frac{11PL^3}{144EI} \quad \leftarrow$$

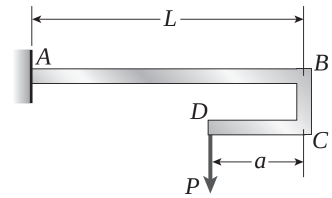
(b) Table G-2, Case 1 $qL = 5P$

$$\delta_2 = \frac{5qL^4}{384EI} = \frac{25PL^3}{384EI} \quad \leftarrow$$

(c) $$\frac{\delta_1}{\delta_2} = \frac{11}{144}\left(\frac{384}{25}\right) = \frac{88}{75} = 1.173 \quad \leftarrow$$

Problem 9.5-3 The cantilever beam *AB* shown in the figure has an extension *BCD* attached to its free end. A force *P* acts at the end of the extension.

(a) Find the ratio a/L so that the vertical deflection of point *B* will be zero.
(b) Find the ratio a/L so that the angle of rotation at point *B* will be zero.



Solution 9.5-3 Cantilever beam with extension

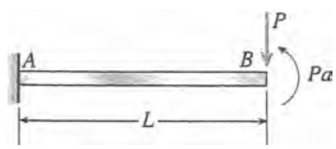


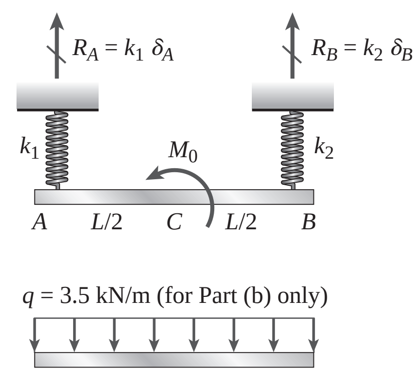
Table G-1, Cases 4 and 6

(a) $\delta_B = \dfrac{PL^3}{3EI} - \dfrac{PaL^2}{2EI} = 0 \qquad \dfrac{a}{L} = \dfrac{2}{3} \quad \leftarrow$

(b) $\theta_B = \dfrac{PL^2}{2EI} - \dfrac{PaL}{EI} = 0 \qquad \dfrac{a}{L} = \dfrac{1}{2} \quad \leftarrow$

Problem 9.5-4 Beam *ACB* hangs from two springs, as shown in the figure. The springs have stiffnesses k_1 and k_2 and the beam has flexural rigidity *EI*.

(a) What is the downward displacement of point *C*, which is at the midpoint of the beam, when the moment M_0 is applied? Data for the structure are as follows: $M_0 = 10.0$ kN·m, $L = 1.8$ m, $EI = 216$ kN·m^2, $k_1 = 250$ kN/m, and $k_2 = 160$ kN/m.
(b) Repeat (a) but remove M_0 and apply uniform load $q = 3.5$ kN/m to the entire beam.



Solution 9.5-4

$M_0 = 10.0\ \text{kN}\cdot\text{m} \qquad L = 1.8\ \text{m} \qquad EI = 216\ \text{kN}\cdot\text{m}^2$

$k_1 = 250\ \text{kN/m} \qquad k_2 = 160\ \text{kN/m}$

$q = 3.5\ \text{kN/m}$

(a) $R_A = \dfrac{M_0}{L} \qquad R_B = -\dfrac{M_0}{L}$

$\delta_A = \dfrac{R_A}{k_1} \qquad \delta_B = \dfrac{R_B}{k_2}$

$\delta_A = 22.22$ mm Downward

$\delta_B = -34.72$ mm Upward

Table G-2, Case 8

$\delta_C = 0 + \dfrac{1}{2}\cdot(\delta_A + \delta_B)$

$\delta_C = -6.25$ mm Upward $\leftarrow$

(b) $R_A = \dfrac{qL}{2}$ $\qquad R_B = R_A$

$\delta_A = \dfrac{R_A}{k_1}$ $\qquad \delta_B = \dfrac{R_B}{k_2}$

$\delta_A = 12.60$ mm

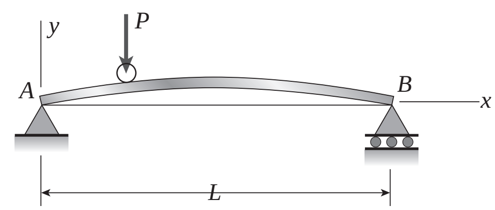
$\delta_B = 19.69$ mm

Table G-2, Case 1

$$\delta_C = \frac{5qL^4}{384EI} + \frac{1}{2}(\delta_A + \delta_B)$$

$\delta_C = 18.36$ mm Downward $\leftarrow$

Problem 9.5-5 What must be the equation $y = f(x)$ of the axis of the slightly curved beam AB (see figure) *before* the load is applied in order that the load P, moving along the bar, always stays at the same level?



Solution 9.5-5 Slightly curved beam

Let x = distance to load P

δ = downward deflection at load P

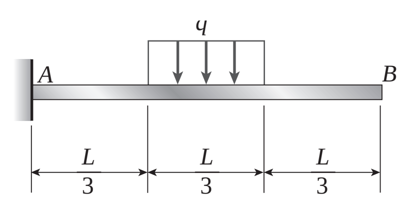
Table G-2, Case 5:

$$\delta = \frac{P(L-x)x}{6LEI}[L^2 - (L-x)^2 - x^2] = \frac{Px^2(L-x)^2}{3LEI}$$

Initial upward displacement of the beam must equal δ.

$$\therefore\ y = \frac{Px^2(L-x)^2}{3LEI} \qquad \leftarrow$$

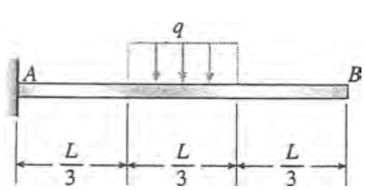
Problem 9.5-6 Determine the angle of rotation θ_B and deflection δ_B at the free end of a cantilever beam AB having a uniform load of intensity q acting over the middle third of its length (see figure).



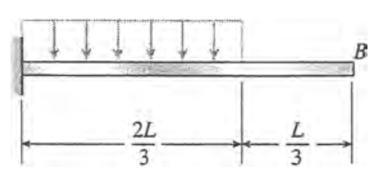
Solution 9.5-6 Cantilever beam (partial uniform load)

q = intensity of uniform load

Original load on the beam:



Load No. 1:



Load No. 2:

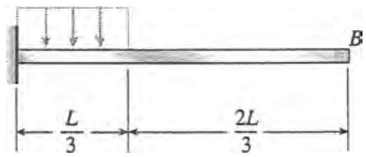


Table G-1, Case 2

$$\theta_B = \frac{q}{6EI}\left(\frac{2L}{3}\right)^3 - \frac{q}{6EI}\left(\frac{L}{3}\right)^3 = \frac{7qL^3}{162EI} \quad \leftarrow$$

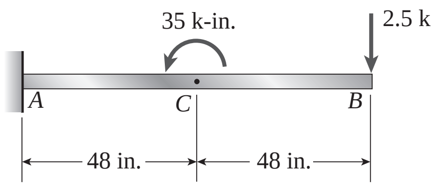
$$\delta_B = \frac{q}{24EI}\left(\frac{2L}{3}\right)^3\left(4L - \frac{2L}{3}\right) - \frac{q}{24EI}\left(\frac{L}{3}\right)^3\left(4L - \frac{L}{3}\right)$$

$$= \frac{23qL^4}{648EI} \quad \leftarrow$$

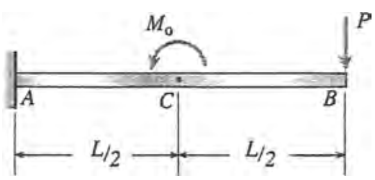
SUPERPOSITION:

Original load = Load No. 1 minus Load No. 2

Problem 9.5-7 The cantilever beam *ACB* shown in the figure has flexural rigidity $EI = 2.1 \times 10^6$ k-in.^{2} Calculate the downward deflections δ_C and δ_B at points *C* and *B*, respectively, due to the simultaneous action of the moment of 35 k-in. applied at point *C* and the concentrated load of 2.5 k applied at the free end *B*.



Solution 9.5-7 Cantilever beam (two loads)



$EI = 2.1 \times 10^6$ k-in.^{2}

$M_0 = 35$ k-in.

$P = 2.5$ k

$L = 96$ in.

Table G-1, Cases 4, 6, and 7

$$\delta_C = -\frac{M_0(L/2)^2}{2EI} + \frac{P(L/2)^2}{6EI}\left(3L - \frac{L}{2}\right)$$

$$= -\frac{M_0L^2}{8EI} + \frac{5PL^3}{48EI} \;(+ = \text{downward deflection})$$

$$\delta_B = -\frac{M_0(L/2)}{2EI}\left(2L - \frac{L}{2}\right) + \frac{PL^3}{3EI}$$

$$= -\frac{3M_0L^2}{8EI} + \frac{PL^3}{3EI} \;(+ = \text{downward deflection})$$

SUBSTITUTE NUMERICAL VALUES:

$\delta_C = -0.01920$ in. $+ 0.10971$ in.

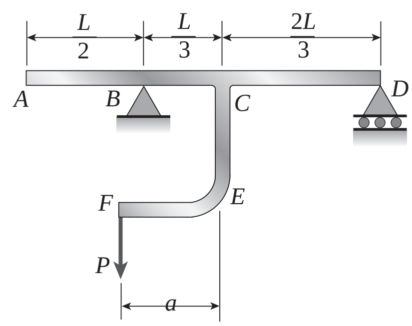
$= 0.0905$ in. $\leftarrow$

$\delta_B = -0.05760$ in. $+ 0.35109$ in.

$= 0.293$ in. $\leftarrow$

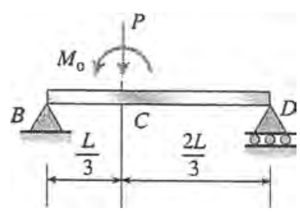
Problem 9.5-8 A beam *ABCD* consisting of a simple span *BD* and an overhang *AB* is loaded by a force *P* acting at the end of the bracket *CEF* (see figure).

(a) Determine the deflection δ_A at the end of the overhang.
(b) Under what conditions is this deflection upward? Under what conditions is it downward?



Solution 9.5-8 Beam with bracket and overhang

Consider part *BD* of the beam.



$$M_0 = Pa$$

Table G-2, Cases 5 and 9

$$\theta_B = \frac{P(L/3)(2L/3)(5L/3)}{6LEI} + \frac{Pa}{6LEI}\left[6\left(\frac{L^2}{3}\right) - 3\left(\frac{L^2}{9}\right) - 2L^2\right]$$

$$= \frac{PL}{162EI}(10L - 9a) \quad (+ = \text{clockwise angle})$$

(a) Deflection at the end of the overhang

$$\delta_A = \theta_B\left(\frac{L}{2}\right) = \frac{PL^2}{324\,EI}(10L - 9a) \quad \leftarrow$$

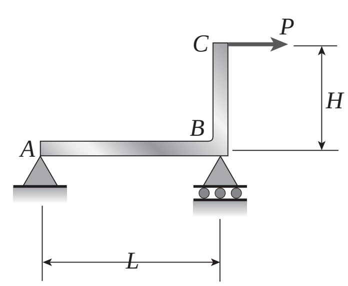
(+ = upward deflection)

(b) Deflection is upward when $\frac{a}{L} < \frac{10}{9}$ and downward when $\frac{a}{L} > \frac{10}{9}$ $\leftarrow$

Problem 9.5-9 A horizontal load *P* acts at end *C* of the bracket *ABC* shown in the figure.

(a) Determine the deflection δ_C of point *C*.
(b) Determine the maximum upward deflection δ_{max} of member *AB*.

Note: Assume that the flexural rigidity *EI* is constant throughout the frame. Also, disregard the effects of axial deformations and consider only the effects of bending due to the load *P*.



Solution 9.5-9 Bracket *ABC*

BEAM *AB*

$M_0 = PH$

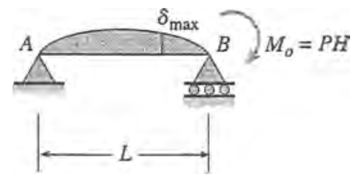


Table G-2, Case 7: $\theta_B = \dfrac{M_0 L}{3EI} = \dfrac{PHL}{3EI}$

(a) ARM *BC* Table G-1, Case 4

$$\delta_C = \frac{PH^3}{3EI} + \theta_B H = \frac{PH^3}{3EI} + \frac{PH^2 L}{3EI}$$

$$= \frac{PH^2}{3EI}(L + H) \quad \leftarrow$$

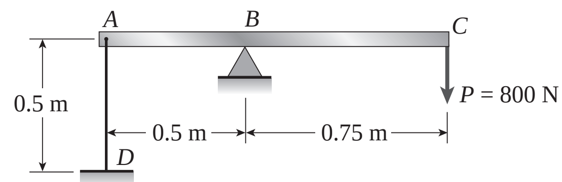
(b) MAXIMUM DEFLECTION OF BEAM *AB*

Table G-2,

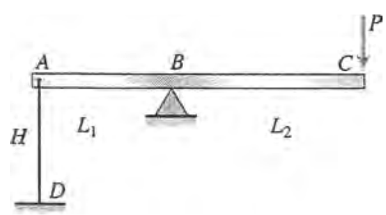
Case 7: $\delta_{max} = \dfrac{M_0 L^2}{9\sqrt{3}EI} = \dfrac{PHL^2}{9\sqrt{3}EI} \quad \leftarrow$

Problem 9.5-10 A beam *ABC* having flexural rigidity $EI = 75 \text{ kN}\cdot\text{m}^2$ is loaded by a force $P = 800$ N at end *C* and tied down at end *A* by a wire having axial rigidity $EA = 900$ kN (see figure).

What is the deflection at point *C* when the load *P* is applied?



Solution 9.5-10 Beam tied down by a wire



$EI = 75 \text{ kN}\cdot\text{m}^2$

$P = 800$ N

$EA = 900$ kN

$H = 0.5$ m $\quad L_1 = 0.5$ m

$L_2 = 0.75$ m

CONSIDER *BC* AS A CANTILEVER BEAM

Table G-1, Case 4: $\delta'_C = \dfrac{PL_2^3}{3EI}$

CONSIDER *AB* AS A SIMPLE BEAM

$M_0 = PL_2$

Table G-2, Case 7: $\theta'_B = \dfrac{M_0 L_1}{3EI} = \dfrac{PL_1 L_2}{3EI}$

CONSIDER THE STRETCHING OF WIRE *AD*

$$\delta'_A = (\text{Force in } AD)\left(\frac{H}{EA}\right) = \left(\frac{PL_2}{L_1}\right)\left(\frac{H}{EA}\right) = \frac{PL_2 H}{EAL_1}$$

DEFLECTION δ_C OF POINT *C*

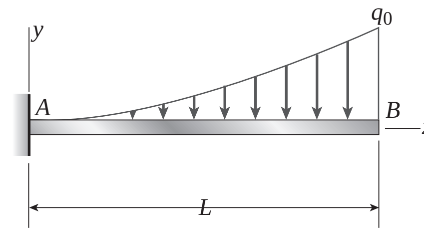
$$\delta_c = \delta'_c + \theta'_B(L_2) + \delta'_B\left(\frac{L_2}{L_1}\right)$$

$$= \frac{PL_2^3}{3EI} + \frac{PL_1L_2^2}{3EI} + \frac{PL_2^2 H}{EAL_1^2} \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$\delta_C = 1.50 \text{ mm} + 1.00 \text{ mm} + 1.00 \text{ mm} = 3.50 \text{ mm} \quad \leftarrow$

Problem 9.5-11 Determine the angle of rotation θ_B and deflection δ_B at the free end of a cantilever beam AB supporting a parabolic load defined by the equation $q = q_0x^2/L^2$ (see figure).



Solution 9.5-11 Cantilever beam (parabolic load)

LOAD: $q = \dfrac{q_0x^2}{L^2}$ $\quad qdx$ = element of load

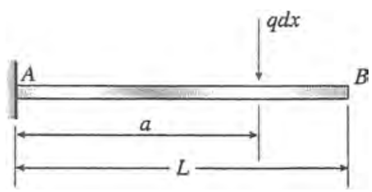


TABLE G-1, CASE 5 (Set a equal to x)

$$\theta_B = \int_0^L \frac{(qdx)(x^2)}{2EI} = \frac{1}{2EI}\int_0^L \left(\frac{q_0x^2}{L^2}\right) x^2dx$$

$$= \frac{q_0}{2EIL^2}\int_0^L x^4dx = \frac{q_0L^3}{10EI} \quad \longleftarrow$$

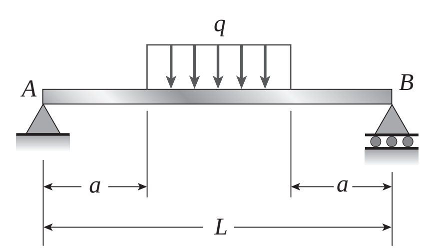
$$\delta_B = \int_0^L \frac{(qdx)(x^2)}{6EI}(3L - x)$$

$$= \frac{1}{6EI}\int_0^L \left(\frac{q_0x^2}{L^2}\right)(x^2)(3L - x)\,dx$$

$$= \frac{q_0}{6EIL^2}\int_0^L (x^4)(3L - x)\,dx = \frac{13q_0L^4}{180EI} \quad \longleftarrow$$

Problem 9.5-12 A simple beam AB supports a uniform load of intensity q acting over the middle region of the span (see figure).

Determine the angle of rotation θ_A at the left-hand support and the deflection δ_{max} at the midpoint.



Solution 9.5-12 Simple beam (partial uniform load)

LOAD: qdx = element of load

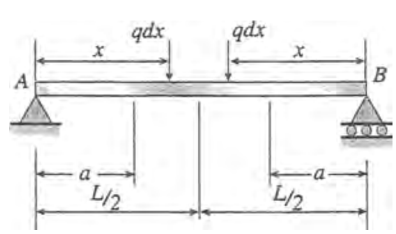


TABLE G-2, CASE 6 $\quad \theta_A = \dfrac{Pa(L - a)}{2EI}$

Replace P by qdx $\quad$ Replace a by x

Integrate x from a to $L/2$

$$\theta_A = \int_a^{L/3} \frac{qdx}{2EI}(x)(L - x) = \frac{q}{2EI}\int_a^{L/2} (xL - x^2)dx$$

$$= \frac{q}{24EI}(L^3 - 6a^2L + 4a^3) \quad \longleftarrow$$

TABLE G-2, CASE 6 $\quad \delta_{max} = \dfrac{Pa}{24EI}(3L^2 - 4a^2)$

Replace P by qdx $\quad$ Replace a by x

Integrate x from a to $L/2$

$$\delta_{max} = \int_a^{L/2} \frac{q dx}{24EI}(x)(3L^2 - 4x^2)$$

$$= \frac{q}{24EI}\int_a^{L/2} (3L^2x - 4x^3)dx$$

$$= \frac{q}{384EI}(5L^4 - 24a^2L^2 + 16a^4) \quad \leftarrow$$

ALTERNATE SOLUTION (not recommended; algebra is extremely lengthy)

Table G-2, Case 3

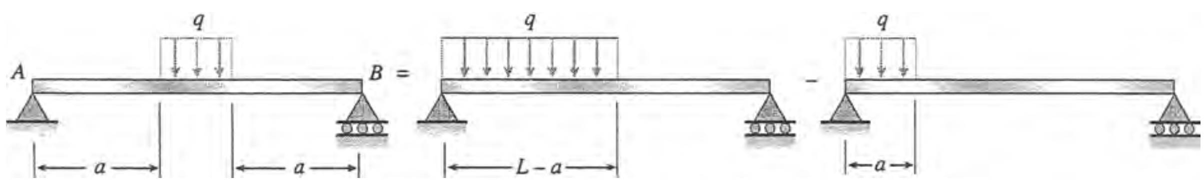
$$\theta_A = \frac{q(L-a)^2}{24LEI}[2L - (L-a)]^2 - \frac{qa^2}{24LEI}(2L-a)^2$$

$$= \frac{q}{24EI}(L^3 - 6La^2 + 4a^3) \quad \leftarrow$$

$$\delta_{max} = \frac{q(L/2)}{24LEI}\Bigg[(L-a)^4 - 4L(L-a)^3 + 4L^2(L-a)^2 + 2(L-a)^2\left(\frac{L}{2}\right)^2 - 4L(L-a)\left(\frac{L}{2}\right)^2 + L\left(\frac{L}{2}\right)^3\Bigg]$$

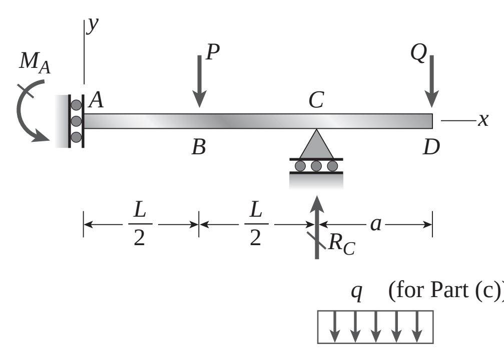
$$= \frac{qa^2}{24LEI}\Bigg[-La^2 + 4L^2\left(\frac{L}{2}\right) + a^2\left(\frac{L}{2}\right) - 6L\left(\frac{L}{2}\right)^2 + 2\left(\frac{L}{2}\right)^3\Bigg]$$

$$\delta_{max} = \frac{q}{384EI}(5L^4 - 24L^2a^2 + 16a^4) \quad \leftarrow$$



Problem 9.5-13 The overhanging beam *ABCD* supports two concentrated loads *P* and *Q* (see figure).

(a) For what ratio *P/Q* will the deflection at point *B* be zero?
(b) For what ratio will the deflection at point *D* be zero?
(c) If *Q* is replaced by uniform load with intensity *q* (on the overhang), repeat (a) and (b) but find ratio *P/(qa)*



Solution 9.5-13

(a) DEFLECTION AT POINT *B*

Table G-2 Cases 6 and 10

$$\delta_B = \frac{P\left(\frac{L}{2}\right)}{6EI}\left[3\left(\frac{L}{2}\right)(2L) - 3\left(\frac{L}{2}\right)^2 - \left(\frac{L}{2}\right)^2\right] - \frac{Qa\left(\frac{L}{2}\right)}{2EI}\left[(2L) - \left(\frac{L}{2}\right)\right]$$

$$\delta_B = 0 \qquad \frac{P}{Q} = \frac{9a}{4L} \quad \leftarrow$$

(b) DEFLECTION AT POINT D

Table G-2 Case 6; Table G-1 Case 4; Table G-2 Case 10

$$\delta_D = -\frac{P\left(\frac{L}{2}\right)\left[(2L) - \frac{L}{2}\right]}{2EI}(a) + \frac{Qa^3}{3EI} + \frac{Qa(2L)}{2EI}(a)$$

$$\delta_D = 0 \qquad \frac{P}{Q} = \frac{8a(3L + a)}{9L^2} \qquad \leftarrow$$

(c.1) DEFLECTION AT POINT B

Table G-2 Cases 6 and 10

$$\delta_B = \frac{P\left(\frac{L}{2}\right)}{6EI}\left[3\left(\frac{L}{2}\right)(2L) - 3\left(\frac{L}{2}\right)^2 - \left(\frac{L}{2}\right)^2\right] - \frac{\left(\frac{qa^2}{2}\right)\left(\frac{L}{2}\right)}{2EI}\left[(2L) - \left(\frac{L}{2}\right)\right]$$

$$\delta_B = 0 \qquad \frac{P}{qa} = \frac{9a}{8L} \qquad \leftarrow$$

(c.2) DEFLECTION AT POINT D

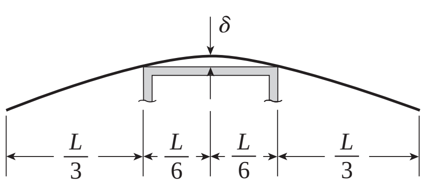
Table G-2 Case 6; Table G-1 Case 1; Table G-2 Case 10

$$\delta_D = -\frac{P\left(\frac{L}{2}\right)\left[(2L) - \frac{L}{2}\right]}{2EI}(a) + \frac{qa^4}{8EI} + \frac{\left(\frac{qa^2}{2}\right)(2L)}{2EI}(a)$$

$$\delta_D = 0 \qquad \frac{P}{qa} = \frac{a(4L + a)}{3L^2} \qquad \leftarrow$$

Problem 9.5-14 A thin metal strip of total weight W and length L is placed across the top of a flat table of width $L/3$ as shown in the figure.

What is the clearance δ between the strip and the middle of the table? (The strip of metal has flexural rigidity EI.)



Solution 9.5-14 Thin metal strip

W = total weight $\quad q = \dfrac{W}{L}$

EI = flexural rigidity

FREE BODY DIAGRAM (the part of the strip above the table)

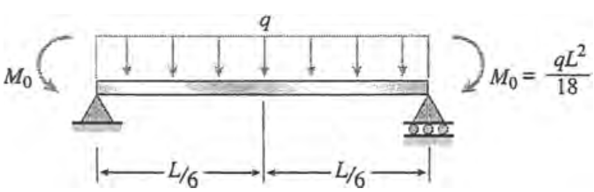


TABLE G-2, CASES 1 AND 10

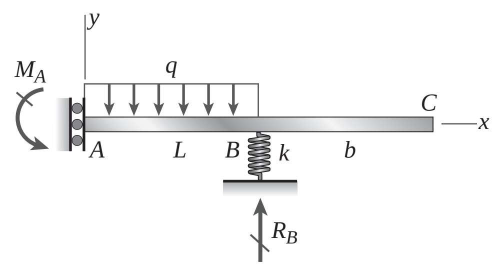
$$\delta = -\frac{5q}{384EI}\left(\frac{L}{3}\right)^4 + \frac{M_0}{8EI}\left(\frac{L}{3}\right)^2$$

$$= -\frac{5qL^4}{31{,}104EI} + \frac{qL^4}{1296EI}$$

$$= \frac{19qL^4}{31{,}104EI}$$

But $q = \dfrac{W}{L}$: $\quad \therefore \delta = \dfrac{19WL^3}{31{,}104EI} \qquad \leftarrow$

Problem 9.5-15 An overhanging beam ABC with flexural rigidity $EI = 15$ k-in.^{2} is supported by a guided support at A and by a spring of stiffness k at point B (see figure). Span AB has length $L = 30$ in. and carries a uniform load. The over-hang BC has length $b = 15$ in. For what stiffness k of the spring will the uniform load produce no deflection at the free end C?



Solution 9.5-15

$EI = 15\text{kip} \cdot \text{in}^2.$ $\quad L = 30$ in. $\quad b = 15$ in.

$R_B = qL$

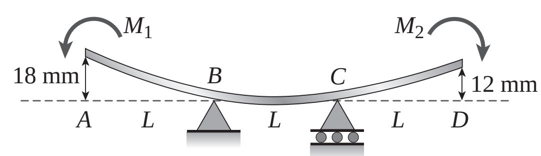
Table G-2, Case 1

$$\delta_C = \theta_B b - \delta_B = \frac{q(2L)^3}{24EI}(b) - \frac{qL}{k}$$

for $\delta_C = 0$ $\quad k = \dfrac{3EI}{bL^2}$

Therefore $\quad k = 3.33\text{lb/in} \quad \leftarrow$

Problem 9.5-16 A beam $ABCD$ rests on simple supports at B and C (see figure). The beam has a slight initial curvature so that end A is 18 mm above the elevation of the supports and end D is 12 mm above. What moments M_1 and M_2, acting at points A and D, respectively, will move points A and D downward to the level of the supports? (The flexural rigidity EI of the beam is 2.5×10^6 N·m^2 and $L = 2.5$ m).



Solution 9.5-16

$EI = 2.5\times10^6\ \text{N}\cdot\text{m}^2$ $\quad \boldsymbol{L} = 2.5$ m $\quad \delta_A = 18$ mm

$\delta_D = 12$ mm

Table G-2, Case 7

$$\theta_B = \frac{M_1 L}{3EI} + \frac{M_2 L}{6EI} \qquad \theta_C = \frac{M_1 L}{6EI} + \frac{M_2 L}{3EI}$$

Deflection at point A and D

Table G-1, Case 6

$$\delta_A = \frac{M_1 L^2}{2EI} + \theta_B L \qquad \delta_D = \frac{M_2 L^2}{2EI} + \theta_C L$$

$$\delta_A = \frac{M_1 L^2}{2EI} + \left(\frac{M_1 L}{3EI} + \frac{M_2 L}{6EI}\right)L$$

$$\delta_D = \frac{M_2 L^2}{2EI} + \left(\frac{M_1 L}{6EI} + \frac{M_2 L}{3EI}\right)L$$

$$5M_1 + M_2 = \frac{6\delta_A EI}{L^2} \qquad (1)$$

$$5M_2 + M_1 = \frac{6\delta_D EI}{L^2} \qquad (2)$$

Solve equation (1) and (2)

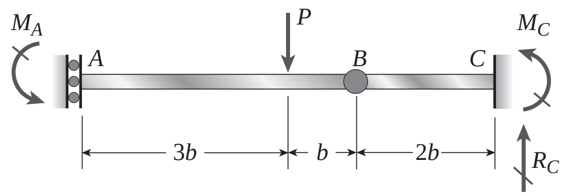
$$M_1 = \frac{EI(5\delta_A - \delta_D)}{4L^2} \qquad M_2 = \frac{EI(5\delta_D - \delta_A)}{4L^2}$$

Therefore

$M_1 = 7800\ \text{N}\cdot\text{m} \quad \leftarrow$

$M_2 = 4200\ \text{N}\cdot\text{m} \quad \leftarrow$

Problem 9.5-17 The compound beam *ABC* shown in the figure has a guided support at *A* and a fixed support at *C*. The beam consists of two members joined by a pin connection (i.e., moment release) at *B*. Find the deflection δ under the load *P*.



Solution 9.5-17

Table G-1, Case 4

$$\delta_B = \frac{P(2b)^3}{3EI}$$

Deflection under the load P

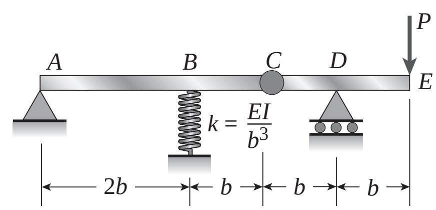
Table G-2, Case 6

$$\delta = \frac{P(b)}{6EI}\left[3(b)(8b) - 3(b)^2 - (b)^2\right] + \delta_B$$

$$\delta = \frac{P(b)}{6EI}\left[3(b)(8b) - 3b^2 - b^2\right] + \frac{P(2b)^3}{3EI}$$

$$\delta = \frac{6Pb^3}{EI} \quad \leftarrow$$

Problem 9.5-18 A compound beam *ABCDE* (see figure) consists of two parts (*ABC* and *CDE*) connected by a hinge (i.e., moment release) at *C*. The elastic support at *B* has stiffness $k = EI/b^3$ Determine the deflection δ_E at the free end *E* due to the load *P* acting at that point.



Solution 9.5-18

Consider beam ABC

$$R_B = \frac{3P}{2} \qquad \delta_B = \frac{R_B}{k} = \frac{3P}{2k} \quad \text{Upward}$$

Table G-2, Case 7; Table G-1, Case 4

$$\delta_C = \frac{Pb(2b)}{3EI}b + \frac{Pb^3}{3EI} + \delta_B\left(\frac{2b + b}{2b}\right)$$

$$= \frac{Pb(2b)}{3EI}b + \frac{Pb^3}{3EI} + \frac{3P}{2k}\left(\frac{2b + b}{2b}\right)$$

$$\delta_C = \frac{P(4b^3k + 9EI)}{4EIk} \quad \text{Upward}$$

Consider beam CDE

Table G-2, Case 7; Table G-1, Case 4

$$\delta_E = \frac{(Pb)(b)}{3EI}b + \frac{Pb^3}{3EI} + \delta_C = \frac{(Pb)(b)}{3EI}b$$

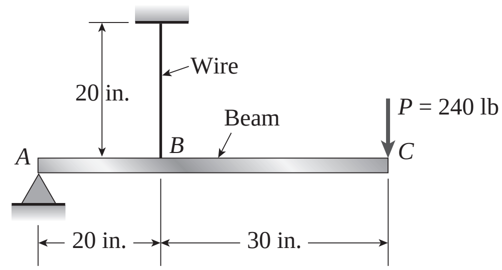
$$+ \frac{Pb^3}{3EI} + \frac{P(4b^3k + 9EI)}{4EIk}$$

for $k = \dfrac{EI}{b^3}$

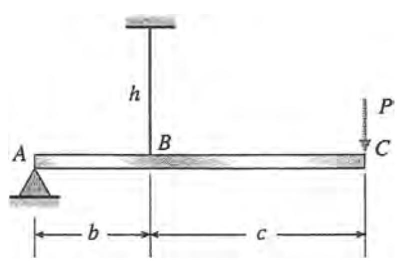
$$\delta_E = \frac{47Pb^3}{12EI} \quad \leftarrow$$

Problem 9.5-19 A stekel beam ABC is simply supported at A and held by a high-strength steel wire at B (see figure). A load $P = 240$ lb acts at the free end C. The wire has axial rigidity $EA = 1500 \times 10^3$ lb, and the beam has flexural rigidity $EI = 36 \times 10^6$ lb-in.^{2}

What is the deflection δ_C of point C due to the load P?



Solution 9.5-19 Beam supported by a wire

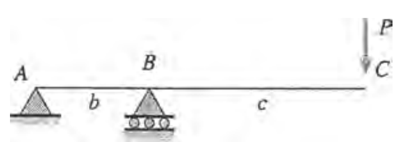


$P = 240$ lb $\quad b = 20$ in. $\quad c = 30$ in. $\quad h = 20$ in.

Beam: $EI = 36 \times 10^6$ lb-in.^{2}

Wire: $EA = 1500 \times 10^3$ lb

(1) ASSUME THAT POINT B IS ON A SIMPLE SUPPORT



$$\delta'_C = \frac{Pc^3}{3EI} + \theta'_B c = \frac{Pc^3}{3EI} + (Pc)\left(\frac{b}{3EI}\right)c$$

$$= \frac{Pc^2}{3EI}(b + c) \quad \text{(downward)}$$

(2) ASSUME THAT THE WIRE STRETCHES

T = tensile force in the wire

$$= \frac{P}{b}(b + c)$$

$$\delta_B = \frac{Th}{EA} = \frac{Ph(b + c)}{EAb}$$

$$\delta''_C = \delta_B\left(\frac{b + c}{b}\right) = \frac{Ph(b + c)^2}{EAb^2} \quad \text{(downward)}$$

(3) DEFLECTION AT POINT C

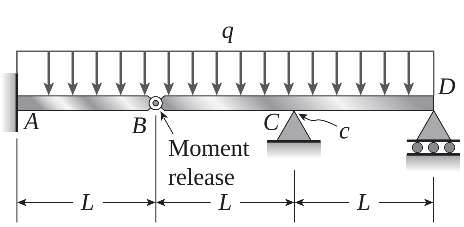
$$\delta_C = \delta'_c + \delta''_c = P(b + c)\left[\frac{c^2}{3EI} + \frac{h(b + c)}{EAb^2}\right] \quad \leftarrow$$

Substitute numerical values:

$$\delta_C = 0.10 \text{ in.} + 0.02 \text{ in.} = 0.12 \text{ in.} \quad \leftarrow$$

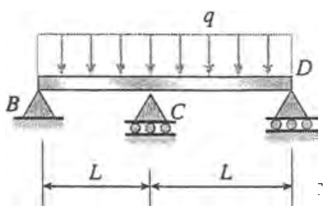
Problem 9.5-20 The compound beam shown in the figure consists of a cantilever beam AB (length L) that is pin-connected to a simple beam BD (length $2L$). After the beam is constructed, a clearance c exists between the beam and a support at C, midway between points B and D. Subsequently, a uniform load is placed along the entire length of the beam.

What intensity q of the load is needed to close the gap at C and bring the beam into contact with the support?



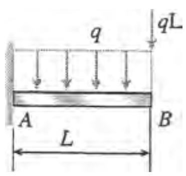
Solution 9.5-20 Compound beam

BEAM *BCD* WITH A SUPPORT AT *B*



$$\delta_c' = \frac{5q(2L)^4}{384EI} = \frac{5qL^4}{24EI}$$

CANTILEVER BEAM *AB*



$$\delta_B = \frac{qL^4}{8EI} + \frac{(qL)L^3}{3EI} = \frac{11qL^4}{24EI} \quad \text{(downward)}$$

δ_c'' = downward displacement of point C due to δ_B

$$\delta_c'' = \frac{1}{2}\delta_B = \frac{11qL^4}{48EI}$$

DOWNWARD DISPLACEMENT OF POINT *C*

$$\delta_c = \delta_c' + \delta_c'' = \frac{5qL^4}{24EI} + \frac{11qL^4}{48EI} = \frac{7qL^4}{16EI}$$

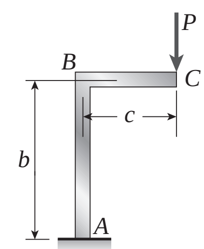
c = clearance $\qquad c = \delta_C = \dfrac{7qL^4}{16EI}$

INTENSITY OF LOAD TO CLOSE THE GAP

$$q = \frac{16EIc}{7L^4} \quad \longleftarrow$$

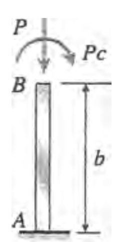
Problem 9.5-21 Find the horizontal deflection δ_h and vertical deflection δ_v at the free end C of the frame *ABC* shown in the figure. (The flexural rigidity *EI* is constant throughout the frame.)

Note: Disregard the effects of axial deformations and consider only the effects of bending due to the load P.



Solution 9.5-21 Frame *ABC*

MEMBER *AB*:



δ_h = horizontal deflection of point B

Table G-1, Case 6:

$$\delta_h = \frac{(Pc)b^2}{2EI} = \frac{Pcb^2}{2EI}$$

$$\theta_B = \frac{Pcb}{EI}$$

Since member *BC* does not change in length, δ_h is also the horizontal displacement of point C.

$$\therefore\ \delta_h = \frac{Pcb^2}{2EI} \quad \longleftarrow$$

MEMBER *BC* WITH *B* FIXED AGAINST ROTATION:

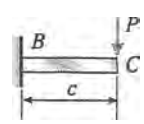


Table G-1, Case 4:

$$\delta_c' = \frac{Pc^3}{3EI}$$

VERTICAL DEFLECTION OF POINT *C*

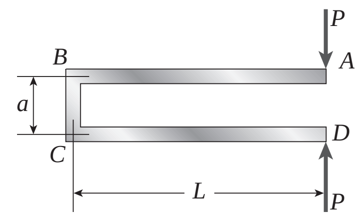
$$\delta_c = \delta_v = \delta_c' + \theta_B c = \frac{Pc^3}{3EI} + \frac{Pcb}{EI}(c)$$

$$= \frac{Pc^2}{3EI}(c + 3b)$$

$$\delta_v = \frac{Pc^2}{3EI}(c + 3b) \quad \longleftarrow$$

Problem 9.5-22 The frame *ABCD* shown in the figure is squeezed by two collinear forces P acting at points A and D. What is the decrease δ in the distance between points A and D when the loads P are applied? (The flexural rigidity EI is constant throughout the frame.)

Note: Disregard the effects of axial deformations and consider only the effects of bending due to the loads P.



Solution 9.5-22 Frame *ABCD*

MEMBER *BC*:

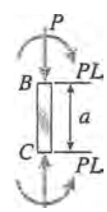


Table G-2, Case 10: $\theta_B = \dfrac{(PL)a}{2EI} = \dfrac{PLa}{2EI}$

MEMBER *BA*:

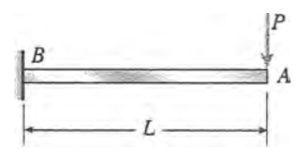


Table G-1, Case 4: $\delta_A = \dfrac{PL^3}{3EI} + \theta_B L$

$$= \frac{PL^3}{3EI} + \frac{PLa}{2EI}(L)$$

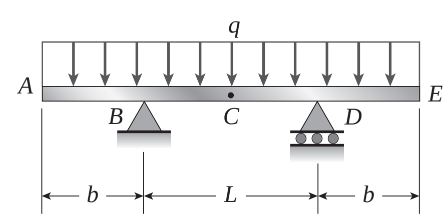
$$= \frac{PL^2}{6EI}(2L + 3a)$$

DECREASE IN DISTANCE BETWEEN POINTS A AND D

$$\delta = 2\delta_A = \frac{PL^2}{3EI}(2L + 3a) \quad \leftarrow$$

Problem 9.5-23 A beam *ABCDE* has simple supports at B and D and symmetrical overhangs at each end (see figure). The center span has length L and each overhang has length b. A uniform load of intensity q acts on the beam.

(a) Determine the ratio b/L so that the deflection δ_C at the midpoint of the beam is equal to the deflections δ_A and δ_E at the ends.
(b) For this value of b/L, what is the deflection δ_C at the midpoint?



Solution 9.5-23 Beam with overhangs

BEAM *BCD*:

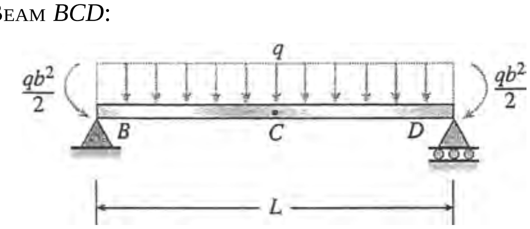


Table G-2, Case 1 and Case 10:

$$\theta_B = \frac{qL^3}{24EI} - \frac{qb^2}{2}\left(\frac{L}{2EI}\right)$$

$$= \frac{qL}{24EI}(L^2 - 6b^2) \quad \text{(clockwise is positive)}$$

$$\delta_C = \frac{5qL^4}{384EI} - \frac{qb^2}{2}\left(\frac{L^2}{8EI}\right) = \frac{qL^2}{384EI}(5L^2 - 24b^2)$$

(downward is positive) (1)

BEAM AB:

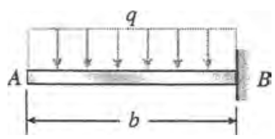


Table G-1, Case 1:

$$\delta_A = \frac{qb^4}{8EI} - \theta_B b = \frac{qb^4}{8EI} - \frac{qL}{24EI}(L^2 - 6b^2)b$$

$$= \frac{qb}{24EI}(3b^3 + 6b^2L - L^3)$$

(downward is positive)

DEFLECTION δ_C EQUALS DEFLECTION δ_A

$$\frac{qL^2}{384EI}(5L^2 - 24b^2) = \frac{qb}{24EI}(3b^3 + 6b^2L - L^3)$$

Rearrange and simplify the equation:

$$48b^4 + 96b^3L + 24b^2L^2 - 16bL^3 - 5L^4 = 0$$

or

$$48\left(\frac{b}{L}\right)^4 + 96\left(\frac{b}{L}\right)^3 + 24\left(\frac{b}{L}\right)^2 - 16\left(\frac{b}{L}\right) - 5 = 0$$

(a) RATIO $\dfrac{b}{L}$

Solve the preceding equation numerically:

$$\frac{b}{L} = 0.40301 \quad \text{Say,} \quad \frac{b}{L} = 0.4030 \quad \leftarrow$$

(b) DEFLECTION δ_C (EQ. 1)

$$\delta_C = \frac{qL^2}{384EI}(5L^2 - 24b^2)$$

$$= \frac{qL^2}{384EI}[5L^2 - 24\,(0.40301\,L)^2]$$

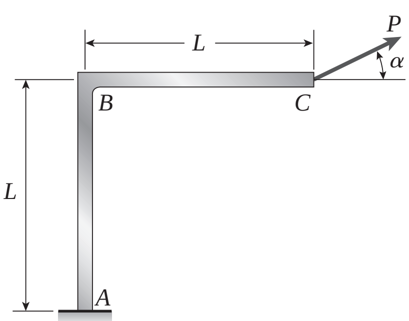
$$= 0.002870\frac{qL^4}{EI}$$

(downward deflection) $\leftarrow$

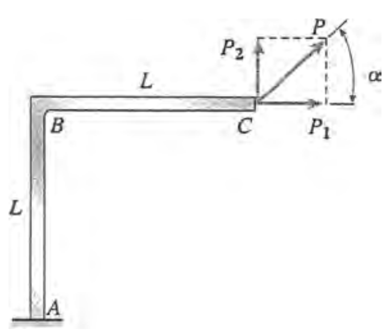
Problem 9.5-24 A frame ABC is loaded at point C by a force P acting at an angle α to the horizontal (see figure). Both members of the frame have the same length and the same flexural rigidity.

Determine the angle α so that the deflection of point C is in the same direction as the load. (Disregard the effects of axial deformations and consider only the effects of bending due to the load P.)

Note: A direction of loading such that the resulting deflection is in the same direction as the load is called a *principal direction*. For a given load on a planar structure, there are two principal directions, perpendicular to each other.



Solution 9.5-24 Principal directions for a frame



P_1 and P_2 are the components of the load P

$P_1 = P\cos\alpha$

$P_2 = P\sin\alpha$

IF P_1 ACTS ALONE $\quad \delta'_H = \dfrac{P_1L^3}{3EI}$ (to the right)

$$\delta'_v = \theta_B L = \left(\frac{P_1L^2}{2EI}\right)L = \frac{P_1L^3}{2EI}$$

(downward)

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If P_2 ACTS ALONE $\quad \delta''_H = \dfrac{P_2L^3}{2EI}$ (to the left)

$$\delta''_v = \frac{P_2L^3}{3EI} + \theta_B L = \frac{P_2L^3}{3EI} + \left(\frac{P_2L^2}{EI}\right)L = \frac{4P_2L^3}{3EI}$$

(upward)

DEFLECTIONS DUE TO THE LOAD P

$$\delta_H = \frac{P_1L^3}{3EI} - \frac{P_2L^3}{2EI} = \frac{L^3}{6EI}(2P_1 - 3P_2)$$

(to the right)

$$\delta_v = -\frac{P_1L^3}{2EI} + \frac{4P_2L^3}{3EI} = \frac{L^3}{6EI}(-3P_1 + 8P_2)$$

(upward)

$$\frac{\delta_v}{\delta_H} = \frac{-3P_1 + 8P_2}{2P_1 - 3P_2}$$

$$= \frac{-3P\cos\alpha + 8P\sin\alpha}{2P\cos\alpha - 3P\sin\alpha} = \frac{-3 + 8\tan\alpha}{2 - 3\tan\alpha}$$

PRINCIPAL DIRECTIONS

The deflection of point C is in the same direction as the load P.

$$\therefore \tan\alpha = \frac{P_2}{P_1} = \frac{\delta_v}{\delta_H} \quad \text{or} \quad \tan\alpha = \frac{-3 + 8\tan\alpha}{2 - 3\tan\alpha}$$

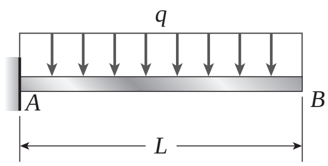
Rearrange and simplity: $\tan^2\alpha + 2\tan\alpha - 1 = 0$ (quadratic equation)

Solving, $\tan\alpha = -1 \pm \sqrt{2}$

$\alpha = 22.5^\circ, \quad 112.5^\circ, \quad -67.5^\circ, \quad -157.5^\circ \quad \leftarrow$

Moment-Area Method

The problems for Section 9.6 are to be solved by the moment-area method. All beams have constant flexural rigidity EI.

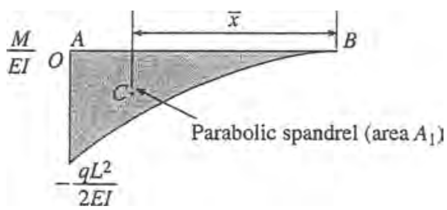


Problem 9.6-1 A cantilever beam AB is subjected to a uniform load of intensity q acting throughout its length (see figure).

Determine the angle of rotation θ_B and the deflection δ_B at the free end.

Solution 9.6-1 Cantilever beam (uniform load)

$\dfrac{M}{EI}$ DIAGRAM:



ANGLE OF ROTATION

Use absolute values of areas.

Appendix D, Case 18: $A_1 = \dfrac{1}{3}(L)\left(\dfrac{qL^2}{2EI}\right) = \dfrac{qL^3}{6EI}$

$$\bar{x} = \frac{3L}{4}$$

$$\theta_{B/A} = \theta_B - \theta_A = A_1 = \frac{qL^3}{6EI}$$

$\theta_A = 0 \qquad \theta_B = \dfrac{qL^3}{6EI}$ (clockwise) $\quad \leftarrow$

DEFLECTION

Q_1 = First moment of area A_1 with respect to B

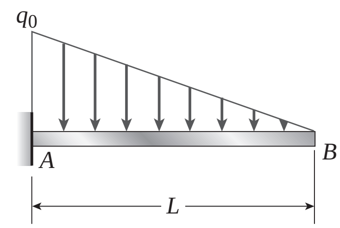
$$Q_1 = A_1\bar{x} = \left(\frac{qL^3}{6EI}\right)\left(\frac{3L}{4}\right) = \frac{qL^4}{8EI}$$

$\delta_B = Q_1 = \dfrac{qL^4}{8EI}$ (Downward) $\quad \leftarrow$

(These results agree with Case 1, Table G-1.)

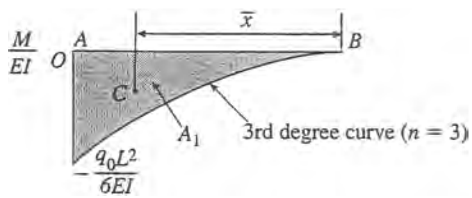
Problem 9.6-2 The load on a cantilever beam *AB* has a triangular distribution with maximum intensity q_0 (see figure).

Determine the angle of rotation θ_B and the deflection δ_B at the free end.



Solution 9.6-2 Cantilever beam (triangular load)

$\dfrac{M}{EI}$ DIAGRAM



ANGLE OF ROTATION

Use absolute values of areas.

Appendix D, Case 20:

$$A_1 = \frac{bh}{n+1} = \frac{1}{4}(L)\left(\frac{q_0L^2}{6EI}\right) = \frac{q_0L^3}{24EI}$$

$$\bar{x} = \frac{b(n+1)}{n+2} = \frac{4L}{5}$$

$$\theta_{B/A} = \theta_B - \theta_A = A_1 = \frac{q_0L^3}{24EI}$$

$$\theta_A = 0 \qquad \theta_B = \frac{q_0L^3}{24EI} \quad \text{(clockwise)} \quad \leftarrow$$

DEFLECTION

Q_1 = First moment of area A_1 with respect to B

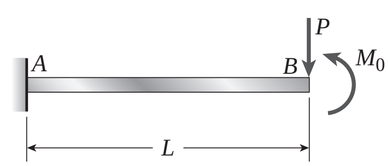
$$Q_1 = A_1\bar{x} = \left(\frac{q_0L^3}{24EI}\right)\left(\frac{4L}{5}\right) = \frac{q_0L^4}{30EI}$$

$$\delta_B = Q_1 = \frac{q_0L^4}{30EI} \quad \text{(Downward)} \quad \leftarrow$$

(These results agree with Case 8, Table G-1.)

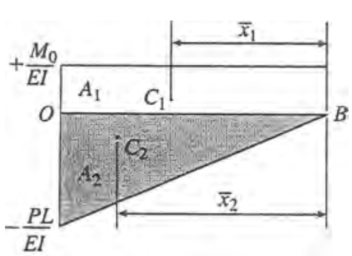
Problem 9.6-3 A cantilever beam *AB* is subjected to a concentrated load *P* and a couple M_0 acting at the free end (see figure).

Obtain formulas for the angle of rotation θ_B and the deflection δ_B at end *B*.



Solution 9.6-3 Cantilever beam (force *P* and couple M_0)

$\dfrac{M}{EI}$ DIAGRAM



NOTE: A_1 is the *M/EI* diagram for M_0 (rectangle). A_2 is the *M/EI* diagram for *P* (triangle).

ANGLE OF ROTATION

Use the sign conventions for the moment-area theorems (page 713 of textbook).

$$A_1 = \frac{M_0L}{EI} \qquad \bar{x}_1 = \frac{L}{2} \qquad A_2 = -\frac{PL^2}{2EI} \qquad \bar{x}_2 = \frac{2L}{3}$$

$$A_0 = A_1 + A_2 = \frac{M_0 L}{EI} - \frac{PL^2}{2EI}$$

$$\theta_{B/A} = \theta_B - \theta_A = A_0 \qquad \theta_A = 0$$

$$\theta_B = A_0 = \frac{M_0 L}{EI} - \frac{PL^2}{2EI}$$

(θ_B is positive when counterclockwise)

DEFLECTION

Q = first moment of areas A_1 and A_2 with respect to point B

$$Q = A_1\bar{x}_1 + A_2\bar{x}_2 = \frac{M_0 L^2}{2EI} - \frac{PL^3}{3EI}$$

$$t_{B/A} = Q = \delta_B \qquad \delta_B = \frac{M_0 L^2}{2EI} - \frac{PL^3}{3EI}$$

(δ_B is positive when upward)

FINAL RESULTS

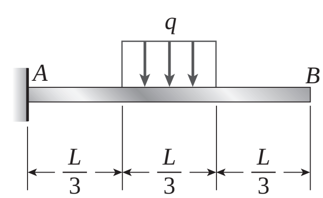
To match the sign conventions for θ_B and δ_B used in Appendix G, change the signs as follows.

$$\theta_B = \frac{PL^2}{2EI} - \frac{M_0 L}{EI} \quad \text{(positive clockwise)} \quad \longleftarrow$$

$$\delta_B = \frac{PL^3}{3EI} - \frac{M_0 L^2}{2EI} \quad \text{(positive downward)} \quad \longleftarrow$$

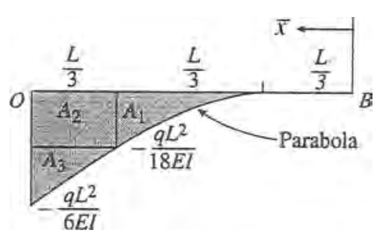
(These results agree with Cases 4 and 6, Table G-1.)

Problem 9.6-4 Determine the angle of rotation θ_B and the deflection δ_B at the free end of a cantilever beam AB with a uniform load of intensity q acting over the middle third of the length (see figure).



Solution 9.6-4 Cantilever beam with partial uniform load

$\frac{M}{EI}$ DIAGRAM



ANGLE OF ROTATION

Use absolute values of areas. Appendix D, Cases 1, 6, and 18:

$$A_1 = \frac{1}{3}\left(\frac{L}{3}\right)\left(\frac{qL^2}{18EI}\right) = \frac{qL^3}{162EI}$$

$$\bar{x}_1 = \frac{L}{3} + \frac{3}{4}\left(\frac{L}{3}\right) = \frac{7L}{12}$$

$$A_2 = \left(\frac{L}{3}\right)\left(\frac{qL^2}{18EI}\right) = \frac{qL^3}{54EI} \qquad \bar{x}_2 = \frac{2L}{3} + \frac{L}{6} = \frac{5L}{6}$$

$$A_3 = \frac{1}{2}\left(\frac{L}{3}\right)\left(\frac{qL^2}{9EI}\right) = \frac{qL^3}{54EI}$$

$$\bar{x}_3 = \frac{2L}{3} + \frac{2}{3}\left(\frac{L}{3}\right) = \frac{8L}{9}$$

$$A_0 = A_1 + A_2 + A_3 = \frac{7qL^3}{162EI}$$

$$\theta_{B/A} = \theta_B - \theta_A = A_0$$

$$\theta_A = 0 \qquad \theta_B = \frac{7qL^3}{162EI} \quad \text{(clockwise)} \quad \longleftarrow$$

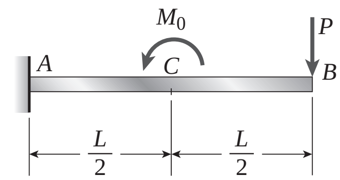
DEFLECTION

Q = first moment of area A_0 with respect to point B

$$Q = A_1\bar{x}_1 + A_2\bar{x}_2 + A_3\bar{x}_3 = \frac{23qL^4}{648EI}$$

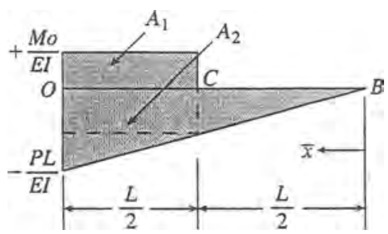
$$\delta_B = Q = \frac{23qL^4}{648EI} \quad \text{(Downward)} \quad \longleftarrow$$

Problem 9.6-5 Calculate the deflections δ_B and δ_C at points B and C, respectively, of the cantilever beam ACB shown in the figure. Assume $M_0 = 36$ k-in., $P = 3.8$ k, $L = 8$ ft, and $EI = 2.25 \times 10^9$ lb-in.^{2}



Solution 9.6-5 Cantilever beam (force P and couple M_0)

$\frac{M}{EI}$ DIAGRAM



NOTE: A_1 is the M/EI diagram for M_0 (rectangle). A_2 is the M/EI diagram for P (triangle).

Use the sign conventions for the moment-area theorems (page 713 of textbook).

DEFLECTION δ_B

Q_B = first moment of areas A_1 and A_2 with respect to point B

$$= A_1\bar{x}_1 + A_2\bar{x}_2 = \left(\frac{M_0}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{3L}{4}\right) - \frac{1}{2}\left(\frac{PL}{EI}\right)(L)\left(\frac{2L}{3}\right)$$

$$= \frac{L^2}{24EI}(9M_0 - 8PL)$$

$$t_{B/A} = Q_B = \delta_B \qquad \delta_B = \frac{L^2}{24EI}(9M_0 - 8PL)$$

(δ_B is positive when upward)

DEFLECTION δ_C

Q_C = first moment of areas A_1 and left-hand part of A_2 with respect to point C

$$= \left(\frac{M_0}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{4}\right) - \left(\frac{PL}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{4}\right) - \frac{1}{2}\left(\frac{PL}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{3}\right)$$

$$= \frac{L^2}{48EI}(6M_0 - 5PL)$$

$$t_{C/A} = Q_C = \delta_C \qquad \delta_C = \frac{L^2}{48EI}(6M_0 - 5PL)$$

(δ_C is positive when upward)

ASSUME DOWNWARD DEFLECTIONS ARE POSITIVE (change the signs of δ_B and δ_C)

$$\delta_B = \frac{L^2}{24EI}(8PL - 9M_0) \quad \leftarrow$$

$$\delta_C = \frac{L^2}{48EI}(5PL - 6M_0) \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$M_0 = 36$ k-in. $\quad P = 3.8$ k

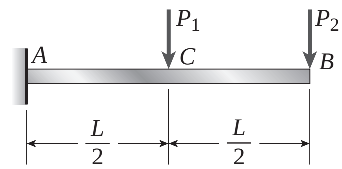
$L = 8$ ft $= 96$ in. $\quad EI = 2.25 \times 10^6$ k-in.^{2}

$\delta_B = 0.4981$ in. $-$ 0.0553 in. $= 0.443$ in. $\leftarrow$

$\delta_C = 0.1556$ in. $-$ 0.0184 in. $= 0.137$ in. $\leftarrow$

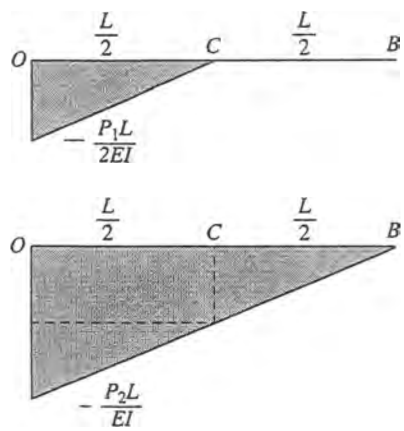
Problem 9.6-6 A cantilever beam *ACB* supports two concentrated loads P_1 and P_2 as shown in the figure.

Calculate the deflections δ_B and δ_C at points *B* and *C*, respectively. Assume $P_1 = 10$ kN, $P_2 = 5$ kN, $L = 2.6$ m, $E = 200$ GPa, and $I = 20.1 \times 10^6$ mm^4.



Solution 9.6-6 Cantilever beam (forces P_1 and P_2)

$\frac{M}{EI}$ DIAGRAMS



$P_1 = 10$ kN $\quad P_2 = 5$ kN $\quad L = 2.6$ m

$E = 200$ GPa $\quad I = 20.1 \times 10^6$ mm^4

Use absolute values of areas.

DEFLECTION δ_B

$\delta_B = t_{B/A} = Q_B =$ first moment of areas with respect to point *B*

$$\delta_B = \frac{1}{2}\left(\frac{P_1L}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{2} + \frac{L}{3}\right) + \frac{1}{2}\left(\frac{P_2L}{EI}\right)(L)\left(\frac{2L}{3}\right)$$

$$= \frac{5P_1L^3}{48EI} + \frac{P_2L^3}{3EI} \quad \text{(downward)} \quad \leftarrow$$

DEFLECTION δ_C

$\delta_C = t_{C/A} = Q_C =$ first moment of areas to the left of point *C* with respect to point *C*

$$\delta_c = \frac{1}{2}\left(\frac{P_1L}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{3}\right) + \left(\frac{P_2L}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{4}\right)$$

$$+ \frac{1}{2}\left(\frac{P_2L}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{3}\right)$$

$$= \frac{P_1L^3}{24EI} + \frac{5P_2L^3}{48EI} \quad \text{(downward)} \quad \leftarrow$$

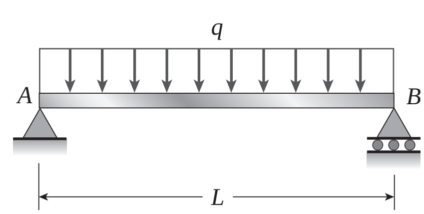
SUBSTITUTE NUMERICAL VALUES:

$\delta_B = 4.554$ mm $+ 7.287$ mm $= 11.84$ mm $\leftarrow$

$\delta_C = 1.822$ mm $+ 2.277$ mm $= 4.10$ mm $\leftarrow$

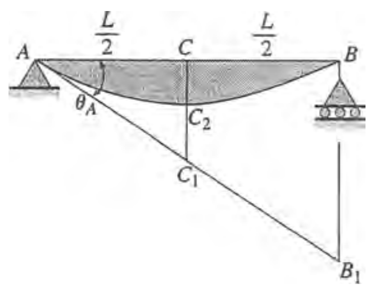
(deflections are downward)

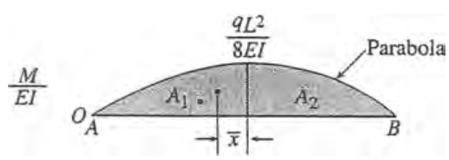
Problem 9.6-7 Obtain formulas for the angle of rotation θ_A at support *A* and the deflection δ_{max} at the midpoint for a simple beam *AB* with a uniform load of intensity *q* (see figure).



Solution 9.6-7 Simple beam with a uniform load

DEFLECTION CURVE AND $\dfrac{M}{EI}$ DIAGRAM





δ_{max} = maximum deflection (distance CC_2)

Use absolute values of areas.

ANGLE OF ROTATION AT END A

Appendix D, Case 17:

$$A_1 = A_2 = \frac{2}{3}\left(\frac{L}{2}\right)\left(\frac{qL^2}{8EI}\right) = \frac{qL^3}{24EI}$$

$$\bar{x}_1 = \frac{3}{8}\left(\frac{L}{2}\right) = \frac{3L}{16}$$

$t_{B/A} = BB_1$ = first moment of areas A_1 and A_2 with respect to point B

$$= (A_1 + A_2)\left(\frac{L}{2}\right) = \frac{qL^4}{24EI}$$

$$\theta_A = \frac{BB_1}{L} = \frac{qL^3}{24EI} \quad \text{(clockwise)} \quad \longleftarrow$$

DEFLECTION δ_{max} AT THE MIDPOINT C

$$\text{Distance } CC_1 = \frac{1}{2}(BB_1) = \frac{qL^4}{48EI}$$

$t_{C_2/A} = C_2C_1$ = first moment of area A_1 with respect to point C

$$= A_1\bar{x}_1 = \left(\frac{qL^3}{24EI}\right)\left(\frac{3L}{16}\right) = \frac{qL^4}{128EI}$$

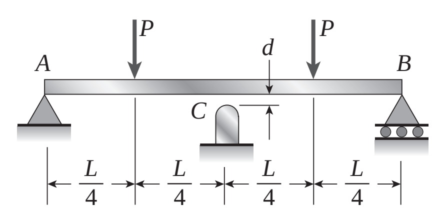
$$\delta_{max} = CC_2 = CC_1 - C_2C_1 = \frac{qL^4}{48EI} - \frac{qL^4}{128EI}$$

$$= \frac{5qL^4}{384EI} \quad \text{(downward)} \quad \longleftarrow$$

(These results agree with Case 1 of Table G-2.)

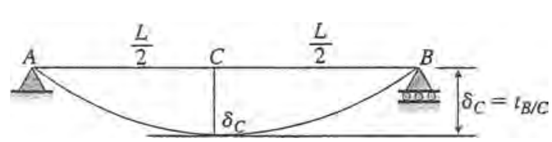
Problem 9.6-8 A simple beam AB supports two concentrated loads P at the positions shown in the figure. A support C at the midpoint of the beam is positioned at distance d below the beam before the loads are applied.

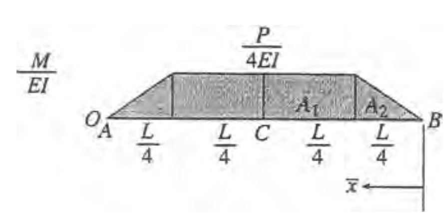
Assuming that $d = 10$ mm, $L = 6$ m, $E = 200$ GPa, and $I = 198 \times 10^6$ mm^4, calculate the magnitude of the loads P so that the beam just touches the support at C.



Solution 9.6-8 Simple beam with two equal loads

DEFLECTION CURVE AND $\dfrac{M}{EI}$ DIAGRAM





δ_C = deflection at the midpoint C

$$A_1 = \frac{PL^2}{16EI} \qquad \bar{x}_1 = \frac{3L}{8}$$

$$A_2 = \frac{PL^2}{32EI} \qquad \bar{x}_2 = \frac{L}{6}$$

Use absolute values of areas.

DEFLECTION δ_C AT MIDPOINT OF BEAM

At point C, the deflection curve is horizontal.

$\delta_C = t_{B/C} =$ first moment of area between B and C with respect to B

$$= A_1\bar{x}_1 + A_2\bar{x}_2 = \frac{PL^2}{16EI}\left(\frac{3L}{8}\right) + \frac{PL^2}{32EI}\left(\frac{L}{6}\right)$$

$$= \frac{11PL^3}{384EI}$$

$d =$ gap between the beam and the support at C

MAGNITUDE OF LOAD TO CLOSE THE GAP

$$\delta = d = \frac{11PL^3}{384EI} \qquad P = \frac{384EId}{11L^3} \qquad \leftarrow$$

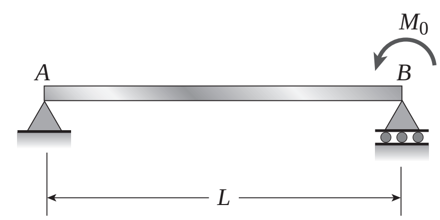
SUBSTITUTE NUMERICAL VALUES:

$d = 10$ mm $\qquad L = 6$ m $\qquad E = 200$ GPa

$I = 198 \times 10^6\ \text{mm}^4 \qquad P = 64$ kN $\qquad \leftarrow$

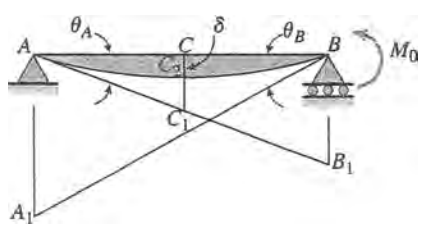
Problem 9.6-9 A simple beam AB is subjected to a load in the form of a couple M_0 acting at end B (see figure).

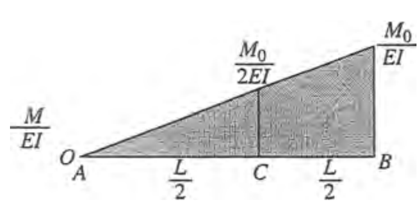
Determine the angles of rotation θ_A and θ_B at the supports and the deflection δ at the midpoint.



Solution 9.6-9 Simple beam with a couple M_0

DEFLECTION CURVE AND $\dfrac{M}{EI}$ DIAGRAM





$\delta =$ deflection at the midpoint C

$\delta =$ distance CC_2

Use absolute values of areas.

ANGLE OF ROTATION θ_A

$t_{B/A} = BB_1 =$ first moment of area between A and B with respect to B

$$= \frac{1}{2}\left(\frac{M_0}{EI}\right)(L)\left(\frac{L}{3}\right) = \frac{M_0L^2}{6EI}$$

$$\theta_A = \frac{BB_1}{L} = \frac{M_0L}{6EI} \quad \text{(clockwise)} \qquad \leftarrow$$

ANGLE OF ROTATION θ_B

$t_{A/B} = AA_1 =$ first moment of area between A and B with respect to A

$$= \frac{1}{2}\left(\frac{M_0}{EI}\right)(L)\left(\frac{2L}{3}\right) = \frac{M_0L^2}{3EI}$$

$$\theta_B = \frac{AA_1}{L} = \frac{M_0L}{3EI} \quad \text{(Counterclockwise)} \qquad \leftarrow$$

DEFLECTION δ AT THE MIDPOINT C

Distance $CC_1 = \dfrac{1}{2}(BB_1) = \dfrac{M_0L^2}{12EI}$

$t_{C_2/A} = C_2C_1 =$ first moment of area between A and C with respect to C

$$= \frac{1}{2}\left(\frac{M_0}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{6}\right) = \frac{M_0L^2}{48EI}$$

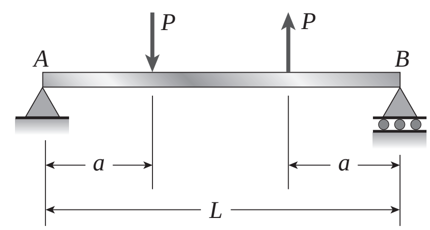
$$\delta = CC_1 - C_2C_1 = \frac{M_0L^2}{12EI} - \frac{M_0L^2}{48EI}$$

$$= \frac{M_0L^2}{16EI} \quad \text{(Downward)} \quad \leftarrow$$

(These results agree with Case 7 of Table G-2.)

Problem 9.6-10 The simple beam AB shown in the figure supports two equal concentrated loads P, one acting downward and the other upward.

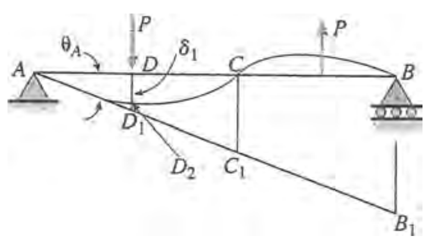
Determine the angle of rotation θ_A at the left-hand end, the deflection δ_1 under the downward load, and the deflection δ_2 at the midpoint of the beam.

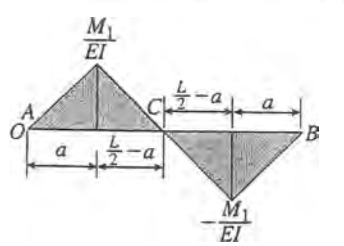


Solution 9.6-10 Simple beam with two loads

Because the beam is symmetric and the load is antisymmetric, the deflection at the midpoint is zero.

$\therefore \delta_2 = 0 \quad \leftarrow$





$$\frac{M_1}{EI} = \frac{Pa(L - 2a)}{LEI}$$

$$A_1 = \frac{1}{2}\left(\frac{M_1}{EI}\right)(a) = \frac{Pa^2(L - 2a)}{2LEI}$$

$$A_2 = \frac{1}{2}\left(\frac{M_1}{EI}\right)\left(\frac{L}{2} - a\right) = \frac{Pa(L - 2a)^2}{4LEI}$$

ANGLE OF ROTATION θ_A AT END A

$t_{C/A} = CC_1 =$ first moment of area between A and C with respect to C

$$= A_1\left(\frac{L}{2} - a + \frac{a}{3}\right) + A_2\left(\frac{2}{3}\right)\left(\frac{L}{2} - a\right)$$

$$= \frac{Pa(L - a)(L - 2a)}{12EI}$$

$$\theta_A = \frac{CC_1}{L/2} = \frac{Pa(L - a)(L - 2a)}{6LEI} \quad \text{(clockwise)} \quad \leftarrow$$

DEFLECTION δ_1 UNDER THE DOWNWARD LOAD

Distance $\quad DD_1 = \left(\dfrac{a}{L/2}\right)(CC_1)$

$$= \frac{Pa^2(L - a)(L - 2a)}{6LEI}$$

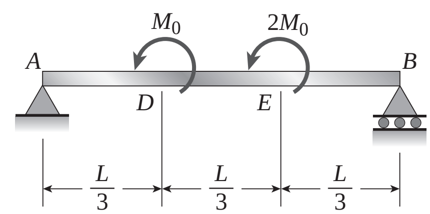
$t_{D_2/A} = D_2D_1 =$ first moment of area between A and D with respect to D

$$= A_1\left(\frac{a}{3}\right) = \frac{Pa^3(L - 2a)}{6LEI}$$

$$\delta_1 = DD_1 - D_2D_1$$

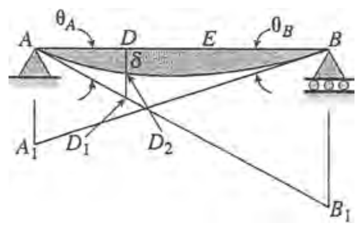
$$= \frac{Pa^2(L - 2a)^2}{6LEI} \quad \text{(Downward)} \quad \leftarrow$$

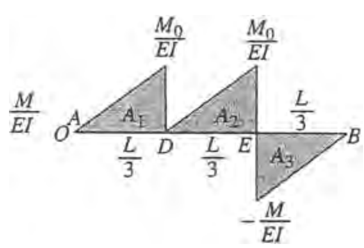
Problem 9.6-11 A simple beam AB is subjected to couples M_0 and $2M_0$ as shown in the figure. Determine the angles of rotation θ_A and θ_B at the beam and the deflection δ at point D where the load M_0 is applied.



Solution 9.6-11 Simple beam with two couples

DEFLECTION CURVE AND $\dfrac{M}{EI}$ DIAGRAM





$$A_1 = A_2 = \frac{1}{2}\left(\frac{M_0}{EI}\right)\left(\frac{L}{3}\right) = \frac{M_0 L}{6EI} \qquad A_3 = -\frac{M_0 L}{6EI}$$

ANGLE OF ROTATION θ_A AT END A

$t_{B/A} = BB_1 =$ first moment of area between A and B with respect to B

$$= A_1\left(\frac{2L}{3} + \frac{L}{9}\right) + A_2\left(\frac{L}{3} + \frac{L}{9}\right) + A_3\left(\frac{2L}{9}\right)$$

$$= \frac{M_0 L^2}{6EI}$$

$$\theta_A = \frac{BB_1}{L} = \frac{M_0 L}{6EI} \quad \text{(clockwise)} \qquad \leftarrow$$

ANGLE OF ROTATION θ_B AT END B

$t_{A/B} = AA_1 =$ first moment of area between A and B with respect to A

$$= A_1\left(\frac{2L}{9}\right) + A_2\left(\frac{L}{3} + \frac{2L}{9}\right) + A_3\left(\frac{2L}{3} + \frac{L}{9}\right) = 0$$

$$\theta_B = \frac{AA_1}{L} = 0 \qquad \leftarrow$$

DEFLECTION δ AT POINT D

Distance $DD_1 = \dfrac{1}{3}(BB_1) = \dfrac{M_0 L^2}{18EI}$

$t_{D_2/A} = D_2D_1 =$ first moment of area between A and D with respect to D

$$= A_1\left(\frac{L}{9}\right) = \frac{M_0 L^2}{54EI}$$

$$\delta = DD_1 - D_2D_1 = \frac{M_0 L^2}{27EI}$$

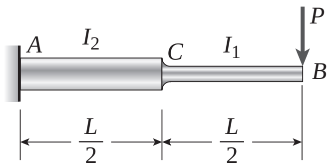
$$\text{(downward)} \qquad \leftarrow$$

NOTE: This deflection is also the maximum deflection.

Nonprismatic Beams

Problem 9.7-1 The cantilever beam *ACB* shown in the figure has moments of inertia I_2 and I_1 in parts *AC* and *CB*, respectively.

(a) Using the method of superposition, determine the deflection δ_B at the free end due to the load *P*.
(b) Determine the ratio *r* of the deflection δ_B to the deflection δ_1 at the free end of a prismatic cantilever with moment of inertia I_1 carrying the same load.
(c) Plot a graph of the deflection ratio *r* versus the ratio I_2/I_1 of the moments of inertia. (Let I_2/I_1 vary from 1 to 5.)

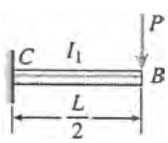


Solution 9.7-1 Cantilever beam (nonprismatic)

Use the method of superposition.

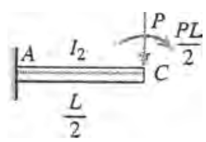
(a) Deflection δ_B at the free end

(1) Part *CB* of the beam:



$$(\delta_B)_1 = \frac{P}{3EI_1}\left(\frac{L}{2}\right)^3 = \frac{PL^3}{24EI_1}$$

(2) Part *AC* of the beam:



$$\delta_C = \frac{P(L/2)^3}{3EI_2} + \frac{(PL/2)(L/2)^2}{2EI_2} = \frac{5PL^3}{48EI_2}$$

$$\theta_C = \frac{P(L/2)^2}{2EI_2} + \frac{(PL/2)(L/2)}{EI_2} = \frac{3PL^2}{8EI_2}$$

$$(\delta_B)_2 = \delta_C + \theta_C\left(\frac{L}{2}\right) = \frac{7PL^3}{24EI_2}$$

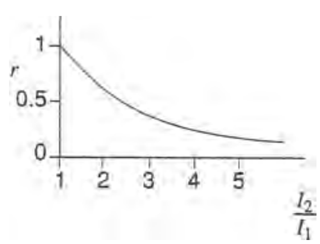
(3) Total deflection at point *B*

$$\delta_B = (\delta_B)_1 + (\delta_B)_2 = \frac{PL^3}{24EI_1}\left(1 + \frac{7I_1}{I_2}\right) \quad \leftarrow$$

(b) Prismatic beam $\delta_1 = \dfrac{PL^3}{3EI_1}$

$$\text{Ratio: } r = \frac{\delta_B}{\delta_1} = \frac{1}{8}\left(1 + \frac{7I_1}{I_2}\right) \quad \leftarrow$$

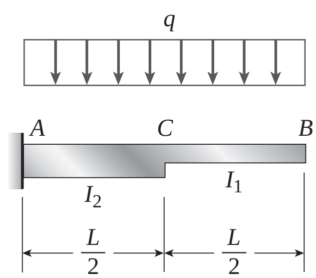
(c) Graph of ratio



$\frac{I_2}{I_1}$	r
1	1.00
2	0.56
3	0.42
4	0.34
5	0.30

Problem 9.7-2 The cantilever beam *ACB* shown in the figure supports a uniform load of intensity q throughout its length. The beam has moments of inertia I_2 and I_1 in parts *AC* and *CB*, respectively.

(a) Using the method of superposition, determine the deflection δ_B at the free end due to the uniform load.
(b) Determine the ratio r of the deflection δ_B to the deflection δ_1 at the free end of a prismatic cantilever with moment of inertia I_1 carrying the same load.
(c) Plot a graph of the deflection ratio r versus the ratio I_2/I_1 of the moments of inertia. (Let I_2/I_1 vary from 1 to 5.)

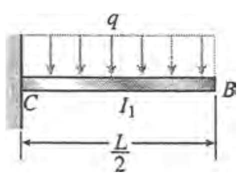


Solution 9.7-2 Cantilever beam (nonprismatic)

Use the method of superposition

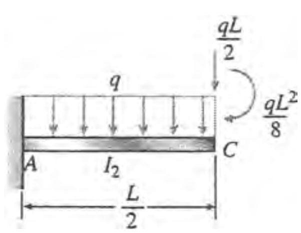
(a) DEFLECTION δ_B AT THE FREE END

(1) Part *CB* of the beam:



$$(\delta_B)_1 = \frac{q}{8EI_1}\left(\frac{L}{2}\right)^4 = \frac{qL^4}{128EI_1}$$

(2) Part *AC* of the beam:



$$\delta_C = \frac{q(L/2)^4}{8EI_2} + \frac{\left(\frac{qL}{2}\right)(L/2)^3}{3EI_2} + \frac{\left(\frac{qL^2}{8}\right)\left(\frac{L}{2}\right)^2}{2EI_2} = \frac{17qL^4}{384EI_2}$$

$$\theta_C = \frac{q(L/2)^3}{6EI_2} + \frac{(qL/2)(L/2)^2}{2EI_2} + \frac{(qL^2/8)(L/2)}{EI_2} = \frac{7qL^3}{48EI_2}$$

$$(\delta_B)_2 = \delta_C + \theta_C\left(\frac{L}{2}\right) = \frac{15qL^4}{128EI_2}$$

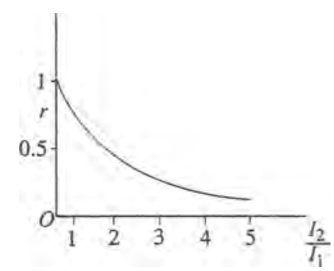
(3) Total deflection at point *B*

$$\delta_B = (\delta_B)_1 + (\delta_B)_2 = \frac{qL^4}{128EI_1}\left(1 + \frac{15I_1}{I_2}\right) \quad \leftarrow$$

(b) PRISMATIC BEAM $\delta_1 = \dfrac{qL^4}{8EI_1}$

$$\text{Ratio: } r = \frac{\delta_B}{\delta_1} = \frac{1}{16}\left(1 + \frac{15I_1}{I_2}\right) \quad \leftarrow$$

(c) GRAPH OF RATIO

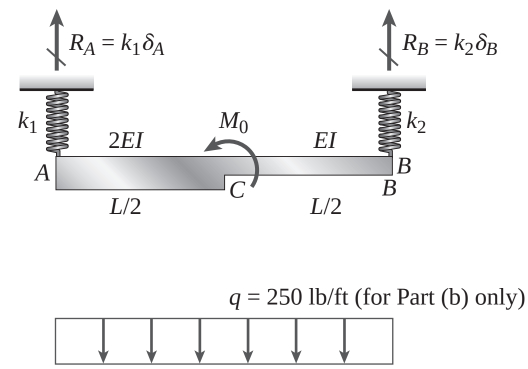


$\frac{I_2}{I_1}$	r
1	1.00
2	0.53
3	0.38
4	0.30
5	0.25

***Problem 9.7-3** Beam *ACB* hangs from two springs, as shown in the figure. The springs have stiffnesses k_1 and k_2 and the beam has flexural rigidity *EI*.

(a) What is the downward displacement of point *C*, which is at the midpoint of the beam, when the moment M_0 is applied? Data for the structure are as follows: $M_0 = 7.5$ k-ft, $L = 6$ ft, $EI = 520$ k-ft^2, $k_1 = 17$ k/ft, and $k_2 = 11$ k/ft.

(b) Repeat (a) but remove M_0 and, instead, apply uniform load q over the entire beam.



*Solution 9.7-3

$M_0 = 7.5$ kip/ft $L = 6$ ft $EI = 520$ kip/ft^2 $k_1 = 17$ kip/ft $k_2 = 11$ kip/ft $q = 250$ lb/ft

(a) BENDING-MOMENT EQUATIONS-MOMENT M_0 AT C

$$2EIv'' = M = \frac{M_0 x}{L} \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$2EIv' = \frac{M_0 x^2}{2L} + C_1 \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$2EIv = \frac{M_0 x^3}{6L} + C_1 x + C_2 \quad \left(0 \le x \le \frac{L}{2}\right)$$

B.C. $v(0) = 0$ $C_2 = 0$ $2EIv = \dfrac{M_0 x^3}{6L} + C_1 x \quad \left(0 \le x \le \dfrac{L}{2}\right)$

$$EIv'' = -\frac{M_0}{2} + \frac{M_0\left(x - \frac{L}{2}\right)}{L} = -M_0 + \frac{M_0 x}{L} \quad \left(\frac{L}{2} \le x \le L\right)$$

$$EIv' = -M_0 x + \frac{M_0 x^2}{2L} + C_3 \quad \left(\frac{L}{2} \le x \le L\right)$$

$$EIv = -\frac{M_0 x^2}{2} + \frac{M_0 x^3}{6L} + C_3 x + C_4 \quad \left(\frac{L}{2} \le x \le L\right)$$

B.C. $v(L) = 0$ $$-\frac{M_0 L^2}{2} + \frac{M_0 L^3}{6L} + C_3 L + C_4 = 0 \qquad (1)$$

B.C. $v'_L\left(\frac{L}{2}\right) = v'_R\left(\frac{L}{2}\right)$ $$\frac{1}{2}\left[\frac{M_0\left(\frac{L}{2}\right)^2}{2L} + C_1\right] = -M_0\frac{L}{2} + \frac{M_0\left(\frac{L}{2}\right)^2}{2L} + C_3 \qquad (2)$$

B.C. $v_L\left(\frac{L}{2}\right) = v_R\left(\frac{L}{2}\right)$ $$\frac{1}{2}\left[\frac{M_0\left(\frac{L}{2}\right)^3}{6L} + C_1\frac{L}{2}\right] = -\frac{M_0\left(\frac{L}{2}\right)^2}{2} + \frac{M_0\left(\frac{L}{2}\right)^3}{6L} + C_3\frac{L}{2} + C_4 \qquad (3)$$

From (1), (2), and (3)

$$C_1 = 0 \qquad C_3 = \frac{7}{16}M_0L \qquad C_4 = \frac{-5}{48}M_0L^2 \qquad \leftarrow$$

Therefore

$$\nu(x) = \frac{M_0x^3}{12EIL} \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$\nu(x) = \frac{M_0}{48EIL}(-24x^2L + 8x^3 + 21L^2x - 5L^3) \quad \left(\frac{L}{2} \le x \le L\right)$$

Deflection at A and B

$$R_A = \frac{M_0}{L} \qquad R_B = -\frac{M_0}{L}$$

$$\delta_A = \frac{R_A}{k_1} \qquad \delta_B = \frac{R_B}{k_2}$$

$\delta_A = 0.88$ in. Downward $\qquad \delta_B = -1.36$ in. Upward

Deflection at point C

$$\delta_C = -\nu\left(\frac{L}{2}\right) + \frac{1}{2}(\delta_A + \delta_B)$$

$$\delta_C = -\frac{M_0\left(\frac{L}{2}\right)^3}{12EIL} + \frac{1}{2}(\delta_A + \delta_B)$$

$\delta_C = -0.31$ in. Upward $\qquad \leftarrow$

(b) Bending-moment equations-uniform load q

$$2EI\nu'' = M = \frac{qLx}{2} - \frac{qx^2}{2} \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$2EI\nu' = \frac{qLx^2}{4} - \frac{qx^3}{6} + C_1 \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$2EI\nu = \frac{qLx^3}{12} - \frac{qx^4}{24} + C_1x + C_2 \quad \left(0 \le x \le \frac{L}{2}\right)$$

B.C. $\nu(0) = 0 \qquad C_2 = 0 \qquad 2EI\nu = \dfrac{qLx^3}{12} - \dfrac{qx^4}{24} + C_1x \quad \left(0 \le x \le \dfrac{L}{2}\right)$

$$EI\nu'' = \frac{qLx}{2} - \frac{qx^2}{2} \quad \left(\frac{L}{2} \le x \le L\right)$$

$$EI\nu' = \frac{qLx^2}{4} - \frac{qx^3}{6} + C_3 \quad \left(\frac{L}{2} \le x \le L\right)$$

$$EI\nu = \frac{qLx^3}{12} - \frac{qx^4}{24} + C_3x + C_4 \quad \left(\frac{L}{2} \le x \le L\right)$$

B.C. $\nu(L) = 0 \qquad \dfrac{qLL^3}{12} - \dfrac{qL^4}{24} + C_3L + C_4 = 0 \qquad (1)$

B.C. $\nu'_L\left(\dfrac{L}{2}\right) = \nu'_R\left(\dfrac{L}{2}\right) \quad \dfrac{1}{2}\left[\dfrac{qL\left(\dfrac{L}{2}\right)^2}{4} - \dfrac{q\left(\dfrac{L}{2}\right)^3}{6} + C_1\right] = \dfrac{qL\left(\dfrac{L}{2}\right)^2}{4} - \dfrac{q\left(\dfrac{L}{2}\right)^3}{6} + C_3 \qquad (2)$

B.C. $\nu_L\left(\dfrac{L}{2}\right) = \nu_R\left(\dfrac{L}{2}\right) \quad \dfrac{1}{2}\left[\dfrac{qL\left(\dfrac{L}{2}\right)^3}{12} - \dfrac{q\left(\dfrac{L}{2}\right)^4}{24} + C_1\dfrac{L}{2}\right]$

$$= \frac{qL\left(\frac{L}{2}\right)^3}{12} - \frac{q\left(\frac{L}{2}\right)^4}{24} + C_3\frac{L}{2} + C_4 \qquad (3)$$

From (1), (2), and (3)

$$C_1 = \frac{-7}{128}qL^3 \qquad C_3 = \frac{-37}{768}qL^3 \qquad C_4 = \frac{5}{768}qL^4$$

Therefore

$$\nu(x) = -\frac{qx}{768EI}(-32Lx^2 + 16x^3 + 21L^3) \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$\nu(x) = \frac{q}{768EI}(64Lx^3 - 32x^4 - 37L^3x + 5L^4) \quad \left(\frac{L}{2} \le x \le L\right)$$

DEFLECTION AT A AND B

$$R_A = \frac{qL}{2} \qquad R_B = \frac{qL}{2}$$

$$\delta_A = \frac{R_A}{k_1} \qquad \delta_B = \frac{R_B}{k_2}$$

$\delta_A = 0.53$ in. Downward $\qquad \delta_B = 0.82$ in. Downward

DEFLECTION AT POINT C

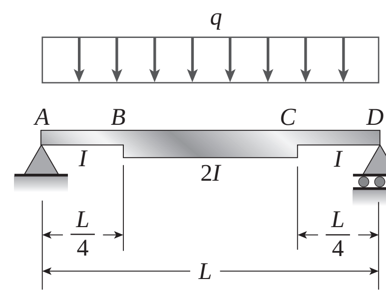
$$\delta_C = -\nu\left(\frac{L}{2}\right) + \frac{1}{2}(\delta_A + \delta_B)$$

$$\delta_C = \frac{q\frac{L}{2}}{768EI}\left[-32L\left(\frac{L}{2}\right)^2 + 16\left(\frac{L}{2}\right)^3 + 21L^3\right] + \frac{1}{2}(\delta_A + \delta_B)$$

$\delta_C = 0.75$ in. Downward $\leftarrow$

Problem 9.7-4 A simple beam *ABCD* has moment of inertia *I* near the supports and moment of inertia $2I$ in the middle region, as shown in the figure. A uniform load of intensity q acts over the entire length of the beam.

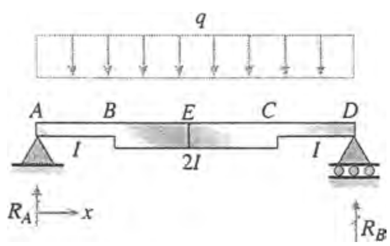
Determine the equations of the deflection curve for the left-hand half of the beam. Also, find the angle of rotation θ_A at the left-hand support and the deflection $\delta_{\max}$ at the midpoint.



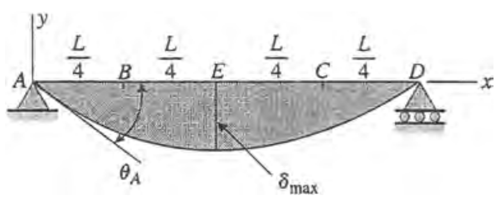
Solution 9.7-4 Simple beam (nonprismatic)

Use the bending-moment equation (Eq. 9-12a).

REACTIONS, BENDING MOMENT, AND DEFLECTION CURVE



$$R_A = R_B = \frac{qL}{2} \qquad M = Rx - \frac{qx^2}{2} = \frac{qLx}{2} - \frac{qx^2}{2}$$



BENDING-MOMENT EQUATIONS FOR THE LEFT-HAND HALF OF THE BEAM

$$EIv'' = M = \frac{qLx}{2} - \frac{qx^2}{2} \quad \left(0 \le x \le \frac{L}{4}\right) \tag{1}$$

$$E(2I)v'' = M = \frac{qLx}{2} - \frac{qx^2}{2} \quad \left(\frac{L}{4} \le x \le \frac{L}{2}\right) \tag{2}$$

INTEGRATE EACH EQUATION

$$EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} + C_1 \quad \left(0 \le x \le \frac{L}{4}\right) \tag{3}$$

$$2EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} + C_2 \quad \left(\frac{L}{4} \le x \le \frac{L}{2}\right) \tag{4}$$

B.C. 1 Symmetry: $v'\left(\frac{L}{2}\right) = 0$

From Eq. (4): $C_2 = -\dfrac{qL^3}{24}$

$$2EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} - \frac{qL^3}{24} \quad \left(\frac{L}{4} \le x \le \frac{L}{2}\right) \tag{5}$$

SLOPE AT POINT *B* (FROM THE RIGHT)

Substitute $x = \dfrac{L}{4}$ into Eq. (5):

$$EIv'_B = -\frac{11qL^3}{768} \tag{6}$$

B.C. 2 CONTINUITY OF SLOPES AT POINT *B*

$(v'_B)_{\text{Left}} = (v'_B)_{\text{Right}}$

From Eqs. (3) and (6):

$$\frac{qL}{4}\left(\frac{L}{4}\right)^2 - \frac{q}{6}\left(\frac{L}{4}\right)^3 + C_1 = -\frac{11qL^3}{768} \qquad \therefore C_1 = -\frac{7qL^3}{256}$$

SLOPE OF THE BEAM (FROM EQS. 3 AND 5)

$$EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} - \frac{7qL^3}{256} \quad \left(0 \le x \le \frac{L}{4}\right) \tag{7}$$

$$EIv' = \frac{qLx^2}{8} - \frac{qx^3}{12} - \frac{qL^3}{48} \quad \left(\frac{L}{4} \le x \le \frac{L}{3}\right) \tag{8}$$

ANGLE OF ROTATION θ_A (FROM EQ. 7)

$$\theta_A = -v'(0) = \frac{7qL^3}{256EI} \text{ (positive clockwise)} \qquad \leftarrow$$

INTEGRATE EQS. (7) AND (8)

$$EIv = \frac{qLx^3}{12} - \frac{qx^4}{24} - \frac{7qL^3x}{256} + C_3 \quad \left(0 \le x \le \frac{L}{4}\right) \qquad (9)$$

$$EIv = \frac{qLx^3}{24} - \frac{qx^4}{48} - \frac{qL^3x}{48} + C_4 \quad \left(\frac{L}{4} \le x \le \frac{L}{2}\right) \qquad (10)$$

B.C. 3 Deflection at support A

$v(0) = 0$ From Eq. (9): $C_3 = 0$

DEFECTION AT POINT B (FROM THE LEFT)

Substitute $x = \frac{L}{4}$ into Eq. (9) with $C_3 = 0$

$$EIv_B = -\frac{35\,qL^4}{6144} \qquad (11)$$

B.C. 4 Continuity of deflections at point B

$(v_B)_{\text{Right}} = (v_B)_{\text{Left}}$

From Eqs. (10) and (11):

$$\frac{qL}{24}\left(\frac{L}{4}\right)^3 - \frac{q}{48}\left(\frac{L}{4}\right)^4 - \frac{qL^3}{48}\left(\frac{L}{4}\right) + C_4 = -\frac{35qL^4}{6144}$$

$$\therefore C_4 = -\frac{13qL^4}{12{,}288}$$

DEFECTION OF THE BEAM (FROM EQS. 9 AND 10)

$$v = -\frac{qx}{768EI}(21L^3 - 64Lx^2 + 32x^3)$$

$$\left(0 \le x \le \frac{L}{4}\right) \quad \leftarrow$$

$$v = -\frac{q}{12{,}288EI}(13L^4 + 256L^3x - 512Lx^3 + 256x^4)$$

$$\left(\frac{L}{4} \le x \le \frac{L}{2}\right) \quad \leftarrow$$

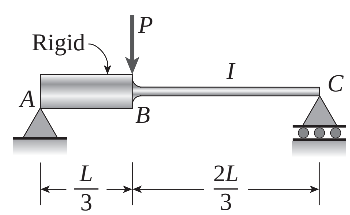
MAXIMUM DEFLECTION (AT THE MIDPOINT E)

(From the preceding equation for v.)

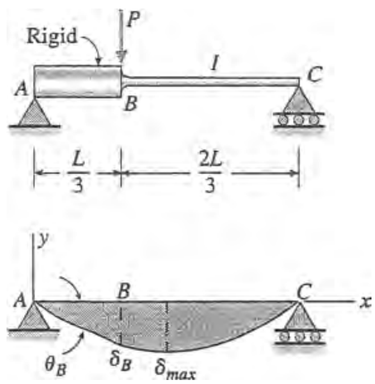
$$\delta_{\max} = -v\left(\frac{L}{2}\right) = \frac{31qL^4}{4096EI} \quad \text{(positive downward)} \quad \leftarrow$$

Problem 9.7-5 A beam ABC has a rigid segment from A to B and a flexible segment with moment of inertia I from B to C (see figure). A concentrated load P acts at point B.

Determine the angle of rotation θ_A of the rigid segment, the deflection δ_B at point B, and the maximum deflection $\delta_{\max}$.



Solution 9.7-5 Simple beam with a rigid segment



FROM A TO B

$$v = -\frac{3\delta_B x}{L} \qquad \left(0 \le x \le \frac{L}{3}\right) \tag{1}$$

$$v' = -\frac{3\delta_B}{L} \qquad \left(0 \le x \le \frac{L}{3}\right) \tag{2}$$

FROM B TO C

$$EIv'' = M = \frac{PL}{3} - \frac{Px}{3} \tag{3}$$

$$EIv' = \frac{PLx}{3} - \frac{Px^2}{6} + C_1$$

B.C. 1 At $x = L/3$, $\quad v' = -\dfrac{3\delta_B}{L}$

$$\therefore C_1 = -\frac{5PL^2}{54} - \frac{3EI\delta_B}{L}$$

$$EIv' = \frac{PLx}{3} - \frac{Px^2}{6} - \frac{5PL^2}{54} - \frac{3EI\delta_B}{L} \qquad \left(\frac{L}{3} \le x \le L\right) \tag{4}$$

$$EIv = \frac{PLx^2}{6} - \frac{Px^3}{18} - \frac{5PL^2x}{54} - \frac{3EI\delta_B x}{L} + C_2 \qquad \left(\frac{L}{3} \le x \le L\right)$$

B.C. 2 $v(L) = 0 \qquad \therefore C_2 = -\dfrac{PL^3}{54} + 3EI\delta_B$

$$EIv = \frac{PLx^2}{6} - \frac{Px^3}{18} - \frac{5PL^2x}{54} - \frac{3EI\delta_B x}{L} - \frac{PL^2}{54} + 3EI\delta_B \qquad \left(\frac{L}{3} \le x \le L\right) \tag{5}$$

B.C. 3 At $x = \dfrac{L}{3}$, $(v_B)_{\text{Left}} = (v_B)_{\text{Right}}$ (Eqs. 1 and 5)

$$\therefore \delta_B = \frac{8PL^3}{729EI} \quad \longleftarrow$$

$$\theta_A = \frac{\delta_B}{L/3} = \frac{8PL^2}{243EI} \quad \longleftarrow$$

Substitute for δ_B in Eq. (5) and simplify:

$$v = \frac{P}{486EI}(7L^3 - 61L^2x + 81Lx^2 - 27x^3) \qquad \left(\frac{L}{3} \le x \le L\right) \tag{6}$$

Also,

$$v' = \frac{P}{486EI}(-61L^2 + 162Lx - 81x^2) \qquad \left(\frac{L}{3} \le x \le L\right) \tag{7}$$

MAXIMUM DEFLECTION

$$v' = 0 \text{ gives } x_1 = \frac{L}{9}(9 - 2\sqrt{5}) = 0.5031L$$

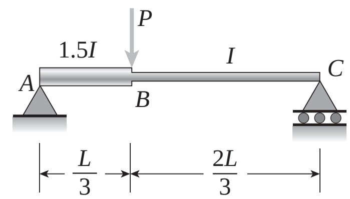
Substitute x_1 in Eq. (6) and simplify:

$$v_{\max} = -\frac{40\sqrt{5}PL^3}{6561EI}$$

$$\delta_{\max} = -v_{\max} = \frac{40\sqrt{5}PL^3}{6561EI} = 0.01363\frac{PL^3}{EI} \quad \longleftarrow$$

Problem 9.7-6 A simple beam ABC has moment of inertia $1.5I$ from A to B and I from B to C (see figure). A concentrated load P acts at point B.

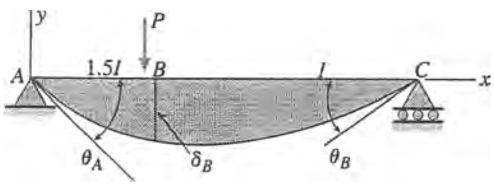
Obtain the equations of the deflection curves for both parts of the beam. From the equations, determine the angles of rotation θ_A and θ_C at the supports and the deflection δ_B at point B.



Solution 9.7-6 Simple beam (nonprismatic)

Use the bending-moment equation (Eq. 9-12a).

DEFLECTION CURVE



BENDING-MOMENT EQUATIONS

$$E\left(\frac{3I}{2}\right)v'' = M = \frac{2Px}{3} \qquad \left(0 \le x \le \frac{L}{3}\right) \tag{1}$$

$$EIv'' = M = \frac{PL}{3} - \frac{Px}{3} \qquad \left(\frac{L}{3} \le x \le L\right) \tag{2}$$

INTEGRATE EACH EQUATION

$$EIv' = \frac{4Px^2}{18} + C_1 \qquad \left(0 \le x \le \frac{L}{3}\right) \tag{3}$$

$$EIv' = \frac{PLx}{3} - \frac{Px^2}{2} + C_2 \qquad \left(\frac{L}{3} \le x \le L\right) \tag{4}$$

B.C. 1 Continuity of slopes at point B

$(v'_B)_{\text{Left}} = (v'_B)_{\text{Right}}$

From Eqs. (3) and (4):

$$\frac{4P}{18}\left(\frac{L}{3}\right)^2 + C_1 = \frac{PL}{3}\left(\frac{L}{3}\right) - \frac{P}{6}\left(\frac{L}{3}\right)^2 + C_2$$

$$C_2 = C_1 - \frac{11PL^2}{162} \tag{5}$$

INTEGRATE EQS. (3) AND (4)

$$EIv = \frac{4Px^3}{54} + C_1x + C_3 \qquad \left(0 \le x \le \frac{L}{3}\right) \tag{6}$$

$$EIv = \frac{PLx^2}{6} - \frac{Px^3}{18} + C_2x + C_4 \qquad \left(\frac{L}{3} \le x \le L\right) \tag{7}$$

B.C. 2 Deflection at support A

$$v(0) = 0 \qquad \text{From Eq. (6):} \qquad C_3 = 0 \tag{8}$$

B.C. 3 Deflection at support C

$$v(L) = 0 \quad \text{From Eq. (7):} \qquad C_4 = -\frac{PL^3}{9} - C_2L \tag{9}$$

B.C. 4 Continuity of deflections at point B

$(v_B)_{\text{Left}} = (v_B)_{\text{Right}}$

From Eqs. (6), (8), and (7):

$$\frac{4P}{54}\left(\frac{L}{3}\right)^3 + C_1\left(\frac{L}{3}\right) = \frac{PL}{6}\left(\frac{L}{3}\right)^2 - \frac{P}{18}\left(\frac{L}{3}\right)^3 + C_2\left(\frac{L}{3}\right) + C_4$$

$$C_1L = \frac{10PL^3}{243} + C_2L + 3C_4 \tag{10}$$

SOLVE EQS. (5), (8), (9), AND (10)

$$C_1 = -\frac{38PL^2}{729} \qquad C_2 = -\frac{175PL^2}{1458} \qquad C_3 = 0$$

$$C_4 = \frac{13PL^3}{1458}$$

SLOPES OF THE BEAM (FROM EQS. 3 AND 4)

$$v' = -\frac{2P}{729EI}(19L^2 - 81x^2) \qquad \left(0 \le x \le \frac{L}{3}\right) \tag{11}$$

$$v' = -\frac{P}{1458EI}(175L^2 - 486Lx + 243x^2)$$

$$\left(\frac{L}{3} \leq x \leq L\right) \qquad (12)$$

ANGLE OF ROTATION θ_A (FROM EQ. 11)

$$\theta_A = -v'(0) = \frac{38PL^2}{729EI} \quad \text{(positive clockwise)} \quad \leftarrow$$

ANGLE OF ROTATION θ_C (FROM EQ. 12)

$$\theta_C = v'(L) = \frac{34PL^2}{729EI} \quad \text{(positive counterclockwise)} \quad \leftarrow$$

DEFLECTIONS OF THE BEAM

Substitute C_1, C_2, C_3, and C_4 into Eqs. (6) and (7):

$$v = -\frac{2Px}{729EI}(19L^2 - 27x^2) \quad \left(0 \leq x \leq \frac{L}{3}\right) \quad \leftarrow$$

$$v = -\frac{P}{1458EI}(-13L^3 + 175L^2x - 243Lx^2 + 81x^3)$$

$$\left(\frac{L}{3} \leq x \leq L\right) \quad \leftarrow$$

DEFLECTION AT POINT B $\left(x = \frac{L}{3}\right)$

$$\delta_B = -v\left(\frac{L}{3}\right) = \frac{32PL^3}{2187EI} \quad \text{(positive downward)} \quad \leftarrow$$

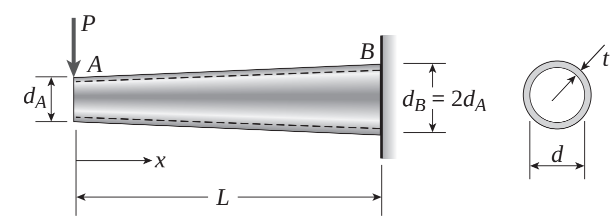
Problem 9.7-7 The tapered cantilever beam AB shown in the figure has thin-walled, hollow circular cross sections of constant thickness t. The diameters at the ends A and B are d_A and $d_B = 2d_A$, respectively. Thus, the diameter d and moment of inertia I at distance x from the free end are, respectively,

$$d = \frac{d_A}{L}(L + x)$$

$$I = \frac{\pi t d^3}{8} = \frac{\pi t d_A^3}{8L^3}(L + x)^3 = \frac{I_A}{L^3}(L + x)^3$$

in which I_A is the moment of inertia at end A of the beam.

Determine the equation of the deflection curve and the deflection δ_A at the free end of the beam due to the load P.



Solution 9.7-7 Tapered cantilever beam

$$M = -Px \qquad EIv'' = -Px \qquad I = \frac{I_A}{L^3}(L + x)^3$$

$$v'' = -\frac{Px}{EI} = -\frac{PL^3}{EI_A}\left[\frac{x}{(L + x)^3}\right] \qquad (1)$$

INTEGRATE EQ. (1)

From Appendix C: $\displaystyle\int \frac{xdx}{(L + x)^3} = -\frac{L + 2x}{2(L + x)^2}$

$$v' = \frac{PL^3}{EI_A}\left[\frac{L + 2x}{2(L + x)^2}\right] + C_1$$

B.C. 1 $v'(L) = 0 \qquad \therefore C_1 = -\frac{3PL^2}{8EI_A}$

$$v' = \frac{PL^3}{EI_A}\left[\frac{L + 2x}{2(L + x)^2}\right] - \frac{3PL^2}{8EI_A}$$

or

$$v' = \frac{PL^3}{EI_A}\left[\frac{L}{2(L + x)^2}\right] + \frac{PL^3}{EI_A}\left[\frac{x}{(L + x)^2}\right] - \frac{3PL^2}{8EI_A} \qquad (2)$$

INTEGRATE EQ. (2)

From Appendix C:

$$\int \frac{dx}{(L + x)^2} = -\frac{1}{L + x}$$

$$\int \frac{xdx}{(L + x)^2} = \frac{L}{L + x} + \ln(L + x)$$

$$v = \frac{PL^3}{EI_A}\left(\frac{L}{2}\right)\left(-\frac{1}{L+x}\right) + \frac{PL^3}{EI_A}\left[\frac{L}{L+x} + \ln(L+x)\right] - \frac{3PL^2}{8EI_A}x + C_2$$

$$= \frac{PL^3}{EI_A}\left[\frac{L}{2(L+x)} + \ln(L+x) - \frac{3x}{8L}\right] + C_2 \qquad (3)$$

B.C. 2 $v(L) = 0 \qquad \therefore C_2 = \frac{PL^3}{EI_A}\left[\frac{1}{8} - \ln(2L)\right]$

Deflection of the beam

Substitute C_2 into Eq. (3).

$$v = \frac{PL^3}{EI_A}\left[\frac{L}{2(L+x)} - \frac{3x}{8L} + \frac{1}{8} + \ln\left(\frac{L+x}{2L}\right)\right] \quad \leftarrow$$

Deflection δ_A at end A of the beam

$$\delta_A = -v(0) = \frac{PL^3}{8EI_A}(8\ln 2 - 5)$$

$$= 0.06815\frac{PL^3}{EI_A} \text{ (positive downward)} \quad \leftarrow$$

Note: $\ln\frac{1}{2} = -\ln 2$

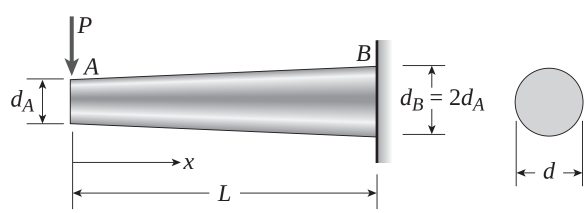
Problem 9.7-8 The tapered cantilever beam AB shown in the figure has a solid circular cross section. The diameters at the ends A and B are d_A and $d_B = 2d_A$, respectively. Thus, the diameter d and moment of inertia I at distance x from the free end are, respectively,

$$d = \frac{d_A}{L}(L+x)$$

$$I = \frac{\pi d^4}{64} = \frac{\pi d_A^4}{64L^4}(L+x)^4 = \frac{I_A}{L^4}(L+x)^4$$

in which I_A is the moment of inertia at end A of the beam.

Determine the equation of the deflection curve and the deflection δ_A at the free end of the beam due to the load P.



Solution 9.7-8 Tapered cantilever beam

$$M = -Px \qquad EIv'' = -Px \qquad I = \frac{I_A}{L^4}(L+x)^4$$

$$v'' = -\frac{Px}{EI} = -\frac{PL^4}{EI_A}\left[\frac{x}{(L+x)^4}\right] \qquad (1)$$

Integrate eq. (1)

From Appendix C: $\displaystyle\int \frac{x\,dx}{(L+x)^4} = -\frac{L+3x}{6(L+x)^3}$

$$v' = \frac{PL^4}{EI_A}\left[\frac{L+3x}{6(L+x)^3}\right] + C_1$$

B.C. 1 $v'(L) = 0 \qquad \therefore C_1 = -\frac{PL^2}{12EI_A}$

$$v' = \frac{PL^4}{EI_A}\left[\frac{L+3x}{6(L+x)^3}\right] - \frac{PL^2}{12EI_A}$$

or

$$v' = \frac{PL^4}{EI_A}\left[\frac{L}{6(L+x)^3}\right] + \frac{PL^4}{EI_A}\left[\frac{x}{2(L+x)^3}\right] - \frac{PL^2}{12EI_A} \qquad (2)$$

INTEGRATE EQ. (2)

From Appendix C: $\displaystyle\int \frac{dx}{(L+x)^3} = -\frac{1}{2(L+x)^2}$

$$\int \frac{xdx}{(L+x)^3} = \frac{-(L+2x)}{2(L+x)^2}$$

$$\nu = \frac{PL^4}{EI_A}\left(\frac{L}{6}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{L+x}\right)^2 + \frac{PL^4}{EI_A}\left(\frac{1}{2}\right)\left[-\frac{L+2x}{2(L+x)^2}\right]$$

$$-\frac{PL^2}{12EI_A}x + C_2$$

$$= \frac{PL^3}{EI_A}\left[-\frac{L^2}{12(L+x)^2} - \frac{L(L+2x)}{4(L+x)^2} - \frac{x}{12L}\right] + C_2 \quad (3)$$

B.C. 2 $\nu(L) = 0$ $\quad \therefore\ C_2 = \dfrac{PL^3}{EI_A}\left(\dfrac{7}{24}\right)$

DEFLECTION OF THE BEAM

Substitute C_2 into Eq. (3)

$$\nu = \frac{PL^3}{24EI_A}\left[7 - \frac{4L(2L+3x)}{(L+x)^2} - \frac{2x}{L}\right] \quad \leftarrow$$

DEFLECTION δ_A AT END A OF THE BEAM

$$\delta_A = -\nu(0) = \frac{PL^3}{24EI_A} \quad \text{(positive downward)} \quad \leftarrow$$

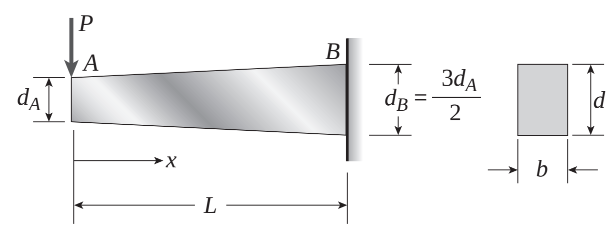
Problem 9.7-9 A tapered cantilever beam AB supports a concentrated load P at the free end (see figure). The cross sections of the beam are rectangular with constant width b, depth d_A at support A, and depth $d_B = 3d_A/2$ at the support. Thus, the depth d and moment of inertia I at distance x from the free end are, respectively,

$$d = \frac{d_A}{2L}(2L + x)$$

$$I = \frac{bd^3}{12} = \frac{bd_A^3}{96L^3}(2L+x)^3 = \frac{I_A}{8L^3}(2L+x)^3$$

in which I_A is the moment of inertia at end A of the beam.

Determine the equation of the deflection curve and the deflection δ_A at the free end of the beam due to the load P.



Solution 9.7-9 Tapered cantilever beam

$M = -Px \qquad EI\nu'' = -Px \qquad I = \dfrac{I_A}{8L^3}(2L+x)^3$

$$\nu'' = -\frac{Px}{EI} = -\frac{8PL^3}{EI_A}\left[\frac{x}{(2L+x)^3}\right] \quad (1)$$

INTEGRATE EQ. (1)

From Appendix C: $\displaystyle\int \frac{xdx}{(2L+x)^3} = -\frac{2L+2x}{2(2L+x)^2}$

$$\nu' = \frac{8PL^3}{EI_A}\left[\frac{L+x}{(2L+x)^3}\right] + C_1$$

B.C. 1 $\nu'(L) = 0$ $\quad \therefore\ C_1 = -\dfrac{16PL^2}{9EI_A}$

$$\nu' = \frac{8PL^3}{EI_A}\left[\frac{L+x}{(2L+x)^2}\right] - \frac{16PL^2}{9EI_A}$$

or

$$\nu' = \frac{8PL^3}{EI_A}\left[\frac{L}{(2L+x)^2}\right] + \frac{8PL^3}{EI_A}\left[\frac{x}{(2L+x)^2}\right]$$

$$-\frac{16PL^2}{9EI_A} \quad (2)$$

INTEGRATE EQ. (2)

From Appendix C: $\displaystyle\int \frac{dx}{(2L+x)^2} = -\frac{1}{2L+x}$

$$\int \frac{xdx}{(2L+x)^2} = \frac{2L}{2L+x} + \ln(2L+x)$$

$$v = \frac{8PL^3}{EI_A}\left(-\frac{L}{2L+x}\right) + \frac{8PL^3}{EI_A}\left[\frac{2L}{2L+x}\right.$$

$$\left. + \ln(2L+x)\right] - \frac{16PL^2}{9EI_A}x + C_2$$

$$= \frac{PL^3}{EI_A}\left[\frac{8L}{2L+x} + 8\ln(2L+x) - \frac{16x}{9L}\right] + C_2 \qquad (3)$$

B.C. 2 $v(L) = 0$ $\quad \therefore\ C_2 = -\dfrac{8PL^3}{EI_A}\left[\dfrac{1}{9} + \ln(3L)\right]$

DEFLECTION OF THE BEAM

Substitute C_2 into Eq. (3).

$$v = \frac{8PL^3}{EI_A}\left[\frac{L}{2L+x} - \frac{2x}{9L} - \frac{1}{9}\right.$$

$$\left. + \ln\left(\frac{2L+x}{3L}\right)\right] \qquad \leftarrow$$

DEFLECTION δ_A AT END A OF THE BEAM

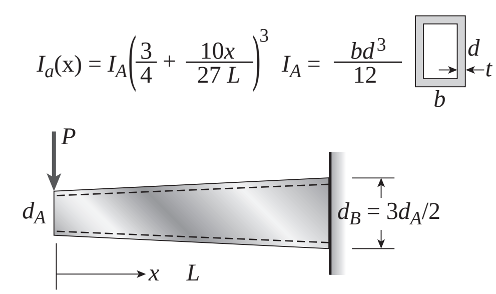
$$\delta_A = -v(0) = \frac{8PL^2}{EI_A}\left[\ln\left(\frac{3}{2}\right) - \frac{7}{18}\right]$$

$$= 0.1326\frac{PL^3}{EI_A} \qquad \text{(positive downward)} \qquad \leftarrow$$

NOTE: $\ln\dfrac{2}{3} = -\ln\dfrac{3}{2}$

Problem 9.7-10 A tapered cantilever beam AB supports a concentrated load P at the free end (see figure). The cross sections of the beam are rectangular tubes with constant width b and *outer tube* depth d_A at A, and *outer tube* depth $d_B = 3d_A/2$ at support B. The tube thickness is constant, $t = d_A/20$. I_A is the moment of inertia of the *outer tube* at end A of the beam.

If the moment of inertia of the tube is approximated as $I_a(x)$ as defined, find the *equation* of the deflection curve and the deflection δ_A at the free end of the beam due to the load P.



Solution 9.7-10

BENDING-MOMENT EQUATIONS

$EIv'' = M = -Px$

$$v'' = \frac{-Px}{EI_a(x)} = \frac{-Px}{EI_A\left(\frac{3}{4} + \frac{10x}{27L}\right)^3} = \frac{-P}{EI_A}\frac{x}{\left(\frac{3}{4} + \frac{10x}{27L}\right)^3}$$

From Appendix C: $\displaystyle\int \frac{x}{(a+bx)^3}dx = -\frac{a+2bx}{2b^2(a+bx)^2}$

$$v' = \frac{-P}{EI_A}\left[\frac{\frac{3}{4} + 2\frac{10}{27L}x}{2\left(\frac{10}{27L}\right)^2\left(\frac{3}{4} + \frac{10}{27L}x\right)^2}\right] + C_1$$

$$v' = \frac{PL^3}{EI_A}\frac{19683}{50}\left[\frac{81L}{(81L+40x)^2} + \frac{80x}{(81L+40x)^2}\right] + C_1$$

From Appendix C: $\displaystyle\int \frac{1}{(a+bx)^2}\,dx = \frac{-1}{b(a+bx)}$

$$\int \frac{x}{(a+bx)^2}\,dx = \frac{1}{b^2}\left(\frac{a}{a+bx} + ln(a+bx)\right)$$

$$v = \frac{PL^3}{EI_A}\frac{19683}{50}\left[\frac{-81L}{40(81L+40x)} + \frac{80}{40^2}\left(\frac{81L}{81L+40x} + \ln(81L+40x)\right)\right] + C_1x + C_2$$

$$v = \frac{19683PL^3}{2000EI_A}\left(\frac{81L + 162\ln(81L+40x)L + 80\ln(81L+40x)x}{81L+40x}\right) + C_1x + C_2$$

B.C. $v'(L) = 0 \qquad C_1 = \dfrac{-3168963}{732050}\dfrac{PL^2}{EI_A}$

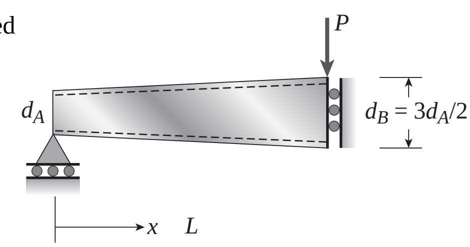
B.C. $v(L) = 0 \qquad C_2 = \dfrac{-19683PL^3}{29282000EI_A}(3361 + 29282\ln(121L))$

$$v(x) = \frac{19683PL^3}{2000EI_A}\left(\frac{81L}{81L+40x} + 2\ln\left(\frac{81}{121} + \frac{40x}{121L}\right) - \frac{6440x}{14641L} - \frac{3361}{14641}\right) \quad \leftarrow$$

$$\delta_A = -v(0) = \frac{19683PL^3}{7320500EI_A}\left(-2820 + 14641\ln\left(\frac{11}{9}\right)\right) = 0.317\frac{PL^3}{EI_A} \quad \leftarrow$$

****Problem 9.7-11** Repeat Problem 9.7-10 but now use the tapered propped cantilever tube AB, with guided support at B, shown in the figure which supports a concentrated load P at the guided end.

Find the equation of the deflection curve and the deflection δ_B at the guided end of the beam due to the load P.



Solution 9.7-11

Bending-moment equations

$EIv'' = M = Px$

$$v'' = \frac{Px}{EI_a(x)} = \frac{Px}{EI_A\left(\frac{3}{4} + \frac{10x}{27L}\right)^3} = \frac{P}{EI_A}\frac{x}{\left(\frac{3}{4} + \frac{10x}{27L}\right)^3}$$

From Appendix C: $\displaystyle\int \frac{x}{(a+bx)^3}\,dx = -\frac{a+2bx}{2b^2(a+bx)^2}$

$$v' = \frac{P}{EI_A}\left[\frac{\frac{3}{4} + 2\frac{10}{27L}x}{2\left(\frac{10}{27L}\right)^2\left(\frac{3}{4} + \frac{10}{27L}x\right)^2}\right] + C_1$$

$$\nu' = -\frac{PL^3}{EI_A}\frac{19683}{50}\left[\frac{81L}{(81L+40x)^2}+\frac{80x}{(81L+40x)^2}\right]+C_1$$

From Appendix C: $\displaystyle\int\frac{1}{(a+bx)^2}\,dx = \frac{-1}{b(a+bx)}$

$$\int\frac{x}{(a+bx)^2}\,dx = \frac{1}{b^2}\left(\frac{a}{a+bx}+\ln(a+bx)\right)$$

$$\nu = -\frac{PL^3}{EI_A}\frac{19683}{50}\left[\frac{-81L}{40(81L+40x)}+\frac{80}{40^2}\left(\frac{81L}{81L+40x}+\ln(81L+40x)\right)\right]+C_1x+C_2$$

$$\nu = -\frac{19683PL^3}{2000EI_A}\left(\frac{81L+162\ln(81L+40x)L+80\ln(81L+40x)x}{81L+40x}\right)+C_1x+C_2$$

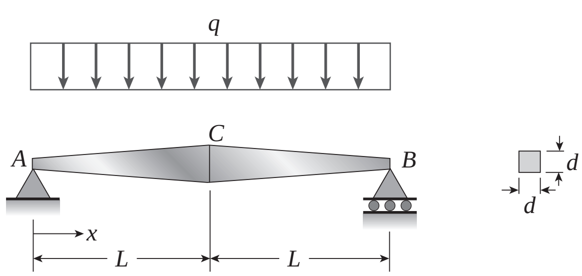
B.C. $\nu'(L) = 0$ $\qquad C_1 = \dfrac{3168963}{732050}\dfrac{PL^2}{EI_A}$

B.C. $\nu(L) = 0$ $\qquad C_2 = \dfrac{19683PL^3}{2000EI_A}(1+2\ln(81L))$

$$\nu(x) = -\frac{19683PL^3}{2000EI_A}\left(\frac{81L}{81L+40x}+2\ln\left(1+\frac{40x}{81L}\right)-\frac{6440x}{14641L}-1\right) \quad \leftarrow$$

$$\delta_B = -\nu(L) = \frac{19683PL^3}{7320500EI_A}\left(-2820+14641\ln\left(\frac{11}{9}\right)\right) = 0.317\frac{PL^3}{EI_A} \quad \leftarrow$$

Problem 9.7-12 A simple beam *ACB* is constructed with square cross sections and a double taper (see figure). The depth of the beam at the supports is d_A and at the midpoint is $d_C = 2d_A$. Each half of the beam has length L. Thus, the depth d and moment of inertia I at distance x from the left-hand end are, respectively,



$$d = \frac{d_A}{L}(L+x)$$

$$I = \frac{d^4}{12} = \frac{d_A^4}{12L^4}(L+x)^4 = \frac{I_A}{L^4}(L+x)^4$$

in which I_A is the moment of inertia at end A of the beam. (These equations are valid for x between 0 and L, that is, for the left-hand half of the beam.)

(a) Obtain equations for the slope and deflection of the left-hand half of the beam due to the uniform load.

(b) From those equations obtain formulas for the angle of rotation θ_A at support A and the deflection δ_C at the midpoint.

Solution 9.7-12 Simple beam with a double taper

L = length of one-half of the beam

$$I = \frac{I_A}{L^4}(L+x)^4 \qquad (0 \le x \le L)$$

(x is measured from the left-hand support A)

Reactions: $R_A = R_B = qL$

Bending moment: $M = R_A x - \dfrac{qx^2}{2} = qLx - \dfrac{qx^2}{2}$

From Eq. (9-12a):

$$EI\nu'' = M = qLx - \frac{qx^2}{2}$$

$$\nu'' = \frac{qL^5 x}{EI_A(L+x)^4} - \frac{qL^4x^2}{2EI_A(L+x)^4} \qquad (0 \le x \le L) \quad (1)$$

INTEGRATE EQ. (1)

From Appendix C: $\displaystyle\int \frac{x\,dx}{(L+x)^4} = -\frac{L+3x}{6(L+x)^3}$

$$\int \frac{x^2dx}{(L+x)^4} = -\frac{L^2 + 3Lx + 3x^2}{3(L+x)^3}$$

$$\nu' = \frac{qL^5}{EI_A}\left[-\frac{L+3x}{6(L+x)^3}\right] - \frac{qL^4}{2EI_A}\left[-\frac{L^2+3Lx+3x^2}{3(L+x)^3}\right] + C_1$$

$$= \frac{qL^4x^2}{2EI_A(L+x)^3} + C_1 \qquad (0 \le x \le L) \qquad (2)$$

B.C. 1 (symmetry) $\nu'(L) = 0 \qquad \therefore C_1 = -\dfrac{qL^3}{16EI_A}$

SLOPE OF THE BEAM

Substitute C_1 into Eq. (2).

$$\nu' = \frac{qL^4x^2}{2EI_A(L+x)^3} - \frac{qL^3}{16EI_A}$$

$$= -\frac{qL^3}{16EI_A}\left[1 - \frac{8Lx^2}{(L+x)^3}\right] \qquad (0 \le x \le L) \qquad (3)$$

ANGLE OF ROTATION AT SUPPORT A

$$\theta_A = -\nu'(0) = \frac{qL^3}{16EI_A} \qquad \text{(positive clockwise)} \quad \leftarrow$$

INTEGRATE EQ. (3)

From Appendix C: $\displaystyle\int \frac{x^2dx}{(L+x)^3} = \frac{L(3L+4x)}{2(L+x)^2} + \ln(L+x)$

$$\nu = -\frac{qL^3}{16EI_A}\left[x - \frac{8L^2(3L+4x)}{2(L+x)^2} - 8L\ln(L+x)\right] + C_2 \qquad (0 \le x \le L) \qquad (4)$$

B.C. 2 $\nu(0) = 0 \qquad \therefore C_2 = -\dfrac{qL^4}{2EI_A}\left(\dfrac{3}{2} + \ln L\right)$

DEFLECTION OF THE BEAM

Substitute C_2 into Eq. (4) and simplify. (The algebra is lengthy.)

$$\nu = -\frac{qL^4}{2EI_A}\left[\frac{(9L^2+14Lx+x^2)x}{8L(L+x)^2} - \ln\left(1+\frac{x}{L}\right)\right]$$

$$(0 \le x \le L) \quad \leftarrow$$

DEFLECTION AT THE MIDPOINT C OF THE BEAM

$$\delta_C = -\nu(L) = \frac{qL^4}{8EI_A}(3 - 4\ln 2) = 0.02843\frac{qL^4}{EI_A}$$

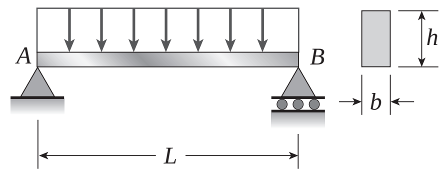
(positive downward) $\leftarrow$

Strain Energy

The beams described in the problems for Section 9.8 have constant flexural rigidity EI.

Problem 9.8-1 A uniformly loaded simple beam AB (see figure) of span length L and rectangular cross section (b = width, h = height) has a maximum bending stress σ_{max} due to the uniform load.

Determine the strain energy U stored in the beam.



Solution 9.8-1 Simple beam with a uniform load

Given: L, b, h, σ_{max} Find: U (strain energy)

Bending moment: $M = \dfrac{qLx}{2} - \dfrac{qx^2}{2}$

Strain energy (Eq. 9-80a): $U = \displaystyle\int_0^L \frac{M^2dx}{2EI}$

$$= \frac{q^2L^5}{240EI} \qquad (1)$$

Maximum stress: $\sigma_{max} = \dfrac{M_{max}c}{I} = \dfrac{M_{max}h}{2I}$

$M_{max} = \dfrac{qL^2}{8} \qquad \sigma_{max} = \dfrac{qL^2h}{16I}$

Solve for q: $q = \dfrac{16I\sigma_{max}}{L^2h}$

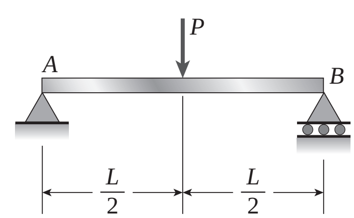
Substitute q into Eq. (1):

$$U = \frac{16I\sigma_{max}^2 L}{15h^2E}$$

Substitute $I = \dfrac{bh^3}{12}$: $\qquad U = \dfrac{4bhL\sigma_{max}^2}{45E} \quad \leftarrow$

Problem 9.8-2 A simple beam AB of length L supports a concentrated load P at the midpoint (see figure).

(a) Evaluate the strain energy of the beam from the bending moment in the beam.
(b) Evaluate the strain energy of the beam from the equation of the deflection curve.
(c) From the strain energy, determine the deflection δ under the load P.



Solution 9.8-2 Simple beam with a concentrated load

(a) Bending moment $\quad M = \dfrac{Px}{2} \quad \left(0 \le x \le \dfrac{L}{2}\right)$

Strain energy (Eq. 9-80a):

$$U = 2\int_0^{L/2} \frac{M^2dx}{2EI} = \frac{P^2L^3}{96EI} \quad \leftarrow$$

(b) Deflection curve

From Table G-2, Case 4:

$$v = -\frac{Px}{48EI}(3L^2 - 4x^2) \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$\frac{dv}{dx} = -\frac{P}{16EI}(L^2 - 4x^2) \qquad \frac{d^2v}{dx^2} = \frac{Px}{2EI}$$

Strain energy (Eq. 9-80b):

$$U = 2\int_0^{L/2} \frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2 dx = EI\int_0^{L/2}\left(\frac{Px}{2EI}\right)^2 dx$$

$$= \frac{P^2L^3}{96EI} \qquad \leftarrow$$

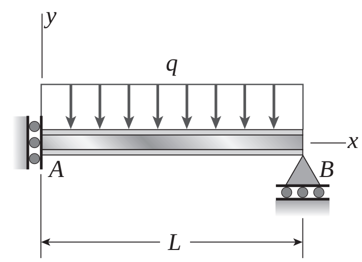
(c) DEFLECTION δ UNDER THE LOAD P

From Eq. (9-82a):

$$\delta = \frac{2U}{P} = \frac{PL^3}{48EI} \qquad \leftarrow$$

Problem 9.8-3 A propped cantilever beam AB of length L, and with guided support at A, supports a uniform load of intensity q (see figure).

(a) Evaluate the strain energy of the beam from the bending moment in the beam.
(b) Evaluate the strain energy of the beam from the equation of the deflection curve.



Solution 9.8-3

(a) BENDING-MOMENT EQUATIONS

Measure x from end B

$$M = qLx - \frac{qx^2}{2}$$

Strain Energy (Eq. 9-80a):

$$U = \int_0^L \frac{M^2}{2EI}dx$$

$$= \int_0^L \frac{1}{2EI}\left(qLx - \frac{qx^2}{2}\right)^2 dx = \frac{q^2L^5}{15EI} \qquad \leftarrow$$

(b) DEFLECTION CURVE

Measure x from end B

$$v = -\frac{qx}{24EI}(8L^3 - 4Lx^2 + x^3)$$

$$\frac{d}{dx}v = -\frac{q}{24EI}(8L^3 - 12Lx^2 + 4x^3)$$

$$\frac{d^2}{dx^2}v = -\frac{q}{2EI}(-2Lx + x^2)$$

Strain energy (Eq. 9-80b):

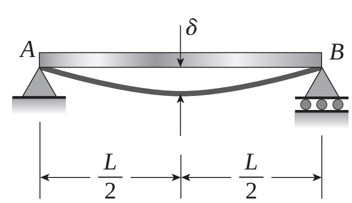
$$U = \int_0^L \frac{EI}{2}\left(\frac{d^2}{dx^2}v\right)^2 dx$$

$$U = \int_0^L \frac{EI}{2}\left[-\frac{q}{2EI}(-2Lx + x^2)\right]^2 dx$$

$$U = \frac{q^2L^5}{15EI} \qquad \leftarrow$$

Problem 9.8-4 A simple beam AB of length L is subjected to loads that produce a symmetric deflection curve with maximum deflection δ at the midpoint of the span (see figure).

How much strain energy U is stored in the beam if the deflection curve is (a) a parabola, and (b) a half wave of a sine curve?



Solution 9.8-4 Simple beam (symmetric deflection curve)

GIVEN: L, EI, δ $\quad\delta$ = maximum deflection at midpoint

Determine the strain energy U.

Assume the deflection ν is positive downward.

(a) DEFLECTION CURVE IS A PARABOLA

$$\nu = \frac{4\delta x}{L^2}(L-x) \qquad \frac{d\nu}{dx} = \frac{4\delta}{L^2}(L-2x)$$

$$\frac{d^2\nu}{dx^2} = -\frac{8\delta}{L^2}$$

Strain energy (Eq. 9-80b):

$$U = \int_0^L \frac{EI}{2}\left(\frac{d^2\nu}{dx^2}\right)^2 dx = \frac{EI}{2}\int_0^L\left(-\frac{8\delta}{L^2}\right)^2 dx$$

$$= \frac{32EI\delta^2}{L^3} \quad \leftarrow$$

(b) DEFLECTION CURVE IS A SINE CURVE

$$\nu = \delta\sin\frac{\pi x}{L} \qquad \frac{d\nu}{dx} = \frac{\pi\delta}{L}\cos\frac{\pi x}{L}$$

$$\frac{d^2\nu}{dx^2} = -\frac{\pi^2\delta}{L^2}\sin\frac{\pi x}{L}$$

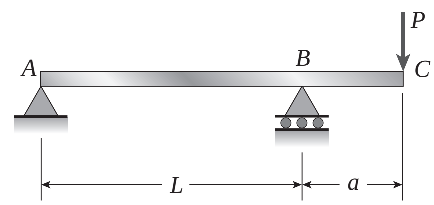
Strain energy (Eq. 9-80b):

$$U = \int_0^L \frac{EI}{2}\left(\frac{d^2\nu}{dx^2}\right)^2 dx = \frac{EI}{2}\int_0^L\left(-\frac{\pi^2\delta}{L^2}\right)^2\sin^2\frac{\pi x}{L}\,dx$$

$$= \frac{\pi^4 EI\delta^2}{4L^3} \quad \leftarrow$$

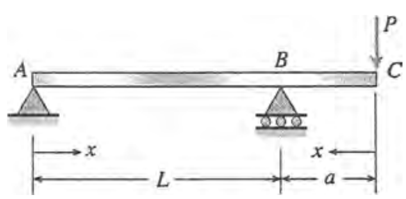
Problem 9.8-5 A beam ABC with simple supports at A and B and an overhang BC supports a concentrated load P at the free end C (see figure).

(a) Determine the strain energy U stored in the beam due to the load P.
(b) From the strain energy, find the deflection δ_C under the load P.
(c) Calculate the numerical values of U and δ_C if the length L is 8 ft, the overhang length a is 3 ft, the beam is a W 10 × 12 steel wide-flange section, and the load P produces a maximum stress of 12,000 psi in the beam. (Use $E = 29 \times 10^6$ psi.)



Solution 9.8-5 Simple beam with an overhang

(a) STRAIN ENERGY (use Eq. 9-80a)



FROM A TO B: $M = -\dfrac{Pax}{L}$

$$U_{AB} = \int\frac{M^2dx}{2EI} = \int_0^L\frac{1}{2EI}\left(-\frac{Pax}{L}\right)^2 dx = \frac{P^2a^2L}{6EI}$$

FROM B TO C: $M = -Px$

$$U_{BC} = \int_0^a \frac{1}{2EI}(-Px)^2\,dx = \frac{P^2a^3}{6EI}$$

TOTAL STRAIN ENERGY:

$$U = U_{AB} + U_{BC} = \frac{P^2a^2}{6EI}(L + a) \quad \leftarrow$$

(b) DEFLECTION δ_C UNDER THE LOAD P

From Eq. (9-82a):

$$\delta_C = \frac{2U}{P} = \frac{Pa^2}{3EI}(L + a) \quad \leftarrow$$

(c) CALCULATE U AND δ_c

Data: $L = 8$ ft $= 96$ in. $\quad a = 3$ ft $= 36$ in.

W $10 \times 12 \quad E = 29 \times 10^6$ psi

$\sigma_{max} = 12{,}000$ psi

$I = 53.8 \text{ in.}^4 \qquad c = \dfrac{d}{2} = \dfrac{9.87}{2} = 4.935$ in.

Express load P in terms of maximum stress:

$$\sigma_{max} = \frac{Mc}{I} = \frac{M_{max}c}{I} = \frac{Pac}{I} \text{□□□□} \therefore P = \frac{\sigma_{max} I}{ac}$$

$$U = \frac{P^2a^2(L+a)}{6EI} = \frac{\sigma_{max}^2 I(L+a)}{6c^2E}$$

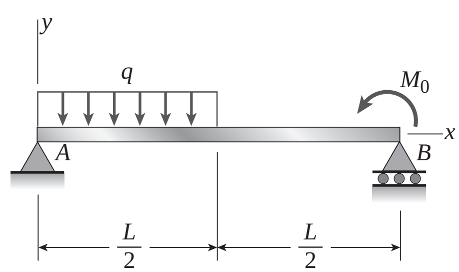
$= 241$ in.-lb $\quad\leftarrow$

$$\delta_c = \frac{Pa^2(L+a)}{3EI} = \frac{\sigma_{max} a(L+a)}{3cE}$$

$= 0.133$ in. $\quad\leftarrow$

Problem 9.8-6 A simple beam ACB supporting a uniform load q over the first half of the beam and a couple of moment M_0 at end B is shown in the figure.

Determine the strain energy U stored in the beam due to the load q and the couple M_0 acting simultaneously.



Solution 9.8-6

FROM A TO MID-SPAN

Bending-Moment Equations

$$M = \left(\frac{3qL}{8} + \frac{M_0}{L}\right)x - \frac{qx^2}{2}$$

STRAIN ENERGY (EQ. 9-80A):

$$U_1 = \int_0^{\frac{L}{2}} \frac{M^2}{2EI}\,dx$$

$$= \int_0^{\frac{L}{2}} \frac{1}{2EI}\left[\left(\frac{3qL}{8} + \frac{M_0}{L}\right)x - \frac{qx^2}{2}\right]^2 dx$$

$$U_1 = \frac{L}{3840EI}\left(3L^4q^2 + 30qL^2M_0 + 80M_0^2\right)$$

FROM MID-SPAN TO B

Bending-Moment Equations

$$M = \left(\frac{3qL}{8} + \frac{M_0}{L}\right)x - \frac{qL}{2}\left(x - \frac{L}{4}\right)$$

STRAIN ENERGY (EQ. 9-80A):

$$U_2 = \int_{\frac{L}{2}}^{L} \frac{M^2}{2EI}\,dx = \int_{\frac{L}{2}}^{L} \frac{1}{2EI}\left[\left(\frac{3qL}{8} + \frac{M_0}{L}\right)x - \frac{qL}{2}\left(x - \frac{L}{4}\right)\right]^2 dx$$

$$U_2 = \frac{L}{3072EI}\left(L^4q^2 + 32qL^2M_0 + 448M_0^2\right)$$

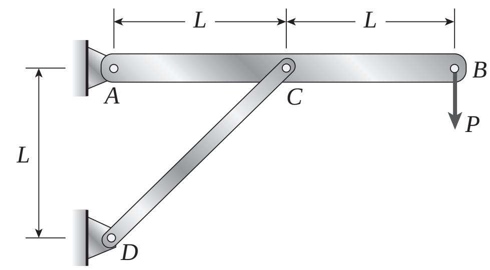
STRAIN ENERGY OF THE ENTIRE BEAM

$$U = U_1 + U_2 = \frac{L}{15360EI}\left(17L^4q^2 + 280qL^2M_0 + 2560M_0^2\right) \quad \leftarrow$$

Problem 9.8-7 The frame shown in the figure consists of a beam *ACB* supported by a struct *CD*. The beam has length $2L$ and is continuous through joint *C*. A concentrated load *P* acts at the free end *B*.

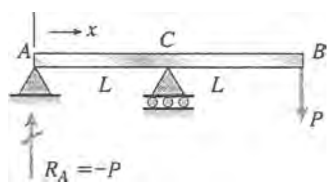
Determine the vertical deflection δ_B at point *B* due to the load *P*.

Note: Let *EI* denote the flexural rigidity of the beam, and let *EA* denote the axial rigidity of the strut. Disregard axial and shearing effects in the beam, and disregard any bending effects in the strut.



Solution 9.8-7 Frame with beam and strut

BEAM *ACB*



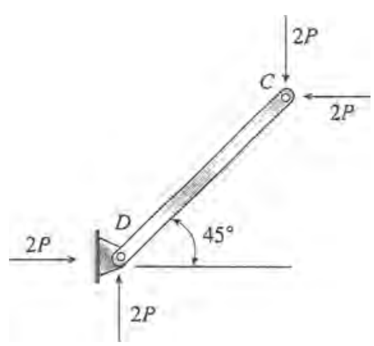
For part *AC* of the beam: $M = -Px$

$$U_{AC} = \int \frac{M^2 dx}{2EI} = \frac{1}{2EI}\int_0^L (-Px)^2 dx = \frac{P^2L^3}{6EI}$$

For part *CB* of the beam: $U_{CB} = U_{AC} = \dfrac{P^2L^3}{6EI}$

Entire beam: $U_{\text{BEAM}} = U_{AC} + U_{CB} = \dfrac{P^2L^3}{3EI}$

SRUCT *CD*



L_{CD} = length of strut

$= \sqrt{2}L$

F = axial force in strut

$= 2\sqrt{2}P$

$$U_{\text{STRUT}} = \frac{F^2 L_{CD}}{2EA} \qquad \text{(Eq. 2-37a)}$$

$$U_{\text{STRUT}} = \frac{(2\sqrt{2}P)^2(\sqrt{2}L)}{2EA} = \frac{4\sqrt{2}P^2L}{EA}$$

FRAME $\quad U = U_{\text{BEAM}} + U_{\text{STRUT}} = \dfrac{P^2L^3}{3EI} + \dfrac{4\sqrt{2}P^2L}{EA}$

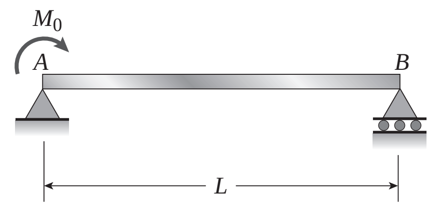
DEFLECTION δ_B AT POINT *B*

From Eq. (9-82a):

$$\delta_B = \frac{2U}{P} = \frac{2PL^3}{3EI} + \frac{8\sqrt{2}PL}{EA} \qquad \leftarrow$$

Castigliano's Theorem

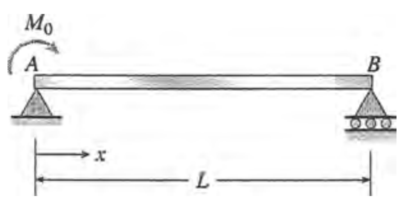
The beams described in the problems for Section 9.9 have constant flexural rigidity EI.



Problem 9.9-1 A simple beam AB of length L is loaded at the left-hand end by a couple of moment M_0 (see figure).

Determine the angle of rotation θ_A at support A. (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)

Solution 9.9-1 Simple beam with couple M_0



$$R_A = \frac{M_0}{L} \quad \text{(downward)}$$

$$M = M_0 - R_A x = M_0 - \frac{M_0 x}{L}$$
$$= M_0\left(1 - \frac{x}{L}\right)$$

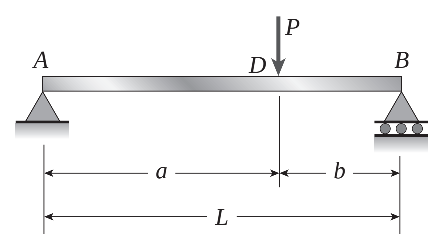
STRAIN ENERGY

$$U = \int \frac{M^2 dx}{2EI} = \frac{M_0^2}{2EI}\int_0^L \left(1 - \frac{x}{L}\right)^2 dx = \frac{M_0^2 L}{6EI}$$

CASTIGLIANO'S THEOREM

$$\theta_A = \frac{dU}{dM_0} = \frac{M_0 L}{3EI} \quad \text{(clockwise)} \qquad \leftarrow$$

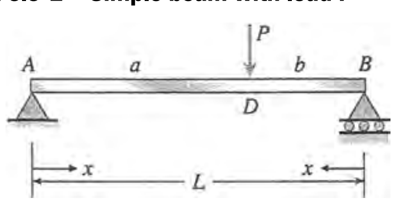
(This result agrees with Case 7, Table G-2)



Problem 9.9-2 The simple beam shown in the figure supports a concentrated load P acting at distance a from the left-hand support and distance b from the right-hand support.

Determine the deflection δ_D at point D where the load is applied. (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)

Solution 9.9-2 Simple beam with load P



$$R_A = \frac{Pb}{L} \qquad R_B = \frac{Pa}{L}$$

$$M_{AD} = R_A x = \frac{Pbx}{L}$$

$$M_{DB} = R_B x = \frac{Pax}{L}$$

STRAIN ENERGY $\quad U = \int \frac{M^2 dx}{2EI}$

$$U_{AD} = \frac{1}{2EI}\int_0^a \left(\frac{Pbx}{L}\right)^2 dx = \frac{P^2 a^3 b^2}{6EIL^2}$$

$$U_{DB} = \frac{1}{2EI}\int_0^b \left(\frac{Pax}{L}\right)^2 dx = \frac{P^2a^2b^3}{6EIL^2}$$

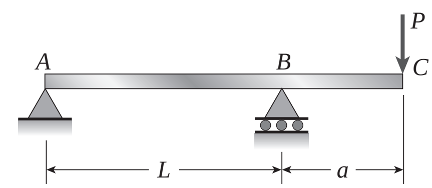
$$U = U_{AD} + U_{DB} = \frac{P^2a^2b^2}{6LEI}$$

CASTIGLIANO'S THEOREM

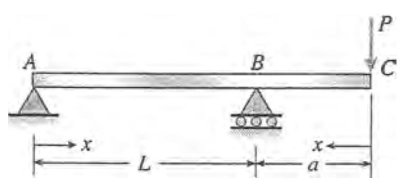
$$\delta_D = \frac{dU}{dP} = \frac{Pa^2b^2}{3LEI} \quad \text{(downward)} \quad \leftarrow$$

Problem 9.9-3 An overhanging beam ABC supports a concentrated load P at the end of the overhang (see figure). Span AB has length L and the overhang has length a.

Determine the deflection δ_C at the end of the overhang. (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)



Solution 9.9-3 Overhanging beam



$$R_A = \frac{Pa}{L} \quad \text{(downward)}$$

$$M_{AB} = -R_A x = -\frac{Pax}{L}$$

$$M_{CB} = -Px$$

STRAIN ENERGY $\quad U = \int \frac{M^2 dx}{2EI}$

$$U_{AB} = \frac{1}{2EI}\int_0^L \left(-\frac{Pax}{L}\right)^2 dx = \frac{P^2a^2L}{6EI}$$

$$U_{CB} = \frac{1}{2EI}\int_0^a (-Px)^2 dx = \frac{P^2a^3}{6EI}$$

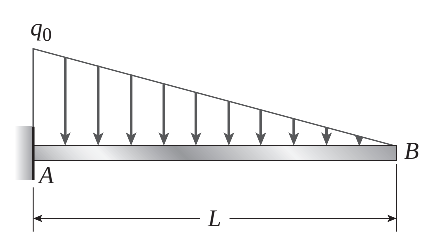
$$U = U_{AB} + U_{CB} = \frac{P^2a^2}{6EI}(L + a)$$

CASTIGLIANO'S THEOREM

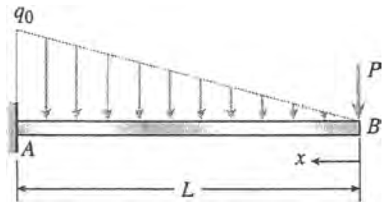
$$\delta_C = \frac{dU}{dP} = \frac{Pa^2}{3EI}(L + a) \quad \text{(downward)} \quad \leftarrow$$

Problem 9.9-4 The cantilever beam shown in the figure supports a triangularly distributed load of maximum intensity q_0.

Determine the deflection δ_B at the free end B. (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)



Solution 9.9-4 Cantilever beam with triangular load



P = fictitious load corresponding to deflection δ_B

$$M = -Px - \frac{q_0 x^3}{6L}$$

STRAIN ENERGY

$$U = \int \frac{M^2 dx}{2EI} = \frac{1}{2EI}\int_0^L \left(-Px - \frac{q_0 x^3}{6L}\right)^2 dx$$

$$= \frac{P^2L^3}{6EI} + \frac{Pq_0L^4}{30EI} + \frac{q_0^2L^5}{42EI}$$

CASTIGLIANO'S THEOREM

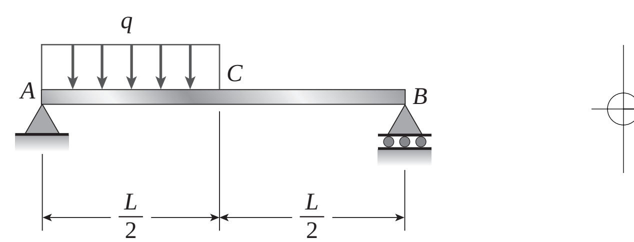
$$\delta_B = \frac{\partial U}{\partial P} = \frac{PL^3}{3EI} + \frac{q_0L^4}{30EI} \quad \text{(downward)}$$

(This result agrees with Cases 1 and 8 of Table G-1.)

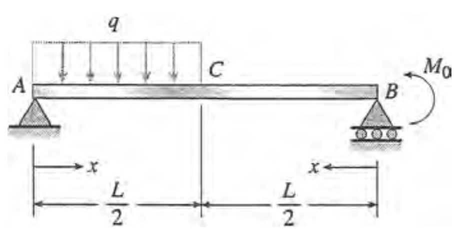
SET $P = 0$: $\delta_B = \dfrac{q_0L^4}{30EI}$ $\leftarrow$

Problem 9.9-5 A simple beam ACB supports a uniform load of intensity q on the left-hand half of the span (see figure).

Determine the angle of rotation θ_B at support B. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-5 Simple beam with partial uniform load



M_0 = fictitious load corresponding to angle of rotation θ_B

$$R_A = \frac{3qL}{8} + \frac{M_0}{L} \qquad R_B = \frac{qL}{8} - \frac{M_0}{L}$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT AC

$$M_{AC} = R_A x - \frac{qx^2}{2} = \left(\frac{3qL}{8} + \frac{M_0}{L}\right)x - \frac{qx^2}{2}$$

$$\left(0 \le x \le \frac{L}{2}\right)$$

$$\frac{\partial M_{AC}}{\partial M_0} = \frac{x}{L}$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT CB

$$M_{CB} = R_B x + M_0 = \left(\frac{qL}{8} - \frac{M_0}{L}\right)x + M_0$$

$$\left(0 \le x \le \frac{L}{2}\right)$$

$$\frac{\partial M_{CB}}{\partial M_0} = -\frac{x}{L} + 1$$

MODIFIED CASTIGLIANO'S THEOREM (EQ. 9-88)

$$\theta_B = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial M_0}\right)dx$$

$$= \frac{1}{EI}\int_0^{L/2}\left[\left(\frac{3qL}{8} + \frac{M_0}{L}\right)x - \frac{qx^2}{2}\right]\left[\frac{x}{L}\right]dx$$

$$+ \frac{1}{EI}\int_0^{L/2}\left[\left(\frac{qL}{8} - \frac{M_0}{L}\right)x + M_0\right]\left[1 - \frac{x}{L}\right]dx$$

SET FICTITIOUS LOAD M_0 EQUAL TO ZERO

$$\theta_B = \frac{1}{EI}\int_0^{L/2}\left(\frac{3qLx}{8} - \frac{qx^2}{2}\right)\left(\frac{x}{L}\right)dx$$

$$+ \frac{1}{EI}\int_0^{L/2}\left(\frac{qLx}{8}\right)\left(1 - \frac{x}{L}\right)dx$$

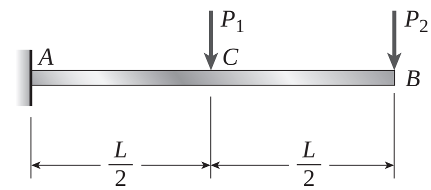
$$= \frac{qL^3}{128EI} + \frac{qL^3}{96EI}$$

$$= \frac{7qL^3}{384EI} \quad \text{(counterclockwise)} \quad \leftarrow$$

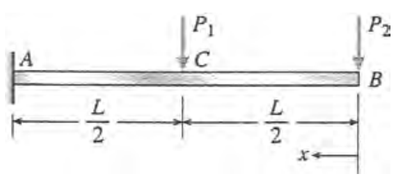
(This result agrees with Case 2, Table G-2.)

Problem 9.9-6 A cantilever beam ACB supports two concentrated loads P_1 and P_2, as shown in the figure.

Determine the deflections δ_C and δ_B at points C and B, respectively. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-6 Cantilever beam with loads P_1 and P_2



BENDING MOMENT AND PARTIAL DERIVATIVES FOR SEGMENT CB

$$M_{CB} = -P_2x \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$\frac{\partial M_{CB}}{\partial P_1} = 0 \qquad \frac{\partial M_{CB}}{\partial P_2} = -x$$

BENDING MOMENT AND PARTIAL DERIVATIVES FOR SEGMENT AC

$$M_{AC} = -P_1\left(x - \frac{L}{2}\right) - P_2x \quad \left(\frac{L}{2} \le x \le L\right)$$

$$\frac{\partial M_{AC}}{\partial P_1} = \frac{L}{2} - x \qquad \frac{\partial M_{AC}}{\partial P_2} = -x$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_C

$$\delta_C = \frac{1}{EI}\int_0^{L/2}(M_{CB})\left(\frac{\partial M_{CB}}{\partial P_1}\right)dx$$

$$+ \frac{1}{EI}\int_{L/2}^{L}(M_{AC})\left(\frac{\partial M_{AC}}{\partial P_1}\right)dx$$

$$= 0 + \frac{1}{EI}\int_{L/2}^{L}\left[-P_1\left(x - \frac{L}{2}\right) - P_2x\right]\left(\frac{L}{2} - x\right)dx$$

$$= \frac{L^3}{48EI}(2P_1 + 5P_2) \quad \leftarrow$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_B

$$\delta_B = \frac{1}{EI}\int_0^{L/2}(M_{CB})\left(\frac{\partial M_{CB}}{\partial P_2}\right)dx$$

$$+ \frac{1}{EI}\int_{L/2}^{L}(M_{AC})\left(\frac{\partial M_{AC}}{\partial P_2}\right)dx$$

$$= \frac{1}{EI}\int_0^{L/2}(-P_2x)(-x)dx$$

$$+ \frac{1}{EI}\int_{L/2}^{L}\left[-P_1\left(x - \frac{L}{2}\right) - P_2x\right](-x)dx$$

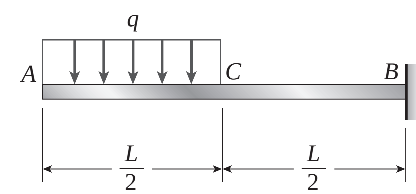
$$= \frac{P_2L^3}{24EI} + \frac{L^3}{48EI}(5P_1 + 14P_2)$$

$$= \frac{L^3}{48EI}(5P_1 + 16P_2) \quad \longleftarrow$$

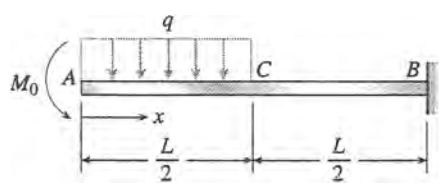
(These results can be verified with the aid of Cases 4 and 5, Table G-1.)

Problem 9.9-7 The cantilever beam ACB shown in the figure is subjected to a uniform load of intensity q acting between points A and C.

Determine the angle of rotation θ_A at the free end A. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-7 Cantilever beam with partial uniform load



M_0 = fictitious load corresponding to the angle of rotation θ_A

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT AC

$$M_{AC} = -M_0 - \frac{qx^2}{2} \qquad \left(0 \le x \le \frac{L}{2}\right)$$

$$\frac{\partial M_{AC}}{\partial M_0} = -1$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT CB

$$M_{CB} = -M_0 - \frac{qL}{2}\left(x - \frac{L}{4}\right) \qquad \left(\frac{L}{2} \le x \le L\right)$$

$$\frac{\partial M_{CB}}{\partial M_0} = -1$$

MODIFIED CASTIGLIANO'S THEOREM (EQ. 9-88)

$$\theta_A = \int\left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial M_0}\right)dx$$

$$= \frac{1}{EI}\int_0^{L/2}\left(-M_0 - \frac{qx^2}{2}\right)(-1)dx$$

$$+ \frac{1}{EI}\int_{L/2}^{L}\left[-M_0 - \frac{qL}{2}\left(x - \frac{L}{4}\right)\right](-1)dx$$

SET FICTITIOUS LOAD M_0 EQUAL TO ZERO

$$\theta_A = \frac{1}{EI}\int_0^{L/2}\frac{qx^2}{2}dx + \frac{1}{EI}\int_{L/2}^{L}\left(\frac{qL}{2}\right)\left(x - \frac{L}{4}\right)dx$$

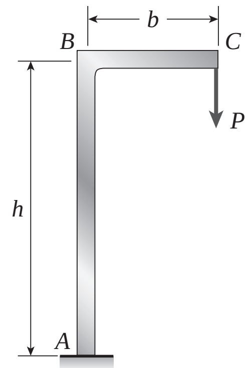
$$= \frac{qL^3}{48EI} + \frac{qL^3}{8EI}$$

$$= \frac{7qL^3}{48EI} \quad \text{(counterclockwise)} \quad \longleftarrow$$

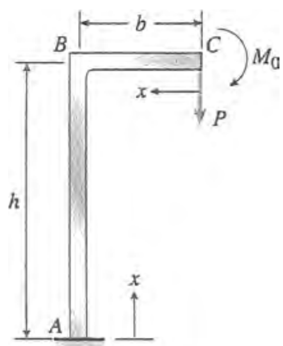
(This result can be verified with the aid of Case 3, Table G-1.)

Problem 9.9-8 The frame ABC supports a concentrated load P at point C (see figure). Members AB and BC have lengths h and b, respectively.

Determine the vertical deflection δ_C and angle of rotation θ_C at end C of the frame. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-8 Frame with concentrated load



P = concentrated load acting at point C (corresponding to the deflection δ_C)

M_0 = fictitious moment corresponding to the angle of rotation θ_C

Bending moment and partial derivatives for member AB

$$M_{AB} = Pb + M_0 \quad (0 \le x \le h)$$

$$\frac{\partial M_{AB}}{\partial P} = b \qquad \frac{\partial M_{AB}}{M_0} = 1$$

Bending moment and partial derivatives for member BC

$$M_{BC} = Px + M_0 \quad (0 \le x \le b)$$

$$\frac{\partial M_{BC}}{\partial P} = x \qquad \frac{\partial M_{BC}}{\partial M_0} = 1$$

Modified Castigliano's theorem for deflection δ_C

$$\delta_C = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P}\right) dx$$

$$= \frac{1}{EI}\int_0^h (Pb + M_0)(b)\,dx + \frac{1}{EI}\int_0^b (Px + M_0)(x)\,dx$$

Set $M_0 = 0$:

$$\delta_C = \frac{1}{EI}\int_0^h Pb^2 dx + \frac{1}{EI}\int_0^b Px^2 dx$$

$$= \frac{Pb^2}{3EI}(3h + b) \quad \text{(downward)} \quad \leftarrow$$

Modified Castigliano's theorem for angle of rotation θ_C

$$\theta_C = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial M_0}\right) dx$$

$$= \frac{1}{EI}\int_0^h (Pb + M_0)(1)dx + \frac{1}{EI}\int_0^b (Px + M_0)(1)\,dx$$

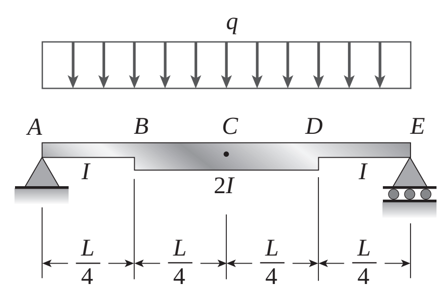
Set $M_0 = 0$:

$$\theta_C = \frac{1}{EI}\int_0^h Pb\,dx + \frac{1}{EI}\int_0^b Pxdx$$

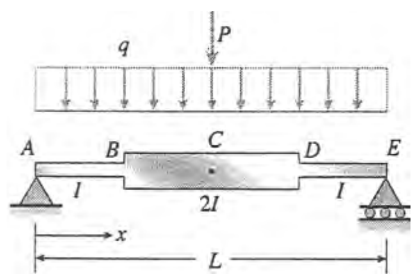
$$= \frac{Pb}{2EI}(2h + b) \quad \text{(clockwise)} \quad \leftarrow$$

Problem 9.9-9 A simple beam *ABCDE* supports a uniform load of intensity q (see figure). The moment of inertia in the central part of the beam (*BCD*) is twice the moment of inertia in the end parts (*AB* and *DE*).

Find the deflection δ_C at the midpoint C of the beam. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-9 Nonprismatic beam



P = fictitiuous load corresponding to the deflection δ_C at the midpoint

$$R_A = \frac{qL}{2} + \frac{P}{2}$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR THE LEFT-HAND HALF OF THE BEAM (A TO C)

$$M_{AC} = \frac{qLx}{2} - \frac{qx^2}{2} + \frac{Px}{2} \qquad \left(0 \le x \le \frac{L}{2}\right)$$

$$\frac{\partial M_{AC}}{\partial P} = \frac{x}{2} \qquad \left(0 \le x \le \frac{L}{2}\right)$$

MODIFIED CASTIGLIANO'S THEOREM (EQ. 9-88)

Integrate from A to C and multiply by 2.

$$\delta_C = 2\int\left(\frac{M_{AC}}{EI}\right)\left(\frac{\partial M_{AC}}{\partial P}\right)dx$$

$$= 2\left(\frac{1}{EI}\right)\int_0^{L/4}\left(\frac{qLx}{2} - \frac{qx^2}{2} + \frac{Px}{2}\right)\left(\frac{x}{2}\right)dx$$

$$+ 2\left(\frac{1}{2EI}\right)\int_{L/4}^{L/2}\left(\frac{qLx}{2} - \frac{qx^2}{2} + \frac{Px}{2}\right)\left(\frac{x}{2}\right)dx$$

SET FICTITIOUS LOAD P EQUAL TO ZERO

$$\delta_C = \frac{2}{EI}\int_0^{L/4}\left(\frac{qLx}{2} - \frac{qx^2}{2}\right)\left(\frac{x}{2}\right)dx$$

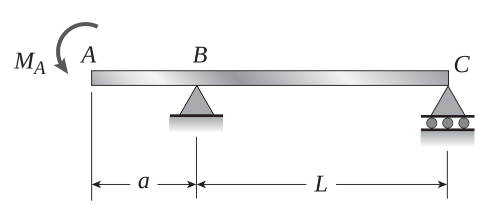
$$+ \frac{1}{EI}\int_{L/4}^{L/2}\left(\frac{qLx}{2} - \frac{qx^2}{2}\right)\left(\frac{x}{2}\right)dx$$

$$= \frac{13qL^4}{6{,}144EI} + \frac{67qL^4}{12{,}288EI}$$

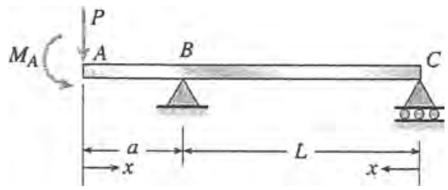
$$\delta_C = \frac{31qL^4}{4096EI} \quad \text{(downward)} \quad \leftarrow$$

Problem 9.9-10 An overhanging beam *ABC* is subjected to a couple M_A at the free end (see figure). The lengths of the overhang and the main span are a and L, respectively.

Determine the angle of rotation θ_A and deflection δ_A at end A. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-10 Overhanging beam *ABC*



M_A = couple acting at the free end A (corresponding to the angle of rotation θ_A)

P = fictitious load corresponding to the deflection δ_A

BENDING MOMENT AND PARTIAL DERIVATIVES FOR SEGMENT AB

$$M_{AB} = -M_A - Px \quad (0 \le x \le a)$$

$$\frac{\partial M_{AB}}{\partial M_A} = -1 \qquad \frac{\partial M_{AB}}{\partial P} = -x$$

BENDING MOMENT AND PARTIAL DERIVATIVES FOR SEGMENT BC

Reaction at support C: $R_C = \dfrac{M_A}{L} + \dfrac{Pa}{L}$ (downward)

$$M_{BC} = -R_C x = -\frac{M_A x}{L} - \frac{Pax}{L} \quad (0 \le x \le L)$$

$$\frac{\partial M_{BC}}{\partial M_A} = -\frac{x}{L} \qquad \frac{\partial M_{BC}}{\partial P} = -\frac{ax}{L}$$

MODIFIED CASTIGLIANO'S THEOREM FOR ANGLE OF ROTATION θ_A

$$\theta_A = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial M_A}\right)dx$$

$$= \frac{1}{EI}\int_0^a (-M_A - Px)(-1)\,dx$$

$$+ \frac{1}{EI}\int_0^L \left(-\frac{M_A x}{L} - \frac{Pax}{L}\right)\left(-\frac{x}{L}\right)dx$$

Set $P = 0$:

$$\theta_A = \frac{1}{EI}\int_0^a M_A dx + \frac{1}{EI}\int_0^L \left(\frac{M_A x}{L}\right)\left(\frac{x}{L}\right)dx$$

$$= \frac{M_A}{3EI}(L + 3a) \quad \text{(counterclockwise)} \quad \leftarrow$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_A

$$\delta_A = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P}\right)dx$$

$$= \frac{1}{EI}\int_0^a (-M_A - Px)(-x)dx$$

$$+ \frac{1}{EI}\int_0^L \left(-\frac{M_A x}{L} - \frac{Pax}{L}\right)\left(-\frac{ax}{L}\right)dx$$

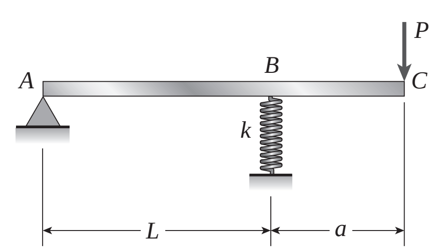
Set $P = 0$:

$$\delta_A = \frac{1}{EI}\int_0^a M_A x dx + \frac{1}{EI}\int_0^L \left(\frac{M_A x}{L}\right)\left(\frac{ax}{L}\right)dx$$

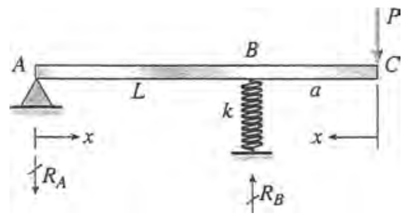
$$= \frac{M_A a}{6EI}(2L + 3a) \quad \text{(downward)} \quad \leftarrow$$

Problem 9.9-11 An overhanging beam ABC rests on a simple support at A and a spring support at B (see figure). A concentrated load P acts at the end of the overhang. Span AB has length L, the overhang has length a, and the spring has stiffness k.

Determine the downward displacement δ_C of the end of the overhang. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-11 Beam with spring support



$$R_A = \frac{Pa}{L} \quad \text{(downward)}$$

$$R_B = \frac{P}{L}(L + a) \quad \text{(upward)}$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT AB

$$M_{AB} = -R_A x = -\frac{Pax}{L} \qquad \frac{dM_{AB}}{dP} = -\frac{ax}{L}$$

$$(0 \le x \le L)$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT BC

$$M_{BC} = -Px \quad \frac{dM_{BC}}{dP} = -x \quad (0 \le x \le a)$$

STRAIN ENERGY OF THE SPRING (EQ. 2-38a)

$$U_S = \frac{R_B^2}{2k} = \frac{P^2(L + a)^2}{2kL^2}$$

STRAIN ENERGY OF THE BEAM (EQ. 9-80a)

$$U_B = \int \frac{M^2 dx}{2EI}$$

TOTAL STRAIN ENERGY U

$$U = U_B + U_S = \int \frac{M^2 dx}{2EI} + \frac{P^2(L + a)^2}{2kL^2}$$

APPLY CASTIGLIANO'S THEOREM (EQ. 9-87)

$$\delta_C = \frac{dU}{dP} = \frac{d}{dP}\int \frac{M^2 dx}{2EI} + \frac{d}{dP}\left[\frac{P^2(L + a)^2}{2kL^2}\right]$$

$$= \frac{d}{dP}\int \frac{M^2 dx}{2EI} + \frac{P(L + a)^2}{kL^2}$$

DIFFERENTIATE UNDER THE INTEGRAL SIGN (MODIFIED CASTIGLIANO'S THEOREM)

$$\delta_C = \int \left(\frac{M}{EI}\right)\left(\frac{dM}{dP}\right)dx + \frac{P(L + a)^2}{kL^2}$$

$$= \frac{1}{EI}\int_0^L \left(-\frac{Pax}{L}\right)\left(-\frac{ax}{L}\right)dx$$

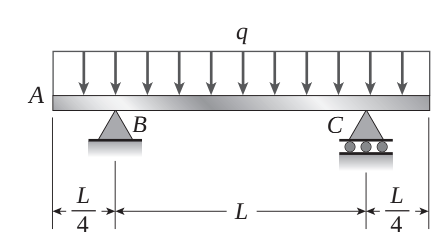
$$+ \frac{1}{EI}\int_0^a (-Px)(-x)\,dx + \frac{P(L + a)^2}{kL^2}$$

$$= \frac{Pa^2 L}{3EI} + \frac{Pa^3}{3EI} + \frac{P(L + a)^2}{kL^2}$$

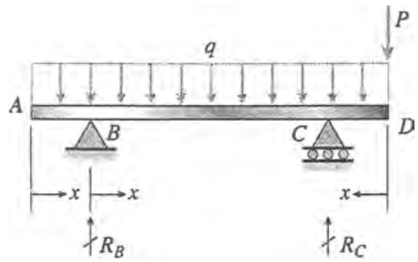
$$\delta_C = \frac{Pa^2(L + a)}{3EI} + \frac{P(L + a)^2}{kL^2} \quad \longleftarrow$$

Problem 9.9-12 A symmetric beam $ABCD$ with overhangs at both ends supports a uniform load of intensity q (see figure).

Determine the deflection δ_D at the end of the overhang. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-12 Beam with overhangs



q = intensity of uniform load

P = fictitious load corresponding to the deflection δ_D

$\frac{L}{4}$ = length of segments AB and CD

L = length of span BC

$$R_B = \frac{3qL}{4} - \frac{P}{4} \qquad R_C = \frac{3qL}{4} + \frac{5P}{4}$$

BENDING MOMENTS AND PARTIAL DERIVATIVES

SEGMENT AB

$$M_{AB} = -\frac{qx^2}{2} \qquad \frac{\partial M_{AB}}{\partial P} = 0 \qquad \left(0 \le x \le \frac{L}{4}\right)$$

SEGMENT BC

$$M_{BC} = -\left[q\left(x + \frac{L}{4}\right)\right]\left[\frac{1}{2}\left(x + \frac{L}{4}\right)\right] + R_B x$$

$$= -\frac{q}{2}\left(x + \frac{L}{4}\right)^2 + \left(\frac{3\,qL}{4} - \frac{P}{4}\right)x$$

$$(0 \le x \le L)$$

$$\frac{\partial M_{BC}}{\partial P} = -\frac{x}{4}$$

SEGMENT CD $\quad M_{CD} = -\frac{qx^2}{2} - Px \quad \left(0 \le x \le \frac{L}{4}\right)$

$$\frac{\partial M_{CD}}{\partial P} = -x$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_D

$$\delta_D = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P}\right)dx$$

$$= \frac{1}{EI}\int_0^{L/4}\left(-\frac{qx^2}{2}\right)(0)dx$$

$$+ \frac{1}{EI}\int_0^{L}\left[-\frac{q}{2}\left(x + \frac{L}{4}\right)^2 + \left(\frac{3qL}{4} - \frac{P}{4}\right)x\right]$$

$$\times\left[-\frac{x}{4}\right]dx + \frac{1}{EI}\int_0^{L/4}\left(-\frac{qx^2}{2} - Px\right)(-x)dx$$

SET $P = 0$:

$$\delta_D = \frac{1}{EI}\int_0^{L}\left[-\frac{q}{2}\left(x + \frac{L}{4}\right)^2 + \frac{3qL}{4}x\right]\left[-\frac{x}{4}\right]dx$$

$$+ \frac{1}{EI}\int_0^{L/4}\left(-\frac{qx^2}{2}\right)(-x)dx$$

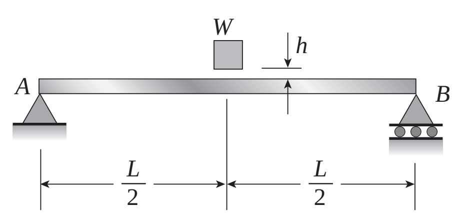
$$= -\frac{5qL^4}{768EI} + \frac{qL^4}{2048EI} = -\frac{37qL^4}{6144EI}$$

(Minus means the deflection is opposite in direction to the fictitious load P.)

$$\therefore \delta_D = \frac{37qL^4}{6144EI} \quad \text{(upward)} \qquad \leftarrow$$

Deflections Produced by Impact

The beams described in the problems for Section 9.10 have constant flexural rigidity EI. Disregard the weights of the beams themselves, and consider only the effects of the given loads.



Problem 9.10-1 A heavy object of weight W is dropped onto the midpoint of a simple beam AB from a height h (see figure).

Obtain a formula for the maximum bending stress σ_{max} due to the falling weight in terms of h, σ_{st}, and δ_{st}, where σ_{st} is the maximum bending stressand δ_{st} is the deflection at the midpoint when the weight W acts on the beam as a statically applied load.

Plot a graph of the ratio σ_{max}/σ_{st} (that is, the ratio of the dynamic stress to the static stress) versus the ratio h/δ_{st}. (Let h/δ_{st} vary from 0 to 10.)

Solution 9.10-1 Weight *W* dropping onto a simple beam

MAXIMUM DEFLECTION (EQ. 9-94)

$\delta_{max} = \delta_{st} + (\delta_{st}^2 + 2h\delta_{st})^{1/2}$

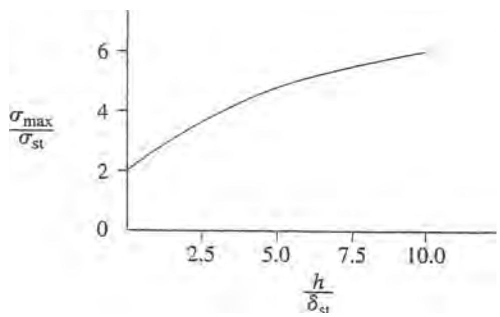
MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ

$$\therefore \frac{\sigma_{max}}{\sigma_{st}} = \frac{\delta_{max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}$$

$$\sigma_{max} = \sigma_{st}\left[1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}\right] \quad \leftarrow$$

GRAPH OF RATIO σ_{max}/σ_{st}

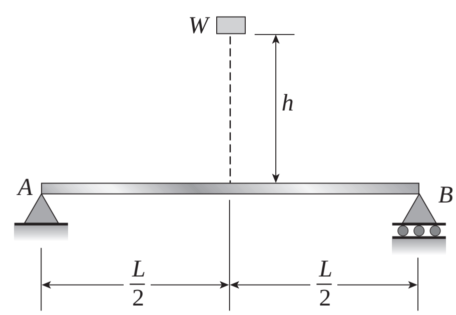


$\frac{h}{\delta_{st}}$	$\frac{\sigma_{max}}{\sigma_{st}}$
0	2.00
2.5	3.45
5.0	4.33
7.5	5.00
10.0	5.58

NOTE: $\delta_{st} = \dfrac{WL^3}{48EI}$ for a simple beam with a load at the midpoint.

Problem 9.10-2 An object of weight W is dropped onto the midpoint of a simple beam AB from a height h (see figure). The beam has a rectangular cross section of area A.

Assuming that h is very large compared to the deflection of the beam when the weight W is applied statically, obtain a formula for the maximum bending stress σ_{max} in the beam due to the falling weight.



Solution 9.10-2 Weight *W* dropping onto a simple beam

Height h is very large.

MAXIMUM DEFLECTION (EQ. 9-95)

$$\delta_{max} = \sqrt{2h\delta_{st}}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ.

$$\therefore \frac{\sigma_{max}}{\sigma_{st}} = \frac{\delta_{max}}{\delta_{st}} = \sqrt{\frac{2h}{\delta_{st}}}$$

$$\sigma_{max} = \sqrt{\frac{2h\sigma_{st}^2}{\delta_{st}}} \qquad (1)$$

$$\sigma_{st} = \frac{M}{S} = \frac{WL}{4S} \qquad \sigma_{st}^2 = \frac{W^2L^2}{16S^2}$$

$$\delta_{st} = \frac{WL^3}{48EI} \qquad \frac{\sigma_{st}^2}{\delta_{st}} = \frac{3WEI}{S^2L} \qquad (2)$$

For a RECTANGULAR BEAM (with b, depth d):

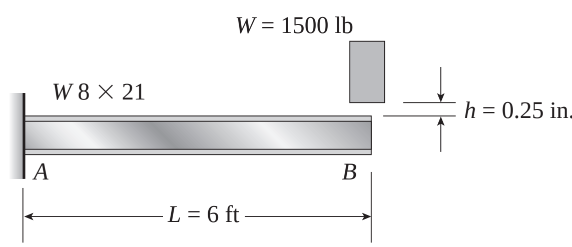
$$I = \frac{bd^3}{12} \qquad S = \frac{bd^2}{6} \qquad \frac{I}{S^2} = \frac{3}{bd} = \frac{3}{A} \qquad (3)$$

Substitute (2) and (3) into (1):

$$\sigma_{max} = \sqrt{\frac{18WhE}{AL}} \qquad \leftarrow$$

Problem 9.10-3 A cantilever beam AB of length $L = 6$ ft is constructed of a W 8 × 21 wide-flange section (see figure). A weight $W = 1500$ lb falls through a height $h = 0.25$ in. onto the end of the beam.

Calculate the maximum deflection δ_{max} of the end of the beam and the maximum bending stress σ_{max} due to the falling weight. (Assume $E = 30 \times 10^6$ psi.)



Solution 9.10-3 Cantilever beam

DATA: $L = 6 \text{ ft} = 72$ in. $\quad W = 1500$ lb

$h = 0.25$ in. $\quad E = 30 \times 10^6$ psi

W 8 × 21 $\quad I = 75.3 \text{ in.}^4 \quad S = 18.2 \text{ in.}^3$

MAXIMUM DEFLECTION (EQ. 9-94)

Equation (9-94) may be used for any linearly elastic structure by substituting $\delta_{st} = W/k$, where k is the stiffness of the particular structure being considered. For instance: Simple beam with load at midpoint:

$$k = \frac{48EI}{L^3}$$

Cantilever beam with load at the free end: $k = \dfrac{3EI}{L^3}$

For the cantilever beam in this problem:

$$\delta_{st} = \frac{WL^3}{3EI} = \frac{(1500 \text{ lb})(72 \text{ in.})^3}{3(30 \times 10^6 \text{ psi})(75.3 \text{ in.}^4)}$$

$$= 0.08261 \text{ in.}$$

Equation (9-94):

$$\delta_{max} = \delta_{st} + (\delta_{st}^2 + 2h\delta_{st})^{1/2} = 0.302 \text{ in.} \qquad \leftarrow$$

MAXIMUM BENDING STRESS

Consider a cantilever beam with load P at the free end:

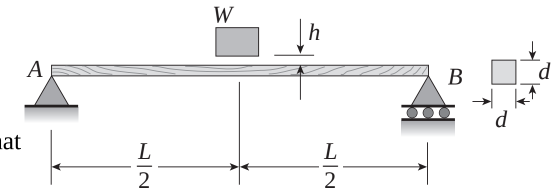
$$\sigma_{max} = \frac{M_{max}}{S} = \frac{PL}{S} \qquad \delta_{max} = \frac{PL^3}{3EI}$$

$$\text{Ratio: } \frac{\sigma_{max}}{\delta_{max}} = \frac{3\,EI}{SL^2}$$

$$\therefore \sigma_{max} = \frac{3EI}{SL^2}\delta_{max} = 21{,}700 \text{ psi} \qquad \leftarrow$$

Problem 9.10-4 A weight $W = 20$ kN falls through a height $h = 1.0$ mm onto the midpoint of a simple beam of length $L = 3$ m (see figure). The beam is made of wood with square cross section (dimension d on each side) and $E = 12$ GPa.

If the allowable bending stress in the wood is $\sigma_{allow} = 10$ MPa, what is the minimum required dimension d?



Solution 9.10-4 Simple beam with falling weight W

DATA: $W = 20$ kN $\quad h = 1.0$ mm $\quad L = 3.0$ m

$E = 12$ GPa $\quad \sigma_{allow} = 10$ MPa

CROSS SECTION OF BEAM (SQUARE)

d = dimension of each side

$$I = \frac{d^4}{12} \qquad S = \frac{d^3}{6}$$

MAXIMUM DEFLECTION (EQ. 9-94)

$$\delta_{max} = \delta_{st} + (\delta_{st}^2 + 2h\delta_{st})^{1/2}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ.

$$\therefore \frac{\sigma_{max}}{\sigma_{st}} = \frac{\delta_{max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2} \tag{1}$$

STATIC TERMS σ_{st} AND δ_{st}

$$\sigma_{st} = \frac{M}{S} = \left(\frac{WL}{4}\right)\left(\frac{6}{d^3}\right) = \frac{3WL}{2d^3} \tag{2}$$

$$\delta_{st} = \frac{WL^3}{48EI} = \frac{WL^3}{48E}\left(\frac{12}{d^4}\right) = \frac{WL^3}{4Ed^4} \tag{3}$$

SUBSTITUTE (2) AND (3) INTO EQ. (1)

$$\frac{2\sigma_{max}d^3}{3WL} = 1 + \left(1 + \frac{8hEd^4}{WL^3}\right)^{1/2}$$

SUBSTITUTE NUMERICAL VALUES:

$$\frac{2(10\text{ MPa})d^3}{3(20\text{ kN})(3.0\text{ m})} = 1 + \left[1 + \frac{8(1.0\text{ mm})(12\text{ GPa})d^4}{(20\text{ kN})(3.0\text{ m})^3}\right]^{1/2}$$

$$\frac{1000}{9}d^3 - 1 = \left[1 + \frac{1600}{9}d^4\right]^{1/2} \quad (d = \text{meters})$$

SQUARE BOTH SIDES, REARRANGE, AND SIMPLIFY

$$\left(\frac{1000}{9}\right)^2 d^3 - \frac{1600}{9}d - \frac{2000}{9} = 0$$

$$2500d^3 - 36d - 45 = 0 \quad (d = \text{meters})$$

SOLVE NUMERICALLY

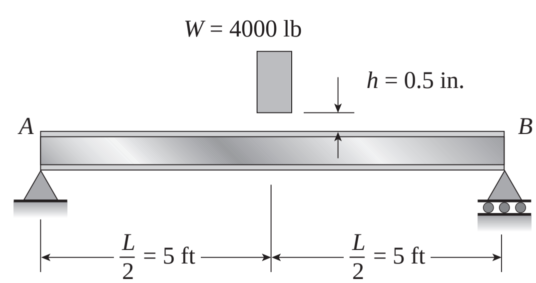
$d = 0.2804$ m $= 280.4$ mm

For minimum value, round upward.

$\therefore d = 281$ mm $\quad\leftarrow$

Problem 9.10-5 A weight $W = 4000$ lb falls through a height $h = 0.5$ in. onto the midpoint of a simple beam of length $L = 10$ ft (see figure).

Assuming that the allowable bending stress in the beam is $\sigma_{allow} = 18{,}000$ psi and $E = 30 \times 10^6$ psi, select the lightest wide-flange beam listed in Table E-1a in Appendix E that will be satisfactory.



Solution 9.10-5 Simple beam of wide-flange shape

DATA: $W = 4000$ lb $\quad h = 0.5$ in.

$L = 10$ ft $= 120$ in.

$\sigma_{allow} = 18{,}000$ psi $\quad E = 30 \times 10^6$ psi

MAXIMUM DEFLECTION (EQ. 9-94)

$$\delta_{max} = \delta_{st} + (\delta_{st}^2 + 2h\delta_{st})^{1/2}$$

or $$\frac{\delta_{max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ.

$$\therefore \frac{\sigma_{max}}{\sigma_{st}} = \frac{\delta_{max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2} \qquad (1)$$

STATIC TERMS σ_{st} AND δ_{st}

$$\sigma_{st} = \frac{M}{S} = \frac{WL}{4S} \qquad \delta_{st} = \frac{WL^3}{48EI}$$

$$\frac{\sigma_{max}}{\sigma_{st}} = \sigma_{allow}\left(\frac{4S}{WL}\right) = \frac{4\sigma_{allow}S}{WL} \qquad (2)$$

$$\frac{2h}{\delta_{st}} = 2h\left(\frac{48EI}{WL^3}\right) = \frac{96hEI}{WL^3} \qquad (3)$$

SUBSTITUTE (2) AND (3) INTO EQ. (1):

$$\frac{4\sigma_{allow}S}{WL} = 1 + \left(1 + \frac{96hEI}{WL^3}\right)^{1/2}$$

REQUIRED SECTION MODULUS

$$S = \frac{WL}{4\sigma_{allow}}\left[1 + \left(1 + \frac{96hEI}{WL^3}\right)^{1/2}\right]$$

SUBSTITUTE NUMERICAL VALUES

$$S = \left(\frac{20}{3}\text{ in.}^3\right)\left[1 + \left(1 + \frac{5I}{24}\right)^{1/2}\right] \qquad (4)$$

($S =$ in.^{3}; $I =$ in.^{4})

PROCEDURE

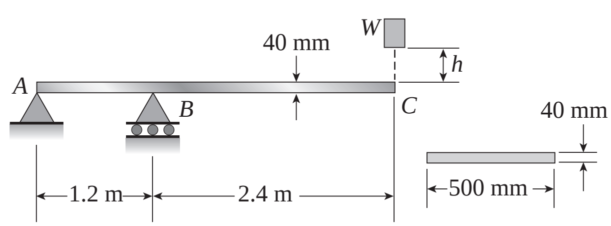
1. Select a trial beam from Table E-1a.
2. Substitute I into Eq. (4) and calculate required S.
3. Compare with actual S for the beam.
4. Continue until the lightest beam is found.

Trial beam	Actual I	Actual S	Required S
W 8 × 35	127	31.2	41.6 (NG)
W 10 × 45	248	49.1	55.0 (NG)
W 10 × 60	341	66.7	63.3 (OK)
W 12 × 50	394	64.7	67.4 (NG)
W 14 × 53	541	77.8	77.8 (OK)
W 16 × 31	375	47.2	66.0 (NG)

Lightest beam is W 14 × 53 ←

Problem 9.10-6 An overhanging beam *ABC* of rectangular cross section has the dimensions shown in the figure. A weight $W = 750$ N drops onto end C of the beam.

If the allowable normal stress in bending is 45 MPa, what is the maximum height h from which the weight may be dropped? (Assume $E = 12$ GPa.)



Solution 9.10-6 Overhanging beam

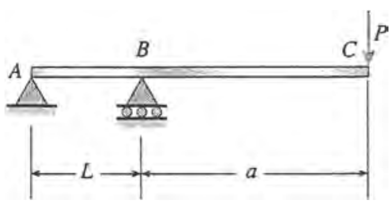
DATA: $W = 750$ N $\quad L_{AB} = 1.2$ m. $\quad L_{BC} = 2.4$ m

$E = 12$ GPa $\quad \sigma_{\text{allow}} = 45$ MPa

$$I = \frac{bd^3}{12} = \frac{1}{12}(500 \text{ mm})(40 \text{ mm})^3$$
$$= 2.6667 \times 10^6 \text{ mm}^4$$
$$= 2.6667 \times 10^{-6} \text{ m}^4$$

$$S = \frac{bd^2}{6} = \frac{1}{6}(500 \text{ mm})(40 \text{ mm})^2$$
$$= 133.33 \times 10^3 \text{ mm}^3$$
$$= 133.33 \times 10^{-6} \text{ m}^3$$

DEFLECTION δ_C AT THE END OF THE OVERHANG



P = load at end C

L = length of span AB

a = length of overhang BC

From the answer to Prob. 9.8-5 or Prob. 9.9-3:

$$\delta_C = \frac{Pa^2(L + a)}{3EI}$$

Stiffness of the beam: $k = \dfrac{P}{\delta_C} = \dfrac{3EI}{a^2(L + a)}$ (1)

MAXIMUM DEFLECTION (EQ. 9-94)

Equation (9-94) may be used for any linearly elastic structure by substituting $\delta_{st} = W/k$, where k is the stiffness of the particular structure being considered. For instance:

Simple beam with load at midpoint: $k = \dfrac{48EI}{L^3}$

Cantilever beam with load at free end: $k = \dfrac{3EI}{L^3}$ Etc.

For the overhanging beam in this problem (see Eq. 1):

$$\delta_{st} = \frac{W}{k} = \frac{Wa^2(L + a)}{3EI} \tag{2}$$

in which $a = L_{BC}$ and $L = L_{AB}$:

$$\delta_{st} = \frac{W(L_{BC}^2)(L_{AB} + L_{BC})}{3EI} \tag{3}$$

EQUATION (9-94):

$$\delta_{\max} = \delta_{st} + (\delta_{st}^2 + 2h\delta_{st})^{1/2}$$

or

$$\frac{\delta_{\max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2} \tag{4}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ.

$$\therefore \frac{\sigma_{\max}}{\sigma_{st}} = \frac{\delta_{\max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2} \tag{5}$$

$$\sigma_{st} = \frac{M}{S} = \frac{WL_{BC}}{S} \tag{6}$$

MAXIMUM HEIGHT h

Solve Eq. (5) for h:

$$\frac{\sigma_{\max}}{\sigma_{st}} - 1 = \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}$$

$$\left(\frac{\sigma_{\max}}{\sigma_{st}}\right)^2 - 2\left(\frac{\sigma_{\max}}{\sigma_{st}}\right) + 1 = 1 + \frac{2h}{\delta_{st}}$$

$$h = \frac{\delta_{st}}{2}\left(\frac{\sigma_{\max}}{\sigma_{st}}\right)\left(\frac{\sigma_{\max}}{\sigma_{st}} - 2\right) \tag{7}$$

Substitute δ_{st} from Eq. (3), σ_{st} from Eq. (6), and σ_{allow} for $\sigma_{\max}$:

$$h = \frac{W(L_{BC}^2)(L_{AB} + L_{BC})}{6EI}\left(\frac{\sigma_{\text{allow}}S}{WL_{BC}}\right)\left(\frac{\sigma_{\text{allow}}S}{WL_{BC}} - 2\right) \tag{8}$$

SUBSTITUTE NUMERICAL VALUES INTO EQ. (8):

$$\frac{W(L_{BC}^2)(L_{AB} + L_{BC})}{6EI} = 0.08100 \text{ m}$$

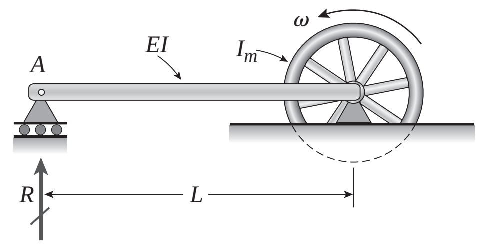
$$\frac{\sigma_{\text{allow}}S}{WL_{BC}} = \frac{10}{3} = 3.3333$$

$$h = (0.08100 \text{ m})\left(\frac{10}{3}\right)\left(\frac{10}{3} - 2\right) = 0.36 \text{ m}$$

or $h = 360$ mm $\quad\leftarrow$

Problem 9.10-7 A heavy flywheel rotates at an angular speed ω (radians per second) around an axle (see figure). The axle is rigidly attached to the end of a simply supported beam of flexural rigidity EI and length L (see figure). The flywheel has mass moment of inertia I_m about its axis of rotation.

If the flywheel suddenly freezes to the axle, what will be the reaction R at support A of the beam?



Solution 9.10-7 Rotating flywheel

NOTE: We will disregard the mass of the beam and all energy losses due to the sudden stopping of the rotating flywheel. Assume that *all* of the kinetic energy of the flywheel is transformed into strain energy of the beam.

KINETIC ENERGY OF ROTATING FLYWHEEL

$$KE = \frac{1}{2} I_m \omega^2$$

STRAIN ENERGY OF BEAM $\qquad U = \int \frac{M^2 dx}{2EI}$

$M = Rx$, where x is measured from support A.

$$U = \frac{1}{2EI}\int_0^L (Rx)^2 dx = \frac{R^2L^3}{6EI}$$

CONSERVATION OF ENERGY

$$KE = U \qquad \frac{1}{2} I_m \mathrm{v}^2 = \frac{R^2L^3}{6EI}$$

$$R = \sqrt{\frac{3EI\, I_m \mathrm{v}^2}{L^3}} \qquad \leftarrow$$

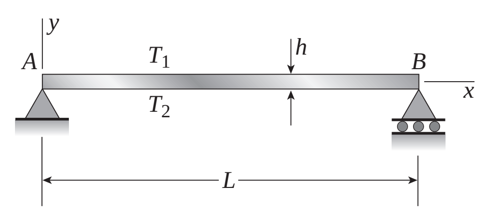
NOTE: The moment of inertia I_m has units of $kg \cdot m^2$ or $N \cdot m \cdot s^2$

Temperature Effects

The beams described in the problems for Section 9.11 have constant flexural rigidity EI. In every problem, the temperature varies linearly between the top and bottom of the beam.

Problem 9.11-1 A simple beam AB of length L and height h undergoes a temperature change such that the bottom of the beam is at temperature T_2 and the top of the beam is at temperature T_1 (see figure).

Determine the equation of the deflection curve of the beam, the angle of rotation θ_A at the left-hand support, and the deflection δ_{max} at the midpoint.



Solution 9.11-1 Simple beam with temperature differential

Eq. (9-147): $v'' = \dfrac{d^2v}{dx^2} = \dfrac{\alpha(T_2 - T_1)}{h}$

$v' = \dfrac{dv}{dx} = \dfrac{\alpha(T_2 - T_1)x}{h} + C_1$

B.C. 1 (Symmetry) $v'\left(\dfrac{L}{2}\right) = 0$

$\therefore\ C_1 = -\dfrac{\alpha L(T_2 - T_1)}{2h}$

$v = \dfrac{\alpha(T_2 - T_1)x^2}{2h} - \dfrac{\alpha L(T_2 - T_1)x}{2h} + C_2$

B.C. 2 $v(0) = 0$ $\quad\therefore C_2 = 0$

$v = -\dfrac{\alpha(T_2 - T_1)(x)(L - x)}{2h} \quad \leftarrow$

(positive v is upward deflection)

$v' = -\dfrac{\alpha(T_2 - T_1)(L - 2x)}{2h}$

$\theta_A = -v'(0) = \dfrac{\alpha L(T_2 - T_1)}{2h} \quad \leftarrow$

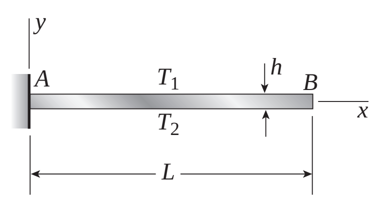
(positive θ_A is clockwise rotation)

$\delta_{max} = -v\left(\dfrac{L}{2}\right) = \dfrac{\alpha L^2(T_2 - T_1)}{8h} \quad \leftarrow$

(positive δ_{max} is downward deflection)

Problem 9.11-2 A cantilever beam AB of length L and height h (see figure) is subjected to a temperature change such that the temperature at the top is T_1 and at the bottom is T_2.

Determine the equation of the deflection curve of the beam, the angle of rotation θ_B at end B, and the deflection δ_B at end B.



Solution 9.11-2 Cantilever beam with temperature differential

Eq. (9-147): $v'' = \dfrac{d^2v}{dx^2} = \dfrac{\alpha(T_2 - T_1)}{h}$

$v' = \dfrac{dv}{dx} = \dfrac{\alpha(T_2 - T_1)}{h}x + C_1$

B.C. 1 $v'(0) = 0$ $\quad\therefore C_1 = 0$

$v' = \dfrac{\alpha(T_2 - T_1)}{h}x$

$v = \dfrac{\alpha(T_2 - T_1)}{h}\left(\dfrac{x^2}{2}\right) + C_2$

B.C. 2 $v(0) = 0$ $\quad\therefore C_2 = 0$

$v = \dfrac{\alpha(T_2 - T_1)x^2}{2h} \quad \leftarrow$

(positive v is upward deflection)

$\theta_B = v'(L) = \dfrac{\alpha L(T_2 - T_1)}{h} \quad \leftarrow$

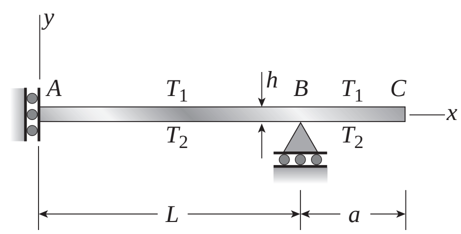
(positive θ_B is counterclockwise rotation)

$\delta_B = v(L) = \dfrac{\alpha L^2(T_2 - T_1)}{2h} \quad \leftarrow$

(positive δ_B is upward deflection)

Problem 9.11-3 An overhanging beam ABC of height h has a guided support at A and a roller at B. The beam is heated to a temperature T_1 on the top and T_2 on the bottom (see figure).

Determine the equation of the deflection curve of the beam, the angle of rotation θ_C at end C, and the deflection δ_C at end C.



Solution 9.11-3

$$v'' = \frac{d^2}{dx^2}v = \frac{\alpha(T_2 - T_1)}{h}$$

$$v' = \frac{\alpha(T_2 - T_1)x}{h} + C_1$$

$$v = \frac{\alpha(T_2 - T_1)x^2}{2h} + C_1 x + C_2$$

B.C. $v'(0) = 0 \qquad C_1 = 0$

B.C. $v(L) = 0 \qquad C_2 = -\frac{\alpha(T_2 - T_1)L^2}{2h}$

$$v(x) = \frac{\alpha(T_2 - T_1)(x^2 - L^2)}{2h}$$

$$\delta_C = v(L + a) = \frac{\alpha(T_2 - T_1)[(L + a)^2 - L^2]}{2h}$$

$$= \frac{\alpha(T_2 - T_1)(2La + a^2)}{2h} \quad \leftarrow$$

Upward

$$\theta_C = v'(L + a) = \frac{\alpha(T_2 - T_1)(L + a)}{h} \quad \leftarrow$$

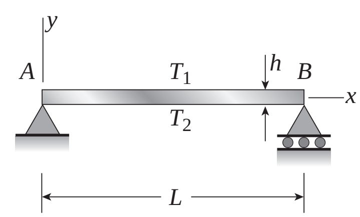
Counter Clockwise

Problem 9.11-4 A simple beam AB of length L and height h (see figure) is heated in such a manner that the temperature difference $T_2 - T_1$ between the bottom and top of the beam is proportional to the distance from support A; that is, assume the temperature difference varies *linearly* along the beam:

$$T_2 - T_1 = T_0 x$$

in which T_0 is a constant having units of temperature (degrees) per unit distance.

(a) Determine the maximum deflection δ_{max} of the beam.
(b) Repeat for *quadratic* temperature variation along the beam, $T_2 - T_1 = T_0 x^2$.



Solution 9.11-4

(a) $(T_2 - T_1) = T_0 x$

$$v'' = \frac{d^2}{dx^2}v = \frac{\alpha T_0 x}{h}$$

$$v' = \frac{\alpha T_0 x^2}{2h} + C_1$$

$$v = \frac{\alpha T_0 x^3}{6h} + C_1 x + C_2$$

B.C. $v(0) = 0 \qquad C_2 = 0$

B.C. $v(L) = 0 \qquad C_1 = -\frac{\alpha T_0 L^2}{6h}$

$$v(x) = \frac{\alpha T_0 (x^3 - L^2 x)}{6h}$$

$$v'(x) = \frac{\alpha T_0}{2h}\left(x^2 - \frac{L^2}{3}\right)$$

MAXIMUM DEFLECTION

Set $v'(x) = 0$ and solve for x

$$0 = \frac{\alpha T_0}{2h}\left(x^2 - \frac{L^2}{3}\right) \qquad x = \frac{L}{\sqrt{3}}$$

$$\delta_{max} = -v\left(\frac{L}{\sqrt{3}}\right)$$

$$= -\frac{\alpha T_0\left[\left(\frac{L}{\sqrt{3}}\right)^3 - L^2\frac{L}{\sqrt{3}}\right]}{6h}$$

$$\delta_{max} = \frac{\alpha T_0 L^3}{9\sqrt{3}h} \quad \text{Downward} \qquad \longleftarrow$$

(b) $(T_2 - T_1) = T_0\,x^2$

$$v'' = \frac{d^2}{dx^2}v = \frac{\alpha T_0 x^2}{h}$$

$$v' = \frac{\alpha T_0 x^3}{3h} + C_1$$

$$v = \frac{\alpha T_0 x^4}{12h} + C_1 x + C_2$$

B.C. $v(0) = 0 \qquad C_2 = 0$

B.C. $v(L) = 0 \qquad C_1 = -\frac{\alpha T_0 L^3}{12h}$

$$v(x) = \frac{\alpha T_0 (x^4 - L^3 x)}{12h}$$

$$v'(x) = \frac{\alpha T_0}{3h}\left(x^3 - \frac{L^3}{4}\right)$$

MAXIMUM DEFLECTION

Set $v'(x) = 0$ and solve for x

$$0 = \frac{\alpha T_0}{3h}\left(x^3 - \frac{L^3}{4}\right) \qquad x = \frac{L}{\sqrt{2}}$$

$$\delta_{max} = -v\left(\frac{L}{\sqrt{2}}\right)$$

$$= -\frac{\alpha T_0\left[\left(\frac{L}{\sqrt{2}}\right)^4 - L^3\frac{L}{\sqrt{2}}\right]}{12h}$$

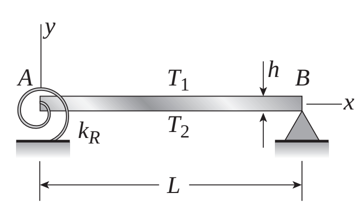
$$\delta_{max} = \frac{\alpha T_0 L^4}{48h}(2\sqrt{2} - 1) \qquad \longleftarrow$$

Downward

Problem 9.11-5 Beam AB, with elastic support k_R at A and pin support at B, of length L and height h (see figure) is heated in such a manner that the temperature difference $T_2 - T_1$ between the bottom and top of the beam is proportional to the distance from support A; that is, assume the temperature difference varies *linearly* along the beam:

$$T_2 - T_1 = T_0 x$$

in which T_0 is a constant having units of temperature (degrees) per unit distance. Assume the spring at A is unaffected by the temperature change.



(a) Determine the maximum deflection δ_{max} of the beam.
(b) Repeat for *quadratic* temperature variation along the beam, $T_2 - T_1 = T_0\,x^2$.
(c) What is δ_{max} for (a) and (b) above if k_R goes to infinity?

Solution 9.11-5

(a) $(T_2 - T_1) = T_0\,x$

$$v'' = \frac{d^2}{dx^2}v = \frac{\alpha T_0 x}{h}$$

$$v' = \frac{\alpha T_0 x^2}{2h} + C_1$$

$$v = \frac{\alpha T_0 x^3}{6h} + C_1 x + C_2$$

B.C. $v'(0) = 0 \qquad C_1 = 0$

B.C. $v(L) = 0 \qquad C_2 = -\frac{\alpha T_0 L^3}{6h}$

$$v(x) = \frac{\alpha T_0 (x^3 - L^3)}{6h}$$

$$v'(x) = \frac{\alpha T_0 x^2}{2h}$$

MAXIMUM DEFLECTION

Set $v'(x) = 0$ and solve for x

$$0 = \frac{\alpha T_0 x^2}{2h} \qquad x = 0$$

$$\delta_{max} = -v(0) = \frac{\alpha T_0 L^3}{6h} \quad \text{Downward} \quad \leftarrow$$

(b) $(T_2 - T_1) = T_0\,x^2$

$$v'' = \frac{d^2}{dx^2}v = \frac{\alpha T_0 x^2}{h}$$

$$v' = \frac{\alpha T_0 x^3}{3h} + C_1$$

$$v = \frac{\alpha T_0 x^4}{12h} + C_1 x + C_2$$

B.C. $v'(0) = 0 \qquad C_1 = 0$

B.C. $v(L) = 0 \qquad C_2 = -\frac{\alpha T_0 L^4}{12h}$

$$v(x) = \frac{\alpha T_0 (x^4 - L^4)}{12h}$$

$$v'(x) = \frac{\alpha T_0 x^3}{3h}$$

MAXIMUM DEFLECTION

Set $v'(x) = 0$ and solve for x

$$0 = \frac{\alpha T_0 x^3}{3h} \qquad x = 0$$

$$\delta_{max} = -v(0) = \frac{\alpha T_0 L^4}{12h} \quad \text{Downward} \quad \leftarrow$$

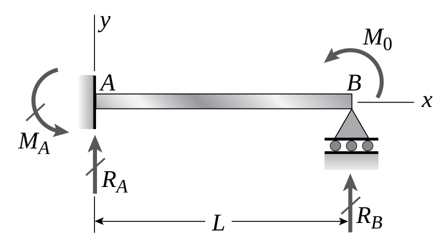
(c) Changing k_R does not change δ_{max} in both cases.

10

Statically Indeterminate Beams

Differential Equations of the Deflection Curve

The problems for Section 10.3 are to be solved by integrating the differential equations of the deflection curve. All beams have constant flexural rigidity EI. When drawing shear-force and bending-moment diagrams, be sure to label all critical ordinates, including maximum and minimum values.



Problem 10.3-1 A propped cantilever beam AB of length L is loaded by a counterclockwise moment M_0 acting at support B (see figure).

Beginning with the second-order differential equation of the deflection curve (the bending-moment equation), obtain the reactions, shear forces, bending moments, slopes, and deflections of the beam. Construct the shear-force and bending-moment diagrams, labeling all critical ordinates.

Solution 10.3-1 Propped cantilever beam

M_0 = applied load

Select M_A as the redundant reaction.

REACTIONS (FROM EQUILIBRIUM)

$$R_A = \frac{M_A}{L} + \frac{M_0}{L} \quad (1) \qquad R_B = -R_A \quad (2)$$

BENDING MOMENT (FROM EQUILIBRIUM)

$$M = R_A x - M_A = \frac{M_A}{L}(x - L) + \frac{M_0 x}{L} \quad (3)$$

DIFFERENTIAL EQUATIONS

$$EIv'' = M = \frac{M_A}{L}(x - L) + \frac{M_0 x}{L}$$

$$EIv' = \frac{M_A}{L}\left(\frac{x^2}{2} - Lx\right) + \frac{M_0 x^2}{2L} + C_1 \quad (4)$$

B.C. 1 $v'(0) = 0 \qquad \therefore C_1 = 0$

$$EIv = \frac{M_A}{L}\left(\frac{x^3}{6} - \frac{Lx^2}{2}\right) + \frac{M_0 x^3}{6L} + C_2 \quad (5)$$

B.C. 2 $v(0) = 0 \qquad \therefore C_2 = 0$

B.C. 3 $v(L) = 0 \qquad \therefore M_A = \dfrac{M_0}{2}$

REACTIONS (SEE EQS. 1 AND 2)

$$M_A = \frac{M_0}{2} \qquad R_A = \frac{3M_0}{2L} \qquad R_B = -\frac{3M_0}{2L} \quad \leftarrow$$

SHEAR FORCE (FROM EQUILIBRIUM)

$$V = R_A = \frac{3M_0}{2L} \quad \leftarrow$$

BENDING MOMENT (FROM EQ. 3)

$$M = \frac{M_0}{2L}(3x - L) \quad \leftarrow$$

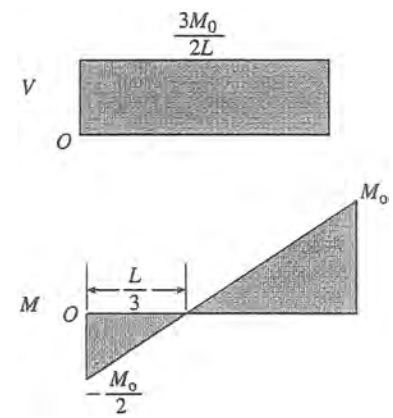
SLOPE (FROM EQ. 4)

$$v' = -\frac{M_0 x}{4LEI}(2L - 3x) \quad \leftarrow$$

DEFLECTION (FROM EQ. 5)

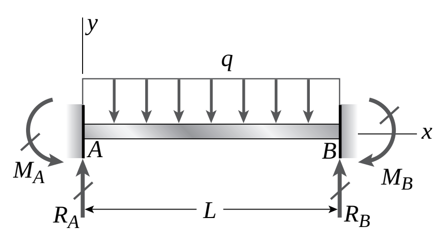
$$v = -\frac{M_0 x^2}{4LEI}(L - x) \quad \leftarrow$$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS



Problem 10.3-2 A fixed-end beam AB of length L supports a uniform load of intensity q (see figure).

Beginning with the second-order differential equation of the deflection curve (the bending-moment equation), obtain the reactions, shear forces, bending moments, slopes, and deflections of the beam. Construct the shear-force and bending-moment diagrams, labeling all critical ordinates.



Solution 10.3-2 Fixed-end beam (uniform load)

Select M_A as the redundant reaction.

REACTIONS (FROM SYMMETRY AND EQUILIBRIUM)

$$R_A = R_B = \frac{qL}{2} \qquad M_B = M_A$$

BENDING MOMENT (FROM EQUILIBRIUM)

$$M = R_A x - M_A - \frac{qx^2}{2} = -M_A + \frac{q}{2}(Lx - x^2) \qquad (1)$$

DIFFERENTIAL EQUATIONS

$$EIv'' = M = -M_A + \frac{q}{2}(Lx - x^2)$$

$$EIv' = -M_A x + \frac{q}{2}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right) + C_1 \qquad (2)$$

B.C. 1 $v'(0) = 0 \qquad \therefore C_1 = 0$

$$EIv = -\frac{M_A x^2}{2} + \frac{q}{2}\left(\frac{Lx^3}{6} - \frac{x^4}{12}\right) + C_2 \qquad (3)$$

B.C. 2 $v(0) = 0 \qquad \therefore C_2 = 0$

B.C. 3 $v(L) = 0 \qquad \therefore M_A = \frac{qL^2}{12}$

REACTIONS

$$R_A = R_B = \frac{qL}{2} \qquad M_A = M_B = \frac{qL^2}{12} \quad \leftarrow$$

SHEAR FORCE (FROM EQUILIBRIUM)

$$V = R_A - qx = \frac{q}{2}(L - 2x) \quad \leftarrow$$

BENDING MOMENT (FROM EQ. 1)

$$M = -\frac{q}{12}(L^2 - 6Lx + 6x^2) \qquad \leftarrow$$

SLOPE (FROM EQ. 2)

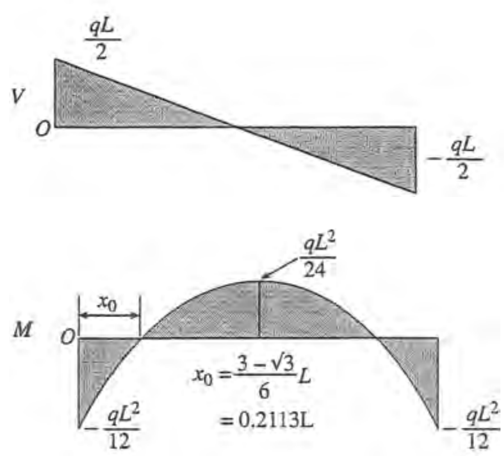
$$v' = -\frac{qx}{12EI}(L^2 - 3Lx + 2x^2) \qquad \leftarrow$$

DEFLECTION (FROM EQ. 3)

$$v = -\frac{qx^2}{24EI}(L - x)^2 \qquad \leftarrow$$

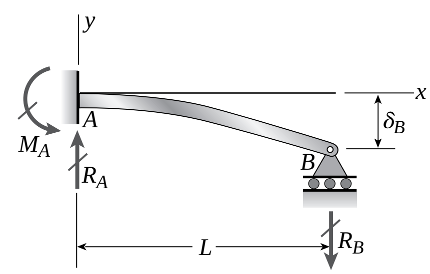
$$\delta_{max} = -v\left(\frac{L}{2}\right) = \frac{qL^4}{384EI}$$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS



Problem 10.3-3 A cantilever beam AB of length L has a fixed support at A and a roller support at B (see figure). The support at B is moved downward through a distance δ_B.

Using the fourth-order differential equation of the deflection curve (the load equation), determine the reactions of the beam and the equation of the deflection curve. (*Note:* Express all results in terms of the imposed displacement δ_B.)



Solution 10.3-3 Cantilever beam with imposed displacement δ_B

REACTIONS (FROM EQUILIBRIUM)

$R_A = R_B \quad (1) \qquad M_A = R_B L \quad (2)$

DIFFERENTIAL EQUATIONS

$$EIv'''' = -q = 0 \tag{3}$$

$$EIv''' = V = C_1 \tag{4}$$

$$EIv'' = M = C_1x + C_2 \tag{5}$$

$$EIv' = C_1x^2/2 + C_2x + C_3 \tag{6}$$

$$EIv = C_1x^3/6 + C_2x^2/2 + C_3x + C_4 \tag{7}$$

B.C. 1 $v(0) = 0 \qquad \therefore C_4 = 0$

B.C. 2 $v'(0) = 0 \qquad \therefore C_3 = 0$

B.C. 3 $v''(L) = 0 \qquad \therefore C_1L + C_2 = 0 \qquad (8)$

B.C. 4 $v(L) = -\delta_B$

$$\therefore C_1L + 3C_2 = -6EI\delta_B/L^2 \tag{9}$$

SOLVE EQUATIONS (8) AND (9):

$$C_1 = \frac{3EI\delta_B}{L^3} \qquad C_2 = -\frac{3EI\delta_B}{L^2}$$

SHEAR FORCE (EQ. 4)

$$V = \frac{3EI\delta_B}{L^3} \qquad R_A = V(0) = \frac{3EI\delta_B}{L^3}$$

Reactions (eqs. 1 and 2)

$$R_A = R_B = \frac{3EI\delta_B}{L^3}$$

$$M_A = R_B L = \frac{3EI\delta_B}{L^2} \quad \leftarrow$$

Deflection (from eq. 7):

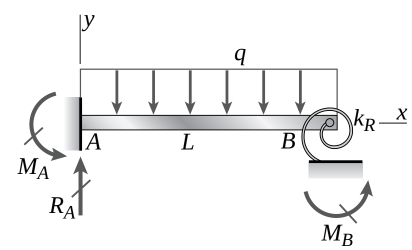
$$v = -\frac{\delta_B x^2}{2L^3}(3L - x) \quad \leftarrow$$

Slope (from eq. 6):

$$v' = -\frac{3\delta_B x}{2L^3}(2L - x)$$

Problem 10.3-4 A cantilever beam of length L and loaded by uniform load of intensity q has a fixed support at A and spring support at B with rotational stiffness k_R. A rotation at B, θ_B, results in a reaction moment $M_B = k_R \times \theta_B$.

Find rotation θ_B and displacement δ_B at end B. Use the second-order differential equation of the deflection curve to solve for displacements at end B.



Solution 10.3-4

q = intensity of uniform load

Equilibrium

$$R_A = qL \qquad (1)$$

$$M_A = \frac{qL^2}{2} - M_B \qquad (2)$$

$$M_B = k_R\theta_B \qquad (3)$$

Bending moment

$$M = R_A x - M_A - \frac{qx^2}{2}$$

Differential equation

$$EIv'' = M = R_A x - M_A - \frac{qx^2}{2}$$

$$EIv' = R_A\frac{x^2}{2} - M_A x - \frac{qx^3}{6} + C_1$$

$$EIv = R_A\frac{x^3}{6} - M_A\frac{x^2}{2} - \frac{qx^4}{24} + C_1 x + C_2$$

b.c. 1 $v'(0) = 0 \qquad \therefore\ C_1 = 0$

b.c. 2 $v(0) = 0 \qquad \therefore\ C_2 = 0$

b.c. 3 $v'(L) = \theta_B$

Substitute R_A and M_A from Eqs. (1) and (2):

$$\therefore\ EI\theta_B = qL\frac{L^2}{2} - \left(\frac{qL^2}{2} - k_R\theta_B\right)L - \frac{qL^3}{6}$$

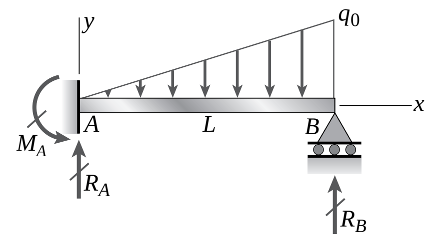
$$\therefore\ \theta_B = \frac{qL^3}{6(k_R L - EI)} \quad \leftarrow$$

$$\therefore\ EI\delta_B = qL\frac{L^3}{6} - \left(\frac{qL^2}{2} - k_R\theta_B\right)\frac{L^2}{2} - \frac{qL^4}{24}$$

$$\therefore\ \delta_B = \frac{1}{EI}\left(-\frac{1}{8}qL^4 + \frac{k_R qL^5}{12\,(k_R L - EI)}\right) \quad \leftarrow$$

Problem 10.3-5 A cantilever beam of length L and loaded by a triangularly distributed load of maximum intensity q_0 at B.

Use the fourth-order differential equation of the deflection curve to solve for reactions at A and B and also the equation of the deflection curve.



Solution 10.3-5

Triangular load $q = q_0 \dfrac{x}{L}$

DIFFERENTIAL EQUATION

$$EIv'''' = -q = -q_0 \frac{x}{L} \qquad (1)$$

$$EIv''' = -q_0 \frac{x^2}{2L} + C_1 \qquad (2)$$

$$EIv'' = M = -q_0 \frac{x^3}{6L} + C_1 x + C_2 \qquad (3)$$

$$EIv' = -q_0 \frac{x^4}{24L} + C_1 \frac{x^2}{2} + C_2 x + C_3 \qquad (4)$$

$$EIv = -q_0 \frac{x^5}{120L} + C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4 \qquad (5)$$

B.C. 1 $v''(L) = 0 \quad \therefore\ C_1 L + C_2 = q_0 \dfrac{L^2}{6}$ (6)

B.C. 2 $v'(0) = 0 \quad \therefore\ C_3 = 0$

B.C. 3 $v(0) = 0 \quad \therefore\ C_4 = 0$

B.C. 4 $v(L) = 0 \quad \therefore\ C_1 \dfrac{L}{3} + C_2 = q_0 \dfrac{L^2}{60}$ (7)

Solve Eqs. (6) and (7):

$$C_1 = \frac{9}{40} q_0 L$$

$$C_2 = -\frac{7}{120} q_0 L^2$$

SHEAR FORCE (EQ. 2)

$$V = -q_0 \frac{x^2}{2L} + \frac{9}{40} q_0 L$$

REACTIONS

$$R_A = V(0) = \frac{9}{40} q_0 L \quad \leftarrow$$

$$R_B = -V(L) = \frac{11}{40} q_0 L \quad \leftarrow$$

FROM EQUILIBRIUM

$$M_A = \frac{7}{120} q_0 L^2 \quad \leftarrow$$

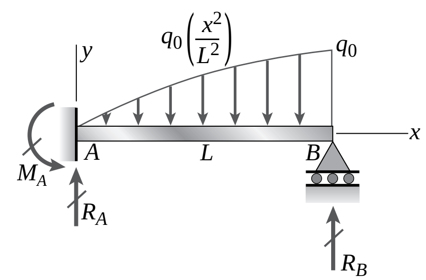
DEFLECTION CURVE (EQ. 5)

$$EIv = -q_0 \frac{x^5}{120L} + \frac{9}{40} q_0 L \frac{x^3}{6} - \frac{7}{20} q_0 L^2 \frac{x^2}{2} \quad \text{or}$$

$$v = \frac{1}{240LEI}(-2q_0 x^5 + 9q_0 L x^3 - 7q_0 L^2 x^2) \quad \leftarrow$$

Problem 10.3-6 A propped cantilever beam of length L is loaded by a parabolically distributed load with maximum intensity q_0 at B.

(a) Use the fourth-order differential equation of the deflection curve to solve for reactions at A and B and also the equation of the deflection curve.
(b) Repeat (a) if the parabolic load is replaced by $q_0 \sin(\pi x/2L)$.



SOLUTION 10.3-6

(a) Parabolic load $\quad q = q_0 \dfrac{x^2}{L^2}$

DIFFERENTIAL EQUATION

$$EIv'''' = -q = -q_0 \frac{x^2}{L^2} \quad (1)$$

$$EIv''' = -q_0 \frac{x^3}{3L^2} + C_1 \quad (2)$$

$$EIv'' = M = -q_0 \frac{x^4}{12L^2} + C_1 x + C_2 \quad (3)$$

$$EIv' = -q_0 \frac{x^5}{60L^2} + C_1 \frac{x^2}{2} + C_2 x + C_3 \quad (4)$$

$$EIv = -q_0 \frac{x^6}{360L^2} + C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4 \quad (5)$$

B.C. 1 $v''(L) = 0 \qquad \therefore\ C_1 L + C_2 = q_0 \dfrac{L^2}{12} \quad (6)$

B.C. 2 $v'(0) = 0 \qquad \therefore\ C_3 = 0$

B.C. 3 $v(0) = 0 \qquad \therefore\ C_4 = 0$

B.C. 4 $v(L) = 0 \qquad \therefore\ C_1 L + 3C_2 = q_0 \dfrac{L^2}{60} \quad (7)$

Solve Eqs. (6) and (7):

$$C_1 = \frac{7}{60} q_0 L \qquad C_2 = -\frac{1}{30} q_0 L^2$$

SHEAR FORCE (EQ. 2)

$$V = -q_0 \frac{x^3}{3L^2} + \frac{7}{60} q_0 L$$

REACTIONS

$$R_A = V(0) = \frac{7}{60} q_0 L \quad \leftarrow$$

$$R_B = -V(L) = \frac{13}{60} q_0 L \quad \leftarrow$$

FROM EQUILIBRIUM

$$M_A = \frac{1}{30} q_0 L^2 \quad \leftarrow$$

DEFLECTION CURVE (EQ. 5)

$$EIv = -q_0 \frac{x^6}{360L^2} + \frac{7}{60} q_0 L \frac{x^3}{6} - \frac{1}{30} q_0 L^2 \frac{x^2}{2} \quad \text{or}$$

$$v = \frac{q_0}{360L^2 EI}\left(-x^6 + 7L^3x^3 - 6q_0 L^4 x^2\right) \quad \leftarrow$$

(b) Loading $q = q_0 \sin\left(\dfrac{\pi x}{2L}\right)$

DIFFERENTIAL EQUATION

$$EIv'''' = -q = -q_0 \sin\left(\frac{\pi x}{2L}\right) \quad (1)$$

$$EIv''' = q_0 \frac{2L}{\pi} \cos\left(\frac{\pi x}{2L}\right) + C_1 \quad (2)$$

$$EIv'' = M = q_0 \left(\frac{2L}{\pi}\right)^2 \sin\left(\frac{\pi x}{2L}\right) + C_1 x + C_2 \quad (3)$$

$$EIv' = -q_0\left(\frac{2L}{\pi}\right)^3\cos\left(\frac{\pi x}{2L}\right) + C_1\frac{x^2}{2}$$
$$+ C_2x + C_3 \qquad (4)$$

$$EIv = -q_0\left(\frac{2L}{\pi}\right)^4\sin\left(\frac{\pi x}{2L}\right) + C_1\frac{x3}{6} + C_2\frac{x^2}{2}$$
$$+ C_3x + C_4 \qquad (5)$$

B.C. 1 $v''(L) = 0$

$$\therefore C_1L + C_2 = -q_0\frac{4L^2}{\pi^2} \qquad (6)$$

B.C. 2 $v'(0) = 0$ $\quad \therefore C_3 = q_0\left(\frac{2L}{\pi}\right)^3$

B.C. 3 $v(0) = 0$ $\quad \therefore C_4 = 0$

B.C. 4 $v(L) = 0$

$$\therefore C_1L + 3C_2 = -q_0\left(\frac{2L}{\pi}\right)^3L\frac{6}{L^2}$$
$$+ q_0\left(\frac{2L}{\pi}\right)^4\frac{6}{L^2} \qquad (7)$$

Solve Eqs. (6) and (7):

$$C_1 = -6q_0L\frac{\pi^2 - 4\pi + 8}{\pi^4}$$

$$C_2 = 2q_0L^2\frac{\pi^2 - 12\pi + 24}{\pi^4}$$

SHEAR FORCE (EQ. 2)

$$V = q_0\frac{2L}{\pi}\cos\left(\frac{\pi x}{2L}\right) - 6q_0L\frac{\pi^2 - 4\pi + 8}{\pi^4}$$

REACTIONS

$$R_A = V(0) = 0.31q_0L$$
$$= \left(\frac{2}{\pi} - 6\frac{\pi^2 - 4\pi + 8}{\pi^4}\right)q_0L \quad \leftarrow$$

$$R_B = -V(L) = 0.327q_0L$$
$$= \left(6\frac{\pi^2 - 4\pi + 8}{\pi^4}\right)q_0L \quad \leftarrow$$

From equilibrium

$$M_A = -C_2 = -2q_0L^2\frac{\pi^2 - 12\pi + 24}{\pi^4} \quad \leftarrow$$

DEFLECTION CURVE (EQ. 5)

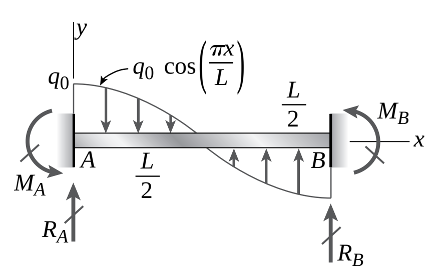
$$EIv = -q_0\left(\frac{2L}{\pi}\right)^4\sin\left(\frac{\pi x}{2L}\right)$$
$$+ C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4\text{, or}$$

$$v = \frac{1}{EI}\left[-q_0\left(\frac{2L}{\pi}\right)^4\sin\left(\frac{\pi x}{2L}\right)\right.$$
$$-6q_0L\frac{\pi^2 - 4\pi + 8}{\pi^4}\frac{x^3}{6}$$
$$+2q_0L^2\cdot\frac{\pi^2 - 12\pi + 24}{\pi^4}\frac{x^2}{2}$$
$$\left.+q_0\left(\frac{2L}{\pi}\right)^3x\right] \quad \leftarrow$$

Problem 10.3-7 A fixed-end beam of length L is loaded by distributed load in the form of a cosine curve with maximum intensity q_0 at A.

(a) Use the fourth-order differential equation of the deflection curve to solve for reactions at A and B and also the equation of the deflection curve.

(b) Repeat (a) using the distributed load $q_0\sin(\pi x/L)$.



Solution 10.3-7

(a) Loading $q = q_0 \cos\left(\frac{\pi x}{L}\right)$

DIFFERENTIAL EQUATION

$$EIv'''' = -q = -q_0 \cos\left(\frac{\pi x}{L}\right) \quad (1)$$

$$EIv''' = -q_0 \frac{L}{\pi} \sin\left(\frac{\pi x}{L}\right) + C_1 \quad (2)$$

$$EIv'' = M = q_0\left(\frac{L}{\pi}\right)^2 \cos\left(\frac{\pi x}{L}\right) + C_1 x + C_2 \quad (3)$$

$$EIv' = q_0\left(\frac{L}{\pi}\right)^3 \sin\left(\frac{\pi x}{L}\right) + C_1 \frac{x^2}{2} + C_2 x + C_3 \quad (4)$$

$$EIv = -q_0\left(\frac{L}{\pi}\right)^4 \cos\left(\frac{\pi x}{L}\right) + C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4 \quad (5)$$

B.C. 1. $v'(0) = 0 \qquad \therefore C_3 = 0$

B.C. 2. $v(0) = 0 \qquad \therefore C_4 = q_0\left(\frac{L}{\pi}\right)^4$

B.C. 3. $v'(L) = 0 \qquad \therefore C_1 \frac{L}{2} + C_2 = 0$

B.C. 4. $v(L) = 0 \qquad \therefore C_1 \frac{L^3}{6} + \left(-C_1 \frac{L}{2}\right)\frac{L^2}{2} = -2q_0\left(\frac{L}{\pi}\right)^4 \quad (6)$

SOLVE EQS. (6):

$$C_1 = \frac{24}{\pi^4} q_0 L$$

$$C_2 = -\frac{12}{\pi^4} q_0 L^2$$

SHEAR FORCE (EQ. 2)

$$V = -\frac{q_0 L}{\pi} \sin\left(\frac{\pi x}{L}\right) + \frac{24}{\pi^4} q_0 L$$

REACTIONS

$$R_A = V(0) = \frac{24}{\pi^4} q_0 L \quad \leftarrow$$

$$R_B = -V(L) = -\frac{24}{\pi^4} q_0 L \quad \leftarrow$$

From equilibrium

$$M_A = \left(\frac{12}{\pi^4} - \frac{1}{\pi^2}\right) q_0 L^2$$

(counter-clockwise) $\leftarrow$

$$M_B = \left(\frac{12}{\pi^4} - \frac{1}{\pi^2}\right) q_0 L^2$$

(counter-clockwise) $\leftarrow$

DEFLECTION CURVE (EQ. 5)

$$EIv = -q_0\left(\frac{L}{\pi}\right)^4 \cos\left(\frac{\pi x}{L}\right) + \frac{24}{\pi^4} q_0 L \frac{x^3}{6} - \frac{12}{\pi^4} q_0 L^2 \frac{x^2}{2} + q_0\left(\frac{L}{\pi}\right)^4, \text{ or}$$

$$v = \frac{1}{\pi^4 EI}\left[-q_0 L^4 \cos\left(\frac{\pi x}{L}\right) + 4q_0 L x^3 - 6q_0 L^2 x^2 + q_0 L^4\right] \quad \leftarrow$$

(b) Loading $q = q_0 \sin \pi x/L$

FROM SYMMETRY: $R_A = R_B \qquad M_A = M_B$

DIFFERENTIAL EQUATIONS

$$EIv'''' = -q = -q_0 \sin \pi x/L \quad (1)$$

$$EIv''' = V = \frac{q_0 L}{\pi} \cos \frac{\pi x}{L} + C_1 \quad (2)$$

$$EIv'' = M = \frac{q_0 L^2}{\pi^2} \sin \frac{\pi x}{L} + C_1 x + C_2 \quad (3)$$

$$EIv' = -\frac{q_0 L^3}{\pi^3} \cos \frac{\pi x}{L} + C_1 \frac{x^2}{2} C_2 x + C_3 \quad (4)$$

$$EIv = -\frac{q_0 L^4}{\pi^4}\sin\frac{\pi x}{L} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4 \quad (5)$$

B.C. 1 From symmetry, $V\left(\frac{L}{2}\right) = 0 \qquad \therefore C_1 = 0$

B.C. 2 $v'(0) = 0 \qquad \therefore C_3 = q_0\left(\frac{L}{\pi}\right)^3$

B.C. 3 $v'(L) = 0 \qquad \therefore C_2 = -2q_0\frac{L^2}{\pi^3}$

B.C. 4 $v(0) = 0 \qquad \therefore C_4 = 0$

SHEAR FORCE (EQ. 2)

$$V = \frac{q_0 L}{\pi}\cos\frac{\pi x}{L} \qquad R_A = V(0) = \frac{q_0 L}{\pi} \quad \leftarrow$$

$$R_B = R_A = \frac{q_0 L}{\pi} \quad \leftarrow$$

BENDING MOMENT (EQ. 3)

$$M = \frac{q_0 L^2}{\pi^3}\left(\pi\sin\frac{\pi x}{L} - 2\right)$$

$$M_A = -M(0) = \frac{2q_0 L^2}{\pi^3}$$

$$M_B = M_A = \frac{2q_0 L^2}{\pi^3} \quad \leftarrow$$

DEFLECTION CURVE (FROM EQ. 5)

$$EIv = -\frac{q_0 L^2}{\pi^4}\sin\frac{\pi x}{L} - \frac{q_0 L^2 x^2}{\pi^3} + \frac{q_0 L^3 x}{\pi^3}$$

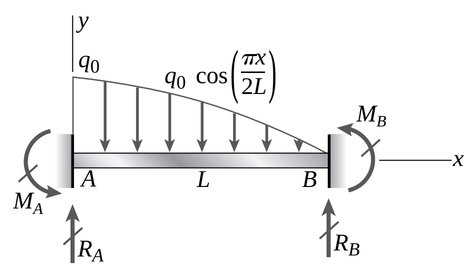
or

$$v = -\frac{q_0 L^2}{\pi^4 EI}\left(L^2\sin\frac{\pi x}{L} + \pi x^2 - \pi L x\right) \quad \leftarrow$$

Problem 10.3-8 A fixed-end beam of length L is loaded by a distributed load in the form of a cosine curve with maximum intensity q_0 at A.

(a) Use the fourth-order differential equation of the deflection curve to solve for reactions at A and B and also the equation of the deflection curve.

(b) Repeat (a) if the distributed load is now $q_0(1 - x^2/L^2)$.



Solution 10.3-8

(a) Loading $q = q_0\cos\left(\frac{\pi x}{2L}\right)$

DIFFERENTIAL EQUATION

$$EIv'''' = -q = -q_0\cos\left(\frac{\pi x}{2L}\right) \quad (1)$$

$$EIv''' = -q_0\frac{2L}{\pi}\sin\left(\frac{\pi x}{2L}\right) + C_1 \quad (2)$$

$$EIv'' = M = q_0\left(\frac{2L}{\pi}\right)^2\cos\left(\frac{\pi x}{2L}\right) + C_1 x + C_2 \quad (3)$$

$$EIv' = q_0\left(\frac{2L}{\pi}\right)^3\sin\left(\frac{\pi x}{2L}\right) + C_1\frac{x^2}{2} + C_2 x + C_3 \quad (4)$$

$$EIv = -q_0\left(\frac{2L}{\pi}\right)^4\cos\left(\frac{\pi x}{2L}\right) + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4 \quad (5)$$

B.C. 1 $v'(0) = 0 \qquad \therefore C_3 = 0$

B.C. 2 $v(0) = 0 \qquad \therefore C_4 = q_0\left(\frac{2L}{\pi}\right)^4$

B.C. 3 $v'(L) = 0$

$$\therefore C_1\frac{L^2}{2} + C_2L = -q_0\left(\frac{2L}{\pi}\right)^3 \qquad (6)$$

B.C. 4 $v(L) = 0$

$$\therefore C_1\frac{L^3}{6} + C_2\frac{L^2}{2} = -q_0\left(\frac{2L}{\pi}\right)^4 \qquad (7)$$

Solve Eqs. (6) and (7):

$$C_1 = \frac{48(4 - \pi)}{\pi^4} q_0 L$$

$$C_2 = -\frac{16(6 - \pi)}{\pi^4} q_0 L^2$$

SHEAR FORCE (EQ. 2)

$$V = -q_0\frac{2L}{\pi}\sin\left(\frac{\pi x}{2L}\right) + \frac{48(4 - \pi)}{\pi^4} q_0 L$$

REACTIONS

$$R_A = V(0) = \frac{48(4 - \pi)}{\pi^4} q_0 L \qquad \leftarrow$$

$$R_B = -V(L) = \left(\frac{2}{\pi} - \frac{48(4 - \pi)}{\pi^4}\right) q_0 L \qquad \leftarrow$$

From equilibrium

$$M_A = -q_0\left(\frac{2L}{\pi}\right)^2 + \frac{16(6 - \pi)}{\pi^4} q_0 L^2 \qquad \leftarrow$$

$$M_B = -\frac{32(\pi - 3)}{\pi^4} q_0 L^2 \qquad \leftarrow$$

DEFLECTION CURVE (EQ. 5)

$$EIv = -q_0\left(\frac{2L}{\pi}\right)^4 \cos\left(\frac{\pi x}{2L}\right) + \frac{48(4 - \pi)}{\pi^4} \times q_0 L\frac{x^3}{6} - \frac{16(6 - \pi)}{\pi^4} \times q_0 L^2\frac{x^2}{2} + q_0\left(\frac{2L}{\pi}\right)^4, \text{ or}$$

$$v = \frac{1}{\pi^4 EI}\left[-16\, q_0 L^4 \cos\left(\frac{\pi x}{2L}\right) + 8(4 - \pi)\, q_0 L x^3 - 8(6 - \pi)\, q_0 L^2 x^2 + 16 q_0 L^4\right] \qquad \leftarrow$$

(b) Loading $q = q_0\left(1 - \frac{x^2}{L^2}\right)$

DIFFERENTIAL EQUATION

$$EIv'''' = -q = -q_0\left(1 - \frac{x^2}{L^2}\right) \qquad (1)$$

$$EIv''' = -q_0\left(x - \frac{x^3}{3L^2}\right) + C_1 \qquad (2)$$

$$EIv'' = M = -q_0\left(\frac{x^2}{2} - \frac{x^4}{12L^2}\right) + C_1x + C_2 \qquad (3)$$

$$EIv' = -q_0\left(\frac{x^3}{6} - \frac{x^5}{60L^2}\right) + C_1\frac{x^2}{2} + C_2 x + C_3 \qquad (4)$$

$$EIv = -q_0\left(\frac{x^4}{24} - \frac{x^6}{360L^2}\right) + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4 \qquad (5)$$

B.C. 1 $v'(0) = 0 \qquad \therefore C_3 = 0$

B.C. 2 $v(0) = 0 \qquad \therefore C_4 = 0$

B.C. 3 $v'(L) = 0$

$$\therefore C_1L + 2C_2 = \frac{3}{10} q_0 L^2 \qquad (6)$$

B.C. 4 $v(L) = 0$

$$\therefore C_1L + 3C_2 = \frac{7}{30} q_0 L^2 \qquad (7)$$

Solve Eqs. (6) and (7):

$$C_1 = \frac{13}{30} q_0 L \qquad C_2 = -\frac{1}{15} q_0 L^2$$

SHEAR FORCE (EQ. 2)

$$V = -q_0\left(x - \frac{x^3}{3L^2}\right) + \frac{13}{30} q_0 L$$

REACTIONS

$$R_A = V(0) = \frac{13}{30} q_0 L \quad \leftarrow$$

$$R_B = -V(L) = \frac{7}{30} q_0 L \quad \leftarrow$$

From equilibrium

$$M_A = -q_0\left(\frac{x^2}{2} - \frac{x^4}{12L^2}\right) + \frac{13}{30} q_0 Lx - \frac{1}{15} q_0 L^2$$

$$M_A = \frac{1}{15} q_0 L^2 \text{ (counter-clockwise)} \quad \leftarrow$$

$$M_B = -\frac{1}{20} q_0 L^2 \text{ (clockwise)} \quad \leftarrow$$

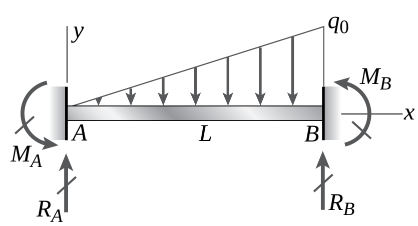
DEFLECTION CURVE (EQ. 5)

$$EIv = -q_0\left(\frac{x^4}{24} - \frac{x^6}{360L^2}\right) + \frac{13}{30} q_0 L \frac{x^3}{6} - \frac{1}{15} q_0 L^2 \frac{x^2}{2}, \text{ or}$$

$$v = \frac{q_0}{360L^2EI}[x^6 - 15L^2x^4 + 26L^3x^3 - 12L^4x^2] \quad \leftarrow$$

Problem 10.3-9 A fixed-end beam of length L is loaded by triangularly distributed load of maximum intensity q_0 at B.

Use the fourth-order differential equation of the deflection curve to solve for reactions at A and B and also the equation of the deflection curve.



Solution 10.3-9

Triangular load $q = q_0\frac{x}{L}$

DIFFERENTIAL EQUATION

$$EIv'''' = -q = -q_0 \frac{x}{L} \tag{1}$$

$$EIv''' = -q_0 \frac{x^2}{2L} + C_1 \tag{2}$$

$$EIv'' = M = -q_0 \frac{x^3}{6L} + C_1x + C_2 \tag{3}$$

$$EIv' = -q_0\frac{x^4}{24L} + C_1\frac{x^2}{2} + C_2x + C_3 \tag{4}$$

$$EIv = -q_0\frac{x^5}{120L} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4 \tag{5}$$

B.C. 1 $v'(0) = 0 \quad \therefore C_3 = 0$

B.C. 2 $v(0) = 0 \quad \therefore C_4 = 0$

B.C. 3 $v'(L) = 0 \quad \therefore C_1L + 2C_2 = \frac{q_0 L^2}{12}$ (6)

B.C. 4 $v(L) = 0 \quad \therefore C_1L + 3C_2 = \frac{q_0 L^2}{20}$ (7)

Solve Eqs. (6) and (7):

$$C_1 = \frac{3}{20} q_0 L$$

$$C_2 = -\frac{1}{30} q_0 L^2$$

SHEAR FORCE (EQ. 2)

$$V = -q_0\frac{x^2}{2L} + \frac{3}{20} q_0 L$$

REACTIONS

$$R_A = V(0) = \frac{3}{20} q_0 L \quad \leftarrow$$

$$R_B = -V(L) = \frac{7}{20} q_0 L \qquad \leftarrow$$

From equilibrium

$$M_A = \frac{1}{30} q_0 L^2 \qquad \leftarrow$$

DEFLECTION CURVE (EQ. 5)

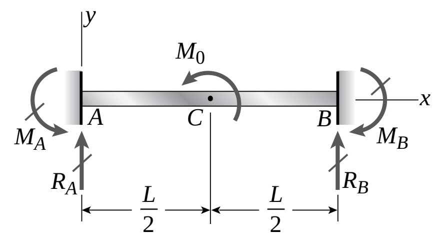
$$EIv = -q_0 \frac{x^5}{120L} + \frac{3}{20} q_0 L \frac{x^3}{6} - \frac{1}{30} q_0 L^2 \frac{x^2}{2} \qquad \text{or}$$

$$v = \frac{1}{120LEI}(-q_0 x^5 + 3q_0 L^2 x^3 - 2q_0 L^3 x^2) \qquad \leftarrow$$

Problem 10.3-10 A counterclockwise moment M_0 acts at the midpoint of a fixed-end beam ACB of length L (see figure).

Beginning with the second-order differential equation of the deflection curve (the bending-moment equation), determine all reactions of the beam and obtain the equation of the deflection curve for the left-hand half of the beam.

Then construct the shear-force and bending-moment diagrams for the entire beam, labeling all critical ordinates. Also, draw the deflections curve for the entire beam.



Solution 10.3-10 Fixed-end beam (M_0 = applied load)

Beam is symmetric; load is antisymmetric.

Therefore, $R_A = -R_B \qquad M_A = -M_B \qquad \delta_C = 0$

DIFFERENTIAL EQUATION $(0 \le x \le L/2)$

$$Elv'' = M = R_A x - M_A \qquad (1)$$

$$Elv' = R_A \frac{x^2}{2} - M_A x + C_1 \qquad (2)$$

$$Elv = R_A \frac{x^3}{6} - M_A \frac{x^2}{2} + C_1 x + C_2 \qquad (3)$$

B.C. 1 $v'(0) = 0 \qquad \therefore\ C_1 = 0$

B.C. 2 $v(0) = 0 \qquad \therefore\ C_2 = 0$

B.C. 3 $v\left(\frac{L}{2}\right) = 0$

$$\therefore\ M_A = \frac{R_A L}{6} \quad \text{Also, } M_B = \frac{-R_A L}{6}$$

EQUILIBRIUM (OF ENTIRE BEAM)

$$\sum M_B = 0 \qquad M_A + M_0 - M_B - R_A L = 0$$

$$\text{or, } \frac{R_A L}{6} + M_0 + \frac{R_A L}{6} - R_A L = 0$$

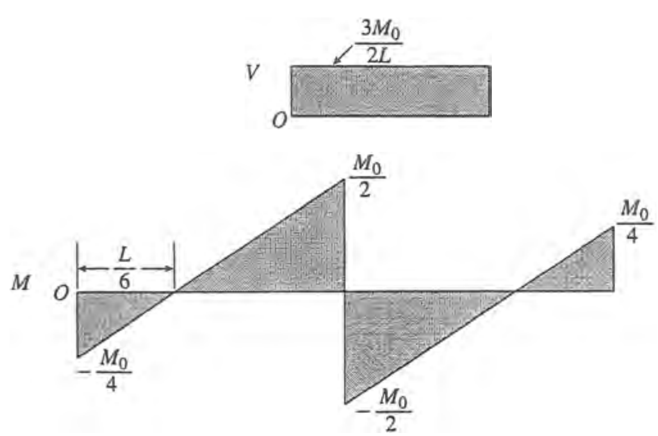
$$\therefore\ R_A = -R_B = \frac{3M_0}{2L} \qquad \leftarrow$$

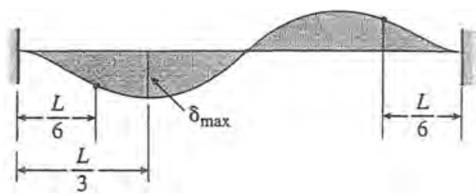
$$M_A = \frac{R_A L}{6} \qquad \therefore\ M_A = -M_B = \frac{M_0}{4} \qquad \leftarrow$$

DEFLECTION CURVE (EQ. 3)

$$v = -\frac{M_0 x^2}{8LEI}(L - 2x) \qquad \left(0 \le x \le \frac{L}{2}\right) \qquad \leftarrow$$

DIAGRAMS

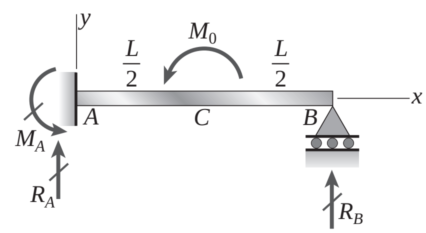




$$\delta_{max} = \frac{M_0 L^2}{216EI}$$

At point of inflection: $\delta = \delta_{max}/2$

Problem 10.3-11 A propped cantilever beam of length L is loaded by a concentrated moment M_0 at midpoint C. Use the second-order differential equation of the deflection curve to solve for reactions at A and B. Draw shear-force and bending-moment diagrams for the entire beam. Also find the equations of the deflection curves for both halves find the equations of the deflection curves for both halves of the beam, and draw the deflection curve for the entire beam.



Solution 10.3-11

Equilibrium

$$R_A = -R_B \qquad (1)$$

$$M_A = -M_0 - R_B L \qquad (2)$$

Bending moments (from equilibrium)

$$M = R_A x - M_A = -R_B x + M_0 + R_B L \qquad \left(0 \le x \le \frac{L}{2}\right)$$

$$M = R_B (L - x) \qquad \left(\frac{L}{2} \le x \le L\right)$$

Differential equations $(0 \le x \le L/2)$

$$EIv'' = M = -R_B x + M_0 + R_B L \qquad (3)$$

$$EIv' = -R_B \frac{x^2}{2} + M_0 x + R_B L x + C_1 \qquad (4)$$

$$EIv = -R_B \frac{x^3}{6} + M_0 \frac{x^2}{2} + R_B L \frac{x^2}{2} + C_1 x + C_2 \qquad (5)$$

B.C. 1 $v'(0) = 0 \qquad \therefore C_1 = 0$

B.C. 2 $v(0) = 0 \qquad \therefore C_2 = 0$

Differential equations $(L/2 \le x \le L)$

$$EIv'' = M = R_B (L - x) \qquad (6)$$

$$EIv' = R_B L x - R_B \frac{x^2}{2} + C_3 \qquad (7)$$

$$EIv = -R_B L \frac{x^2}{2} - R_B \frac{x^3}{6} + C_3 x + C_4 \qquad (8)$$

B.C. 3 $v(L) = 0 \qquad \therefore C_3 L + C_4 = -\frac{R_B L^3}{3} \qquad (9)$

B.C. 4 continuity condition at point C

At $x = \frac{L}{2}$: $(v')_{left} = (v')_{right}$

$$-R_B \frac{L^2}{8} + M_0 \frac{L}{2} + R_B L \frac{L}{2}$$

$$= R_B L \frac{L}{2} - R_B \frac{L^2}{8} + C_3$$

$$C_3 = M_0 \frac{L}{2}$$

From eq. (9): $C_4 = -\frac{R_B L^3}{3} - M_0 \frac{L^2}{2}$

B.C. 5 continuity condition at point C

At $x = \frac{L}{2}$: $(v)_{left} = (v)_{right}$

$$EIv = -R_B\frac{x^3}{6} + M_0\frac{x^2}{2} + R_BL\frac{x^2}{2} + C_1x + C_2$$

$$-R_B\frac{L^3}{48} + M_0\frac{L^2}{8} + R_BL\frac{L^2}{8} = R_BL\frac{L^2}{8}$$

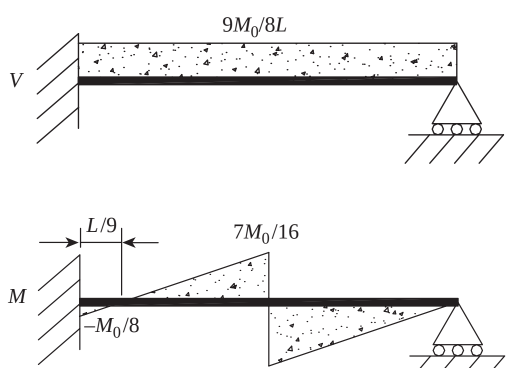
$$-R_B\frac{L^3}{48} + M_0\frac{L}{2}\frac{L}{2} - R_B\frac{L^3}{3} - M_0\frac{L^2}{2}$$

$$R_B = -\frac{9}{8}\frac{M_0}{L} \quad \leftarrow$$

From eq. (1) $\quad R_A = \dfrac{9}{8}\dfrac{M_0}{L} \quad \leftarrow$

From eq. (2) $\quad M_A = -M_0 + \dfrac{9}{8}\dfrac{M_0}{L}L = \dfrac{1}{8}\dfrac{M_0}{L} \quad \leftarrow$

Shear force and bending moment diagrams



Deflection curve for $(0 \le x \le L/2)$

$$EIv = -R_B\frac{x^3}{6} + M_0\frac{x^2}{2} + R_BL\frac{x^2}{2} + C_1x + C_2$$

$$v = \frac{1}{EI}\left(\frac{9M_0}{48L}x^3 - \frac{M_0}{16}x^2\right) \quad \left(0 \le x \le \frac{L}{2}\right) \quad \leftarrow$$

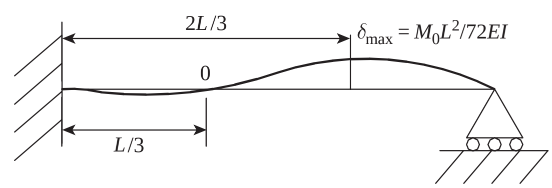
Deflection curve for $(L/2 \le x \le L)$

$$EIv = R_BL\frac{x^2}{2} - R_B\frac{x^3}{6} + C_3x + C_4$$

$$EIv = R_BL\frac{x^2}{2} - R_B\frac{x^3}{6} + M_0\frac{L}{2}x + \left(-\frac{R_BL^3}{3} - M_0\frac{L^2}{2}\right)$$

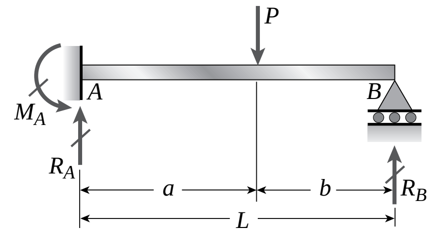
$$v = \frac{1}{EI}\left(\frac{9M_0}{48L}x^3 - \frac{9M_0}{16}x^2 + \frac{M_0L}{2}x - \frac{M_0L^2}{8}\right) \quad \left(\frac{L}{2} \le x \le L\right) \quad \leftarrow$$

Deflection curve



Method of Superposition

The problems for Section 10.4 are to be solved by the method of superposition. All beams have constant flexural rigidity EI unless otherwise stated. When drawing shear-force and bending-moment diagrams, be sure to label all critical ordinates, including maximum and minimum values.



Problem 10.4-1 A proposed cantilever beam AB of length L carries a concentrated load P acting at the position shown in the figure.

Determine the reactions R_A, R_B, and M_A for this beam. Also, draw the shear-force and bending-moment diagrams, labeling all critical ordinates.

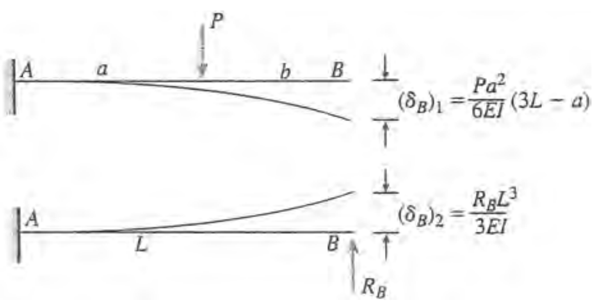
Solution 10.4-1 Propped cantilever beam

Select R_B as redundant.

EQUILIBRIUM

$R_A = P - R_B \qquad M_A = Pa - R_B L$

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS



COMPATIBILITY

$$\delta_B = (\delta_B)_1 - (\delta_B)_2 = 0$$

$$\delta_B = \frac{Pa^2}{6EI}(3L - a) - \frac{R_B L^3}{3EI} = 0$$

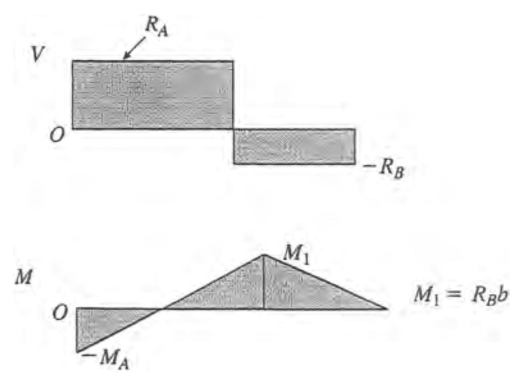
$$R_B = \frac{Pa^2}{2L^3}(3L - a) \qquad \leftarrow$$

OTHER REACTIONS (FROM EQUILIBRIUM)

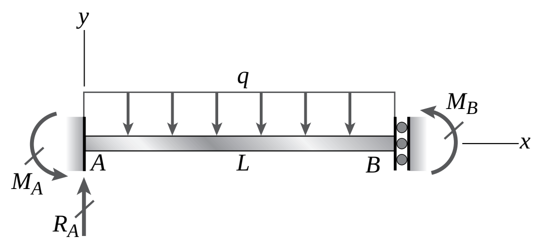
$$R_A = \frac{Pb}{2L^3}(3L^2 - b^2)$$

$$M_A = \frac{Pab}{2L^2}(L + b) \qquad \leftarrow$$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS



Problem 10.4-2 A beam with a guided support at B is loaded by a uniformly distributed load with intensity q. Use the method of superposition to solve for all reactions. Also draw shear-force and bending-moment diagrams, labeling all critical ordinates.



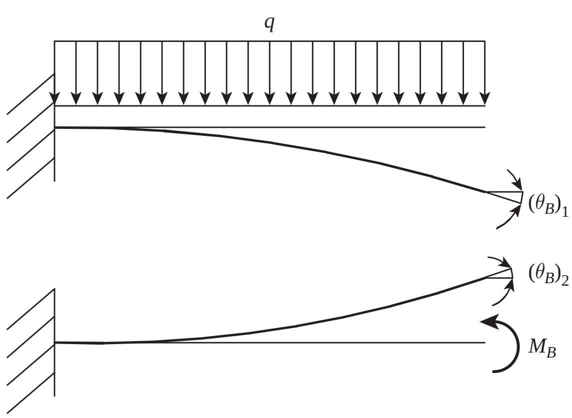
Solution 10.4-2

Select M_B as redundant.

EQUILIBRIUM

$$R_A = qL \qquad (1) \qquad\qquad M_A = \frac{qL^2}{2} - M_B \qquad (2)$$

RELEASED STRUCTURE AND FORCE OR MOMENT-ROTATION RELATIONS



$(\theta_B)_1$ = Rotation at B due to uniform load q

$$(\theta_B)_1 = \frac{qL^3}{6EI}$$

$(\theta_B)_2$ = Rotation at B due to Moment M_B

$$(\theta_B)_2 = \frac{M_B L}{EI}$$

FROM COMPATIBILITY EQUATION

$$(\theta_B)_1 = (\theta_B)_2 \qquad \therefore\ M_B = \frac{qL^2}{6}$$

FROM EQUILIBRIUM EQS.

$$R_A = qL \qquad M_A = \frac{qL^2}{2} - M_B = \frac{qL^2}{3}$$

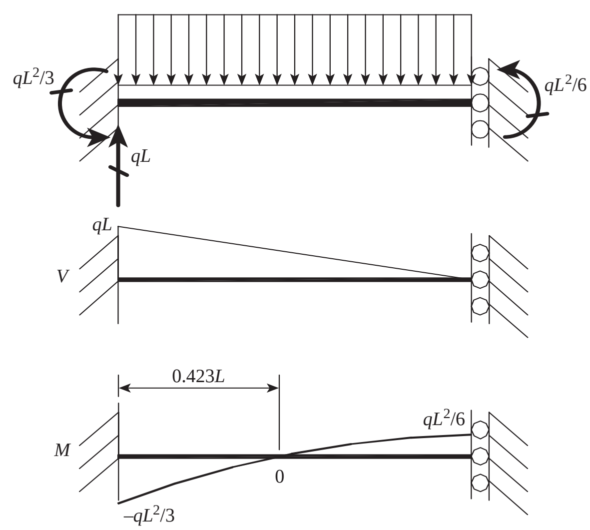
SHEAR FORCE

$$V = R_A - qx = q(L - x)$$

BENDING MOMENTS

$$M = R_A x - M_A - \frac{qx^2}{2} = -\frac{qx^2}{2} + qLx - \frac{qL^2}{3}$$

SHEAR FORCE AND BENDING MOMENT DIAGRAMS



Another solution by the 2nd order differential equation.

DIFFERENTIAL EQUATIONS

$$EIv'' = M = R_A x - M_A - \frac{qx^2}{2}$$

$$EIv' = R_A\frac{x^2}{2} - M_A x - \frac{qx^3}{6} + C_1$$

$$EIv = R_A\frac{x^3}{6} - M_A\frac{x^2}{2} - \frac{qx^4}{24} + C_1 x + C_2$$

B.C. 1 $v'(0) = 0$ $\quad \therefore C_1 = 0$

B.C. 2 $v'(0) = 0$ $\quad \therefore C_2 = 0$

B.C. 3 $v'(L) = 0$ $\quad M_A L = qL\frac{L^2}{2} - \frac{qL^3}{6} = \frac{qL^3}{3}$

$$\therefore M_A = \frac{qL^2}{3} \quad \leftarrow$$

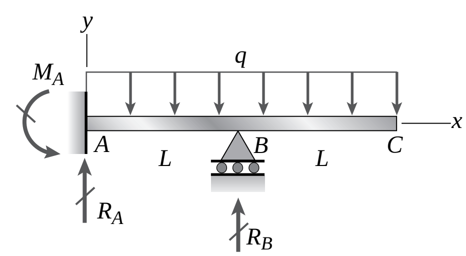
$$M_B = \frac{qL^2}{2} - \frac{qL^2}{3} = \frac{qL^2}{6} \quad \leftarrow$$

DEFLECTION CURVE

$$EIv = R_A\frac{x^3}{6} - M_A\frac{x^2}{2} - \frac{qx^4}{24} = qL\,\frac{x^3}{6} - \frac{qL^2}{3}\,\frac{x^2}{2} - \frac{qx^4}{24}$$

or $v = \dfrac{q}{24EI}(-x^4 + 4Lx^3 - 4L^2x^2) \quad \leftarrow$

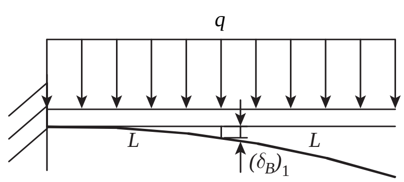
Problem 10.4-3 A propped cantilever beam of length $2L$ with support at B is loaded by a uniformly distributed load with intensity q. Use the method of superposition to solve for all reactions. Also draw shear-force and bending-moment diagrams, labeling all critical ordinates.

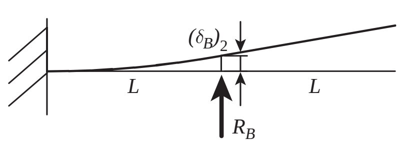


Solution 10.4-3

Select R_B as redundant.

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS





$(\delta_B)_1$ = Deflection at B due to uniform load q

$$(\delta_B)_1 = \frac{17qL^4}{24EI}$$

$(\delta_B)_2$ = Deflection at B due to Force R_B

$$(\delta_B)_2 = \frac{R_B L^3}{3EI}$$

FROM COMPATIBILITY EQUATION

$$(\delta_B)_1 = (\delta_B)_2 \qquad \therefore R_B = \frac{17}{8}qL$$

EQUILIBRIUM

$$R_A = 2qL - R_B = -\frac{1}{8}qL \tag{1}$$

$$M_A = 2qL^2 - R_B L = -\frac{1}{8}qL^2 \tag{2}$$

SHEAR FORCE

$$V = R_A - qx = -\frac{qL}{8} - qx \qquad (0 \le x \le L)$$

$$V = 2qL - qx \qquad (L \le x \le 2L)$$

BENDING MOMENTS

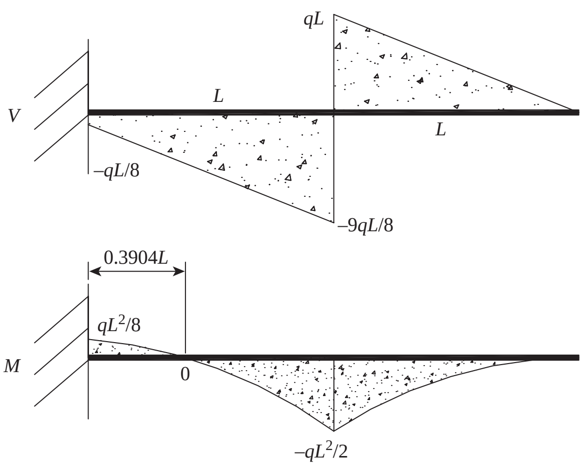
$$M = R_A x - M_A - q\frac{x^2}{2} = -\frac{qx^2}{2} - \frac{qLx}{8} + \frac{qL^2}{8}$$

$$(0 \le x \le L)$$

$$M = R_A x - M_A - q\frac{x^2}{2} + R_B x = \frac{-qx^2}{2} + 2qLx - 2qL^2$$

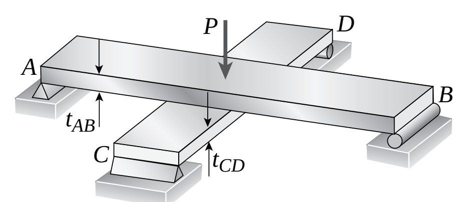
$$(L \le x \le 2L)$$

SHEAR FORCE AND BENDING MOMENT DIAGRAMS



Problem 10.4-4 Two flat beams *AB* and *CD*, lying in horizontal planes, cross at right angles and jointly support a vertical load *P* at their midpoints (see figure). Before the load *P* is applied, the beams just touch each other. Both beams are made of the same material and have the same widths. Also, the ends of both beams are simply supported. The lengths of beams *AB* and *CD* are L_{AB} and L_{CD}, respectively.

What should be the ratio t_{AB}/t_{CD} of the thicknesses of the beams if all four reactions are to be the same?



Solution 10.4-4 Two beams supporting a load *P*

For all four reactions to be the same, each beam must support one-half of the load *P*.

DEFLECTIONS

$$\delta_{AB} = \frac{(P/2)\,L_{AB}^3}{48EI_{AB}} \qquad \delta_{CD} = \frac{(P/2)L_{CD}^3}{48EI_{CD}}$$

COMPATIBILITY

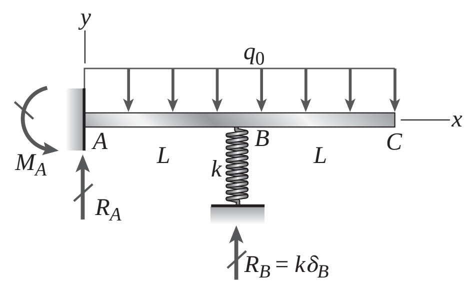
$$\delta_{AB} = \delta_{CD} \qquad \text{or} \qquad \frac{L_{AB}^3}{I_{AB}} = \frac{L_{CD}^3}{I_{CD}}$$

MOMENT OF INERTIA

$$I_{AB} = \frac{1}{12}bt_{AB}^3 \qquad I_{CD} = \frac{1}{12}bt_{CD}^3$$

$$\therefore \frac{L_{AB}^3}{t_{AB}^3} = \frac{L_{CD}^3}{t_{CD}^3} \qquad \frac{t_{AB}}{t_{CD}} = \frac{L_{AB}}{L_{CD}} \quad \longleftarrow$$

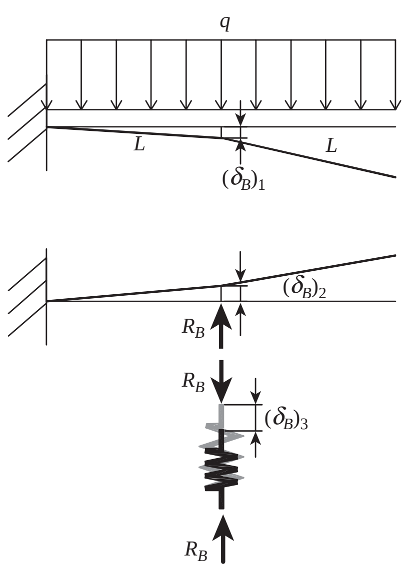
Problem 10.4-5 A propped cantilever beam of length $2L$ is loaded by a uniformly distributed load with intensity q. The beam is supported at B by a linearly elastic spring with stiffness k. Use the method of superposition to solve for all reactions. Also draw shear-force and bending-moment diagrams, labeling all critical ordinates. Let $k = 6EI/L^3$.



Solution 10.4-5

Select R_B as the redundant.

Released structure and force-displacement relations



$(\delta_B)_1$ = Deflection at B due to uniform load q

$$(\delta_B)_1 = \frac{17qL^4}{24EI}$$

$(\delta_B)_2$ = Deflection at B due to Force R_B

$$(\delta_B)_2 = \frac{R_B L^3}{3EI}$$

$(\delta_B)_3$ = Shortening in spring due to Force R_B

$$(\delta_B)_3 = \frac{R_B}{k} = \frac{R_B L^3}{6EI}$$

From compatibility equation

$$(\delta_B)_1 - (\delta_B)_2 = (\delta_B)_3 \qquad \therefore\ R_B = \frac{17}{12}qL$$

Equilibrium

$$R_A = 2qL - R_B = \frac{7}{12}qL \tag{1}$$

$$M_A = 2qL^2 - R_B L = \frac{7}{12}qL^2 \tag{2}$$

Shear force

$$V = R_A - qx = \frac{7}{12}qL - qx \qquad (0 \le x \le L)$$

$$V = R_A - qx + R_B = 2qL - qx \qquad (L \le x \le 2L)$$

BENDING MOMENTS

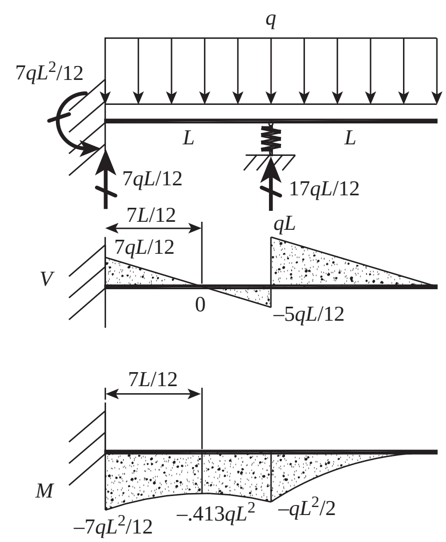
$$M = R_A x - M_A - q\frac{x^2}{2} = -q\left[\frac{1}{2}x^2 - \frac{7L}{12}x + \frac{7L^2}{12}\right]$$

$$(0 \leq x \leq L)$$

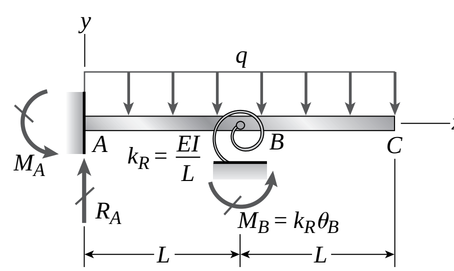
$$M = R_A x - M_A - q\frac{x^2}{2} + R_B(x - L)$$

$$= -q\left[\frac{1}{2}x^2 - 2Lx + 2L^2\right] \qquad (L \leq x \leq 2L)$$

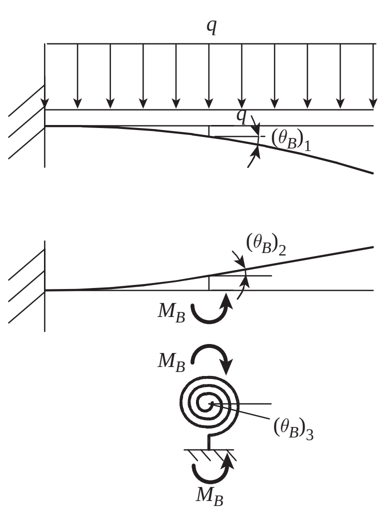
SHEAR FORCE AND BENDING MOMENT DIAGRAMS



Problem 10.4-6 A propped cantilever beam of length $2L$ is loaded by a uniformly distributed load with intensity q. The beam is supported at B by a linearly elastic rotational spring with stiffness k_R, which provides a resisting moment M_B due to rotation θ_B. Use the method of superposition to solve for all reactions. Also draw shear-force and bending-moment diagrams, labeling all critical ordinates. Let $k_R = EI/L$.



Solution 10.4-6



Select M_B as the redundant.

RELEASED STRUCTURE AND FORCE-SLOPE RELATIONS

$(\theta_B)_1 =$ Slope at B due to uniform load q

$$(\theta_B)_1 = \frac{7qL^3}{6EI}$$

$(\theta_B)_2 =$ Slope at B due to Moment

$$(\theta_B)_2 = \frac{M_B L}{EI}$$

$(\theta_B)_3 =$ Spring rotation at B due to Moment

$$(\theta_B)_3 = \frac{M_B}{k_R} = \frac{M_B L}{EI}$$

FROM COMPATIBILITY EQUATION

$$(\theta_B)_1 - (\theta_B)_2 = (\theta_B)_3 \qquad \therefore\ M_B = \frac{7}{12}\,qL^2$$

FROM EQUILIBRIUM EQS.

$$R_A = 2qL \tag{1}$$

$$M_A = 2qL^2 - M_B = \frac{17}{12}qL^2 \tag{2}$$

SHEAR FORCE

$$V = R_A - qx = 2qL - qx \qquad (0 \le x \le 2L)$$

BENDING MOMENTS

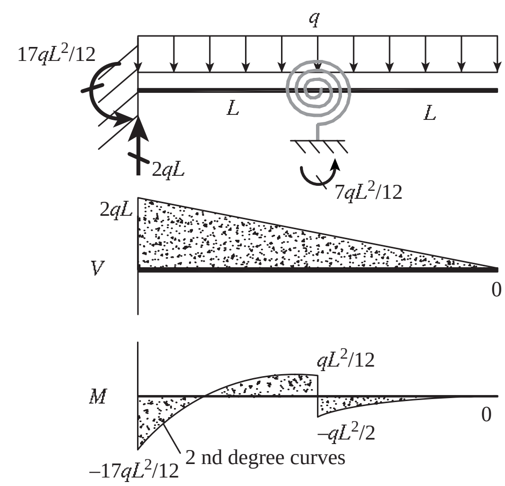
$$M = R_A x - M_A - q\frac{x^2}{2} = -q\left[\frac{1}{2}x^2 - 2Lx + \frac{17L^2}{12}\right]$$

$$(0 \le x \le L)$$

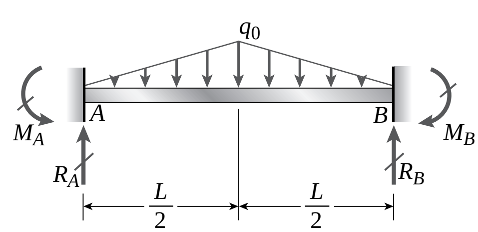
$$M = R_A x - M_A - q\frac{x^2}{2} - M_B$$

$$= -q\left[\frac{1}{2}x^2 - 2Lx + 2L^2\right] \qquad (L \le x \le 2L)$$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS



Problem 10.4-7 Determine the fixed-end moments (M_A and M_B) and fixed-end forces (R_A and R_B) for a beam of length L supporting a triangular load of maximum intensity q_0 (see figure). Then draw the shear-force and bending-moment diagrams, labeling all critical ordinates.



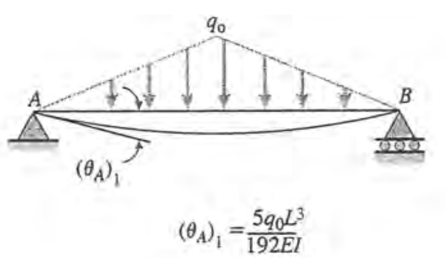
Solution 10.4-7 Fixed-end beam (triangular load)

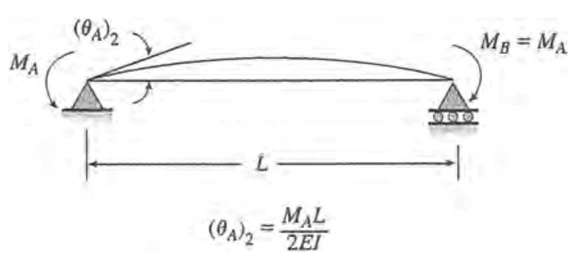
Select M_A and M_B as redundants.

SYMMETRY $M_A = M_B \quad R_A = R_B$

EQUILIBRIUM $R_A = R_B = q_0 L/4 \quad \leftarrow$

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS



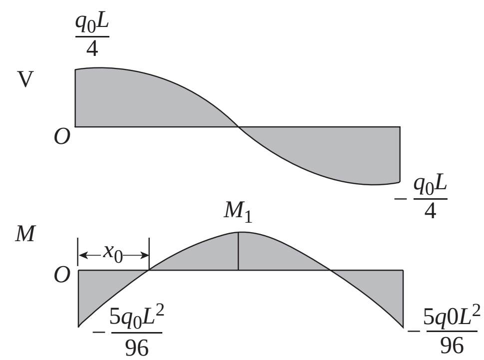


COMPATIBILITY $\theta_A = (\theta_A)_1 - (\theta_A)_2 = 0$

Substitute for $(\theta_A)_1$ and $(\theta_A)_2$ and solve for M_A:

$$M_A = M_B = \frac{5q_0L^2}{96} \quad \leftarrow$$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS

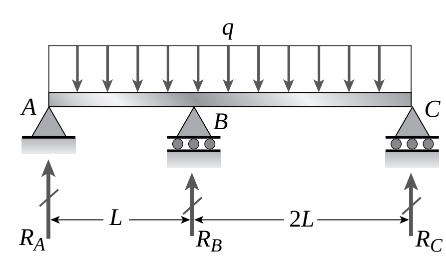


$$M_1 = \frac{q_0L^2}{32}$$

$$x_0 = 0.2231L$$

Problem 10.4-8 A continuous beam ABC with two unequal spans, one of length L and one of length $2L$, supports a uniform load of intensity q (see figure).

Determine the reactions R_A, R_B, and R_C for this beam. Also, draw the shear-force and bending-moment diagrams, labeling all critical ordinates.



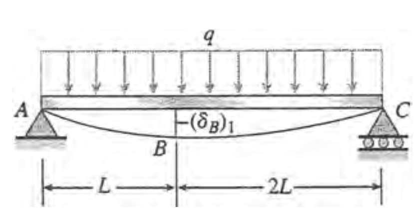
Solution 10.4-8 Continuous beam with two spans

Select R_B as the redundant.

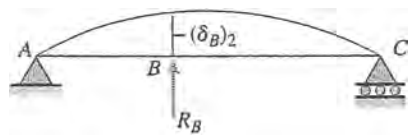
EQUILIBRIUM

$$R_A = \frac{3qL}{2} - \frac{2}{3}R_B \qquad R_c = \frac{3qL}{2} - \frac{1}{3}R_B$$

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS



$$(\delta_B)_1 = \frac{11qL^4}{12EI}$$



$$(\delta_B)_2 = \frac{4R_BL^3}{9EI}$$

COMPATIBILITY

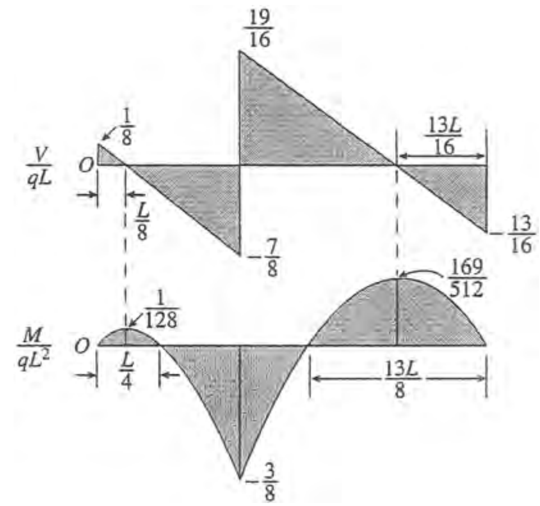
$$\delta_B = (\delta_B)_1 - (\delta_B)_2 = 0$$

$$\frac{11qL^4}{12EI} - \frac{4R_BL^3}{9EI} = 0 \qquad R_B = \frac{33qL}{16} \qquad \leftarrow$$

OTHER REACTIONS (FROM EQUILIBRIUM)

$$R_A = \frac{qL}{8} \qquad R_c = \frac{13qL}{16} \qquad \leftarrow$$

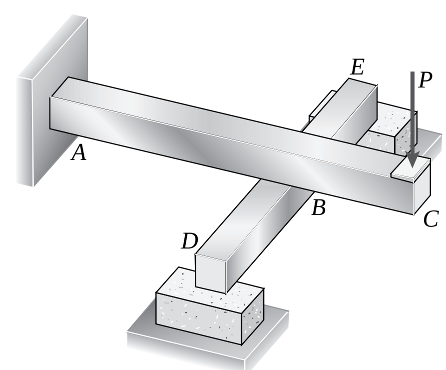
SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS

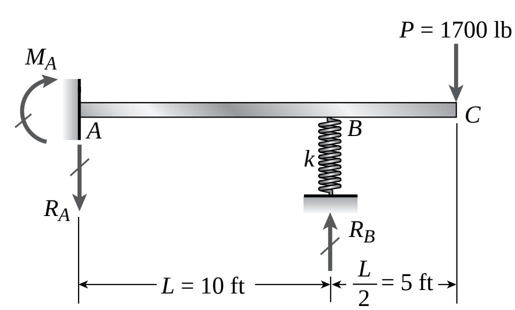


Problem 10.4-9 Beam ABC is fixed at support A and rests (at point B) upon the midpoint of beam DE (see the first part of the figure). Thus, beam ABC may be represented as a propped cantilever beam with an overhang BC and a linearly elastic support of stiffness k at point B (see the second part of the figure).

The distance from A to B is $L = 10$ ft, the distance from B to C is $L/2 = 5$ ft, and the length of beam DE is $L = 10$ ft. Both beams have the same flexural rigidity EI. A concentrated load $P = 1700$ lb acts at the free end of beam ABC.

Determine the reactions R_A, R_B, and M_A for beam ABC. Also, draw the shear-force and bending-moment diagrams for beam ABC, labeling all critical ordinates.





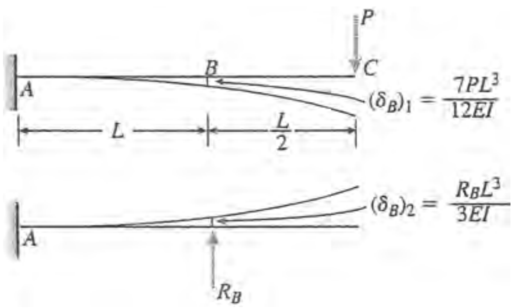
Solution 10.4-9 Beam with spring support

Select R_B as the redundant.

EQUILIBRIUM

$R_A = R_B - P \qquad M_A = R_B L - 3PL/2$

RELEASED STRUCTURE AND FORCE-DISPL. EQS.



COMPATIBILITY $\delta_B = (\delta_B)_1 - (\delta_B)_2 = \dfrac{R_B}{k}$

Beam DE: $k = \dfrac{48EI}{L^3}$

$$\frac{7PL^3}{12EI} - \frac{R_B L^3}{3EI} = \frac{R_B L^3}{48EI} \qquad R_B = \frac{28P}{17} \quad \longleftarrow$$

OTHER REACTIONS (FROM EQUILIBRIUM)

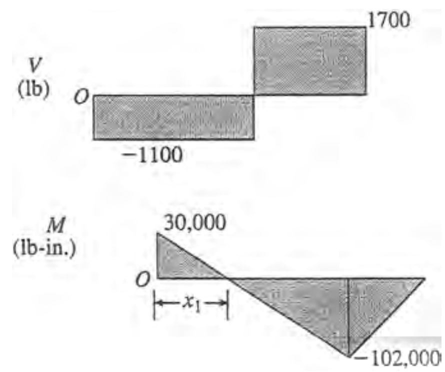
$$R_A = \frac{11P}{17} \qquad M_A = \frac{5PL}{34} \quad \longleftarrow$$

NUMERICAL VALUES

$P = 1700$ lb $\qquad L = 10$ ft $= 120$ in.

$$\left.\begin{array}{l} R_A = 1100 \text{ lb} \qquad R_B = 2800 \text{ lb} \\ M_A = 30{,}000 \text{ lb-in.} \end{array}\right\} \quad \longleftarrow$$

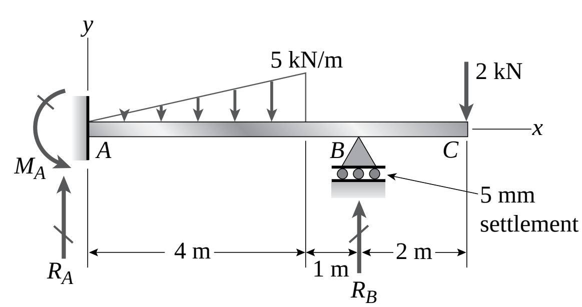
SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS



$$x_1 = \frac{300}{11} \text{ in.}$$

$$= 27.27 \text{ in.}$$

Problem 10.4-10 A propped cantilever beam has flexural rigidity $EI = 4.5$ MN · m^2. When the loads shown are applied to the beam, it settles at joint B by 5 mm. Find the reaction at joint B.

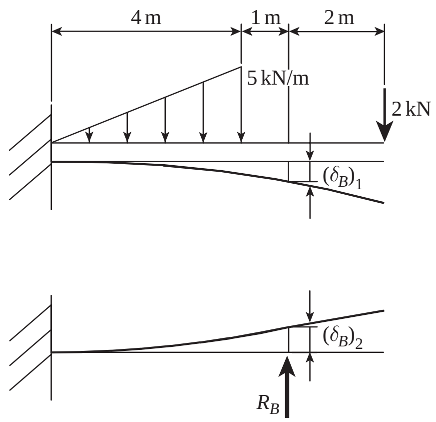


Solution 10.4-10

$EI = 4.5\ \text{MN}\cdot\text{m}^2 = 4500\ \text{kN}\cdot\text{m}^2$

Select R_B as the redundant.

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS



$(\delta_B)_1$ = Deflection at B due to distributed and concentrated loads

$$(\delta_B)_1 = -\int_{2\text{m}}^{7\text{m}} 2x\frac{(x-2)}{EI}\,dx$$

$$-\int_{1\text{ m}}^{5\text{ m}}\left[\frac{5}{2}(x-1)^2 - \frac{5}{24}(x-1)^3\right]\frac{x}{EI}\,dx$$

$$= -(29.63 + 34.963)\times 10^3$$

$$= -64.593\times 10^3 = -64.593\text{ mm}$$

$(\delta_B)_2$ = Deflection at B due to Force R_B

$$(\delta_B)_2 = \int_0^{5\text{m}} R_B x\frac{x}{EI}\,dx = R_B\,9.259\times 10^{-3}$$

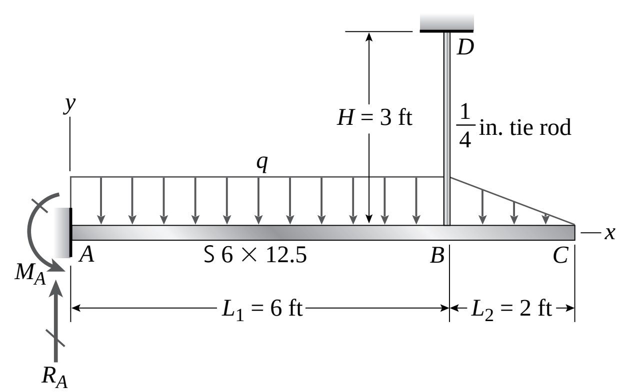
$$= R_B\,9.259\text{ mm}$$

COMPATIBILITY (SETTLEMENT AT B = 5 mm)

$(\delta_B)_1 = (\delta_B)_2 - 5\text{ mm}$ $\quad \therefore\ R_B = 6.44\text{ kN}$ ←

Problem 10.4-11 A cantilever beam is supported by a tie rod at B as shown. Both the tie rod and the beam are steel with $E = 30\times 10^6$ psi. The tie rod is just taut before the distributed load $q = 200$ lb/ft is applied.

(a) Find the tension force in the tie rod.
(b) Draw shear-force and bending-moment diagrams for the beam, labeling all critical ordinates.



Solution 10.4-11

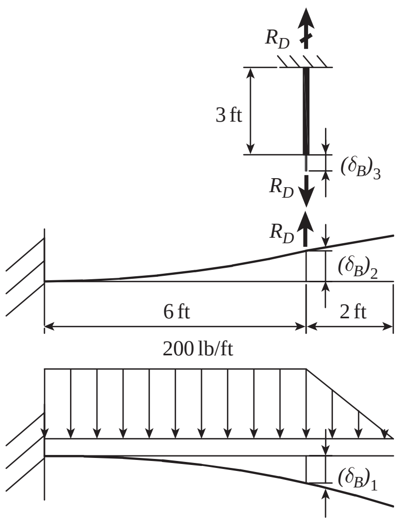
$E = 30 \times 10^6$ psi

$A = \dfrac{(1/4)^2\pi}{4} = 0.0491 \text{ in.}^2$

$I = 22.1 \text{ in.}^4$ From S 6 × 12.5

Select R_D as the redundant.

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS



$(\delta_B)_1$ = Deflection at B due to distributed load q

$q = 200 \text{ lb/ft} = \dfrac{200}{12} \text{ lb/in.}$

$$(\delta_B)_1 = \int_0^{72} \left(\frac{200}{12}\frac{x^2}{2} + \frac{200}{12}\frac{2 \cdot 12}{2}\frac{2 \cdot 12}{3} \right) \frac{x}{EI} dx$$

$= 0.1282$ in.

$(\delta_B)_2$ = Deflection at B due to Force R_D

$$(\delta_B)_2 = \int_0^{72} \frac{R_D x^2}{EI} dx = R_D\, 1.877 \times 10^{-4} \text{ in.}$$

$(\delta_B)_3$ = Extension in tie rod due to Force R_D

$$(\delta_B)_3 = \frac{R_D L}{AE} = \frac{R_D(3 \cdot 12)}{0.0491 \cdot 30 \times 10^6}$$

$= R_D\, 2.445 \times 10^{-5}$ in.

COMPATIBILITY

$$(\delta_B)_1 - (\delta_B)_2 = (\delta_B)_3$$

$0.1282 - R_D\, 1.877 \times 10^{-4} = R_D\, 2.445 \times 10^{-5}$

$\therefore R_D = 604.3$ lb

(a) THE TENSION FORCE IN THE TIE ROD $= R_D$
$= 604$ lb ←

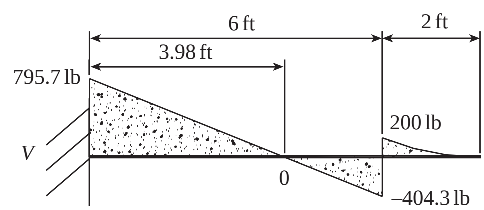
FROM EQUILIBRIUM EQS.

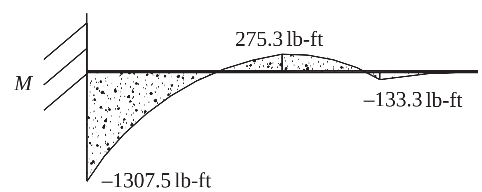
$$R_A = 200 \cdot 6 + 200\frac{2}{2} - R_D = 795.7 \text{ lb} \qquad (1)$$

$$M_A = \frac{200 \cdot 2}{2}\left(6 + \frac{2}{3}\right) + 200\frac{6^2}{2} - R_D 6$$

$= 1308 \text{ lb-ft} = 1.569 \times 10^4 \text{ lb} \cdot \text{in.}$ ←

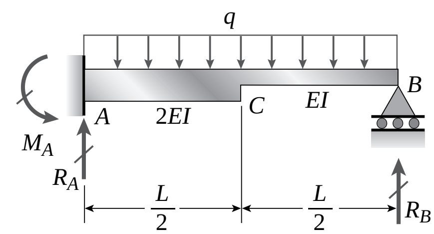
(b) SHEAR FORCE AND BENDING MOMENT DIAGRAMS





Problem 10.4-12 The figure shows a nonprismatic, propped cantilever beam AB with flexural rigidity $2EI$ from A to C and EI from C to B.

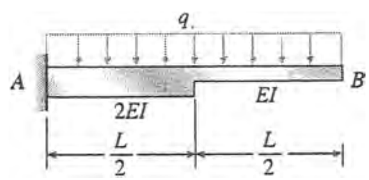
Determine all reactions of the beam due to the uniform load of intensity q. (*Hint:* Use the results of Problems 9.7-1 and 9.7-2.)



Solution 10.4-12 Nonprismatic beam

Select R_B as the redundant.

RELEASED STRUCTURE



$(\delta_B)_1$ = downward deflection of end B due to load q

$(\delta_B)_2$ = upward deflection due to reaction R_B

FORCE-DISPLACEMENT RELATIONS

From Prob. 9.7-2: $\delta_B = \dfrac{qL^4}{128EI_1}\left(1 + 15\dfrac{I_1}{I_2}\right)$

$I_1 \rightarrow I \quad I_2 \rightarrow 2I \qquad \therefore\ (\delta_B)_1 = \dfrac{17qL^4}{256EI}$

From Prob. 9.7-1:

$\delta_B = \dfrac{PL^3}{24EI_1}\left(1 + 7\dfrac{I_1}{I_2}\right) \qquad \therefore\ (\delta_B)_2 = \dfrac{3R_BL^3}{16EI}$

COMPATIBILITY

$\delta_B = (\delta_B)_1 - (\delta_B)_2 = 0$

or

$\dfrac{17qL^4}{256EI} - \dfrac{3R_BL^3}{16EI} = 0 \qquad R_B = \dfrac{17qL}{48} \quad \leftarrow$

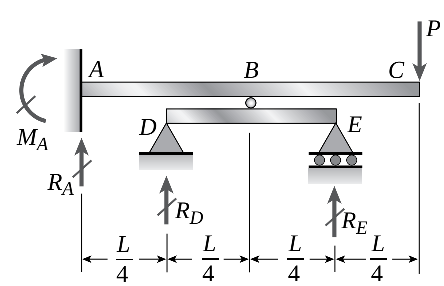
EQUILIBRIUM

$R_A = qL - R_B = \dfrac{31qL}{48}$

$M_A = \dfrac{qL^2}{2} - R_BL = \dfrac{7qL^2}{48} \quad \leftarrow$

Problem 10.4-13 A beam ABC is fixed at end A and supported by beam DE at point B (see figure). Both beams have the same cross section and are made of the same material.

(a) Determine all reactions due to the load P.
(b) What is the numerically largest bending moment in either beam?

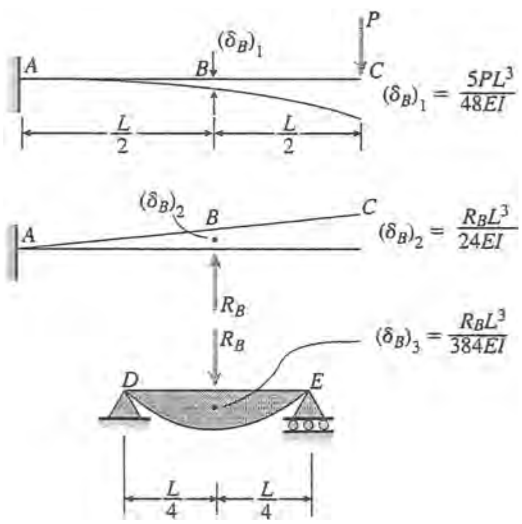


Solution 10.4-13 Beam supported by a beam

Let R_B = interaction force between beams

Select R_B as the redundant.

RELEASED STRUCTURE AND FORCE-DISPL. EQS.



COMPATIBILITY $(\delta_B)_1 - (\delta_B)_2 = (\delta_B)_3$

Substitute and solve: $R_B = \dfrac{40P}{17}$ $\longleftarrow$

SYMMETRY AND EQUILIBRIUM

$$R_D = R_E = \frac{R_B}{2} = \frac{20P}{17} \quad \longleftarrow$$

$$R_A = P - R_D - R_E = -\frac{23P}{17} \quad \longleftarrow$$

(minus means downward)

$$M_A = R_B\left(\frac{L}{2}\right) - PL = \frac{3PL}{17} \quad \longleftarrow$$

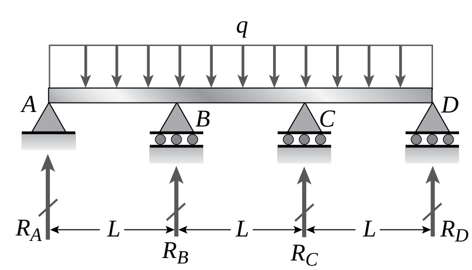
BEAM ABC: $M_{\text{max}} = M_B = -\dfrac{PL}{2}$

BEAM DE: $M_{\text{max}} = M_B = \dfrac{5PL}{17}$

$$|M_{\text{max}}| = \frac{PL}{2} \quad \longleftarrow$$

Problem 10.4-14 A three-span continuous beam $ABCD$ with three equal spans supports a uniform load of intensity q (see figure).

Determine all reactions of this beam and drawn the shear-force and bending-moment diagrams, labeling all critical ordinates.



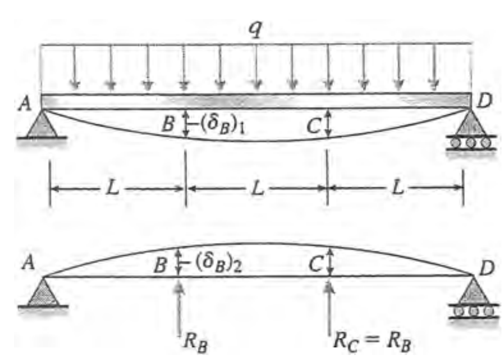
Solution 10.4-14 Three-span continuous beam

SELECT R_B AND R_C AS THE REDUNDANTS.

SYMMENTRY AND EQUILIBRIUM

$$R_C = R_B \qquad R_A = R_D = \frac{3qL}{2} - R_B$$

RELEASED STRUCTURE



FORCE-DISPLACEMENT RELATIONS

$$(\delta_B)_1 = \frac{11qL^4}{12EI} \qquad (\delta_B)_2 = \frac{5R_BL^3}{6EI}$$

COMPATIBIITY

$$\delta_B = (\delta_B)_1 - (\delta_B)_2 = 0 \qquad \therefore\ R_B = \frac{11qL}{10} \quad \leftarrow$$

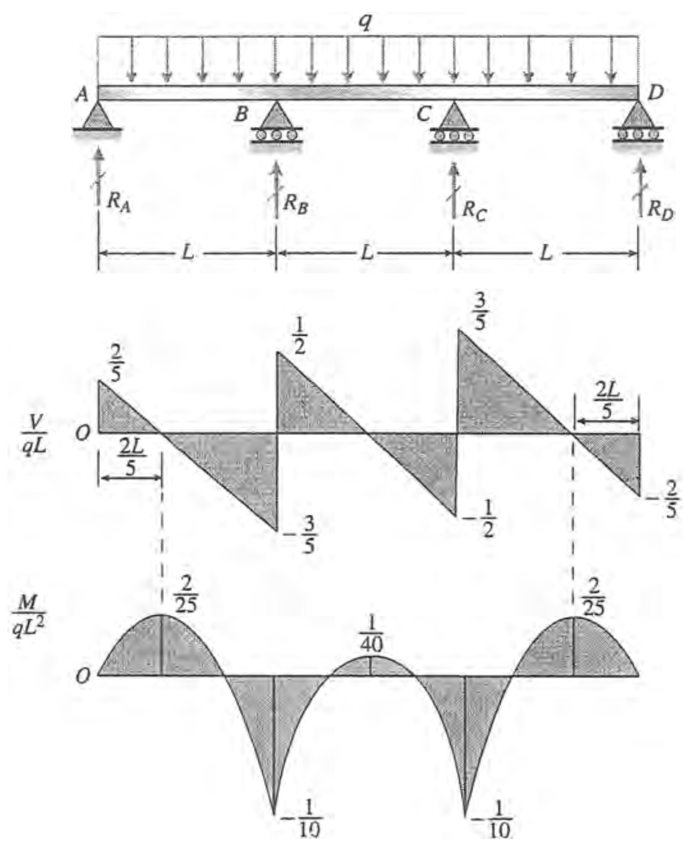
OTHER REACTIONS

From symmetry and equilibrium:

$$R_C = R_B = \frac{11qL}{10} \quad \leftarrow$$

$$R_A = R_D = \frac{2qL}{5} \quad \leftarrow$$

LOADING, SHEAR-FORCE, AND BENDING-MOMENT DIAGRAMS

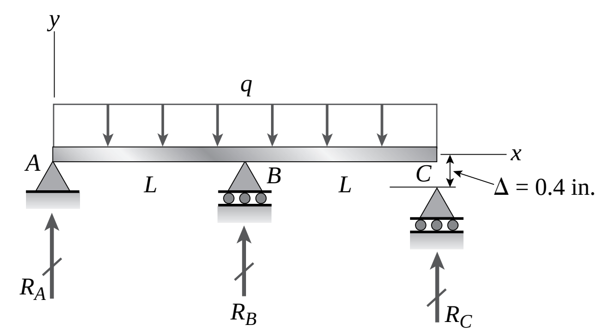


$$M_B = M_C = -\frac{qL^2}{10}$$

$$M_{\max} = \frac{2qL^2}{25}$$

Problem 10.4-15 A beam rests on supports at A and B and is loaded by a distributed load with intensity q as shown. A small gap Δ exists between the unloaded beam and the support at C. Assume that span length $L = 40$ in. and flexural rigidity of the beam $EI = 0.4 \times 10^9$ lb-in.^{2} Plot a graph of the bending moment at B as a function of the load intensity q.

(*HINT:* See Example 9-9 for guidance on computing the deflection at C.)



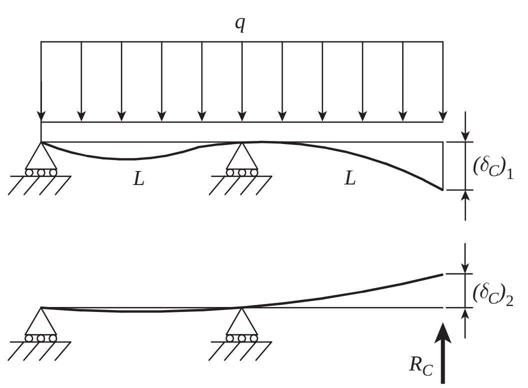
Solution 10.4-15

Select R_C as the redundant.

EQUILIBRIUM

$$R_C = 2qL - R_A - R_B$$

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS



$$(\delta_C)_1 = \frac{qL^4}{4EI}$$

$$(\delta_C)_2 = \frac{2L^3}{3EI}R_C$$

COMPATIBILITY

1) $\delta_C = (\delta_C)_1$ for $(\delta_C)_1 < 0.4$ in. $\therefore R_C(q) = 0$

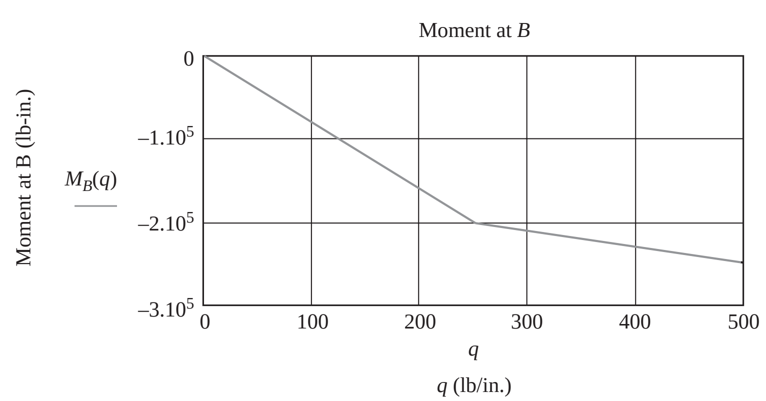
$$M_B(q) = -\frac{qL^2}{2} = -800q \text{ lb} \cdot \text{in. for } q < 250 \text{ lb/in.}$$

2) $\delta_C = (\delta_C)_1 - (\delta_C)_2 = 0.4$ in. for $(\delta_C)_1 = 0.4$ in.

$\therefore R_C(q) = 15q - 3750$

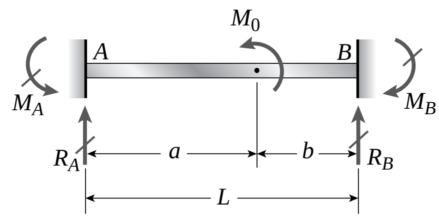
$$M_B(q) = R_A L - \frac{qL^2}{2} = -200q - 150000 \text{ lb} \cdot \text{in.}$$

for $q \geq 250$ lb/in.



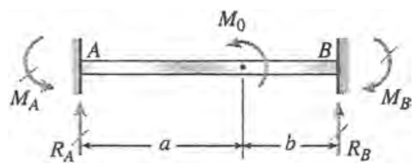
Problem 10.4-16 A fixed-end beam AB of length L is subjected to a moment M_0 acting at the position shown in the figure.

(a) Determine all reactions for this beam.
(b) Draw shear-force and bending-moment diagrams for the special case in which $a = b = L/2$.



Solution 10.4-16 Fixed-end beam (M_0 = applied load)

Select R_B and M_B as redundants.

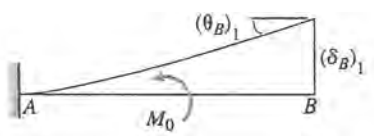


$L = a + b$

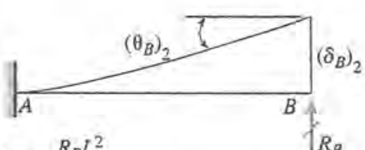
EQUILIBRIUM

$R_A = -R_B \qquad M_A = M_B - R_BL - M_0$

RELEASED STRUCTURE AND FORCE-DISPL. EQS.

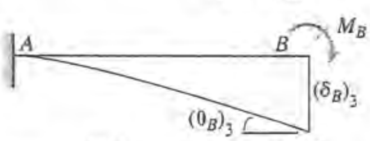


$(\theta_B)_1 = \dfrac{M_0a}{EI} \qquad (\delta_B)_1 = \dfrac{M_0a}{2EI}(a + 2b)$



$(\theta_B)_2 = \dfrac{R_BL^2}{2EI}$

$(\delta_B)_2 = \dfrac{R_BL^3}{3EI}$



$(\theta_B)_3 = \dfrac{M_BL}{EI} \qquad (\delta_B)_3 = \dfrac{M_BL^2}{2EI}$

$$(\theta_B)_1 = \frac{M_0a}{EI} \qquad (\delta_B)_1 = \frac{M_0a}{2EI}(a + 2b)$$

$$(\theta_B)_2 = \frac{R_BL^2}{2EI} \qquad (\delta_B)_2 = \frac{R_BL^3}{3EI}$$

$$(\theta_B)_3 = \frac{M_BL}{EI} \qquad (\delta_B)_3 = \frac{M_BL^3}{2EI}$$

COMPATIBILITY

$\delta_B = -(\delta_B)_1 - (\delta_B)_2 + (\delta_B)_3 = 0$

or $2R_BL^3 - 3M_BL^2 = -3M_0a(a + 2b)$ (1)

$\theta_B = (\theta_B)_1 + (\theta_B)_2 - (\theta_B)_3 = 0$

or $R_BL^2 - 2M_BL = -2M_0a$ (2)

SOLVE EQS. (1) AND (2):

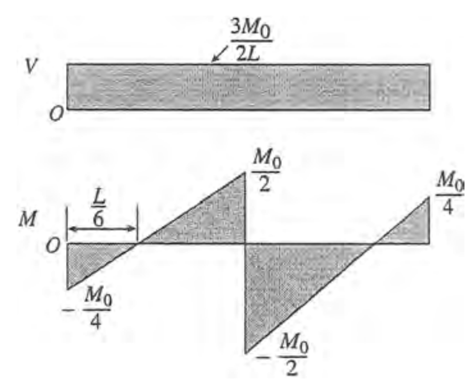
$$R_B = -\frac{6M_0ab}{L^3} \qquad M_B = -\frac{M_0a}{L^2}(3b - L) \qquad \leftarrow$$

FROM EQUILIBRIUM:

$$R_A = \frac{6M_0ab}{L^3} \qquad M_A = \frac{M_0b}{L^2}(3a - L) \qquad \leftarrow$$

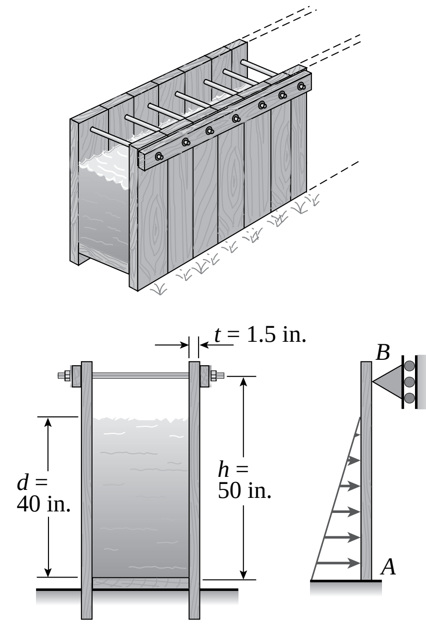
SPECIAL CASE $a = b = L/2$

$$R_A = -R_B = \frac{3M_0}{2L} \qquad M_A = -M_B = \frac{M_0}{4}$$

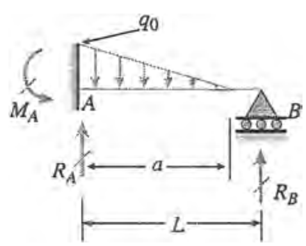


Problem 10.4-17 A temporary wood flume serving as a channel for irrigation water is shown in the figure. The vertical boards forming the sides of the flume are sunk in the ground, which provides a fixed support. The top of the flume is held by tie rods that are tightened so that there is no deflection of the boards at the point. Thus, the vertical boards may be modeled as a beam *AB*, supported and loaded as shown in the last part of the figure.

Assuming that the thickness *t* of the boards is 1.5 in., the depth *d* of the water is 40 in., and the height *h* to the tie rods is 50 in., what is the maximum bending stress σ in the boards? (*Hint:* The numerically largest bending moment occurs at the fixed support.)



Solution 10.4-17 Side wall of a wood flume



Select R_B as the redundant.

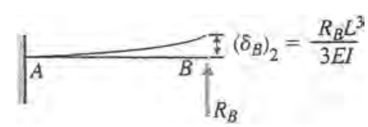
Equilibrium: $M_A = \dfrac{q_0 a^2}{6} - R_B L$

RELEASED STRUCTURE AND FORCE-DISPL. EQS.

From Table G-1, Case B:

$$(\delta_B)_1 = \frac{q_0 a^4}{30EI} + \frac{q_0 a^3}{24EI}(L-a) = \frac{q_0 a^3}{120EI}(5L-a)$$

$$(\delta_B)_2 = \frac{R_B L^3}{3EI}$$



COMPATIBILITY

$$\delta_B = (\delta_B)_1 - (\delta_B)_2 = 0 \qquad \therefore\ R_B = \frac{q_0 a^3(5L-a)}{40L^3}$$

MAXIMUM BENDING MOMENT

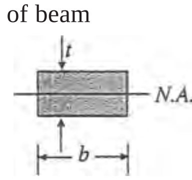
$$M_{max} = M_A = \frac{1}{6} q_0 a^2 - R_B L$$

$$= \frac{q_0 a^2}{120L^2}(20L^2 - 15aL + 3a^2)$$

NUMERICAL VALUES

$a = 40$ in. $\quad L = 50$ in. $\quad t = 1.5$ in.

$b =$ width of beam



$$S = \frac{bt^2}{6} \qquad \sigma = \frac{M_{max}}{S}$$

$\gamma = 62.4 \text{ lb/ft}^3 = 0.03611 \text{ lb/in.}^3$

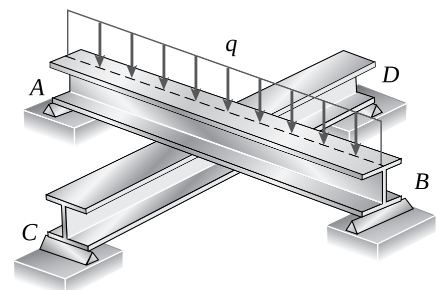
Pressure $p = \gamma a \qquad q_0 = pb = \gamma ab$

$$M_{max} = \frac{\gamma a^3 b}{120L^2}(20L^2 - 15aL + 3a^2) = 191.05\ b$$

$$S = \frac{bt^2}{6} = 0.3750\ b \qquad \sigma = \frac{M_{max}}{S} = 509 \text{ psi} \qquad \leftarrow$$

Problem 10.4-18 Two identical, simply supported beams *AB* and *CD* are placed so that they cross each other at their midpoints (see figure). Before the uniform load is applied, the beams just touch each other at the crossing point.

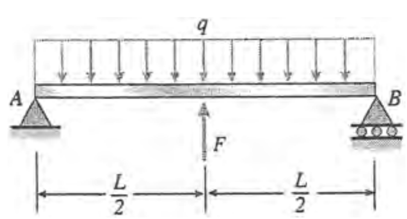
Determine the maximum bending moments $(M_{AB})_{max}$ and $(M_{CD})_{max}$ in beams *AB* and *CD*, respectively, due to the uniform load it the intensity of the load is $q = 6.4$ kN/m and the length of each beam is $L = 4$ m .



Solution 10.4-18 Two beams that cross

F = interaction force between the beams

UPPER BEAM



$(\delta_B)_1$ = downward deflection due to q

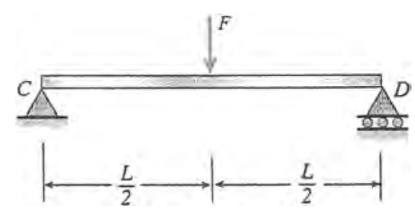
$$= \frac{5qL^4}{384EI}$$

$(\delta_B)_2$ = downward deflection due to F

$$= \frac{FL^3}{48EI}$$

$$\delta_{AB} = (\delta_B)_1 - (\delta_B)_2 = \frac{5qL^4}{384EI} - \frac{FL^3}{48EI}$$

LOWER BEAM

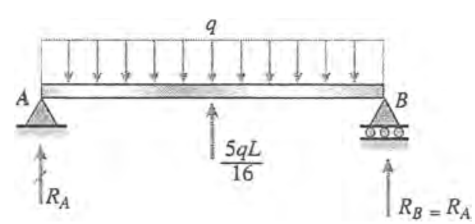


$$\delta_{CD} = \frac{FL^3}{48EI}$$

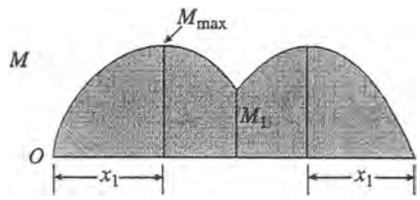
COMPATIBILITY $\delta_{AB} = \delta_{CD}$

$$\frac{5qL^4}{384EI} - \frac{FL^3}{48EI} = \frac{FL^3}{48EI} \qquad \therefore F = \frac{5qL}{16}$$

UPPER BEAM



$$R_A = \frac{11qL}{32}$$



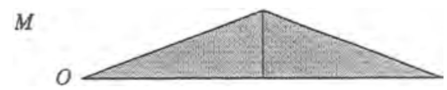
$$x_1 = \frac{11L}{32}$$

$$M_{max} = \frac{121qL^2}{2048}$$

$$M_1 = \frac{3qL^2}{64} \qquad (M_{AB})_{max} = \frac{121qL^2}{2048} \quad \leftarrow$$

Lower beam

$$M_{max} = \frac{FL}{4} = \frac{5qL^2}{64}$$



$$(M_{CD})_{max} = \frac{5qL^2}{64} \quad \leftarrow$$

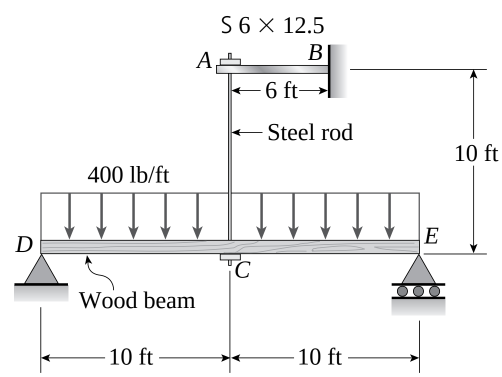
Numerical values

$q = 6.4$ kN/m $\qquad (M_{AB})_{max} = 6.05$ kN · m $\quad \leftarrow$

$L = 4$ m $\qquad (M_{CD})_{max} = 8.0$ kN · m $\quad \leftarrow$

Problem 10.4-19 The cantilever beam AB shown in the figure is an S 6 × 12.5 steel I-beam with $E = 30 \times 10^6$ psi. The simple beam DE is a wood beam 4 in. × 12 in. (normal dimension) in cross section with $E = 1.5 \times 10^6$ psi. A steel rod AC of diameter 0.25 in., length 10 ft, and $E = 30 \times 10^6$ psi serves as a hanger joining the two beams. The hanger fits snugly between the beam before the uniform load is applied to beam DE.

Determine the tensile force F in the hanger and the maximum bending moments M_{AB} and M_{DE} in the two beams due to the uniform load, which has intensity $q = 400$ lb/ft . (*Hint:* To aid in obtaining the maximum bending moment in beam DE, draw the shear-force and bending-moment diagrams.)

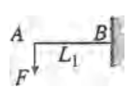


Solution 10.4-19 Beams joined by a hanger

F = tensile force in hanger

Select F as redundant.

(1) Cantilever beam AB



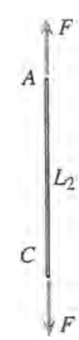
S 6 × 12.5 $\qquad I_1 = 22.1$ in.^{4}

$L_1 = 6$ ft $= 72$ in.

$E_1 = 30 \times 10^6$ psi

$$(\delta_A)_1 = \frac{FL_1^3}{3E_1I_1} = 187.66 \times 10^{-6} F \qquad \begin{cases} F = \text{lb} \\ \delta = \text{in.} \end{cases}$$

(2) Hanger AC



$d = 0.25$ in. $\qquad L_2 = 10$ ft $= 120$ in.

$E_2 = 30 \times 10^6$ psi

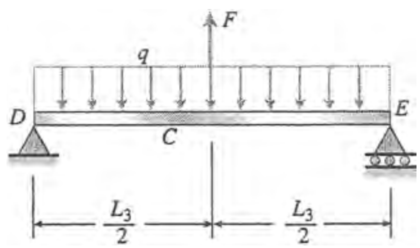
$$A_2 = \frac{\pi d^2}{4} = 0.049087 \text{ in.}^2$$

Δ = elongation of AC

$$\Delta = \frac{FL_2}{E_2 A_2} = 81.488 \times 10^{-6} F$$

(F = lb, Δ = in.)

(3) BEAM DCE



$L_3 = 20 \text{ ft} = 240 \text{ in.}$

$q = 400 \text{ lb/ft}$

$= 33.333 \text{ lb/in.}$

$E_3 = 1.5 \times 10^6 \text{ psi}$

4 in. × 12 in. (nominal)

$I_3 = 415.28 \text{ in.}^4$

$$(\delta_C)_3 = \frac{5qL_3^4}{384E_3I_3} - \frac{FL_3^3}{48E_3I_3}$$

$$= 2.3117 \text{ in.} - 462.34 \times 10^{-6} F \quad \begin{cases} F = \text{lb} \\ \delta = \text{in.} \end{cases}$$

COMPATIBILITY

$(\delta_A)_1 + \Delta = (\delta_C)_3$

$187.66 \times 10^{-6} F + 81.488 \times 10^{-6} F$

$= 2.3117 - 462.34 \times 10^{-6} F$

$F = 3160 \text{ lb}$ ←

(1) MAX. MOMENT IN AB

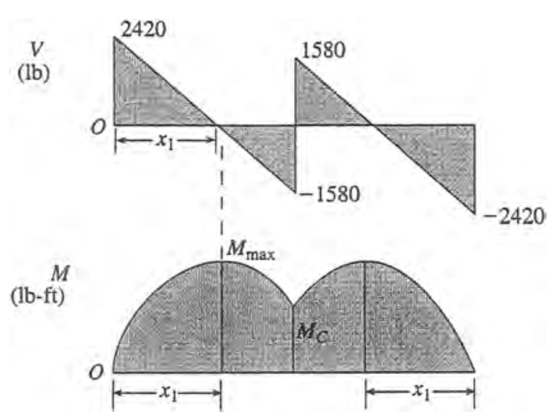
$M_{AB} = FL_1 = (3160 \text{ lb})(6 \text{ ft})$

$= 18{,}960 \text{ lb-ft}$ ←

(3) MAX. MOMENT IN DCE

$$R_D = \frac{qL_3}{2} - \frac{F}{2} = 2420 \text{ lb}$$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS



$x_1 = 6.050 \text{ ft}$

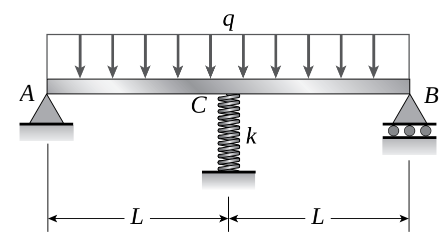
$M_{max} = 7320 \text{ lb-ft}$

$M_C = 4200 \text{ lb-ft}$

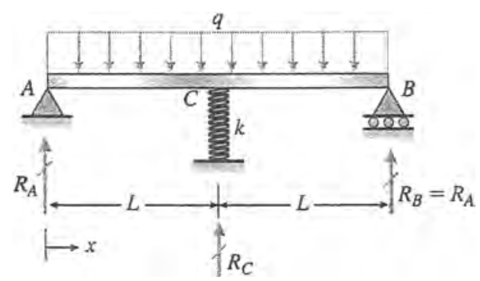
$M_{DE} = 7320 \text{ lb-ft}$ ←

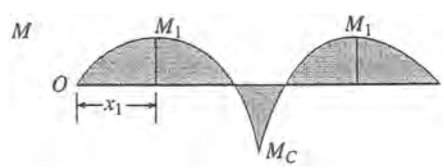
Problem 10.4-20 The beam AB shown in the figure is simply supported at A and B and supported on a spring of stiffness k at its midpoint C. The beam has flexural rigidity EI and length $2L$.

What should be the stiffness k of the spring in order that the maximum bending moment in the beam (due to the uniform load) will have the smallest possible value?



Solution 10.4-20 Beam supported by a spring





Bending moment $M = R_A x - \dfrac{qx^2}{2}$

Location of maximum positive moment

$$\frac{dM}{dx} = 0 \qquad R_A - qx = 0 \qquad x_1 = \frac{R_A}{q}$$

Maximum positive moment

$$M_1 = (M)_{x=x_1} = \frac{R_A^2}{2q}$$

Maximum negative moment

$$M_C = (M)_{x=L} = R_A L - \frac{qL^2}{2}$$

For the smallest maximum moment:

$|M_1| = |M_C|$ or $M_1 = -M_C$

$$\frac{R_A^2}{2q} = -R_A L + \frac{qL^2}{2}$$

Solve for R_A:

$$R_A = qL(\sqrt{2} - 1)$$

Equilibrium

$$\sum F_{\text{vert}} = 0 \qquad 2R_A + R_C - 2qL = 0$$

$$R_C = 2qL(2 - \sqrt{2})$$

Downward deflection of beam

$$(\delta_C)_1 = \frac{5qL^4}{24EI} - \frac{R_C L^3}{6EI} = \frac{qL^4}{24EI}(8\sqrt{2} - 11)$$

Downward displacement of spring

$$(\delta_C)_2 = \frac{R_C}{k} = \frac{2qL}{k}(2 - \sqrt{2})$$

Compatibility $(\delta_C)_1 = (\delta_C)_2$

Solve for k:

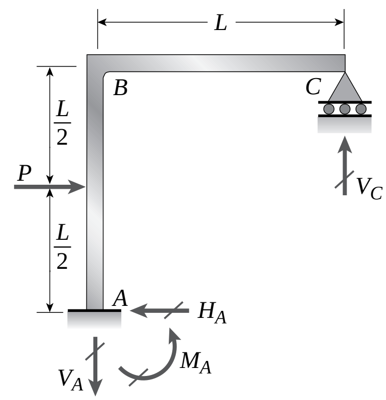
$$k = \frac{48EI}{7L^3}(6 + 5\sqrt{2})$$

$$= 89.63\frac{EI}{L^3} \quad \leftarrow$$

Problem 10.4-21 The continuous frame ABC has a fixed support at A, a roller support at C, and a rigid corner connection at B (see figure). Members AB and BC each have length L and flexural rigidity EI. A horizontal force P acts at midheight of member AB.

(a) Find all reactions of the frame.
(b) What is the largest bending moment M_{max} in the frame?

(*Note:* Disregard axial deformations in member AB and consider only the effects of bending.)



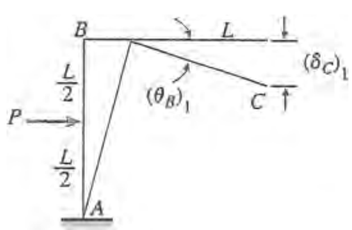
Solution 10.4-21 Frame *ABC* with fixed support

Slect V_C as the redundant.

EQUILIBRIUM $V_A = V_C \qquad H_A = P$

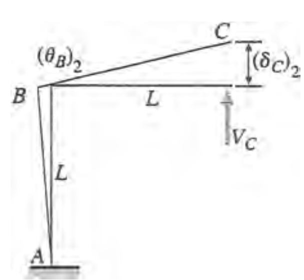
$$M_A = PL/2 - V_C L$$

RELEASED STRUCTURE AND FORCE-DISPL. EQS.



$$(\theta_B)_1 = \frac{PL^2}{8EI}$$

$$(\delta_C)_1 = (\theta_B)_1 L = \frac{PL^3}{8EI}$$



$$(\theta_B)_2 = \frac{V_C L^2}{EI}$$

$$(\delta_C)_2 = (\theta_B)_2 L + \frac{V_C L^3}{3EI} = \frac{4V_C L^3}{3EI}$$

COMPATIBILITY $(\delta_C)_1 = (\delta_C)_2$

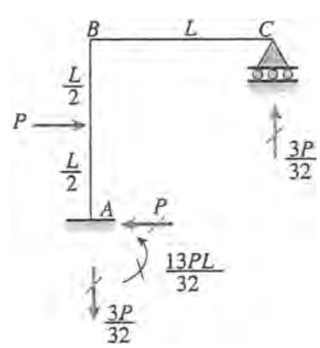
Substitute for $(\delta_C)_1$ and $(\delta_C)_2$ and solve:

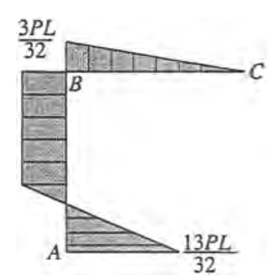
$$V_C = \frac{3P}{32} \qquad \leftarrow$$

FROM EQUILIBRIUM:

$$V_A = \frac{3P}{32} \qquad H_A = P \qquad M_A = \frac{13PL}{32} \qquad \leftarrow$$

REACTIONS AND BENDING MOMENTS

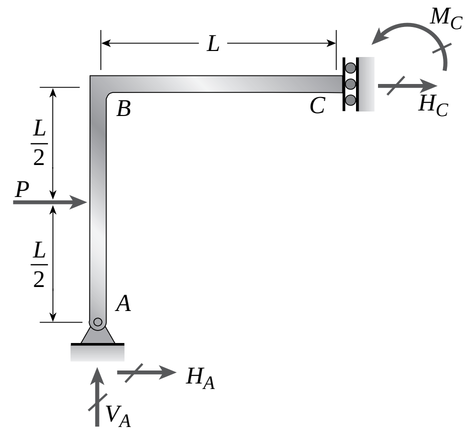




$$M_{max} = \frac{13PL}{32} \qquad \leftarrow$$

Problem 10.4-22 The continuous frame *ABC* has a pinned support at *A*, a guided support at *C*, and a rigid corner connection at *B* (see figure). Members *AB* and *BC* each have length *L* and flexural rigidity *EI*. A horizontal force *P* acts at midheight of member *AB*.

(a) Find all reactions of the frame.
(b) What is the largest bending moment M_{max} in the frame? (*Note*: Disregard axial deformations in members *AB* and *BC* and consider only the effects of bending.)



Solution 10.4-22

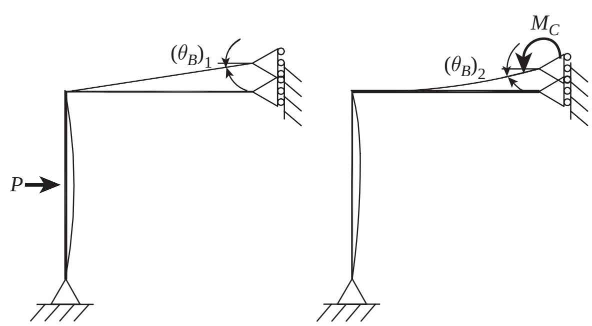
Select M_C as the redundant.

EQUILIBRIUM

$$V_A = 0$$

$$H_A = -\frac{P}{2} - \frac{M_C}{L} \qquad H_C = -\frac{P}{2} + \frac{M_C}{L}$$

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS



$$(\theta_C)_1 = \frac{PL^2}{16EI}$$

$$(\theta_C)_1 = \frac{4L}{3EI} M_C$$

COMPATIBILITY

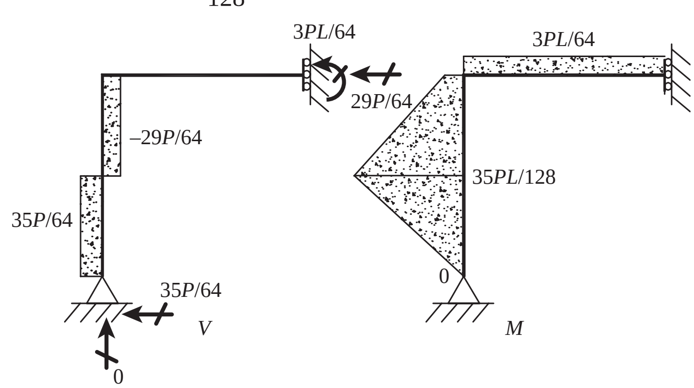
$$\theta_C = (\theta_C)_1 - (\theta_C)_2 = 0 \qquad \therefore M_C = \frac{3}{64} PL$$

FROM EQUILIBRIUM

$$H_A = -\frac{P}{2} - \frac{M_C}{L} = -\frac{35}{64}P$$

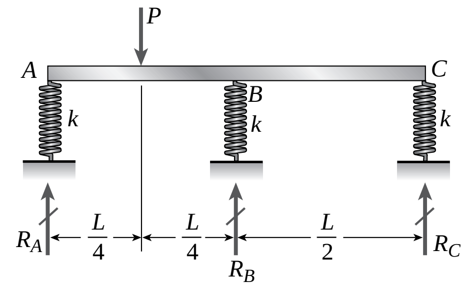
$$H_C = -\frac{P}{2} + \frac{M_C}{L} = -\frac{29}{64}P$$

$$M_{max} = \frac{35}{128}PL$$

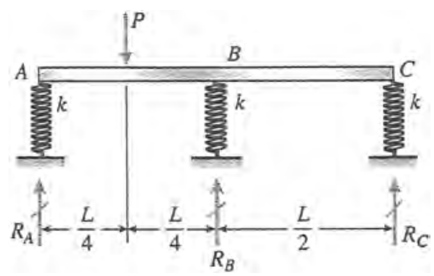


Problem 10.4-23 A wide-flange beam ABC rests on three identical spring supports A, B and C (see figure). The flexural rigidity of the beam is $EI = 6912 \times 10^6$ 1b-in.^{2} and each spring has stiffness $k = 62{,}500$ 1b/in. The length of the beam is $L = 16$ ft.

If the load P is 6000 1b, what are the reactions R_A, R_B, and R_C? Also, draw the shear-force and bending-moment diagrams for the beam, labeling all critical ordinates.



Solution 10.4-23 Beam on three springs

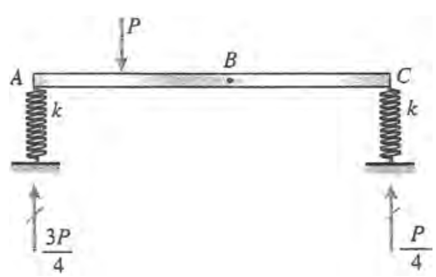


Select R_B as the redundant.

EQUILIBRIUM

$$R_A = \frac{3P}{4} - \frac{R_B}{2} \qquad R_C = \frac{P}{4} - \frac{R_B}{2}$$

RELEASED STRUCTURE AND FORCE-DISPL. EQS.



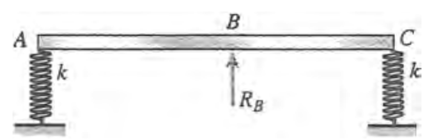
$$(\delta_A)_1 = \frac{3P}{4k}$$

$$(\delta_C)_1 = \frac{P}{4k}$$

$$(\delta_B)_1 = \frac{1}{2}[(\delta_A)_1 + (\delta_C)_1] + \frac{P\left(\frac{L}{4}\right)\left[3L^2 - 4\left(\frac{L}{4}\right)^2\right]}{48EI}$$

(Case 5, Table G-2)

$$(\delta_B)_1 = \frac{P}{2k} + \frac{11PL^3}{768EI} \qquad \text{(downward)}$$



$$(\delta_A)_2 = \frac{R_B}{2k}$$

$$(\delta_C)_2 = \frac{R_B}{2k}$$

$$(\delta_B)_2 = \frac{1}{2}[(\delta_A)_2 + (\delta_C)_2] + \frac{R_BL^3}{48EI}$$

$$= \frac{R_B}{2k} + \frac{R_BL^3}{48EI} \qquad \text{(upward)}$$

COMPATIBILITY $(\delta_B)_1 - (\delta_B)_2 = \dfrac{R_B}{k}$

Substitute and solve:

$$R_B = P\left(\frac{384EI + 11kL^3}{1152EI + 16kL}\right)$$

$$\text{Let } k^* = \frac{kL^3}{EI} \quad \text{(nondimensional)} \qquad \leftarrow$$

$$R_B = \frac{P}{16}\left(\frac{384 + 11k^*}{72 + k^*}\right) \qquad \leftarrow$$

FROM EQUILIBRIUM:

$$R_A = \frac{P}{32}\left(\frac{1344 + 13k^*}{72 + k^*}\right) \quad \leftarrow$$

$$R_C = \frac{3P}{32}\left(\frac{64 - k^*}{72 + k^*}\right) \quad \leftarrow$$

NUMERICAL VALUES

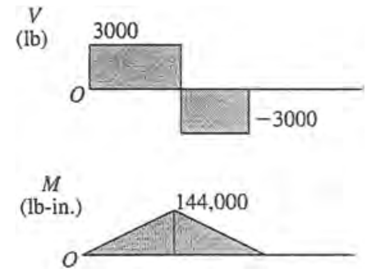
$EI = 6912 \times 10^6$ lb-in.^{2} $\qquad k = 62{,}500$ lb/in.

$L = 16$ ft $= 192$ in. $\qquad P = 6000$ lb

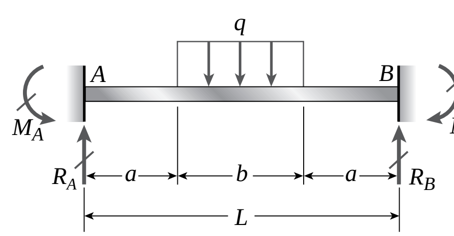
$$k^* = \frac{kL^3}{EI} = 64 \qquad R_B = 3000\,\text{1b} \quad \leftarrow$$

$R_A = 3000$ lb $\qquad R_C = 0 \quad \leftarrow$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS



Problem 10.4-24 A fixed-end beam AB of length L is subjected to a uniform load of intensity q acting over the middle region of the beam (see figure).



(a) Obtain a formula for the fixed-end moments M_A and M_B in terms of the load q, the length L, and the length b of the loaded part of the beam.

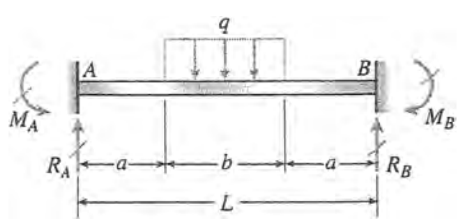
(b) Plot a graph of the fixed-end moment M_A versus the length b of the loaded part of the beam. For convenience, plot the graph in the following nondimensional form:

$$\frac{M_A}{qL^2/12} \quad \text{versus} \quad \frac{b}{L}$$

with the raio b/L varying between its extreme values of 0 and 1.

(c) For the special case in which $a = b = L/3$, draw the shear-force and bending-moment diagrams for the beam, labeling all critical ordinates.

Solution 10.4-24 Fixed-end Beam

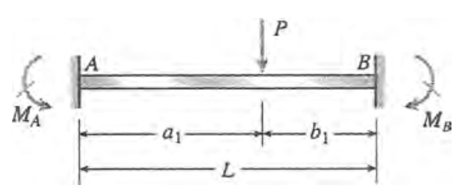


$M_A = M_B$

$$R_A = R_B = \frac{qb}{2}$$

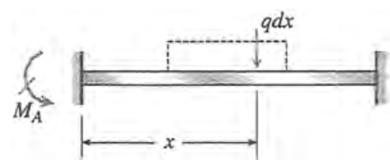
$$a = \frac{L - b}{2}$$

FROM EXAMPLE 10-4, EQ. (10-25a):



$$M_A = \frac{Pa_1b_1^2}{L^2}$$

FOR THE PARTIAL UNIFORM LOAD



$$dM_A = \frac{(qdx)(x)(L-x)^2}{L^2}$$

$$M_A = \int_a^{a+b} dM_A = \int_{(L-b)/2}^{(L+b)/2} dM_A$$

$$= \frac{q}{L^2}\int_{(L-b)/2}^{(L+b)/2} x(L-x)^2 dx$$

$$= \frac{q}{L^2}\int_{(L-b)/2}^{(L+b)/2} (L^2x - 2Lx^2 + x^3)\, dx$$

$$= \frac{q}{L^2}\left[\frac{L^2x^2}{2} - \frac{2Lx^3}{3} + \frac{x^4}{4}\right]_{(L-b)/2}^{(L+b)/2}$$

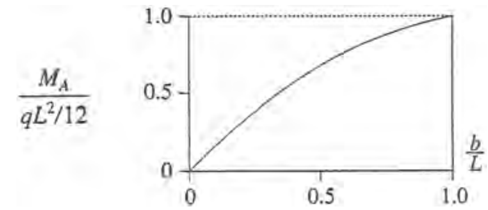
. . . (lenghty substitution) . . .

$$= \frac{qb}{24L}(3L^2 - b^2)$$

(a) $M_A = M_B = \dfrac{qb}{24L}(3L^2 - b^2)$ $\leftarrow$

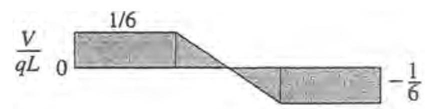
(b) GRAPH OF FIXED-END MOMENT

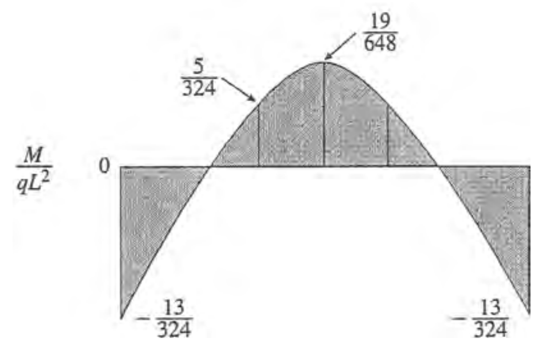
$$\frac{M_A}{qL^2/12} = \frac{b}{2L}\left(3 - \frac{b^2}{L^2}\right)$$



(c) SPECIAL CASE $a = b = L/3$

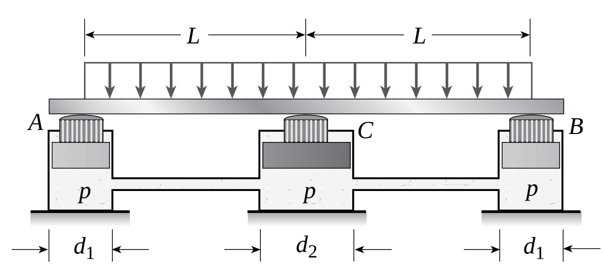
$$R_A = R_B = \frac{qL}{6} \qquad M_A = M_B = \frac{13qL^2}{324}$$



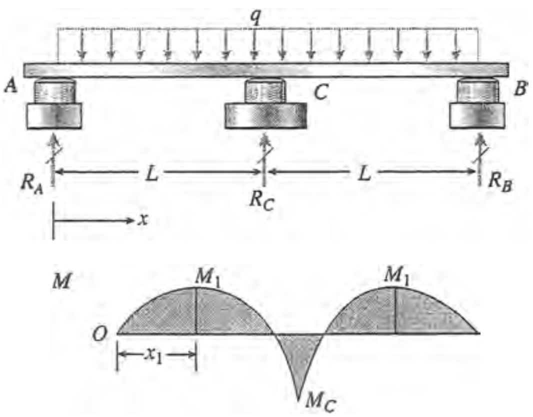


Problem 10.4-25 A beam supporting a uniform load of intensity q throughout its length rests on pistons at points A, C and B (see figure). The cylinders are filled with oil and are connected by a tube so that the oil pressure on each piston is the same. The pistons at A and B have diameter d_1, and the pistons at C has diameter d_2.

(a) Determine the ratio of d_2 to d_1 so that the largest bending moment in the beam is as small as possible.
(b) Under these optimum conditions, what is the largest bending moment M_{max} in the beam?
(c) What is the difference in elevation between point C and the end supports?



Solution 10.4-25 Beam supported by pistons



Bending moment $M = R_A x - \dfrac{qx^2}{2}$

Location of maximum positive moment

$$\frac{dM}{dx} = 0 \qquad R_A - qx = 0 \qquad x_1 = \frac{R_A}{q}$$

Maximum positive moment

$$M_1 = (M)_{x=x_1} = \frac{R_A^2}{2q}$$

Maximum negative moment

$$M_C = (M)_{x=L} = R_A L - \frac{qL^2}{2}$$

For the smallest maximum moment:

$|M_1| = |M_C|$ or $M_1 = -M_C$

$$\frac{R_A^2}{2q} = -R_A L + \frac{qL^2}{2}$$

Solve for R_A: $R_A = qL(\sqrt{2} - 1)$

Equilibrium

$$\sum F_{vert} = 0 \qquad 2R_A + R_C - 2qL = 0$$

$$R_C = 2qL\,(2 - \sqrt{2})$$

Reactions based upon pressure

$$R_A = R_B = p\left(\frac{\pi d_1^2}{4}\right) \qquad R_C = p\left(\frac{\pi d_2^2}{4}\right)$$

(a) $\therefore \dfrac{d_2}{d_1} = \sqrt{\dfrac{R_C}{R_A}} = \sqrt{\dfrac{2(2 - \sqrt{2})}{\sqrt{2} - 1}} = \sqrt[4]{8}$

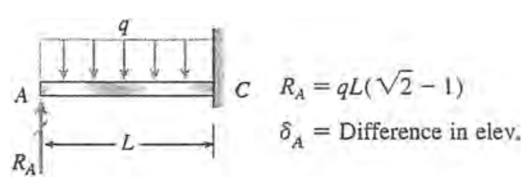
$= 1.682$ ←

(b) $M_{max} = M_1 = \dfrac{R_A^2}{2q} = \dfrac{qL^2}{2}(3 - 2\sqrt{2})$

$= 0.08579\, qL^2$ ←

(c) Difference in elevation

By symmetry, beam has zero slope at C.



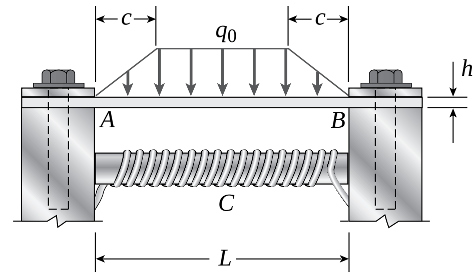
$$\delta_A = \frac{R_A L^3}{3EI} - \frac{qL^4}{8EI} = \frac{qL^4}{24EI}(8\sqrt{2} - 11)$$

$= 0.01307\, qL^4/EI$ ←

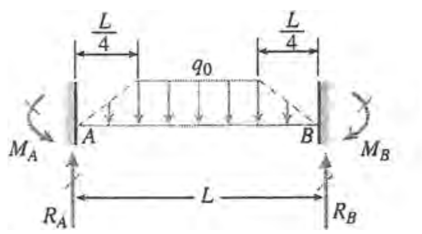
Point C is below points A and B by the amount $0.01307\, qL^4/EI$.

Problem 10.4-26 A thin steel beam AB used in conjunction with an electromagnet in a high-energy physics experiment is securely bolted to rigid supports (see figure). A magnetic field produced by coils C results in a force acting on the beam. The force is trapezoidally distributed with maximum intensity $q_0 = 18$ kN/m. The length of the beam between supports is $L = 200$ mm and the dimension c of the trapezoidal load is 50 mm. The beam has a rectangular cross section with width $b = 60$ mm and height $h = 20$ mm.

Determine the maximum bending stress σ_{max} and the maximum deflection δ_{max} for the beam. (Disregard any effects of axial deformations and consider only the effects of bending. Use $E = 200$ GPa.)



Solution 10.4-26 Fixed-end beam (trapezoidal load)



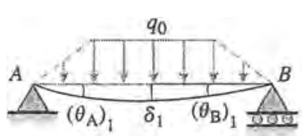
$c = \dfrac{L}{4}$

FROM SYMMETRY AND EQUILIBRIUM

$$M_A = M_B \qquad R_A = R_B = \frac{3q_0L}{8}$$

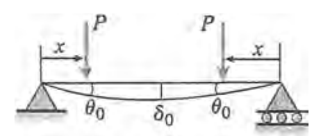
SELECT M_A AND M_B AS REDUNDANTS

RELEASED STRUCTURE WITH APPLIED LOAD



$(\theta_A)_1 = (\theta_B)_1$

Consider the following beam from Case 6, Table G-2:



$$\theta_0 = \frac{Px(L - x)}{2EI} \qquad \delta_0 = \frac{Px}{24EI}(3L^2 - 4x^2)$$

Consider the load P as an element of the distributed load.

Replace P by $q\,dx$, where

$q = \dfrac{4q_0x}{L}$ x from 0 to $L/4$

$q = q_0$ x from $L/4$ to $L/2$

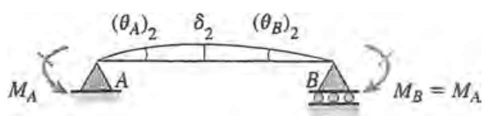
$$(\theta_A)_1 = \frac{1}{2EI}\int_0^{L/2}\left(\frac{4q_0x}{L}\right)(x)(L - x)\,dx + \frac{1}{2EI}\int_{L/4}^{L/2} q_0x(L - x)\,dx$$

$$= \frac{13q_0L^3}{1536EI} + \frac{11q_0L^3}{384EI} = \frac{19q_0L^3}{512EI}$$

$$\delta_1 = \frac{1}{24EI}\int_0^{L/4}\left(\frac{4q_0x}{L}\right)(x)(3L^2 - 4x^2)dx$$

$$+ \frac{1}{24EI}\int_{L/4}^{L/2} q_0 x(3L^2 - 4x^2)dx$$

$$= \frac{19q_0L^4}{7680EI} + \frac{19q_0L^4}{2048EI} = \frac{361q_0L^4}{30{,}720EI}$$



RELEASED STRUCTURE WITH REDUNDANTS

$(\theta_A)_2 = (\theta_B)_2 \qquad M_B = M_A$

From Case 10, Table G-2:

$$(\theta_A)_2 = \frac{M_AL}{2EI} \qquad \delta_2 = \frac{M_AL^2}{8EI}$$

COMPATIBILITY

$$\theta_A = (\theta_A)_1 - (\theta_A)_2 = 0$$

$$\frac{19q_0L^3}{512EI} - \frac{M_AL}{2EI} = 0 \qquad M_A = \frac{19q_0L^2}{256}$$

DEFLECTION AT THE MIDPOINT

$$\delta_{max} = \delta_1 - \delta_2 = \frac{361q_0L^4}{30{,}720EI} - \frac{M_AL^2}{8EI}$$

$$= \frac{361q_0L^4}{30{,}720EI} - \left(\frac{19q_0L^2}{256}\right)\left(\frac{L^2}{8EI}\right)$$

$$= \frac{19q_0L^4}{7680EI}$$

BENDING MOMENT AT THE MIDPOINT

$$M_C = R_A\left(\frac{L}{2}\right) - M_A - \frac{q_0L^2}{24} - \frac{q_0L^2}{32}$$

$$= \frac{3q_0L}{8}\left(\frac{L}{2}\right) - \frac{19q_0L^2}{256} - \frac{7q_0L^2}{96} = \frac{31q_0L^2}{768}$$

MAXIMUM BENDING MOMENT

$$M_A > M_C \qquad \therefore M_{max} = M_A = \frac{19q_0L^2}{256}$$

NUMERICAL VALUES

$q_0 = 18$ kN/m $\qquad L = 200$ mm $\qquad b = 60$ mm

$h = 20$ mm $\qquad E = 200$ GPa

$$S = \frac{bh^2}{6} = 4.0 \times 10^{-6}\ \text{m}^3$$

$$I = \frac{bh^3}{12} = 40 \times 10^{-9}\ \text{m}^4$$

$$M_{max} = \frac{19q_0L^2}{256} = 53.44\ \text{N}\cdot\text{m}$$

$$\sigma_{max} = \frac{M_{max}}{S} = 13.4\ \text{MPa} \qquad \leftarrow$$

$$\delta_{max} = \frac{19q_0L^4}{7680EI} = 0.00891\ \text{mm} \qquad \leftarrow$$

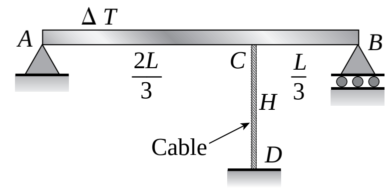
Temperature Effects

The beams described in the problems for Section 10.5 have constant flexural rigidity EI.

Problem 10.5-1 A cable CD of length H is attached to the third point of a simple beam AB of length L (see figure). The moment of inertia of the beam is I, and the effective cross-sectional area of the cable is A. The cable is initially taut but without any initial tension.

(a) Obtain a formula for the tensile force S in the cable when the temperature drops uniformly by ΔT degrees, assuming that the beam and cable are made of the same material (modulus of elasticity E and coefficient of thermal expansion α). Use the method of superposition in the solution.

(b) Repeat part (a) assuming a wood beam and steel cable.



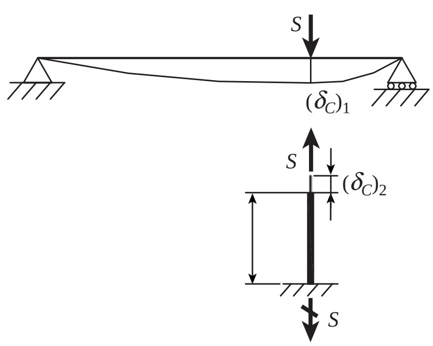
Solution 10.5-1

ΔT = Decrease in temperature. Use method of superposition. Select tensile force S in the cable as redundant.

I = Moment of inertia of beam

A = Cross-sectional area of cable

RELEASED STRUCTURE



(a) BEAM & CABLE ARE SAME MATERIAL

$$(\delta_c)_1 = \frac{4SL^3}{243EI} \quad \text{(downward)}$$

CABLE $\quad (\delta_c)_2 = \alpha H(\Delta T) - \dfrac{SH}{EA} \quad$ (downward)

COMPATIBILITY

$$(\delta_c)_1 = (\delta_c)_2 \qquad \frac{4SL^3}{243EI} = \alpha H(\Delta T) - \frac{SH}{EA}$$

SOLVE FOR S: $S = \dfrac{243EIAH\alpha(\Delta T)}{4AL^3 + 243IH} \quad \leftarrow$

(b) WOOD BEAM, STEEL CABLE

$$(\delta_c)_1 = \frac{4SL^3}{243E_W I} \quad \text{(downward)}$$

CABLE $\quad (\delta_c)_2 = \alpha_s H(\Delta T) - \dfrac{SH}{E_s A} \quad$ (downward)

COMPATIBILITY

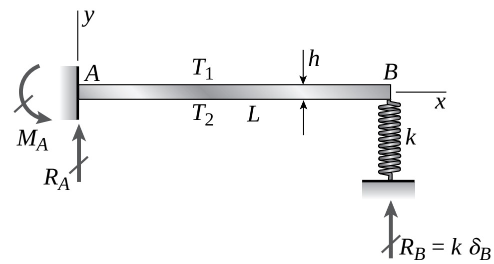
$$(\delta_c)_1 = (\delta_c)_2 \qquad \frac{4SL^3}{243E_W I} = \alpha_s H(\Delta T) - \frac{SH}{E_S A}$$

SOLVE FOR S:

$$S = \frac{243E_S E_W IAH\alpha_s(\Delta T)}{4AL^3E_S + 243IHE_W} \quad \leftarrow$$

Problem 10.5-2 A propped cantilever beam, fixed at the left-hand end A and simply supported at the right-hand end B, is subjected to a temperature differential with temperature T_1 on its upper surface and T_2 on its lower surface (see figure).

(a) Find all reactions for this beam. Use the method of superposition in the solution. Assume the spring support is unaffected by temperature.
(b) What are the reactions when $k \to \infty$?



Probs. 10.5-2 and 10.5-3

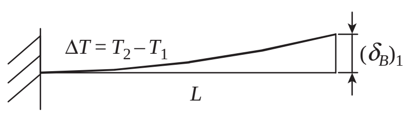
Solution 10.5-2

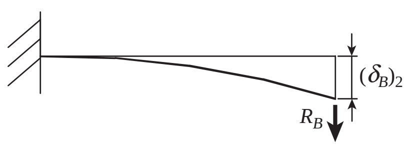
(a) Reactions assuming an elastic spring at B

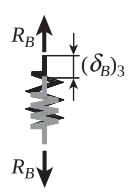
Use the method of superposition.

Select R_B as the redundant.

Released structure







$$(\delta_B)_1 = \frac{\alpha(T_2 - T_1)L^2}{2h}$$

$$(\delta_B)_2 = \frac{R_B L^3}{3EI}$$

$$(\delta_B)_3 = (\delta_B)_1 - (\delta_B)_2 = \frac{R_B}{k}$$

$$R_B = \frac{\alpha(T_2 - T_1)L^2}{2h}\left(\frac{3EIk}{3EI + L^3k}\right) \quad \text{(downward)}$$

From equilibrium

$$R_A = -R_B = -\frac{\alpha(T_2 - T_1)L^2}{2h}\left(\frac{3EIk}{3EI + L^3k}\right) \quad \text{(upward)}$$

$$M_A = R_B L = \frac{\alpha(T_2 - T_1)L^3}{2h}\left(\frac{3EIk}{3EI + L^3k}\right)$$

(counter-clockwise) $\leftarrow$

(b) Reactions assuming an spring at B is rigid

$$R_B = \frac{3EI\alpha(T_2 - T_1)}{2hL} \quad \text{(downward)}$$

$$R_A = -R_B = -\frac{3EI\alpha(T_2 - T_1)}{2hL} \quad \text{(upward)}$$

$$M_A = R_B L = \frac{3EI\alpha(T_2 - T_1)}{2h}$$

(counter-clockwise) $\leftarrow$

Problem 10.5-3 Solve the preceding problem by integrating the differential equation of the deflection curve.

Solution 10.5-3

(a) DIFFERENTIAL EQUATION (EQ. 10-39b)

$$EIv'' = M + \frac{\alpha EI(T_2 - T_1)}{h}$$

$$EIv'' = -R_B(L - x) + \frac{\alpha EI(T_2 - T_1)}{h}$$

$$EIv' = -R_B Lx + R_B\frac{x^2}{2} + \frac{\alpha EI(T_2 - T_1)}{h}x + C_1$$

B.C. 1 $v'(0) = 0 \qquad \therefore C_1 = 0$

$$EIv = -R_B L\frac{x^2}{2} + R_B\frac{x^3}{6} + \frac{\alpha EI(T_2 - T_1)}{2h}x^2 + C_2$$

B.C. 2 $v(0) = 0 \qquad \therefore C_2 = 0$

B.C. 3 $v(L) = \delta_B = \dfrac{R_B}{k}$

$$\therefore R_B = \frac{\alpha EI(T_2 - T_1)L^2}{2h}\left(\frac{3k}{3EI + L^3k}\right)$$

(downward) $\longleftarrow$

FROM EQUILIBRIUM

$$R_A = -R_B = -\frac{\alpha EI(T_2 - T_1)L^2}{2h}\left(\frac{3k}{3EI + L^3k}\right)$$

(upward) $\longleftarrow$

$$M_A = R_B L = \frac{\alpha EI(T_2 - T_1)L^3}{2h}\left(\frac{3k}{3EI + L^3k}\right)$$

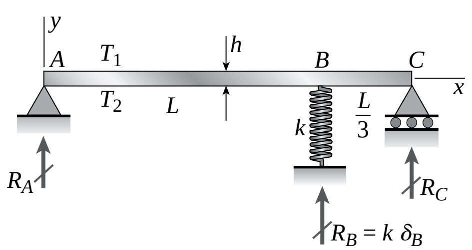
(counter-clockwise) $\longleftarrow$

(b) SAME REACTIONS AS IN 10.5-2(b) WHEN $k \to \infty$

Problem 10.5-4 A two-span beam with spans of lengths L and $L/3$ is subjected to a temperature differential with temperature T_1 on its upper surface and T_2 on its lower surface (see figure).

(a) Determine all reactions for this beam. Use the method of superposition in the solution. Assume the spring support is unaffected by temperature.

(b) What are the reactions when $k \to \infty$?



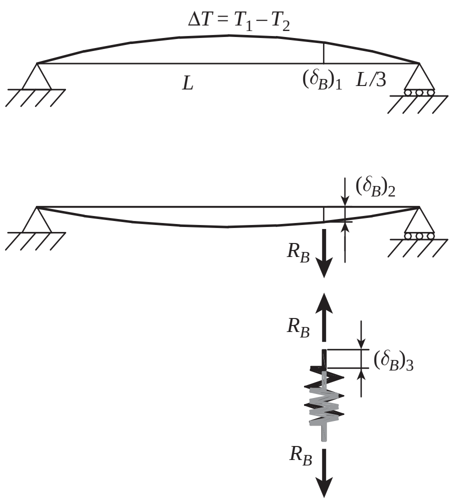
Probs. 10.5-4 and 10.5-5

Solution 10.5-4

(a) Use the method of superposition.

Select R_B as the redundant.

RELEASED STRUCTURE



$$(\delta_B)_1 = \frac{\alpha(T_1 - T_2)L^2}{6h}$$

$$(\delta_B)_2 = \frac{R_B L^3}{36EI}$$

COMPATIBILITY

$$(\delta_B)_3 = (\delta_B)_1 - (\delta_B)_2 = \frac{R_B}{k}$$

$$R_B = -\frac{\alpha(T_1 - T_2)L^2}{h}\left(\frac{6EIk}{36EI + L^3k}\right) \quad \text{(downward)}$$

FROM EQUILIBRIUM

$$R_A + R_B + R_C = 0$$

$$\sum M_C = 0:$$

$$R_A = \frac{1}{4}R_B = \frac{\alpha(T_1 - T_2)L^2}{2h}\left(\frac{3EIk}{36EI + L^3k}\right) \quad \text{(upward)}$$

$$R_C = \frac{3}{4}R_B = \frac{\alpha(T_1 - T_2)L^2}{2h}\left(\frac{9EIk}{36EI + L^3k}\right) \quad \text{(upward)}$$

(b) $R_B = -\dfrac{6EI\alpha(T_1 - T_2)}{Lh}$ (downward)

$$R_A = \frac{3EI\alpha(T_1 - T_2)}{2Lh} \quad \text{(upward)}$$

$$R_C = \frac{9EI\alpha(T_1 - T_2)}{2Lh} \quad \text{(upward)}$$

Problem 10.5-5 Solve the preceding problem by integrating the differential equation of the deflection curve

Solution 10.5-5

(a) EQUILIBRIUM

$$R_A + R_B + R_C = 0$$

$$\sum M_C = 0: R_A = -\frac{1}{4}R_B$$

$$\sum M_A = 0: R_C = -\frac{3}{4}R_B$$

DIFFERENTIAL EQUATION (EQ. 10-39b)

For $0 \le x \le L$

$$EIv'' = R_A x + \frac{\alpha EI(T_1 - T_2)}{h}$$

$$EIv' = \frac{1}{2}R_A x^2 + \frac{\alpha EI(T_1 - T_2)}{h}x + C_1 \qquad (1)$$

$$EIv = \frac{1}{6}R_A x^3 + \frac{\alpha EI(T_1 - T_2)}{2h}x^2 + C_1 x + C_2 \qquad (2)$$

B.C.1 $v(0) = 0$ $\therefore C_2 = 0$

For $L \le x \le \dfrac{4}{3}L$

$$EIv'' = R_A x + \frac{\alpha EI(T_1 - T_2)}{h} - 4R_A(x - L)$$

$$EIv'' = -3R_A x + \frac{\alpha EI(T_1 - T_2)}{h} + 4R_A L$$

$$EIv' = -\frac{3}{2}R_A x^2 + \frac{\alpha EI(T_1 - T_2)}{h}x$$
$$+ 4R_A Lx + C_3 \qquad (3)$$

$$EIv = -\frac{1}{2}R_A x^3 + \frac{\alpha EI(T_1 - T_2)}{2h}x^2$$
$$+ 2R_A Lx^2 + C_3 x + C_4 \qquad (4)$$

B.C. 2 continuity condition at point B

At $x = L$: $(v')_{\text{left}} = (v')_{\text{right}}$

$C_1 = 2R_A L^2 + C_3$

B.C. 3 continuity condition at point B

At $x = L$: $(v)_{\text{left}} = (v)_{\text{right}}$

B.C. 4 $v\left(\frac{4}{3}L\right) = 0$

B.C. 5 $v(L)_{\text{left}} = v(L)_{\text{right}} = \frac{R_B}{k}$

$$R_A = \frac{\alpha(T_1 - T_2)L^2}{2h}\left(\frac{3EIk}{36EI + L^3k}\right) \quad \text{(upward)}$$

$$R_B = -\frac{\alpha(T_1 - T_2)L^2}{h}\left(\frac{6EIk}{36EI + L^3k}\right) \quad \text{(downward)}$$

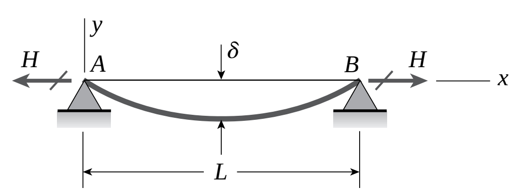
$$R_C = \frac{\alpha(T_1 - T_2)L^2}{2h}\left(\frac{9EIk}{36EI + L^3k}\right) \quad \text{(upward)}$$

(b) SAME REACTIONS AS IN 10.5-4(b) WHEN $k \to \infty$

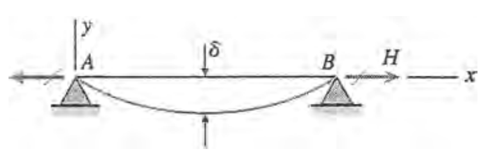
Longitudinal Displacements at the Ends of Beams

Problem 10.6-1 Assume that the deflected shape of a beam AB with *immovable* pinned supports (see figure) is given by the equation $v = -\delta \sin \pi x/L$, where δ is the deflection at the midpoint of the beam and L is the length. Also, assume that the beam has constant axial rigidity EA.

(a) Obtain formulas for the longitudinal force H at the ends of the beam and the corresponding axial tensile stress σ_t.
(b) for an aluminum-alloy beam with $E = 10 \times 10^6$ psi, calculate the tensile stress σ_t when the ratio of the deflection δ to the length L equals 1/200, 1/400, and 1/600.



Solution 10.6-1 Beam with immovable supports



(a) $v = -\delta \sin \frac{\pi x}{L} \qquad \frac{dv}{dx} = -\frac{\pi\delta}{L}\cos\frac{\pi x}{L}$

Eq. (10-42): $\lambda = \frac{1}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx = \frac{\pi^2\delta^2}{4L}$

Eq. (10-45): $H = \frac{EA\lambda}{L} = \frac{\pi^2 EA\delta^2}{4L^2} \qquad \leftarrow$

Eq. (10-46): $\sigma_1 = \frac{H}{A} = \frac{\pi^2 E\delta^2}{4L^2} \qquad \leftarrow$

(b) ALUMINUM ALLOY

$E = 10 \times 10^6$ psi

$\sigma_1 = 24.67 \times 10^6 \left(\frac{\delta}{L}\right)^2$ (psi)

$\frac{\delta}{L}$	$\frac{1}{200}$	$\frac{1}{400}$	$\frac{1}{600}$
σ_t (psi)	617	154	69

NOTE: The axial stress increases as the deflection increases.

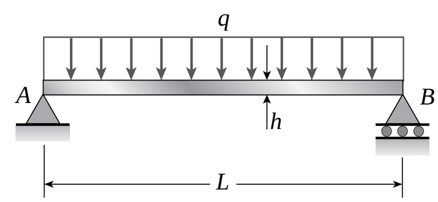
Problem 10.6-2

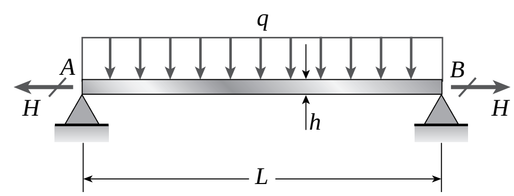
(a) A simple beam AB with length L and height h supports a uniform load of intensity q (see the *first part* of the figure). Obtain a formula for the curvature shortening λ of this beam. Also, obtain a formula for the maximum bending stress σ_b in the beam due to the load q.

(b) Now assume that the ends of the beam are pinned so that curvature shortening is prevented and a horizontal force H develops at the supports (see the *second part* of the figure). Obtain a formula for the corresponding axial tensile stress σ_t.

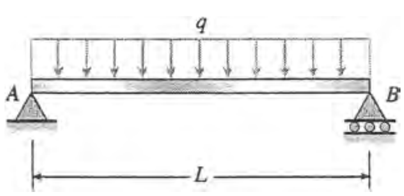
(c) Using the formulas in parts (a) and (b), calculate the curvature shortening λ, the maximum bending stress σ_b, and the tensile stress σ_t for the following steel beam: length $L = 3$ m, height $h = 300$ mm, modulus of elasticity $E = 200$ GPa, and moment of inertia $I = 36 \times 10^6\ \text{mm}^4$. Also, the load on the beam has intensity $q = 25$ kN/m.

Compare the tensile stress σ_t produced by the axial forces with the maximum bending stress σ_b produced by the uniform load.





Solution 10.6-2 Beam with uniform load



h = height of beam

(a) Curvature shortening

From Case 1, Table G-2:

$$\frac{dv}{dx} = -\frac{q}{24EI}(L^3 - 6Lx^2 - 4x^3)$$

Eq. (10-42): $$\lambda = \frac{1}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx$$

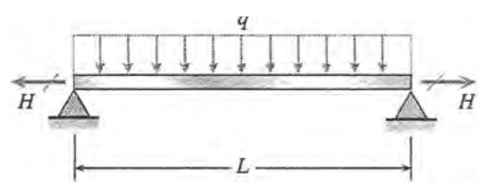
$$= \frac{17q^2L^7}{40{,}320E^2I^2} \quad \leftarrow$$

Bending stress

$$M_{\max} = \frac{qL^2}{8} \qquad c = \frac{h}{2}$$

$$\sigma_b = \frac{Mc}{I} = \frac{qhL^2}{16I} \quad \leftarrow$$

(b) Immovable supports



Eq. (10-45): $$H = \frac{EA\lambda}{L}$$

Eq. (10-46): $$\sigma_t = \frac{H}{A} = \frac{E\lambda}{L} = \frac{17q^2L^6}{40{,}320EI^2} \quad \leftarrow$$

(c) Numerical values $\quad q = 25$ kN/m

$L = 3$ m $\quad h = 300$ mm $\quad E = 200$ GPa

$I = 36 \times 10^6\ \text{mm}^4 \quad \lambda = 0.01112$ mm $\leftarrow$

$\sigma_b = 117.2$ MPa $\quad \sigma_t = 0.7411$ MPa $\leftarrow$

The bending stress is much larger than the axial tensile stress due to curvature shortening.

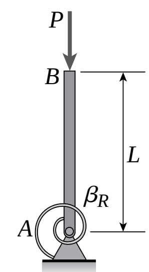
11

Columns

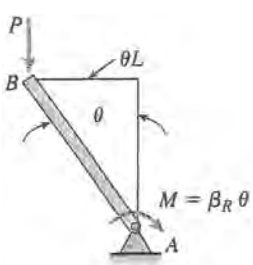
Idealized Buckling Models

Problem 11.2-1 The figure shows an idealized structure consisting of one or more **rigid bars** with pinned connections and linearly elastic springs. Rotational stiffness is denoted β_R, and translational stiffness is denoted β.

Determine the critical load P_{cr} for the structure.



Solution 11.2-1 **Rigid bar** *AB*



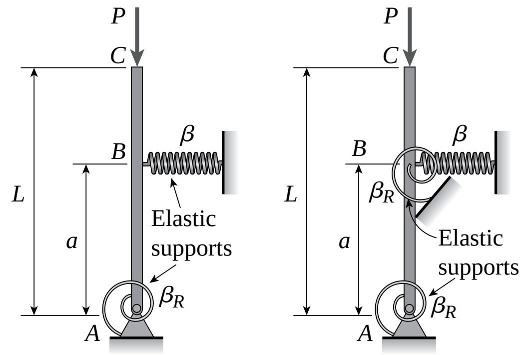
$$\sum M_A = 0$$

$$P(\theta L) - \beta_R \theta = 0$$

$$P_{cr} = \frac{\beta_R}{L} \quad \leftarrow$$

Problem 11.2-2 The figure shows an idealized structure consisting of one or more **rigid bars** with pinned connections and linearly elastic springs. Rotational stiffness is denoted β_R, and translational stiffness is denoted β.

(a) Determine the critical load P_{cr} for the structure from figure part (a).
(b) Find P_{cr} if another rotational spring is added at B from figure part (b).



(a) (b)

Solution 11.2-2

(a) $\Sigma M_A = 0$

$$P\theta L - \beta\theta a^2 - \beta_R\theta = 0$$

$$P_{cr} = \frac{\beta a^2 + \beta_R}{L} \quad \leftarrow$$

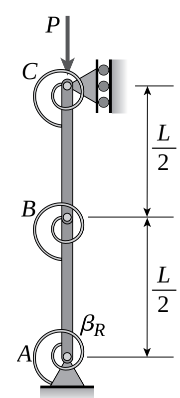
(b) $\Sigma M_A = 0$

$$P\theta L - \beta\theta a^2 - 2\beta_R\theta = 0$$

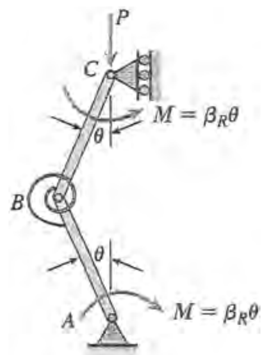
$$P_{cr} = \frac{\beta a^2 + 2\beta_R}{L} \quad \leftarrow$$

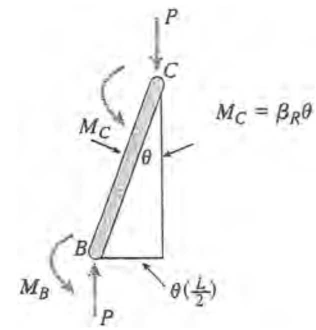
Problem 11.2-3 The figure shows an idealized structure consisting of one or more **rigid bars** with pinned connections and linearly elastic springs. Rotational stiffness is denoted β_R and translational stiffness is denoted β.

Determine the critical load P_{cr} for the structure.



Solution 11.2-3 Two rigid bars with a pin connection





$\sum M_A = 0$ Shows that there are no horizontal reactions at the supports.

FREE-BODY DIAGRAM OF BAR *BC*

$M_C = \beta_R \theta$

$M_B = \beta_R (2\theta)$

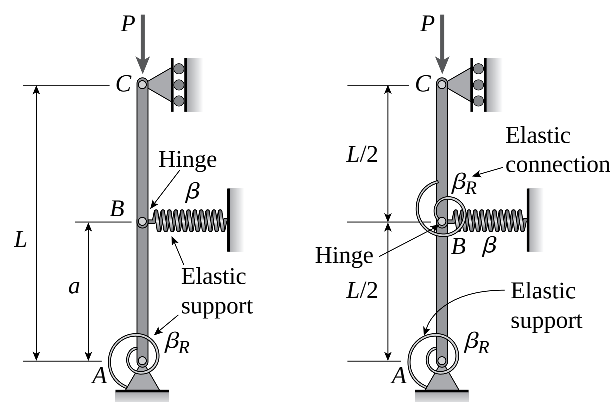
$\sum M_B = 0 \quad M_B + M_C - P\theta\left(\frac{L}{2}\right) = 0$

$\beta_R(2\theta) + \beta_R\theta = \frac{PL\theta}{2}$

$P_{cr} = \frac{6\beta_R}{L} \quad \leftarrow$

Problem 11.2-4 The figure shows an idealized structure consisting of bars *AB* and *BC* which are connected using a hinge at *B* and linearly elastic springs at *A* and *B*. Rotational stiffness is denoted β_R and translational stiffness is denoted β.

(a) Determine the critical load P_{cr} for the structure from figure part (a).
(b) Find P_{cr} if an elastic connection is now used to connect bar segments *AB* and *BC* from figure part (b).



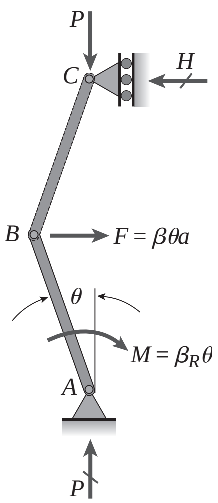
(a) (b)

Solution 11.2-4

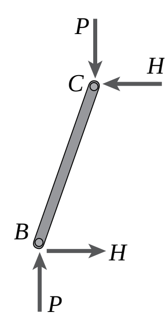
(a) $\sum M_A = 0$

$$\beta\theta a^2 + \beta_R\theta - HL = 0$$

$$H = \frac{\beta\theta a^2 + \beta_R\theta}{L}$$



FREE-BODY DIAGRAM OF BAR BC



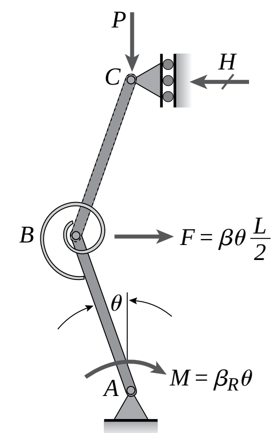
$\sum M_B = 0 \qquad H(L - a) - P(\theta a) = 0$

$$P_{cr} = \frac{(\beta a^2 + \beta_R)(L - a)}{aL} \qquad \longleftarrow$$

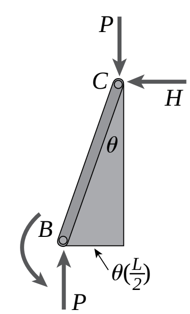
(b) $\sum M_A = 0$

$$\beta\theta\left(\frac{L}{2}\right)^2 + \beta_R\theta - HL = 0$$

$$H = \frac{\beta\theta L^2 + 4\beta_R\theta}{4L}$$



FREE-BODY DIAGRAM OF BAR BC



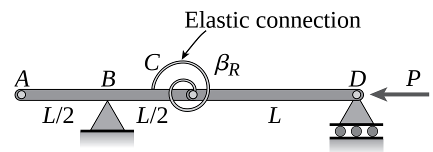
$$M_B = \beta_R(2\theta)$$

$$\Sigma M_B = 0 \qquad H\left(\frac{L}{2}\right) - P\left(\theta\frac{L}{2}\right) + M_B = 0$$

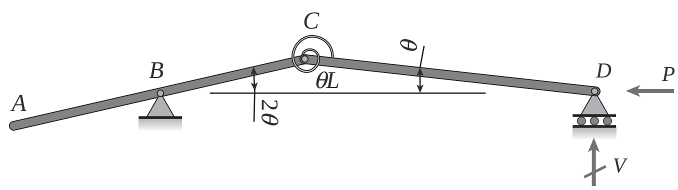
$$\frac{\beta\theta L^2 + 4\beta_R\theta}{4L}\left(\frac{L}{2}\right) - P\left(\theta\frac{L}{2}\right) + \beta_R(2\theta) = 0$$

$$P_{cr} = \frac{\beta L^2 + 20\beta_R}{4L} \qquad \longleftarrow$$

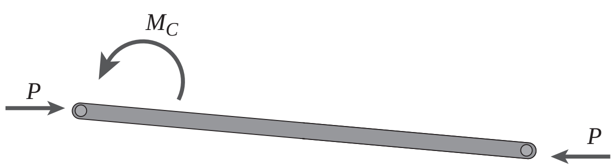
Problem 11.2-5 The figure shows an idealized structure consisting of two rigid bars joined by an elastic connection with rotational stiffness β_R. Determine the critical load P_{cr} for the structure.



Solution 11.2-5



$\Sigma M_B = 0 \qquad V = 0$



FREE-BODY DIAGRAM OF BAR *CD*

Moment in elastic connection

$= \beta_R \times$ total relative rotation $(\theta + 2\theta)$

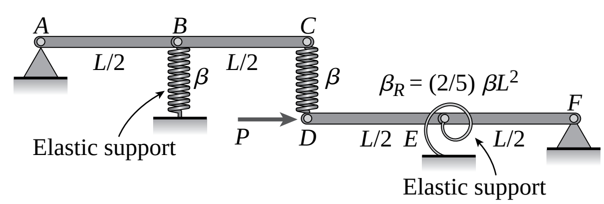
$$M_C = \beta_R(3\theta)$$

$$\Sigma M_C = 0 \quad P\theta L - \beta_R(3\theta) = 0$$

$$P_{cr} = \frac{3\beta_R}{L} \quad \leftarrow$$

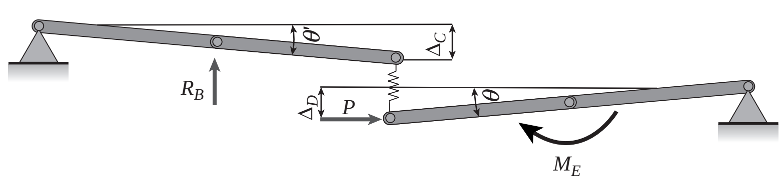
Problem 11.2-6 The figure shows an idealized structure consisting of rigid bars *ABC* and *DEF* joined by linearly elastic spring β between *C* and *D*. The structure is also supported by translational elastic support β at *B* and rotational elastic support at β_R at *E*.

Determine the critical load P_{cr} for the structure.



Solution 11.2-6

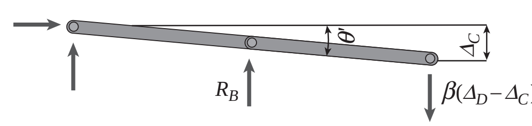
FREE-BODY DIAGRAM OF DEFORMED STRUCTURE



$$\Delta_C = L\theta' \qquad \Delta_D = L\theta \qquad R_B = \beta\left(\theta' \frac{L}{2}\right)$$

$$M_E = \beta_R\theta = \frac{2}{5}\beta L^2\theta$$

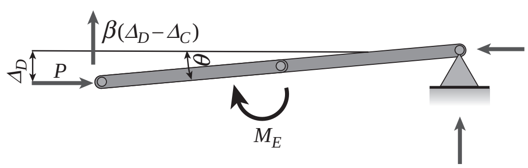
FREE-BODY DIAGRAM OF MEMBER *ABC*



$$\Sigma M_A = 0 \qquad R_B\frac{L}{2} - \beta(\Delta_D - \Delta_C)L = 0$$

$$\beta\left(\theta'\frac{L}{2}\right)\frac{L}{2} - \beta(L\theta - L\theta')L = 0$$

FREE-BODY DIAGRAM OF MEMBER DEF



$$\theta' = \frac{4}{5}\theta$$

$$\Sigma M_F = 0$$

$$P\Delta_D - \beta(\Delta_D - \Delta_C)L - M_E = 0$$

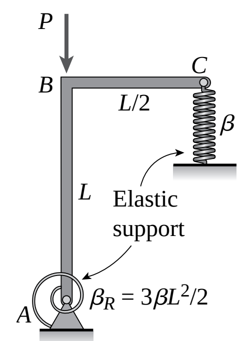
$$P(L\theta) - \beta\left[L\theta - L\left(\frac{4}{5}\theta\right)\right]$$

$$L - \frac{2}{5}\beta L^2\theta = 0$$

$$P_{cr} = \frac{3}{5}\beta L \qquad \leftarrow$$

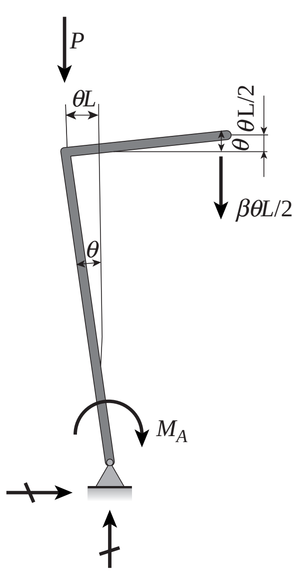
Problem 11.2-7 The figure shown an idealized structure consisting of an L-shaped **rigid bar** structure supported by linearly elastic springs at A and C. Rotational stiffness in denoted β_R and translational stiffness is denoted β.

Determine the critical load P_{cr} for the structure.



Solution 11.2-7

FREE-BODY DIAGRAM OF DEFORMED STRUCTURE



$$M_A = \beta_R\theta = \frac{3}{2}\beta L^2\theta$$

$$\Sigma M_A = 0$$

$$M_A - P(\theta L) + \beta\theta\left(\frac{L}{2}\right)^2 = 0$$

$$\frac{3}{2}\beta L^2\theta - P(\theta L) + \beta\theta\left(\frac{L}{2}\right)^2 = 0$$

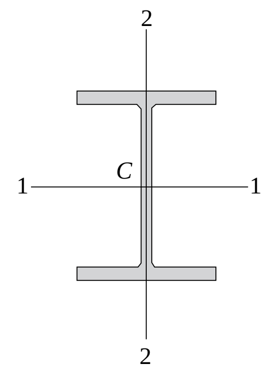
$$P_{cr} = \frac{7}{2}\beta L \qquad \leftarrow$$

Critical Loads of Columns with Pinned Supports

The problems for Section 11.3 are to be solved using the assumptions of ideal, slender, prismatic, linearly elastic columns (Euler buckling). Buckling occurs in the plane of the figure unless stated otherwise.

Problem 11.3-1 Calculate the critical load P_{cr} for a W 8 $\times$ 35 steel column (see figure) having length $L = 24$ ft and $E = 30 \times 10^6$ psi under the following conditions:

(a) The column buckles by bending about it's strong axis (axis 1-1), and (b) the column buckles by bending about its weak axis (axis 2-2). In both cases, assume that the column has pinned ends.



Problem 11.3.1-3.3

Solution 11.3-1 Column with pinned supports

W 8 $\times$ 35 steel column

$L = 24 \text{ ft} = 288 \text{ in.} \quad E = 30 \times 10^6 \text{ psi}$

$I_1 = 127 \text{ in.}^4 \quad I_2 = 42.6 \text{ in.}^4 \quad A = 10.3 \text{ in.}^2$

(a) Buckling about strong axis

$$P_{cr} = \frac{\pi^2 E I_1}{L^2} = 453 \text{ k} \quad \leftarrow$$

(b) Buckling about weak axis

$$P_{cr} = \frac{\pi^2 E I_2}{L^2} = 152 \text{ k} \quad \leftarrow$$

NOTE: $\sigma_{cr} = \dfrac{P_{cr}}{A} = \dfrac{453 \text{ k}}{10.3 \text{ in.}^2} = 44 \text{ ksi}$

$\therefore$ Solution is satisfactory if $\sigma_{PL} \geq 44$ ksi

Problem 11.3-2 Solve the preceding problem for a W 250 $\times$ 89 steel column having length $L = 10$ m. Let $E = 200$ GPa

Solution 11.3-2

W 250 $\times$ 89

$E = 200 \text{ GPa} \qquad L = 10.0 \text{ m} \qquad I_1 = 142 \times 10^6 \text{ mm}^4$

$I_2 = 48.3 \times 10^6 \text{ mm}^4 \qquad A = 11400 \text{ mm}^2$

Buckling about strong axis

$$P_{cr1} = \frac{\pi^2 E I_1}{L^2} \qquad P_{cr1} = 2803 \text{ kN} \quad \leftarrow$$

Buckling about weak axis

$$P_{cr2} = \frac{\pi^2 E I_2}{L^2} \qquad P_{cr2} = 953 \text{ kN} \quad \leftarrow$$

Note: $\sigma_{cr} = \dfrac{P_{cr1}}{A} \qquad \sigma_{cr} = 246 \text{ MPa}$

Solution is satisfactory if $\sigma_{PL} \geq 246$ MPa

Problem 11.3-3 Solve Problem 11.3-1 for a W 10 $\times$ 45 steel column having length $L = 28$ ft.

Solution 11.3-3 Column with pinned supports

W 10 × 45 steel column

$L = 28 \text{ ft} = 336 \text{ in.} \quad E = 30 \times 10^6 \text{ psi}$

$I_1 = 248 \text{ in.}^4 \quad I_2 = 53.4 \text{ in.}^4 \quad A = 13.3 \text{ in.}^2$

(a) Buckling about strong axis

$$P_{cr} = \frac{\pi^2 E I_1}{L^2} = 650 \text{ k} \quad \leftarrow$$

(b) Buckling about weak axis

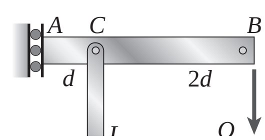
$$P_{cr} = \frac{\pi^2 E I_2}{L^2} = 140 \text{ k} \quad \leftarrow$$

NOTE: $\sigma_{cr} = \dfrac{P_{cr}}{A} = \dfrac{650 \text{ k}}{13.3 \text{ in.}^2} = 49 \text{ ksi}$

$\therefore$ Solution is satisfactory if $\sigma_{PL} \geq 49$ ksi

---

Problem 11.3-4 A horizontal beam AB is pin-supported at end A and carries a load Q at end B, as shown in the figure. The beam is supported at C by a pinned-end column of length L; the column is restrained laterally at $6.0L$ from the base at D. Assume the column can only buckle in the plane of the frame. The column is a solid steel bar ($E = 200$ GPa) of square cross section having length $L = 2.4$ m side dimensions $b = 70$ mm. Let dimensions $d = L/2$. Based upon the critical load of the column, determine the allowable moment M if the factor of safety with respect to buckling is $n = 2.0$.



Solution 11.3-4

Column CD (steel)

$E = 200 \text{ GPa} \quad L = 2.4 \text{ m} \quad d = \dfrac{L}{2} \quad d = 1.2 \text{ m}$

Square cross section: $\quad b = 70$ mm

Factor of safety: $\quad n = 2.0$

$I = \dfrac{b^4}{12} \quad I = 2.00 \times 10^6 \text{ mm}^4$

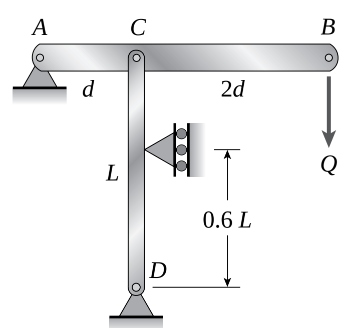
$P_{cr} = \dfrac{\pi^2 E I}{(0.6\, L)^2} \quad P_{cr} = 1905 \text{ kN}$

$P_{allow} = \dfrac{P_{cr}}{n} \quad P_{allow} = 952.3 \text{ kN}$

Beam $ACB \quad \Sigma M_A = 0 \quad M = Pd$

$M_{allow} = P_{allow} d \qquad M_{allow} = 1143 \text{ kN} \cdot \text{m} \quad \leftarrow$

Problem 11.3-5 A horizontal beam AB is pin-supported at end A and carries a load Q at joint B, as shown in the figure. The beam is also supported at C by a pinned-end column of length L; the column is restrained laterally at $0.6L$ from the base at D. Assume the column can only buckle in the plane of the frame. The column is a solid aluminium bar ($E = 10 \times 10^6$ psi) of square cross section having length $L = 30$ in. and side dimensions $b = 1.5$ in. Let dimension $d = L/2$. Based upon the critical load of the column, determine the allowable force Q if the factor of safety with respect to buckling is $n = 1.8$.



Solution 11.3-5

COLUMN CD (STEEL)

$E = 10 \times 10^6$ psi $\qquad L = 30$ in.

$d = \dfrac{L}{2} \qquad d = 15$ in.

Square cross section: $\quad b = 1.5$ in.

Factor of safety: $\quad n = 1.8$

$I = \dfrac{b^4}{12} \quad I = 0.422$ in.^{4}

$$P_{cr} = \frac{\pi^2 EI}{(0.6L)^2} \qquad P_{cr} = 129 \text{ k}$$

$$P_{allow} = \frac{P_{cr}}{n} \qquad P_{allow} = 71 \text{ k}$$

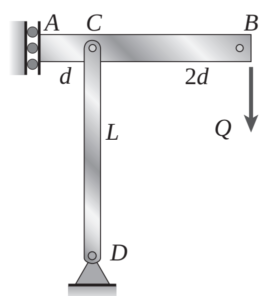
Beam ACB $\qquad \Sigma M_A = 0 \qquad Q = \dfrac{P}{3}$

$$Q_{allow} = \frac{P_{allow}}{3} \qquad Q_{allow} = 23.8 \text{ k} \quad \leftarrow$$

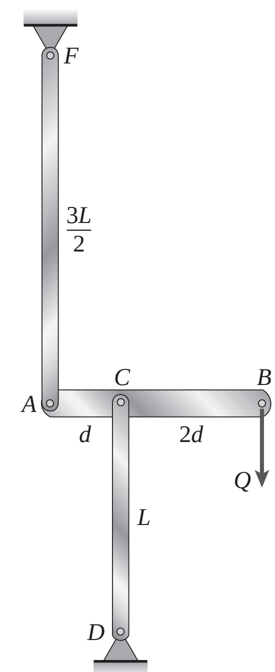
Problem 11.3-6 A horizontal beam AB is pin-supported at end A and carries a load Q at joind B, as shown in the figure part (a). The beam is also supported at C by a pinned-end column of length L. The column has flexural rigidity EI.

(a) For the case of a guided support at A (figure pare (a)), what is the critical load Q_{cr}? (In other words, at what load Q_{cr}? does the system collapse because of Euler buckling of the column DC?)

(b) Repeat (a) if the guided support at A is replaced by column AF with length $3/L2$ and flexural rigidity EI (see figure part (b)).



(a)



(b)

Solution 11.3-6

(a) $P_{cr} = \dfrac{\pi^2 EI}{L^2}$

FROM FREE-BODY DIAGRAM OF THE SYSTEM

$\Sigma F_y = 0 \qquad Q = P$

Therefore $Q_{cr} = P_{cr} = \dfrac{\pi^2 EI}{L^2} \qquad \leftarrow$

(b) Q_{cr} BASED UPON P_{cr} IN COLUMN AF

$$P_{cr,\,AF} = \frac{\pi^2 EI}{\left(\dfrac{3L}{2}\right)^2} = \frac{4\pi^2 EI}{9L^2}$$

FROM FREE-BODY DIAGRAM OF BEAM ACB

$\Sigma M_C = 0 \qquad Q = \dfrac{P_{AF}}{2}$

therefore $\qquad Q_{cr,\,AF} = \dfrac{P_{cr,\,AF}}{2} = \dfrac{2\pi^2 EI}{9L^2}$

Q_{cr} based upon P_{cr} in column CD

$P_{cr,CD} = \dfrac{\pi^2 EI}{L^2} \qquad \leftarrow$

FROM FREE-BODY DIAGRAM OF BEAM ACB

$\Sigma M_A = 0 \qquad Q = \dfrac{P_{CD}}{3}$

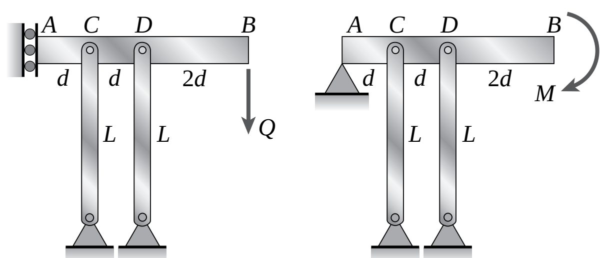
therefore $Q_{cr,\,CD} = \dfrac{P_{cr,\,CD}}{2} = \dfrac{\pi^2 EI}{3L^2}$

$Q_{cr} = Q_{cr,\,AF} = \dfrac{2\pi^2 EI}{9L^2} \qquad \leftarrow$

Column AF governs

Problem 11.3-7 A horizontal beam AB has a guided support at end A and carries a load Q at end B, as show in the figure part (a). The beam is supported at C and D by two identical pinned-end columns of length L. Each column has flexural rigidity EI.

(a) Find an expression for the critical load Q_{cr}. (In other words, at what load Q_{cr} does the system collapses because of Euler buckling of the columns?)
(b) Repeats (a) but assume spin support at A. Find an expression for the critical moment M_{cr} (i.e., find the moment M at B at which the system collapses because of Euler buckling of the columns).

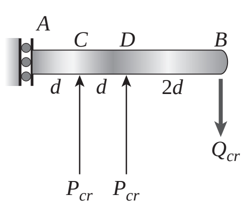


(a) (b)

Solution 11.3-7

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

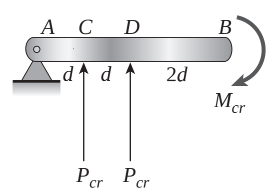
(a) COLLAPSE OCCURS WHEN BOTH COLUMNS REACH THE CRITICAL LOAD.



$\Sigma F_y = 0 \qquad Q_{cr} = 2P_{cr}$

Thus $Q_{cr} = \dfrac{2\pi^2 EI}{L^2} \qquad \leftarrow$

(b) COLLAPSE OCCURS WHEN BOTH COLUMNS REACH THE CRITICAL LOAD.



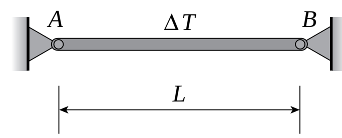
$\Sigma M_A = 0$

$M_{cr} = (2d)P_{cr} + (d)P_{cr} = 3dP_{cr}$

Thus $M_{cr} = \dfrac{3d\pi^2 EI}{L^2} \qquad \leftarrow$

Problem 11.3-8 A slender bar AB with pinned ends and length L is held between immovable supports (see figure).

What increase ΔT in the temperature of the bar will produce bucking at the Euler load?



Solution 11.3-8 Bar with immovable pin supports

L = length $\quad A$ = cross-sectional area

I = moment of inertia $\quad E$ = modules of elasticity

α = coefficient of increase in temperature

ΔT = uniform increase in temperature

AXIAL COMPRESSIVE FORCE IN BAR (EQ. 2-17)

$$P = EA\alpha\,(\Delta T)$$

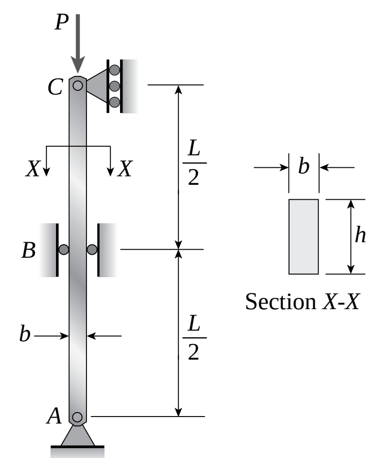
EULER LOAD $P_{cr} = \dfrac{\pi^2 EI}{L^2}$

INCREASE IN TEMPERATURE TO PRODUCE BUCKLING

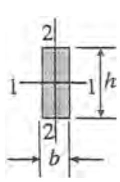
$$P = P_{cr} \quad EA\alpha\,(\Delta T) = \frac{\pi^2 EI}{L^2} \quad \Delta T = \frac{\pi^2 I}{\alpha A L^2} \quad \leftarrow$$

Problem 11.3-9 A rectangular column with cross-sectional dimensions b and h is pin-supported at ends A and C (see figure). AT midheight, the column is restrained in the plane of the figure but is free to deflect perpendicular to the plane of the figure.

Determine the ratio h/b such that the critical load is the same for buckling in the two principal planes of the column.



Solution 11.3-9 Column with restraint at midheight



Critical loads for buckling about axes 1-1 and 2-2:

$$P_1 = \frac{\pi^2 EI_1}{L^2} \quad P_2 = \frac{\pi^2 EI_2}{(L/2)^2} = \frac{4\pi^2 EI_2}{L^2}$$

FOR EQUAL CRITICAL LOADS

$$P_1 = P_2 \quad \therefore I_1 = 4I_2$$

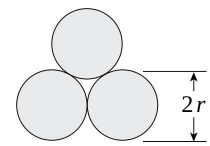
$$I_1 = \frac{bh^3}{12} \quad I_2 = \frac{hb^3}{12}$$

$$bh^3 = 4hb^3 \quad \frac{h}{b} = 2 \quad \leftarrow$$

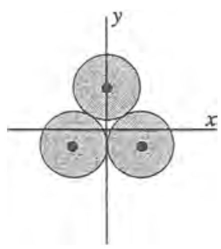
Problem 11.3-10 Three identical, solid circular rods, each of radius r and length L, are placed together to from a compression member (see the cross section shown in the figure).

Assuming pinned-end conditions, determine the critical load P_{cr} as follows: (a) The rods act independently as individual columns, and (b) the rods are bonded by epoxy throughout their lengths so that they function as a single member.

What is the effect on the critical load when the rods act as a single member?



Solution 11.3-10 Three solid circular rods



R= Radius L= Length

(A) Rods act independently

$$P_{cr} = \frac{\pi^2 EI}{L^2}(3) \quad I = \frac{\pi r^4}{4}$$

$$P_{cr} = \frac{3\pi^3 E r^4}{4L^2} \quad \leftarrow$$

(b) Rods are bonded together

The x and y axes have their origin at the centroid of the cross section. Because there are three different centroidal axes of symmetry, all centroidal axes are principal axes and all centroidal moments of inertia are equal (see Section 12.9).

From Case 9, Appendix D:

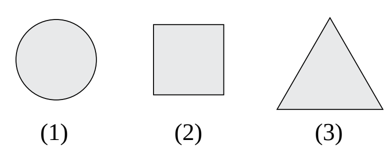
$$I = I_Y = \frac{\pi r^4}{4} + 2\left(\frac{5\pi r^4}{4}\right) = \frac{11\pi r^4}{4}$$

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{11\pi^3 E r^4}{4L^2} \quad \leftarrow$$

NOTE: Joining the rods so that they act as a single member increases the critical load by a factor of 11/3, or 3.67. $\leftarrow$

Problem 11.3-11 Three pinned-end columns of the same material have the same length and the same cross-sectional area (see figure). The columns are free to buckle in any direction. The columns have cross section as follows: (1) a circle, (2) a square, and (3) an equilateral triangle.

Determine the ratios $P_1 : P_2 : P_{\{3\}}$ of the critical loads for these columns.



Solution 11.3-11 Three pinned-end columns

E,L and A are the same for all three columns.

$$P_{cr} = \frac{\pi^2 EI}{L^2} \quad \therefore P_1 : P_2 : P_3 = I_1 : I_2 : I_3$$

(1) Circle Case 9, Appendix D

$$I = \frac{\pi d^4}{64} \quad A = \frac{\pi d^2}{4} \quad \therefore I_1 = \frac{A^2}{4\pi}$$

(2) Square Case 1, Appendix D

$$I = \frac{b^4}{12} \quad A = b^2 \quad \therefore I_2 = \frac{A^2}{12}$$

(3) Equilateral triangle Case 5, Appendix D

$$I = \frac{b^4\sqrt{3}}{96} \quad A = \frac{b^2\sqrt{3}}{4} \quad \therefore I_3 = \frac{A^2\sqrt{3}}{18}$$

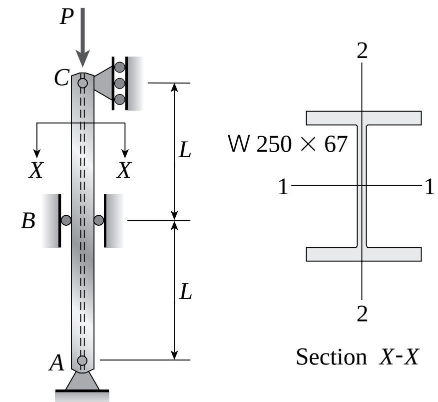
$$P_1 : P_2 : P_3 = I_1 : I_2 : I_3 = 1 : \frac{\pi}{3} : \frac{2\pi\sqrt{3}}{9}$$

$$= 1.000 : 1.047 : 1.209 \quad \leftarrow$$

NOTE: For each of the above cross sections, every centroidal axis has the same moment of inertia (see Section 12.9)

Problem 11.3-12 A long slender column *ABC* is pinned at ends *A* and *C* and compressed by an axial force *P* (see figure). At the midpoint *B*, lateral support is provided to prevent deflection in the plane of the figure. The column is a steel wide-flange section (W 250 × 67) with $E = 200$ GPa. The distance between lateral supports is $L = 5.5$ m.

Calculate the allowable load *P* using a factor of safety $n = 2.4$, taking into account the possibility of Euler buckling about either principal centroidal axis (i.e., axis 1-1 or axis 2-2).



Solution 11.3-12

W 250 × 67 $\quad E = 200$ GPa

$L = 5.5$ m $\qquad I_1 = 103 \times 10^6$ mm^4

$I_2 = 22.2 \times 10^6$ mm^4 $\qquad n = 2.4$

Buckling about axis 1-1

$$P_{cr1} = \frac{\pi^2 E I_1}{(2L)^2} \qquad P_{cr1} = 1680 \text{ kN}$$

Buckling about axis 2-2

$$P_{cr2} = \frac{\pi^2 E I_2}{(L)^2} \qquad P_{cr2} = 1449 \text{ kN}$$

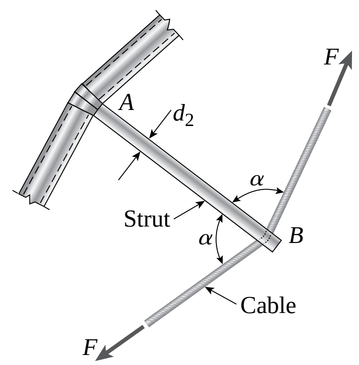
$P_{cr} = P_{cr2}$ $\qquad$ axis 2-2 governs

Allowable Load

$$P_{allow} = \frac{P_{cr}}{n} \qquad P_{allow} = 604 \text{ kN} \qquad \leftarrow$$

Problem 11.3-13 The roof over a concourse at an airport is supported by the use of pretensioned cables. At a typical joint in the roof structure, a strut *AB* is compressed by the action of tensile forces *F* in a cable that makes an angle $\alpha = 75°$ with the strut (see figure and photo). The strut is a circular tube of steel ($E = 30{,}000$ ksi) with outer diameter $d_2 = 2.5$ in. and inner diameter $d_1 = 2.0$ in. The strut is 5.75 ft long and is assumed to be pin-connected at both ends.

Using a factor of safety $n = 2.5$ with respect to the critical load, determine the allowable force *F* in the cable.



Solution 11.3-13

$E = 30000 \text{ ksi} \qquad d_2 = 2.5 \text{ in.} \qquad d_1 = 2.0 \text{ in.}$

$L = 5.75 \text{ ft} \quad n = 2.5 \quad \alpha = 75°$

$I = \frac{\pi}{64}(d_2^4 - d_1^4) \qquad I = 1.132 \text{ in.}^4$

$P_{cr} = \frac{\pi^2 EI}{L^2} \qquad P_{cr} = 70.40 \text{ k}$

Allowable load

$P_{allow} = \frac{P_{cr}}{n} \qquad P_{allow} = 28.16 \text{ k}$

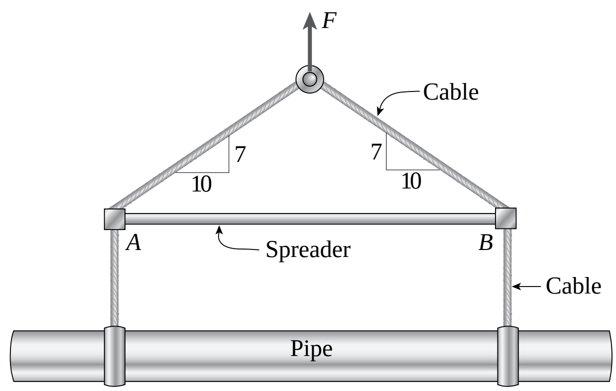
Equilibrium of joint B

$P = 2F\cos(\alpha)$

Thus $F_{allow} = \frac{P_{allow}}{2\cos(\alpha)} \qquad F_{allow} = 54.4 \text{ k} \quad \leftarrow$

Problem 11.3-14 The hoisting arrangement for lifting a large pipe is shown in the figure. The spreader is a steel tubular section with outer diameter 70 mm and inner diameter 57 mm. Its length is 2.6 m and its modulus of elasticity is 200 GPa.

Based upon a factor of safety of 2.25 with respect ot Euler buckling of the spreader, what is the maximum weight of pipe that can be lifted?(Assume pinned conditions at the ends of the spreader.)



Solution 11.3-14

$E = 30000 \text{ ksi} \qquad d_2 = 70 \text{ mm}$

$d_1 = 57 \text{ mm} \qquad L = 2.6 \text{ m}$

$n = 2.25 \qquad \alpha = \text{atan}\left(\frac{7}{10}\right)$

$I = \frac{\pi}{64}(d_2{}^4 - d_1{}^4) \qquad I = 660.4 \times 10^3 \text{ mm}^4$

$P_{cr} = \frac{\pi^2 EI}{L^2} \qquad P_{cr} = 199 \text{ kN}$

Allowable Load

$P_{allow} = \frac{P_{cr}}{n} \qquad P_{allow} = 88.6 \text{ kN}$

Equilibrium of joint A

$\Sigma F_{horiz} = 0 \quad -P + T\cos(\alpha) = 0$

$\Sigma F_{vert} = 0 \quad T\sin(\alpha) - \frac{w}{2} = 0$

Solve the equation

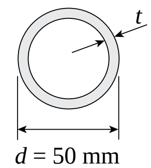
$W = 2P\tan(\alpha)$

Maximum weight of pipe

$W_{max} = 2P_{allow}\tan(\alpha) \qquad W_{max} = 124 \text{ kN} \quad \leftarrow$

Problem 11.3-15 A pinned-end strut of aluminium ($E = 10{,}400$ ksi) with length $L = 6$ ft is constructed of circular tubing with outside diameter $d = 2$ in. (see figure) The strut must resist an axial load $P = 4$ kips with a factor of safety $n = 2.0$ with respect to the critical load.

Determine the required thickness t of the tube.



Solution 11.3-15

$E = 10400$ ksi $\quad L = 6$ ft

$d = 2$ in. $\quad n = 2.0 \quad P = 4$ k

$P_{cr} = nP \quad P_{cr} = 8.0$ k

$$P_{cr} = \frac{\pi^2 EI}{L^2} \qquad I = \frac{P_{cr}L^2}{\pi^2 E} \qquad I = 0.404\ \text{in.}^4$$

MOMENT OF INERTIA

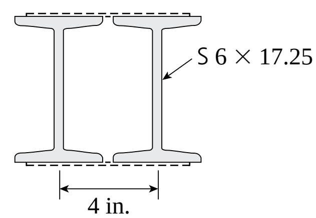
$$I = \frac{\pi}{64}\left[d^4 - (d - 2t)^4\right]$$

$$d^4 - (d - 2t)^4 = I\,\frac{64}{\pi}$$

$t_{min} = 0.165$ in.

Problem 11.3-16 The cross section of a column built up of two steel I-beams (S 150 × 25.7 sections) is shown in the figure. The beams are connected by spacer bars, or *lacing*, to ensure that they act together as a single column. (The lacing is represented by dashed lines in the figure.)

The column is assumed to have pinned ends and may buckle in any direction. Assuming $E = 200$ GPa and $L = 8.5$ m, calculate the critical load P_{cr} for the column.



Solution 11.3-16

S 150 × 25.7 $\quad E = 200$ GPa

$L = 8.5$ m $\quad I_1 = 10.9 \times 10^6\ \text{mm}^4$

$I_2 = 0.953 \times 10^6 \text{mm}^4$

$$A = 3260\ \text{mm}^2 \qquad d = \frac{100\ \text{mm}}{2}$$

COMPOSITE COLUMN

$I_x = 2I_1 \quad I_x = 21.80 \times 10^6\ \text{mm}^4$

$I_y = 2(I_2 + Ad^2) \quad I_y = 18.21 \times 10^6\ \text{mm}^4$

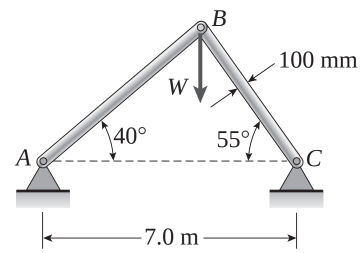
Buckling occurs about the y axis since $I_y < I_x$

$$\text{Critical load} \quad P_{cr} = \frac{\pi^2 EI_y}{L^2}$$

$P_{cr} = 497$ kN $\leftarrow$

Problem 11.3-17 The truss *ABC* shown in the figure supports a vertical load *W* at joint *B*. Each member is a slender circular steel pipe ($E = 30{,}000$ ksi) with outside diameter 4 in. And wall thickness 0.25 in. The distance between supports is 23 ft. Joint *B* is restrained against displacement perpendicular to the plane of the truss.

Determine the critical value W_{cr} of the load.



Solution 11.3-17

$E = 30000$ ksi $\qquad L = 23$ ft

$d_2 = 4$ in. $\qquad t = 0.25$ in.

$d_1 = d_2 - 2t \qquad d_1 = 3.50$ in.

$$I = \frac{\pi}{64}\left(d_2^4 - d_1^4\right) \qquad I = 5.200 \text{ in.}^4$$

$\theta_1 = 40° \qquad \theta_2 = 55°$

$$L_{AB} = L\left(\frac{\sin\left(\theta_2\right)}{\sin(180° - \theta_1 - \theta_2)}\right)$$

$L_{AB} = 18.912$ ft

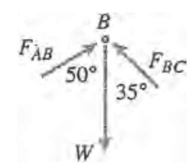
$$L_{BC} = L\left(\frac{\sin\left(\theta_1\right)}{\sin(180° - \theta_1 - \theta_2)}\right)$$

$L_{BC} = 14.841$ ft

Critical loads $\displaystyle P_{cr\_AB} = \frac{\pi^2 EI}{L_{AB}^2} \qquad P_{cr\_AB} = 29.89$ k

$$P_{cr\_BC} = \frac{\pi^2 EI}{L_{BC}^2} \qquad P_{cr\_BC} = 48.55 \text{ k}$$

Free-body diagram of joint *B*



$\Sigma F_{horiz} = 0 \quad F_{AB}\cos(\theta_1) - F_{BC}\cos(\theta_2) = 0$

$\Sigma F_{horiz} = 0 \quad F_{AB}\sin(\theta_1) - F_{BC}\sin(\theta_2) - W = 0$

Solve the two equations

$W = 1.7361 F_{AB} \qquad W = 1.3004\, F_{BC}$

Critical value of the load *W*
Based on member *AB*:

$W_{cr\_AB} = 1.7361 P_{cr\_AB} \qquad W_{cr\_AB} = 51.90$ k

Based on member *BC*:

$W_{cr\_BC} = 1.3004 P_{cr\_BC} \qquad W_{cr\_BC} = 63.13$ k

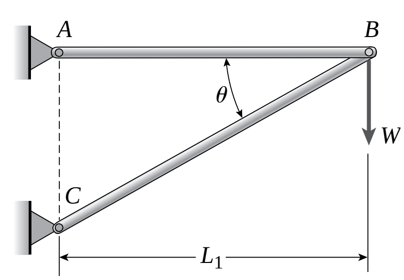
$W_{cr} = \min(W_{cr\_AB}, W_{cr\_BC})$

$W_{cr} = 51.9$ k $\quad \leftarrow$

Member *AB* governs

Problem 11.3-18 A truss *ABC* supports a load *W* at joint *B*, as shown in the figure. The length L_1 of member *AB* is fixed, but the length of strut *BC* varies as the angle θ is changed. Strut *BC* has a solid circular cross section. Joint *B* restrained against displacement perpendicular to the plane of the truss.

Assuming the collapse occurs by Euler buckling of the strut, determine the angle θ for minimum weight of the strut.



Solution 11.3-18 Truss ABC (minimum weight)

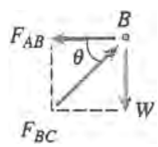
LENGTHS OF MEMBERS

$L_{AB} = L_1$ (a constant)

$L_{BC} = \dfrac{L_1}{\cos\theta}$ (angle θ is variable)

Strut BC may buckle.

FREE-BODY DIGRAM OF JOINT B



$\sum F_{\text{vert}} = 0 \qquad F_{BC}\sin\theta - W = 0$

STRUT BC (SOLID CIRCULAR BAR)

$A = \dfrac{\pi d^2}{4} \qquad I = \dfrac{\pi d^4}{64} \qquad \therefore I = \dfrac{A^2}{4\pi}$

$$P_{cr} = \frac{\pi^2 EI}{L_{BC}^2} = \frac{\pi E A^2 \cos^2\theta}{4\,L_1^2}$$

$$F_{BC} = P_{cr} \qquad \text{or} \qquad \frac{W}{\sin\theta} = \frac{\pi E A^2 \cos^2\theta}{4\,L_1^2}$$

Solve for area A: $A = \dfrac{2L_1}{\cos\theta}\left(\dfrac{W}{\pi E\sin\theta}\right)^{1/2}$

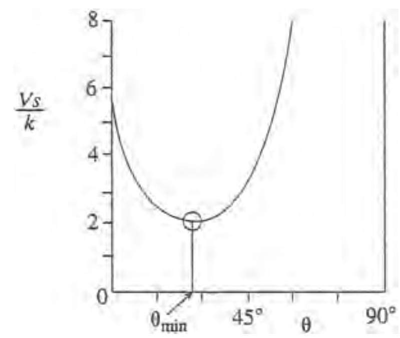
For minimum weight, the volume V_S of the strut must be a minimum.

$$V_s = AL_{BC} = \frac{AL_1}{\cos\theta} = \frac{2L_1^2}{\cos^2\theta}\left(\frac{W}{\pi E\sin\theta}\right)^{1/2}$$

All the terms are constants except $\cos\theta$ and $\sin\theta$ Therefore, we can write V_S in the following form:

$$V_S = \frac{k}{\cos^2\theta\sqrt{\sin\theta}} \quad \text{where } k \text{ is a constant.}$$

GRAPH OF $\dfrac{Vs}{k}$



$\theta_{\min}$ = angle for minimum volume (and minimum weight)

For minimum weight, the term $\cos^2\theta\sqrt{\sin\theta}$ must be a a maximum.

For minimum value, the derivative with respect to θ equals zero.

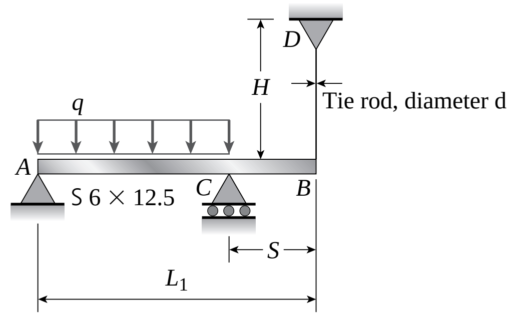
Therefore, $\dfrac{d}{d\theta}\left(\cos^2\theta\sqrt{\sin\theta}\right) = 0$

Taking the derivative and simplifying, we get $\cos^2\theta - 4\sin^2\theta = 0$

or $1 - 4\tan^2\theta = 0$ and $\tan\theta = \dfrac{1}{2}$

$$\therefore \theta_{\min} = \arctan\frac{1}{2} = 26.57° \quad \leftarrow$$

Problem 11.3-19 An S 6 × 12.5 steel cantilever beam *AB* is supported by a steel tie rod at *B* has shown. The tie rod is just taut when a roller support is added at *C* at a distances *S* to the left of *B*, then the distributed load *q* is applied to beam segment *AC*. Assume $E = 30 \times 10^6$ psi and neglect the self weight of the beam and tie rod. See Table E-2(a) in Appendix *E* for the properties of the *S*-Shape beam.



(a) What value of uniform load *q* will, if exceeded, result in buckling of the tie rod if L_1 =6 ft, *S* = 2 ft, *H* = 3 ft, *d* = 0.25 in.?
(b) What minimum beam moment of inertia I_b is required to prevent buckling of the tie rod if $q = 200$ lb/ft, $L_1 = 6$ ft, $H = 3$ ft, $d = 0.25$ in., $S = 2$ ft?
(c) For what distances *S* will the tie rod be just on the verge of buckling if $q = 200$ lb/ft, $L_1 = 6$ ft, $H = 3$ ft, $d = 0.25$ in.?

Solution 11.3-19

$E = 30 \times 10^6$ psi $\qquad L_1 = 6$ ft $\qquad d = \dfrac{1}{4}$ in.

$H = 3$ ft $\qquad q = 200\,\dfrac{lb}{ft}$

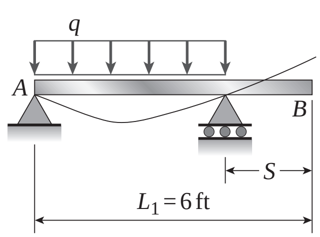
$I_r = \dfrac{\pi}{64}d^4 \qquad I_r = 191.7 \times 10^{-6}$ in.^{4}

$A_r = \dfrac{\pi}{4}d^2 \qquad A_r = 0.049$ in.^{2}

$P_{cr} = \dfrac{\pi^2 E I_r}{H^2} \qquad P_{cr} = 43.81\ lb$

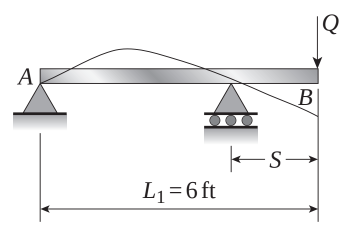
S 6 × 12.5 $\quad I_b = 22.0$ in.^{4}

Analyze 1st degree statically-indeterminate beam by letting force in tie rod be a redundant
Released beam AB with the uniform load *q*



Released beam AB with the redundant *Q*
From appendix G

$$\delta_B = \frac{q(L_1 - s)^3}{24EI_b}$$



FROM APPENDIX G

$$\delta_B'' = \frac{Qs^3}{3EI_b} + \frac{Qs(L_1 - s)s}{3EI_b}$$

Shortening in tie rod $\quad \delta_r = \dfrac{QH}{EA_r}$

Compatibility equation $\quad \delta_B' - \delta_B'' = \delta_r$

$$\frac{q(L_1 - s)^3}{24EI_b} - \left[\frac{Qs^3}{3EI_b} + \frac{Qs(L_1 - s)s}{3EI_b}\right] = \frac{QH}{EA_r}$$

$$Q = \frac{sqA_r(L_1 - s)^3}{8(s^2A_rL_1 + 3HI_b)} \tag{1}$$

(a) FOR $\quad Q = P_{cr} \quad s = 2.0$ ft

From (1) $\qquad q_{max} = \dfrac{8Q}{sA_r(L_1 - s)^3} \times (s^2A_rL_1 + 3HI_b)$

$q_{max} = 142.4\,\dfrac{\text{lb}}{\text{ft}} \quad \leftarrow$

(b) For $q = 200\,\dfrac{\text{lb}}{\text{ft}}$

From (1) $$I_{b\_\min} = \frac{-1}{24} sA_r \frac{8QsL_1 - qL_1^3 + 3sqL_1{}^2 - 3s^2qL_1 + s^3q}{QH}$$

$$l_{b\_\min} = 38.5\text{ in.}^4 \quad \leftarrow$$

(c) From (1) numerically solve for s when $Q = P_{cr}$ $\rightarrow$ 2 solutions are possible:

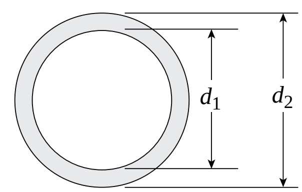
$s = 0.264$ ft and $s = 2.42$ ft $\leftarrow$

Columns with Other Support Conditions

The problems for Section 11.4 are to be solved using the assumptions of ideal, slender, prismatic, linearly elastic columns (Euler buckling). Buckling occurs in the plane of the figure unless stated otherwise.

Problem 11.4-1 An aluminum pipe column ($E = 10{,}400$ ksi) with length $L = 10.0$ ft has inside and outside diameters $d_1 = 5.0$ in. and $d_2 = 6.0$ in., respectively (see figure). The column is supported only at the ends and may buckle in any direction.

Calculate the critical load P_{cr} for the following end conditions: (1) pinned-pinned, (2) fixed-free, (3) fixed-pinned, and (4) fixed-fixed.



Probs.11.4-1 and 11.4-2

Solution 11.4-1 Aluminum pipe column

$d_2 = 6.0$ in. $\quad d_1 = 5.0$ in. $\quad E = 10{,}400$ ksi

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 32.94\text{ in.}^4$$

$L = 10.0$ ft $= 120$ in.

(1) Pinned-pinned

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(10{,}400\text{ ksi})(32.94\text{ in.}^4)}{(120\text{ in.})^2} = 235\text{ k} \quad \leftarrow$$

(2) Fixed-free $$P_{cr} = \frac{\pi^2 EI}{4L^2} = 58.7\text{ k} \quad \leftarrow$$

(3) Fixed-pinned $$P_{cr} = \frac{2.046\pi^2 EI}{L^2} = 480\text{ k} \quad \leftarrow$$

(4) Fixed-fixed $$P_{cr} = \frac{4\pi^2 EI}{L^2} = 939\text{ k} \quad \leftarrow$$

Problem 11.4-2 Solve the preceding problem for a steel pipe column ($E = 210$ GPa) with length $L = 1.2$ m, inner diameter $d_1 = 36$ mm, and outer diameter $d_2 = 40$ mm.

Solution 11.4-2 Steel pipe column

$d_2 = 40$ mm $\quad d_1 = 36$ mm $\quad E = 210$ GPa

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 43.22 \times 10^3 \text{ mm}^4 \quad L = 1.2 \text{ m}$$

(1) PINNED-PINNED $\quad P_{cr} = \dfrac{\pi^2 EI}{L^2} = 62.2 \text{ kN} \quad \leftarrow$

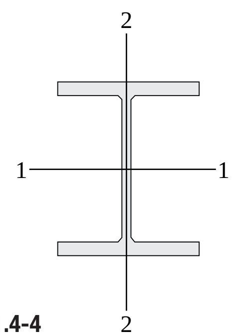
(2) FIXED-FREE $\quad P_{cr} = \dfrac{\pi^2 EI}{4L^2} = 15.6 \text{ kN} \quad \leftarrow$

(3) FIXED-PINNED $\quad P_{cr} = \dfrac{2.046\pi^2 EI}{L^2} = 127 \text{ kN} \quad \leftarrow$

(4) FIXED-FIXED $\quad P_{cr} = \dfrac{4\pi^2 EI}{L^2} = 249 \text{ kN} \quad \leftarrow$

Problem 11.4-3 A wide-flange steel column ($E = 30 \times 10^6$ psi) of W 12 × 87 shape (see figure) has length $L = 28$ ft. It is supported only at the ends and may buckle in any direction.

Calculate the allowable load P_{allow} based upon the critical load with a factor of safety $n = 2.5$. Consider the following end conditions: (1) pinned-pinned, (2) fixed-free, (3) fixed-pinned, and (4) fixed-fixed.



Probs. 11.4-3 and 11.4-4

Solution 11.4-3 Wide-flange column

W12 × 87 $\quad E = 30 \times 10^6$ psi

$L = 28$ ft $= 336$ in. $\quad n = 2.5 \quad I_2 = 241$ in.^{4}

(1) PINNED-PINNED

$$P_{allow} = \frac{P_{cr}}{n} = \frac{\pi^2 EI_2}{nL^2} = 253 \text{ k} \quad \leftarrow$$

(2) FIXED-FREE

$$P_{allow} = \frac{\pi^2 EI_2}{4nL^2} = 63.2 \text{ k} \quad \leftarrow$$

(3) FIXED-PINNED

$$P_{allow} = \frac{2.046\pi^2 EI_2}{nL^2} = 517 \text{ k} \quad \leftarrow$$

(4) FIXED-FIXED

$$P_{allow} = \frac{4\pi^2 EI_2}{nL^2} = 1011 \text{ k} \quad \leftarrow$$

Problem 11.4-4 Solve the preceding problem for a W 250 × 89 shape with length $L = 7.5$ m and $L = 200$ GPa.

Solution 11.4-4

$W\,250 \times 89 \qquad E = 200\text{ GPa}$

$L = 7.5\text{ m} \qquad n = 2.5$

$I_2 = 48.3 \times 10^6\text{ mm}^4$

(1) Pinned-pinned

$$P_{allow} = \frac{\pi^2 EI_2}{nL^2}$$

$P_{allow} = 678\text{ kN} \quad \leftarrow$

(2) Fixed-free

$$P_{allow} = \frac{\pi^2 EI_2}{4nL^2}$$

$P_{allow} = 169.5\text{ kN} \quad \leftarrow$

(3) Fixed-pinned

$$P_{allow} = \frac{2.046\pi^2 EI_2}{nL^2}$$

$P_{allow} = 1387\text{ kN} \quad \leftarrow$

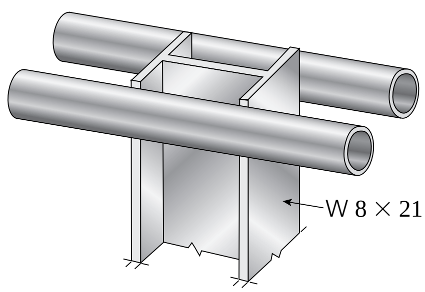
(4) Fixed-fied

$$P_{allow} = \frac{4\pi^2 EI_2}{nL^2}$$

$P_{allow} = 2712\text{ kN} \quad \leftarrow$

Problem 11.4-5 The upper end of a W 8 × 21 wide-flange steel column ($E = 30 \times 10^3$ ksi) is supported laterally between two pipes (see figure). The pipes are not attached to the column, and friction between the pipes and the column is unreliable. The base of the column provides a fixed support, and the column is 13 ft long.

Determine the critical load for the column, considering Euler buckling in the plane of the web and also perpendicular to the plane of the web.

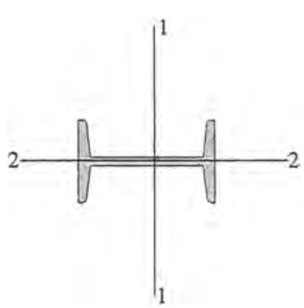


Solution 11.4-5 Wide-flange steel column

W 8 × 21 $\qquad E = 30 \times 10^3$ ksi

$L = 13\text{ ft} = 156\text{ in.} \qquad I_1 = 75.3\text{ in.}^4$

$I_2 = 9.77\text{ in.}^4$



Axis 1-1 (fixed-free)

$$P_{cr} = \frac{\pi^2 EI_1}{4L^2} = 229\text{ k}$$

Axis 2-2 (fixed-pinned)

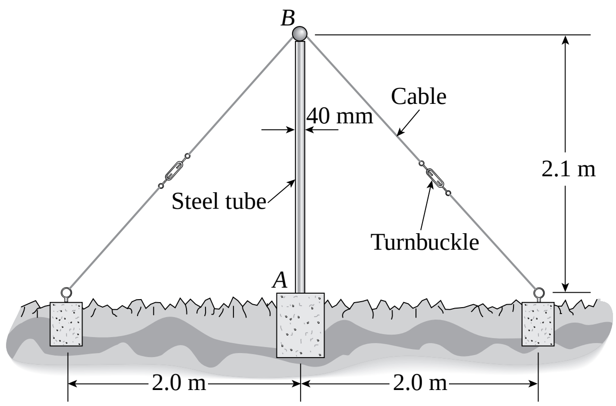
$$P_{cr} = \frac{2.046\pi^2 EI_2}{L^2} = 243\text{ k}$$

Buckling about axis 1-1 governs.

$P_{cr} = 229\text{ k} \quad \leftarrow$

Problem 11.4-6 A vertical post AB is embedded in a concrete foundation and held at the top by two cables (see figure). The post is a hollow steel tube with modulus of elasticity 200 GPa, outer diameter 40 mm, and thickness 5 mm. The cables are tightened equally by turnbuckles.

If a factor of safety of 3.0 against Euler buckling in the plane of the figure is desired, what is the maximum allowable tensile force T_{allow} in the cables?



Solution 11.4-6 Steel tube

$E = 200$ GPa $\quad d_2 = 40$ mm $\quad d_1 = 30$ mm

$L = 2.1$ m $\quad n = 3.0$

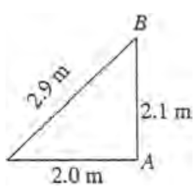
$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 85{,}903 \text{ mm}^4$$

Buckling in the plane of the figure means fixed-pinned end conditions.

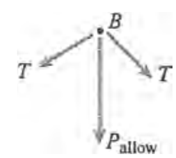
$$P_{cr} = \frac{2.046\pi^2 EI}{L^2} = 78.67 \text{ kN}$$

$$P_{allow} = \frac{P_{cr}}{n} = \frac{78.67 \text{ kN}}{3.0} = 26.22 \text{ kN}$$

DIMENSIONS



FREE-BODY DIAGRAM OF JOINT B



T = tensile force in each cable

P_{allow} = compressive force in tube

EQUILIBRIUM

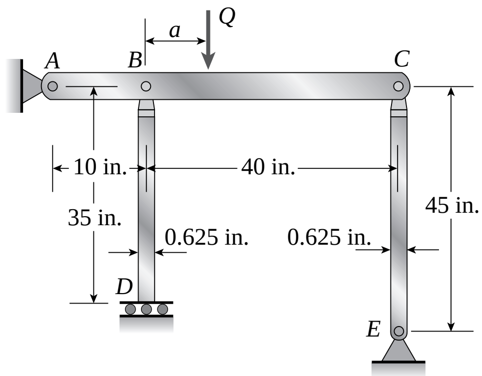
$$\sum F_{vert} = 0 \quad P_{allow} - 2T\left(\frac{2.1 \text{ m}}{2.9 \text{ m}}\right) = 0$$

ALLOWAPLE FORCE IN CABLES

$$T_{allow} = (P_{allow})\left(\frac{1}{2}\right)\left(\frac{2.9 \text{ m}}{2.1 \text{ m}}\right) = 18.1 \text{ kN} \quad \leftarrow$$

Problem 11.4-7 The horizontal beam ABC shown in the figure is supported by column BD and CE. The beam is prevented from moving horizontally by the pin support at end A. Each column is pinned at its upper end to the beam, but at the lower ends, support D is a guided support and support E is pinned. Both columns are solid steel bars ($E = 30 \times 10^6$ psi) of square cross section with width equal to 0.625 in. A load Q acts at distance a from column BD.

(a) If the distance $a = 12$ in., what is the critical value Q_{cr} of the load?
(b) If the distance a can be varied between 0 and 40 in., what is the maximum possible value of Q_{cr}? What is the corresponding value of the distance a?



Solution

$E = 30 \cdot 10^6$ psi $\qquad b = 0.625$ in.

$I = \dfrac{b^4}{12} \qquad I = 0.01272 \text{ in.}^4$

Column BD $\qquad L = 35$ in. $\qquad P_{cr\_BD} = \dfrac{\pi^2 EI}{4L^2}$

$P_{cr\_BD} = 768.4$ lb

Column CE $\qquad L = 45$ in. $\qquad P_{cr\_CE} = \dfrac{\pi^2 EI}{L^2}$

$P_{cr\_CE} = 1859$ lb

(a) Find Q_{cr} if $a = 12$ in.

The system collapses when both columns buckle.

$\Sigma M_A = 0 \qquad P_{cr\_BD}(10 \text{ in.}) + P_{cr\_CE}(50 \text{ in.}) - Q_{cr}(a + 10\text{in.}) = 0$

$$Q_{cr} = \frac{P_{cr\_BD}(10 \text{ in.}) + P_{cr\_CE}(50 \text{ in.})}{a + 10 \text{ in.}}$$

$Q_{cr} = 4575$ lb $\leftarrow$

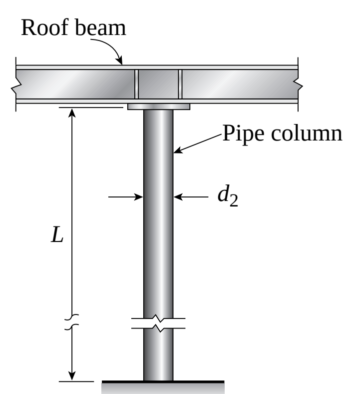
(b) Q_{cr} is maximum when $a = 0$ in.

$$Q_{cr} = \frac{P_{cr\_BD}(10 \text{ in.}) + P_{cr\_CE}(50 \text{ in.})}{a + 10 \text{ in.}}$$

$Q_{cr} = 10065$ lb $\leftarrow$

Problem 11.4-8 The roof beams of a warehouse are supported by pipe columns (see figure on the next page) having outer diameter $d_2 = 100$ mm and inner diameter $d_1 = 90$ mm. The columns have length $L = 4.0$ m, modulus $E = 210$ GPa, and fixed supports at the base.

Calculate the critical load P_{cr} of one of the columns using the following assumptions: (1) the upper end is pinned and the beam prevents horizontal displacement; (2) the upper end is fixed against rotation and the beam prevents horizontal displacement; (3) the upper end is pinned but the beam is free to move horizontally; and (4) the upper end is fixed against rotation but the beam is free to move horizontally.



Solution 11.4-8 Pipe column (with fixed base)

$E = 210$ GPa $\quad L = 4.0$ m

$d_2 = 100$ mm $\quad I = \dfrac{\pi}{64}(d_2^4 - d_1^4) = 1688 \times 10^3\ \text{mm}^4$

$d_1 = 90$ mm

(1) Upper end is pinned (with no horizontal displacement)

$$P_{cr} = \frac{2.046\pi^2 EI}{L^2} = 447\ \text{kN} \quad \longleftarrow$$

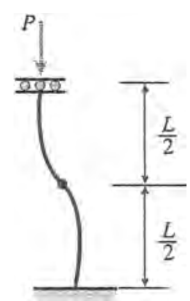
(2) Upper end is fixed (with no horizontal displacement)

$$P_{cr} = \frac{4\pi^2 EI}{L^2} = 875\ \text{kN} \quad \longleftarrow$$

(3) Upper end is pinned (but no horizontal restraint)

$$P_{cr} = \frac{\pi^2 EI}{4L^2} = 54.7\ \text{kN} \quad \longleftarrow$$

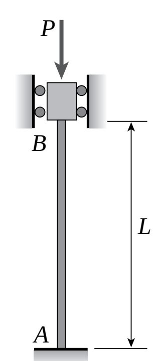
(4) Upper end is guided (no rotation; no horizontal restraint)



The lower half of the column is in the same condition as Case (3) above.

$$P_{cr} = \frac{\pi^2 EI}{4(L/2)^2} = \frac{\pi^2 EI}{L^2} = 219\ \text{kN} \quad \longleftarrow$$

Problem 11.4-9 Determine the critical load P_{cr} and the equation of the buckled shape for an ideal column with ends fixed against rotation (see figure) by solving the differential equation of the deflection curve. (See also Fig. 11-17.)



Solution 11.4-9 Fixed-end column

$\nu =$ deflection in the y direction

DIFFERENTIAL EQUATION (EQ.11-3)

$$EI\nu'' = M = M_0 - P\nu \qquad k^2 = \frac{P}{EI}$$

$$\nu'' + k^2\nu = \frac{M_0}{EI}$$

GENERAL SOLUTION

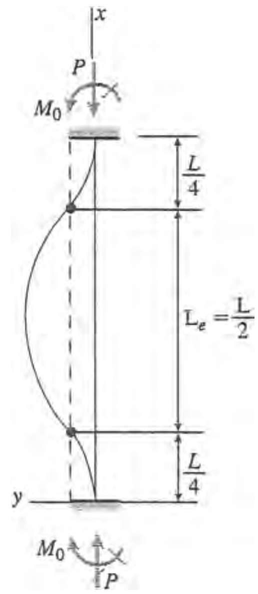
$$\nu = C_1 \sin kx + C_2 \cos kx + \frac{M_0}{P}$$

B.C. 1 $\nu(0) = 0 \qquad \therefore C_2 = -\dfrac{M_0}{P}$

$$\nu = C_1 k \cos kx - C_2 k \sin kx$$

B.C. 2 $\nu'(0) = 0 \qquad \therefore C_1 = 0$

$$\nu = \frac{M_0}{P}(1 - \cos kx)$$



BUCKLING EQUATION

B.C. 3 $\nu(L) = 0 \quad 0 = \dfrac{M_0}{P}(1 - \cos kL)$

$\therefore \cos kL = 1 \quad$ and $\quad kL = 2\pi$

CRITICAL LOAD

$$k^2 = \left(\frac{2\pi}{L}\right)^2 = \frac{4\pi^2}{L^2} \qquad \frac{P}{EI} = \frac{4\pi^2}{L^2}$$

$$P_{cr} = \frac{4\pi^2 EI}{L^2} \quad \leftarrow$$

BUCKLED MODE SHAPE

Let $\delta =$ deflection at midpoint $\left(x = \dfrac{L}{2}\right)$

$$\nu\left(\frac{L}{2}\right) = \delta = \frac{M_0}{P}\left(1 - \cos\frac{kL}{2}\right)$$

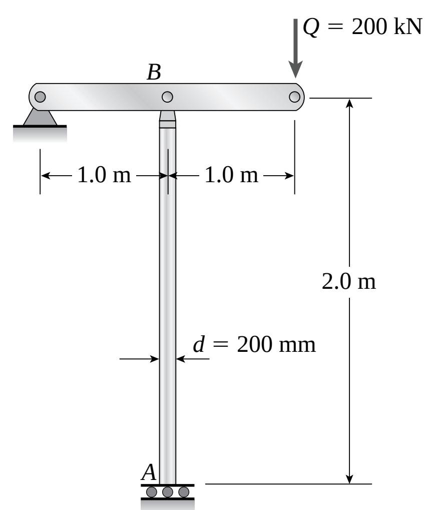
$$\frac{kL}{2} = \pi \qquad \therefore \delta = \frac{M_0}{P}(1 - \cos \pi)$$

$$= \frac{2M_0}{P} \qquad \frac{M_0}{P} = \frac{\delta}{2}$$

$$\nu = \frac{\delta}{2}\left(1 - \cos\frac{2\pi x}{L}\right) \quad \leftarrow$$

Problem 11.4-10 An aluminum tube *AB* of circular cross section has a guided support at the base and is pinned at the top to a horizontal beam supporting a load $Q = 200$ kN (see figure).

Determine the required thickness t of the tube if its outside diameter d is 200 mm and the desired factor of safety with respect to Euler buckling is $n = 3.0$. (Assume $E = 72$ GPa.)



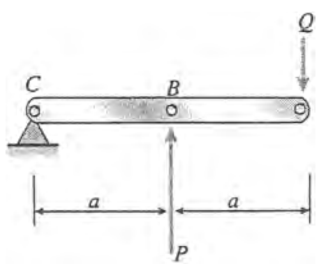
Solution 11.4-10

$E = 72$ GPa $\quad a = 1.0$ m

$L = 2.0$ m $\quad n = 3.0$

$Q = 200$ kN $\quad d = 200$ mm

FREE-BODY DIAGRAM OF THE BEAM



$\Sigma M_c = 0 \quad P = 2Q$

$P = 400$ kN

CRITICAL LOAD

$P_{cr} = P \cdot n \qquad P_{cr} = 1200$ kN

$$P_{cr} = \frac{\pi^2 EI}{4L^2}$$

$$I = \frac{4P_{cr}L^2}{\pi^2 E} \qquad I = 27.019 \times 10^6 \text{ mm}^4$$

MOMENT OF INERTIA

$$I = \frac{\pi}{64}\left[d^4 - (d - 2t)^4\right]$$

$$t_{min} = \frac{d - \sqrt[4]{d^4 - I\dfrac{64}{\pi}}}{2}$$

$t_{min} = 10.0$ mm $\quad \leftarrow$

Problem 11.4-11 The frame *ABC* consists of two members *AB* and *BC* that are rigidly connected at joint *B*, as shown in part (a) of the figure. The frame has pin supports at *A* and *C*. A concentrated load *P* acts at joint *B*, thereby placing member *AB* in direct compression.

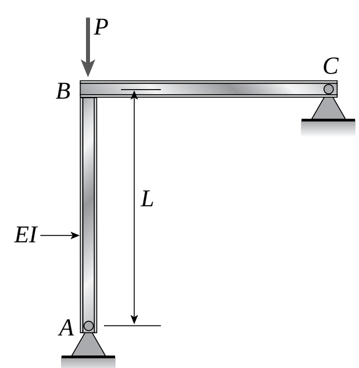
To assist in determining the buckling load for member *AB*, we represents it as a pinned-end column, as shown in part (b) of the figure. At the top of the column, a rotational spring of stiffness β_R represents the restraining action of the horizontal beam *BC* on the column (note that the horizontal beam provides resistance to rotation of joint *B* when the column buckles). Also, consider only bending effects in the analysis (i.e., disregard the effects of axial deformations).

(a) By solving the differential equation of the deflection curve, derive the following buckling equation for this column:

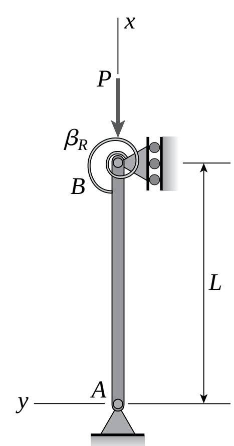
$$\frac{\beta_R L}{EI}(kL \cot kL - 1) - k^2L^2 = 0$$

in which *L* is the length of the column and *EI* is its flexural rigidity.

(b) For the particular case when member *BC* is identical to member *AB*, the rotation stiffness β_R equals $3EI/L$ (see Case 7, Table G-2, Appendix G). For this special case, determine the critical load P_{cr}.



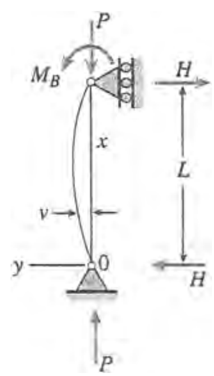
(a)



(b)

Solution 11.4-11 Column *AB* with elastic support at *B*

FREE-BODY DIAGRAM OF COLUMN



ν = deflection in the y direction
M_B = moment at end B
θ_B = angle of rotation at end B (positive clockwise)
$M_B = \beta_R\theta_B$
H = horizontal reactions at ends A and B

EQUILIBRIUM

$$\sum M_0 = \sum M_A = 0 \quad M_B - HL = 0$$

$$H = \frac{M_B}{L} = \frac{\beta_R\theta_B}{L}$$

DIFFERENTIAL EQUATION (EQ. 11-3)

$$EI\nu'' = M = Hx - P\nu \quad k^2 = \frac{P}{EI}$$

$$\nu'' + k^2\nu = \frac{\beta_R\theta_B}{LEI}x$$

GENERAL SOLUTION

$$\nu = C_1 \sin kx + C_2 \cos kx + \frac{\beta_R\theta_B}{PL}x$$

B.C. 1 $\nu(0) = 0 \qquad \therefore C_2 = 0$

B.C. 2 $\nu(L) = 0 \qquad \therefore C_1 = -\dfrac{\beta_R\theta_B}{P\sin kL}$

$$\nu = C_1 \sin kx + \frac{\beta_R\theta_B}{PL}x$$

$$\nu' = C_1 k \cos kx + \frac{\beta_R\theta_B}{PL}$$

(a) BUCKLING EQUATION

B.C. 3 $\nu'(L) = -\theta_B$

$$\therefore\ -\theta_B = -\frac{\beta_R\theta_B}{P\sin kL}(k\cos kL) + \frac{\beta_R\theta_B}{PL}$$

Cancel θ_B and multiply by PL:

$$-PL = -\beta_R\, kL \cot kL + \beta_R$$

Substitute $P = k^2EI$ and rearrange:

$$\frac{\beta_R L}{EI}(kL \cot kL - 1) - k^2L^2 = 0 \quad \leftarrow$$

(b) CRITICAL LOAD FOR $\beta_R = 3EI/L$

$$3(kL \cot kL - 1) - (kL)^2 = 0$$

Solve numerically for kL : $kL = 3.7264$

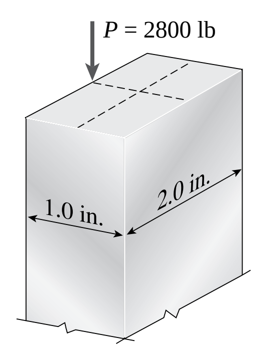
$$P_{cr} = k^2EI = (kL)^2\left(\frac{EI}{L^2}\right) = 13.89\frac{EI}{L^2} \quad \leftarrow$$

Columns with Eccentric Axial Loads

When solving the problems for Section 11.5, assume that bending occurs in the principal plane containing the eccentric axial load.

Problem 11.5-1 An aluminum bar having a rectangular cross section (2.0 in. $\times$ 1.0 in.) and length $L = 30$ in. is compressed by axial loads that have a resultant $P = 2800$ lb acting at the midpoint of the long side of the cross section (see figure).

Assuming that the modulus of elasticity E is equal to 10×10^6 psi and that the ends of the bar are pinned, calculate the maximum deflection δ and the maximum bending moment M_{max}.



Solution 11.5-1 Bar with rectangular cross section

$b = 2.0$ in. $h = 1.0$ in. $L = 30$ in.

$P = 2800$ lb $e = 0.5$ in. $E = 10 \times 10^6$ psi

$$I = \frac{bh^3}{12} = 0.1667 \text{ in.}^4 \qquad kL = L\sqrt{\frac{P}{EI}} = 1.230$$

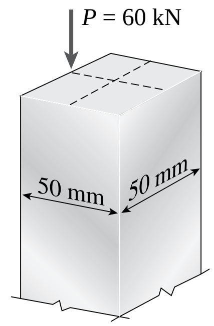
$$\text{Eq.(11-51):} \quad \delta = e\left(\sec\frac{kL}{2} - 1\right) = 0.112 \text{ in.} \quad \leftarrow$$

$$\text{Eq.(11 - 56):} \quad M_{max} = Pe\sec\frac{kL}{2}$$

$$= 1710 \text{ lb-in.} \quad \leftarrow$$

Problem 11.5-2 A steel bar having a square cross section (50 mm $\times$ 50 mm 50 mm $=$ 50 mm) and length $L = 2.0$ m is compressed by axial loads that have a resultant $P = 60$ kN acting at the midpoint of one side of the cross section (see figure).

Assuming that the modulus of elasticity E is equal to 210 GPa and that the ends of the bar are pinned, calculate the maximum deflection δ and the maximum bending moment M_{max}.



Solution 11.5-2 Bar with square cross section

$b = 50$ mm. $L = 2$ m. $P = 60$ kN $e = 25$ mm

$$E = 210 \text{ GPa} \qquad I = \frac{b^4}{12} = 520.8 \times 10^3 \text{ mm}^4$$

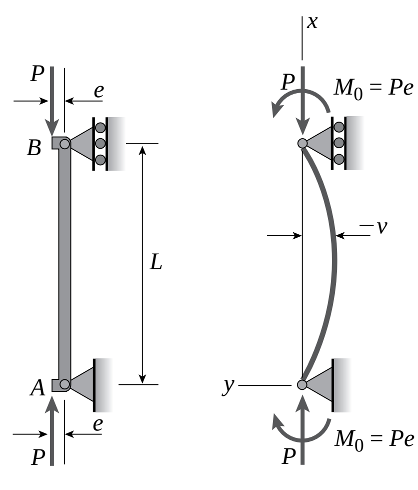
$$kL = L\sqrt{\frac{P}{EI}} = 1.481$$

$$\text{Eq. (11-51):} \ \delta = e\left(\sec\frac{kL}{2} - 1\right) = 8.87 \text{ mm} \quad \leftarrow$$

$$\text{Eq. (11-56):} \ M_{max} = Pe\sec\frac{kL}{2} = 2.03 \text{ kN}\cdot\text{m} \quad \leftarrow$$

Problem 11.5-3 Determine the bending moment M in the pinned-end column with eccentric axial loads shown in the figure. Then plot the bending-moment diagram for an axial load $P = 0.3P_{cr}$.

Note: Express the moment as a function of the distance x from the end of the column, and plot the diagram in nondimensional form with M/Pe as ordinate and x/L as abscissa.



Probs.11.5-3, 11.5-4 and 11.5-5

Solution 11.5-3 Column with eccentric loads

Column has pinned ends.

Use Eq. (11-49):

$$\nu = -e\left(\tan\frac{kL}{2}\sin kx + \cos kx - 1\right)$$

From Eq. (11-45): $M = Pe - P\nu$

$$\therefore M = Pe\left(\tan\frac{kL}{2}\sin kx + \cos kx\right) \quad \leftarrow$$

For $P = 0.3\, P_{cr}$:

From Eq. (11-52): $kL = \pi\sqrt{\frac{P}{P_{cr}}} = \pi\sqrt{0.3}$

$= 1.7207$

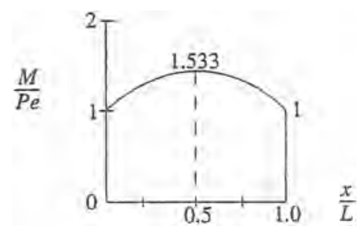
$$\frac{M}{Pe} = \left(\tan\frac{1.7207}{2}\right)\left(\sin 1.7207\frac{x}{L}\right) + \cos 1.7207\frac{x}{L}$$

or

$$\frac{M}{Pe} = 1.162\left(\sin 1.721\frac{x}{L}\right) + \cos 1.721\frac{x}{L} \quad \leftarrow$$

(**NOTE:** kL and kx are in radians)

Bending-moment diagram for $P = 0.3\, P_{cr}$



Problem 11.5-4 Plot the load-deflection diagram for a pinned-end column with eccentric axial loads (see figure) if the eccentricity e of the load is 5 mm and the column has length $L = 3.6$ m, moment of inertia $I = 9.0 \times 10^6$ mm^4, and modulus of elasticity $E = 210$ GPa.

Note: Plot the axial load as ordinate and the deflection at the midpoint as abscissa.

Solution 11.5-4 Column with eccentric loads

Column has pinned ends.

Use Eq. (11-54) for the deflection at the midpoint (maximum deflection):

$$\delta = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] \qquad (1)$$

Data

$e = 5.0$ mm $\quad L = 3.6$ m $\quad E = 210$ GPa

$I = 9.0 \times 10^6$ mm^4

Critical load

$$P_{cr} = \frac{\pi^2 EI}{L^2} = 1439.3 \text{ kN}$$

Maximum deflection (from Eq. 1)

$$\delta = (5.0)\left[\sec\left(0.041404\sqrt{P}\right) - 1\right] \qquad (2)$$

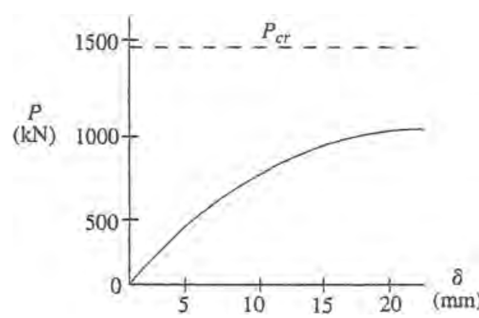
Units: P = kN $\qquad \delta$ = mm

Angles are in radians.

Solve Eq. (2) for P:

$$P = 583.3\left[\arccos\left(\frac{5.0}{5.0 + \delta}\right)\right]^2 \quad \leftarrow$$

Load-deflection diagram



Problem 11.5-5 Solve the preceding problem for a column with $e = 0.20$ in., $L = 12$ ft, $I = 21.7$ in.^{4}, and $E = 30 \times 10^6$ psi.

Solution 11.5-5 Column with eccentric loads

Column has pinned ends
Use Eq. (11-54) for the deflection at the midpoint (maximum deflection):

$$\delta = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] \quad (1)$$

DATA

$e = 0.20$ in. $\quad L = 12$ ft $= 144$ in.
$E = 30 \times 10^6$ psi
$I = 21.7$ in.^{4}

CRITICAL LOAD

$$P_{cr} = \frac{\pi^2 EI}{L^2} = 309.9 \text{ k}$$

MAXIMUM DEFLECTION (FROM EQ. 1)

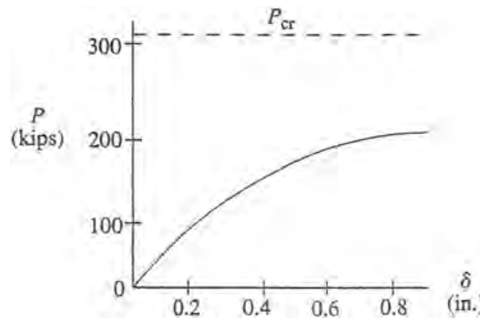
$$\delta = (0.20)\,[\sec(0.08924\sqrt{P}) - 1] \quad (2)$$

Units: P = kips $\quad \delta$ = inches
Angles are in radians.

SOLVE EQ. (2) FOR P:

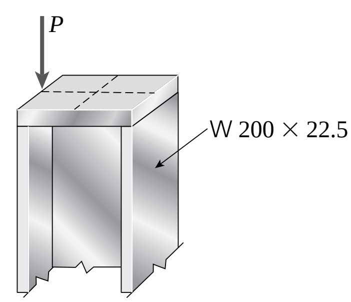
$$P = 125.6\left[\arccos\left(\frac{0.2}{0.2 + \delta}\right)\right]^2 \quad \longleftarrow$$

LOAD-DEFLECTION DIAGRAM



Problem 11.5-6 A wide-flange member (W 200 × 22.5) is compressed by axial loads that have a resultant P acting at the point shown in the figure. The member has modulus of elasticity $E = 2000$ GPa and pinned conditions at the ends. Lateral supports prevent any bending about the weak axis of the cross section.

If the length of the member is 6.2 mm, and the deflection is limited to 6.5 mm, what is the maximum allowable load P_{allow}?



Solution 11.5-6

W 200 × 22.5 $\quad E = 200$ GPa $\quad L = 6.2$ m
$\delta = 6.5$ mm $\quad I = 20 \times 10^6$ mm^4 $\quad d = 206$ mm

$$e = \frac{d}{2} \qquad e = 103 \text{ mm}$$

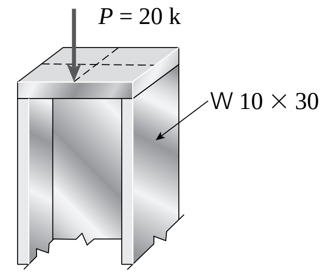
CRITICAL LOAD

$$P_{cr} = \frac{\pi^2 EI}{L^2} \qquad P_{cr} = 1027 \text{ kN}$$

Mamimum deflection $\delta = e\left(\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right)$

Solve for P $\quad P_{allow} = 49.9$ kN $\longleftarrow$

Problem 11.5-7 A wide-flange member (W 10 × 30) is compressed by axial loads that have a resultant $P = 20$ k acting at the point shown in the figure. The material is steel with modulus of elasticity $E = 29{,}000$ ksi. Assuming pinned-end conditions, determine the maximum permissible length L_{max} if the deflection is not to exceed 1/400th of the length.



Solution 11.5-7 Column with eccentric axial load

Wide-flange member: W 10 × 30
Pinned-end conditions.
Bending occurs about the weak axis (axis 2-2).
$P = 20$ k $\quad E = 29{,}000$ ksi $\quad L =$ length (inches)

$$\text{Maximum allowable deflection} = \frac{L}{400} \; (= \delta)$$

From Table E-1: $\quad I = 16.7 \text{ in.}^4$

$$e = \frac{5.810 \text{ in.}}{2} = 2.905 \text{ in.}$$

$$k = \sqrt{\frac{P}{EI}} = 0.006426 \text{ in.}^{-1}$$

DEFLECTION AT MIDPOINT (EQ. 11-51)

$$\delta = e\left(\sec\frac{kL}{2} - 1\right)$$

$$\frac{L}{400} = (2.905 \text{ in.})\,[\sec(0.003213\,\text{L}) - 1]$$

Rearrange terms and simplify:

$$\sec(0.003213\,L) - 1 - \frac{L}{1162 \text{ in.}} = 0$$

(**NOTE:** angles are in radians)
Solve the equation numerically for the length L:
$L = 150.5$ in.

MAXIMUM ALLOWABLE LENGTH

$$L_{max} = 150.5 \text{ in.} = 12.5 \text{ ft} \quad \leftarrow$$

Problem 11.5-8 Solve the preceding problem (W 250 × 44.8) if the resultant force P equals 110 kN and $E = 200$ GPa.

Solution 11.5-8

W 250 × 44.8 $\qquad E = 200$ GPa

$$P = 110 \text{ kN} \qquad \delta = \frac{L}{400}$$

Bending occur about the weak axis (axis 2-2)

$I = 6.95 \times 10^6 \text{ mm}^4 \qquad b = 148$ mm

$$e = \frac{b}{2} \qquad e = 74 \text{ mm}$$

$$k = \sqrt{\frac{P}{EI}} \qquad k = 0.000281 \text{ mm}^{-1}$$

Deflection at midpoint

$$\delta = e\left(\sec\left(\frac{kL}{2}\right) - 1\right)$$

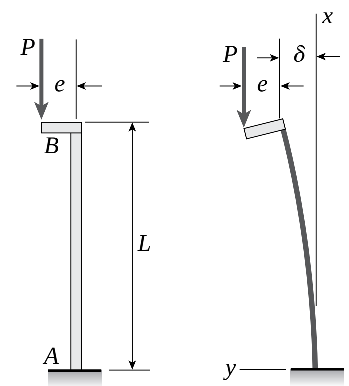
$$\frac{L}{400} = e\left(\sec\left(\frac{kL}{2}\right) - 1\right)$$

Solve for the length L

$$L_{max} = 3.14 \text{ m} \quad \leftarrow$$

Problem 11.5-9 The column shown in the figure is fixed at the base and free at the upper end. A compressive load P acts at the top of the column with an eccentricity e from the axis of the column.

Beginning with the differential equation of the deflection curve, derive formulas for the maximum deflection δ of the column and the maximum bending moment M_{max} in the column.



(a) (b)

Solution 11.5-9 Fixed-free column

e = eccentricity of load P
δ = deflection at the end of the column
v = deflection of the column at distance x from base

DIFFERENTIAL EQUATION (EQ. 11.3)

$$EIv'' = M = P(e + \delta - v) \quad k^2 = \frac{P}{EI}$$

$$v'' = k^2(e + \delta - v)$$

$$v'' + k^2 v = k^2(e + \delta)$$

GENERAL SOLUTION

$$v = C_1 \sin kx + C_2 \cos kx + e + \delta$$

$$v' = C_1 k \cos kx - C_2 k \sin kx$$

B.C. 1 $\quad v(0) = 0 \quad \therefore C_2 = -e - \delta$

B.C. 2 $\quad v'(0) = 0 \quad \therefore C_1 = 0$

$$v = (e + \delta)(1 - \cos kx)$$

B.C. 3 $\quad v(L) = \delta \quad \therefore \delta = (e + \delta)(1 - \cos kL)$

or $\quad \delta = e\,(\sec kL - 1)$

MAXIMUM DEFLECTION $\quad \delta = e(\sec kL - 1) \quad \leftarrow$

MAXIMUM BENDING MOMENT (AT BASE OF COLUMN)

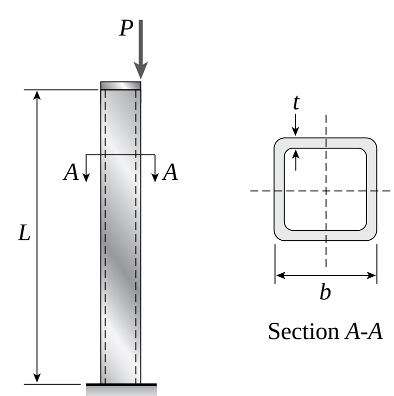
$M_{max} = P(e + \delta) = Pe \sec kL \quad \leftarrow$

NOTE: $\quad v = (e + \delta)(1 - \cos kx)$

$\quad\quad = e(\sec kL)(1 - \cos kx)$

Problem 11.5-10 An aluminum box column of square cross section is fixed at the base and free at the top (see figure). The outside dimension b of each side is 100 mm and the thickness t of the wall is 8 mm. The resultant of the compressive loads acting on the top of the column is a force P = 50 kN acting at the outer edge of the column at the midpoint of one side.

What is the longest permissible length L_{max} of the column if the deflection at the top is not to exceed 30 mm? (Assume E = 73 GPa.)



Probs. 11.5-10 and 11.5-11

Solution 11.5-10 Fixed-free column

δ = deflection at the top

Use Eq. (11-51) with $L/2$ replaced by L:

$$\delta = e(\sec kL - 1) \qquad (1)$$

(This same equation is obtained in Prob. 11.5-9.)

Solve for L from Eq. (1)

$$\sec kL = 1 + \frac{\delta}{e} = \frac{e + \delta}{e}$$

$$\cos kL = \frac{e}{e + \delta} \qquad kL = \arccos\frac{e}{e + \delta}$$

$$L = \frac{1}{k}\arccos\frac{e}{e + \delta} \qquad k = \sqrt{\frac{P}{EI}}$$

$$L = \sqrt{\frac{EI}{P}}\arccos\frac{e}{e + \delta} \qquad (2)$$

Numerical data

$E = 73$ GPa $\quad b = 100$ mm $\quad t = 8$ mm

$P = 50$ kN $\quad \delta = 30$ mm $\quad e = \frac{b}{2} = 50$ mm

$$I = \frac{1}{12}[b^4 - (b - 2t)^4] = 4.1844 \times 10^6 \text{ mm}^4$$

Maximum allowable length

Substitute numerical data into Eq.(2).

$$\sqrt{\frac{EI}{P}} = 2.4717 \text{ m} \qquad \frac{e}{e + \delta} = 0.625$$

$$\arccos\frac{e}{e + \delta} = 0.89566 \text{ radians}$$

$$L_{max} = (2.4717 \text{ m})(0.89566) = 2.21 \text{ m} \quad \leftarrow$$

Problem 11.5-11 Solve the preceding problem for an aluminum column with $b = 6.0$ in., $t = 0.5$ in., $P = 30$ k, and $E = 10.6 \times 10^3$ ksi. The deflection at the top is limited to 2.0 in.

Solution 11.5-11 Fixed-free column

δ = deflection at the top

Use Eq. (11-51) with $L/2$ replaced by L:

$$\delta = e(\sec kL - 1) \qquad (1)$$

(This same equation is obtained in Prob. 11.5-9.)

Solve for L from Eq. (1)

$$\sec kL = 1 + \frac{\delta}{e} = \frac{e + \delta}{e}$$

$$\cos kL = \frac{e}{e + \delta} \qquad kL = \arccos\frac{e}{e + \delta}$$

$$L = \frac{1}{k}\arccos\frac{e}{e + \delta} \qquad k = \sqrt{\frac{P}{EI}}$$

$$L = \sqrt{\frac{EI}{P}}\arccos\frac{e}{e + \delta}$$

Numerical data

$E = 10.6 \times 10^3$ ksi $\quad b = 6.0$ in. $\quad t = 0.5$ in.

$P = 30$ k $\quad \delta = 2.0$ in. $\quad e = \frac{b}{2} = 3.0$ in.

$$I = \frac{1}{12}[b^4 - (b - 2t)^4] = 55.917 \text{ in.}^4.$$

Maximum allowable length

Substitute numerical data into Eq. (2).

$$\sqrt{\frac{EI}{P}} = 140.56 \text{ in.} \qquad \frac{e}{e + \delta} = 0.60$$

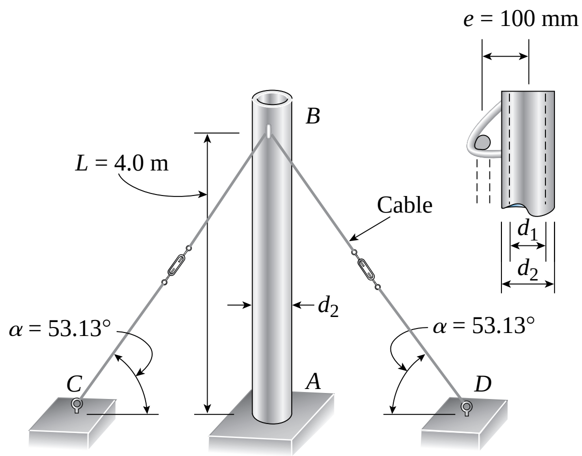
$$\arccos\frac{e}{e + \delta} = 0.92730 \text{ radians}$$

$$L_{max} = (140.56 \text{ in.})(0.92730) = 130.3 \text{ in.} = 10.9 \text{ ft} \quad \leftarrow$$

Problem 11.5-12 A steel post AB of hollow circular cross section is fixed at the base and free at the top (see figure). The inner and outer diameters are $d_1 = 96$ mm and $d_2 = 110$ mm, respectively, and the length $L = 4.0$ m.

A cable CBD passes through a fitting that is welded to the side of the post. The distance between the plane of the cable (plane CBD) and the axis of the post is $e = 100$ mm, and the angles between the cable and the ground are $\alpha = 53.13°$. The cable is pretensioned by tightening the turnbuckles.

If the deflection at the top of the post is limited to $\delta = 20$ mm, what is the maximum allowable tensile force T in the cable? (Assume $E = 205$ GPa.)



Solution 11.5-12 Fixed-free column

δ = deflection at the top

P = compressive force in post $\quad k = \sqrt{\dfrac{P}{EI}}$

Use Eq. (11-51) with $L/2$ replaced by L:

$$\delta = e(\sec kL - 1) \qquad (1)$$

(This same equation is obtained in Prob. 11.5-9.)

SOLVE FOR P FROM EQ. (1)

$$\sec kL = 1 + \frac{\delta}{e} = \frac{e + \delta}{e}$$

$$\cos kL = \frac{e}{e + \delta} \qquad kL = \arccos\frac{e}{e + \delta}$$

$$kL = \sqrt{\frac{PL^2}{EI}} \qquad \sqrt{\frac{PL^2}{EI}} = \arccos\frac{e}{e + \delta}$$

Square both sides and solve for P:

$$P = \frac{EI}{L^2}\left(\arccos\frac{e}{e + \delta}\right)^2 \qquad (2)$$

NUMERICAL DATA

$E = 205$ GPa $\quad L = 4.0$ m $\quad e = 100$ mm
$\delta = 20$ mm $\quad d_2 = 110$ mm $\quad d_1 = 96$ mm

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 3.0177 \times 10^6 \text{ mm}^4$$

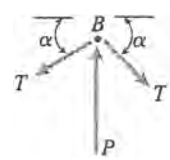
MAXIMUM ALLOWABLE COMPRESSIVE FORCE P

Substitute numerical data into Eq. (2).

$P_{allow} = 13{,}263$ N $= 13.263$ kN

MAXIMUM ALLOWABLE TENSILE FORCE T IN THE CABLE

Free-body diagram of joint B:



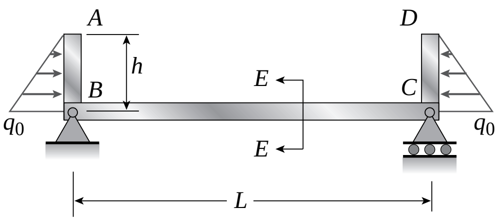
$\alpha = 53.13°$
$\sum F_{vert} = 0 \quad P - 2T\sin\alpha = 0$

$$T = \frac{P}{2\sin\alpha} = \frac{5P}{8} = 8289 \text{ N}$$

$\therefore T_{max} = 8.29$ kN $\quad \longleftarrow$

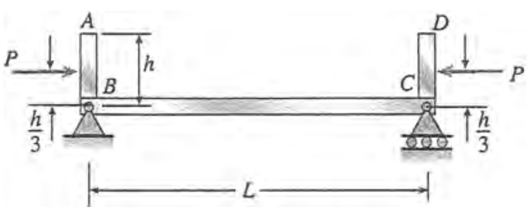
Problem 11.5-13 A frame *ABCD* is constructed of steel wide-flange members (W 8 × 21; $E = 30 \times 10^6$ psi) and subjected to triangularly distributed loads of maximum intensity q_0 acting along the vertical members (see figure). The distance between supports is $L = 20$ ft and the height of the frame is $h = 4$ ft. The members are rigidly connected at *B* and *C*.

(a) Calculate the intensity of load q_0 required to produce a maximum bending moment of 80 k-in. in the horizontal member of *BC*.
(b) If the load q_0 is reduced to one-half of the value calculated in part (a), what is the maximum bending moment in member *BC*? What is the ratio of this moment to the moment of 80 k-in. in part (a)?




Section *E*-*E*

Solution 11.5-13 Frame with triangular loads



P = resultant force
e = eccentricity

$$P = \frac{q_0 h}{2} \quad e = \frac{h}{3}$$

MAXIMUM BENDING MOMENT IN BEAM *BC*

From Eq. (11-56): $M_{max} = Pe \sec \frac{kL}{2}$

$$k = \sqrt{\frac{P}{EI}} \quad \therefore M_{max} = Pe \sec \sqrt{\frac{PL^2}{4EI}} \quad (1)$$

NUMERICAL DATA

W 8 × 21 $I = I_2 = 9.77$ in.^{4} (from Table E-1a)

$E = 30 \times 10^6$ psi $L = 20$ ft $= 240$ in.

$h = 4$ ft $= 48$ in.

$$e = \frac{h}{3} = 16 \text{ in.}$$

(a) LOAD q_0 TO PRODUCE $M_{max} = 80$ k-in.

Substitute numerical values into Eq. (1).
Units: pounds and inches

$$M_{max} = 80{,}000 \text{ lb-in.} \sqrt{\frac{PL^2}{4EI}}$$

$$= 0.1170093\sqrt{P} \quad \text{(radians)}$$

$$80{,}000 = P(16 \text{ in.}) [\sec (0.0070093\sqrt{P})]$$

$$5{,}000 = P \sec (0.0070093\sqrt{P})$$

$$P - 5{,}000 [\cos (0.0070093\sqrt{P})] = 0 \quad (2)$$

SOLVE EQ. (2) NUMERICALLY

$P = 4461.9$ lb

$$q_0 = \frac{2P}{h} = 186 \text{ lb/in.} = 2230 \text{ lb/ft} \quad \leftarrow$$

(b) LOAD q_0 IS REDUCED TO ONE-HALF ITS VALUE

$\therefore P$ is reduced to one-half its value.

$$P = \frac{1}{2}(4461.9 \text{ lb}) = 2231.0 \text{ lb}$$

Substitute numerical values into Eq. (1) and solve for M_{max}.

$$M_{max} = 37.75 \text{ k-in.} \quad \leftarrow$$

$$\text{Ratio: } \frac{M_{max}}{80 \text{ k-in.}} = \frac{37.7}{80} = 0.47 \quad \leftarrow$$

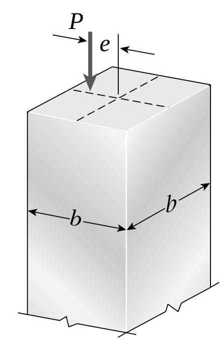
This result shows that the bending moment varies nonlinearly with the load.

The Secant Formula

When solving the problems for Section 11.6, assume that bending occurs in the principal plane containing the eccentric axial load.

Problem 11.6-1 A steel bar has a square cross section of width $b = 2.0$ in. (see figure). The bar has pinned supports at the ends and is 3.0 ft long. The axial forces acting at the end of the bar have a resultant $P = 20$ k located at distance $e = 0.75$ in. from the center of the cross section. Also, the modulus of elasticity of the steel is 29,000 ksi.

(a) Determine the maximum compressive stress σ_{max} in the bar.
(b) If the allowable stress in the steel is 18,000 psi, what is the maximum permissible length L_{max} of the bar?



Probs. 11.6-1 through 11.6-3

Solution 11.6-1 Bar with square cross section

Pinned supports.

DATA
$b = 2.0$ in. $L = 3.0$ ft $= 36$ in. $P = 20$ k
$e = 0.75$ in. $E = 29{,}000$ ksi

(a) MAXIMUM COMPRESSIVE STRESS

Secant formula (Eq. 11-59):

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right] \qquad (1)$$

$$\frac{P}{A} = \frac{P}{b^2} = 5.0 \text{ ksi} \qquad c = \frac{b}{2} = 1.0 \text{ in.}$$

$$I = \frac{b^4}{12} = 1.333 \text{ in.}^4 \qquad r^2 = \frac{I}{A} = 0.3333 \text{ in.}^2$$

$$\frac{ec}{r^2} = 2.25 \qquad \frac{L}{r} = 62.354 \qquad \frac{P}{EA} = 0.00017241$$

Substitute into Eq. (1):

$\sigma_{max} = 17.3$ ksi ←

(b) MAXIMUM PERMISSIBLE LENGTH

$\sigma_{allow} = 18{,}000$ psi

Solve Eq. (1) for the length L:

$$L = 2\sqrt{\frac{EI}{P}}\arccos\left[\frac{P(ec/r^2)}{\sigma_{max}A - P}\right] \qquad (2)$$

Substitute numerical values:

$L_{max} = 46.2$ in. ←

Problem 11.6-2 A brass bar ($E = 100$ GPa) with a square cross section is subjected to axial force having a resultant P acting at distance e from the center (see figure). The bar is pin supported at the ends and is 0.6 m in length. The side dimension b of the bar is 30 mm and the eccentricity e of the load is 10 mm.

If the allowable stress in the brass is 150 MPa, what is the allowable axial force P_{allow}?

Solution 11.6-2 Bar with square cross section

Pinned supports.

DATA
$b = 30$ mm $L = 0.6$ m $\sigma_{allow} = 150$ MPa
$e = 10$ mm $E = 100$ GPa

SECANT FORMULA (EQ. 11-59):

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right] \qquad (1)$$

Units: Newtons and meters

$\sigma_{max} = 150 \times 10^6 \text{ N/m}^2$

$A = b^2 = 900 \times 10^{-6} \text{ m}^2$

$$c = \frac{b}{2} = 0.015 \text{ m} \quad r^2 = \frac{I}{A} = \frac{b^2}{12} = 75 \times 10^{-6} \text{ m}^2$$

$$\frac{ec}{r^2} = 2.0 \quad P = \text{newtons} \quad \frac{L}{2r}\sqrt{\frac{P}{EA}} = 0.0036515\sqrt{P}$$

SUBSTITUTE NUMERICAL VALUES INTO EQ. (1):

$$150 \times 10^6 = \frac{P}{900 \times 10^{-6}}[1 + 2\sec(0.0036515\sqrt{P})]$$

or

$$P[1 + 2\sec(0.0036515\sqrt{P})] - 135{,}000 = 0 \qquad (2)$$

SOLVE EQ. (2) NUMERICALLY:

$P_{allow} = 37{,}200 \text{ N} = 37.2 \text{ kN}$ $\leftarrow$

Problem 11.6-3 A square aluminum bar with pinned ends carries a load $P = 25$ k acting at distance $e = 2.0$ in. from the center (see figure on the previous page). The bar has length $L = 54$ in. and modulus of elasticity $E = 10{,}600$ ksi.

If the stress in the bar is not exceed 6 ksi, what is the minimum permissible width b_{min} of the bar?

Solution 11.6-3 Square aluminum bar

Pinned ends.

DATA

Units: pounds and inches

$P = 25 \text{ k} = 25{,}000 \text{ lb} \qquad e = 2.0 \text{ in.}$

$L = 54 \text{ in.} \qquad E = 10{,}600 \text{ ksi} = 10{,}600{,}000 \text{ psi}$

$\sigma_{max} = 6.0 \text{ ksi} = 6{,}000 \text{ psi}$

SECANT FORMULA (EQ. 11-59):

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right] \qquad (1)$$

$$A = b^2 \qquad c = \frac{b}{2} \qquad r^2 = \frac{I}{A} = \frac{b^2}{12}$$

$$\frac{ec}{r^2} = \frac{12}{b} \qquad \frac{L}{2r}\sqrt{\frac{P}{EA}} = \frac{4.5423}{b^2}$$

SUBSTITUTE TERMS INTO EQ. (1):

$$6{,}000 = \frac{25{,}000}{b^2}\left[1 + \frac{12}{b}\sec\left(\frac{4.5423}{b^2}\right)\right]$$

or

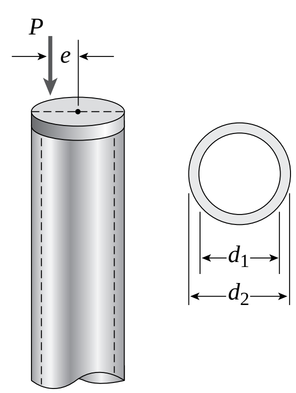
$$1 + \frac{12}{b}\sec\left(\frac{4.5423}{b^2}\right) - 0.24\,b^2 = 0 \qquad (2)$$

SOLVE EQ. (2) NUMERICALLY:

$b_{min} = 4.10 \text{ in.}$ $\leftarrow$

Problem 11.6-4 A pinned-end column of length $L = 2.1$ m is constructed of steel pipe ($E = 210$ GPa) having inside diameter $d_1 = 60$ mm and outside diameter $d_2 = 68$ mm (see figure). A compressive load $P = 10$ kN acts with eccentricity $e = 30$ mm.

(a) What is the maximum compressive stress σ_{max} in the column?
(b) If the allowable stress in the steel is 50 MPa, what is the maximum permissible length L_{max} of the column?



Probs. 11.6-4 throught 11.6-6

Solution 11.6-4 Steel pipe column

Pinned ends.

DATA Units: Newtons and meters

$L = 2.1$ m $\quad E = 210$ GPa $= 210 \times 10^9$ N/m^2

$d_1 = 60$ mm $= 0.06$ m $\quad d_2 = 68$ mm $= 0.068$ m

$P = 10$ kN $= 10{,}000$ N $\quad e = 30$ mm $= 0.03$ m

TUBULAR CROSS SECTION

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 804.25 \times 10^{-6}\ \text{m}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 413.38 \times 10^{-9}\ \text{m}^4$$

(a) MAXIMUM COMPRESSIVE STRESS

Secant formula (Eq. 11-59):

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right] \qquad (1)$$

$$\frac{P}{A} = 12.434 \times 10^6\ \text{N/m}^2$$

$$r^2 = \frac{I}{A} = 513.99 \times 10^{-6}\ \text{m}^2$$

$$r = 22.671 \times 10^{-3}\ \text{m} \qquad c = \frac{d_2}{2} = 0.034\ \text{m}$$

$$\frac{ec}{r^2} = 1.9845 \qquad \frac{L}{2r}\sqrt{\frac{P}{EA}} = 0.35638$$

Substitute into Eq. (1):

$$\sigma_{max} = 38.8 \times 10^6\ \text{N/m}^2 = 38.8\ \text{MPa} \quad \leftarrow$$

(b) MAXIMUM PERMISSIBLE LENGTH

$$\sigma_{allow} = 50\ \text{MPa}$$

Solve Eq. (1) for the length L:

$$L = 2\sqrt{\frac{EI}{P}}\arccos\left[\frac{P(ec/r^2)}{\sigma_{max}A - P}\right] \qquad (2)$$

Substitute numerical values:

$$L_{max} = 5.03\ \text{m} \quad \leftarrow$$

Problem 11.6-5 A pinned-end strut of length $L = 5.2$ ft is constructed of steel pipe ($E = 30 \times 10^3$ ksi) having inside diameter $d_1 = 2.0$ in. and outside diameter $d_2 = 2.2$ in. (see figure). A compressive load $P = 2.0$ k is applied with eccentricity $e = 1.0$ in.

(a) What is the maximum compressive stress σ_{max} in the strut?
(b) What is the allowable load P_{allow} if a factor of safety $n = 2$ with respect to yielding is required? (Assume that the yield stress σ_Y of the steel is 42 ksi.)

Solution 11.6-5 Pinned-end strut

Steel pipe.

DATA Units: kips and inches

$L = 5.2$ ft $= 62.4$ in. $\quad E = 30 \times 10^3$ ksi

$d_1 = 2.0$ in. $\quad d_2 = 2.2$ in.

$P = 2.0$ k $\quad e = 1.0$ in.

TUBULAR CROSS SECTION

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 0.65973\ \text{in.}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 0.36450\ \text{in.}^4$$

(a) MAXIMUM COMPRESSIVE STRESS

Secant formula (Eq. 11-59):

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right] \qquad (1)$$

$$\frac{P}{A} = 3.0315\ \text{ksi} \qquad c = \frac{d_2}{2} = 1.1\ \text{in.}$$

$$r^2 = \frac{I}{A} = 0.55250\ \text{in.}^2 \qquad \frac{ec}{r^2} = 1.9910$$

$$r = 0.74330\ \text{in.} \qquad \frac{L}{2r}\sqrt{\frac{P}{EA}} = 0.42195$$

Substitute into Eq. (1):

$\sigma_{max} = 9.65 \text{ ksi} \quad \leftarrow$

(b) ALLOWABLE LOAD

$\sigma_Y = 42 \text{ ksi} \quad n = 2 \quad \text{Find } P_{allow}$

Substitute numerical values into Eq. (1):

$$42 = \frac{P}{0.65973}[1 + 1.9910\sec(0.29836\sqrt{P})] \qquad (2)$$

Solve Eq. (2) numerically: $P = P_Y = 7.184$ k

$$P_{allow} = \frac{P_Y}{n} = 3.59 \text{ k} \quad \leftarrow$$

Problem 11.6-6 A circular aluminum tube with pinned ends supports a load $P = 18$ kN acting at distance $e = 50$ mm from the center (see figure). The length of the tube is 3.5m and its modulus of elasticity is 73 GPa.

If the maximum permissible stress in the tube is 20 MPa,what is the required outer diameter d_2 if the ratio of diameter is to be $d_1/d_2 = 0.9$?

Solution 11.6-6 Aluminum tube

Pinned ends.

DATA $\quad P = 18 \text{ kN} \quad e = 50 \text{ mm}$

$L = 3.5 \text{ m} \quad E = 73 \text{ GPa}$

$\sigma_{max} = 20 \text{ MPa} \quad d_1/d_2 = 0.9$

SECANT FORMULA (EQ. 11-59)

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right]$$

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = \frac{\pi}{4}[d_2^2 - (0.9d_2)^2] = 0.14923d_2^2$$

$(d_2 = \text{mm}; \quad A = \text{mm}^2)$

$$\frac{P}{A} = \frac{18{,}000 \text{ N}}{0.14923\, d_2^2} = \frac{120{,}620}{d_2^2}\left(\frac{P}{A} = \text{MPa}\right)$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = \frac{\pi}{64}[d_2^4 - (0.9d_2)^4] = 0.016881d_2^4$$

$(d_2 = \text{mm}; \quad I = \text{mm}^4)$

$$r^2 = \frac{I}{A} = 0.11313d_2^2 \quad (d_2 = \text{mm}; r^2 = \text{mm}^2)$$

$r = 0.33634\, d_2 \quad (r = \text{mm})$

$$c = \frac{d_2}{2} \quad \frac{ec}{r^2} = \frac{(50 \text{ mm})(d_2/2)}{0.11313\, d_2^2} = \frac{220.99}{d_2}$$

$$\frac{L}{2r} = \frac{3500 \text{ mm}}{2(0.33634\, d_2)} = \frac{5{,}203.1}{d_2}$$

$$\frac{P}{EA} = \frac{18{,}000 \text{ N}}{(73{,}000 \text{ N/ mm}^2)(0.14923\, d_2^2)} = \frac{1.6524}{d_2^2}$$

$$\frac{L}{2r}\sqrt{\frac{P}{EA}} = \frac{5{,}203.1}{d_2}\sqrt{\frac{1.6524}{d_2^2}} = \frac{6688.2}{d_2^2}$$

SUBSTITUTE THE ABOVE EXPRESSIONS INTO EQ. (1):

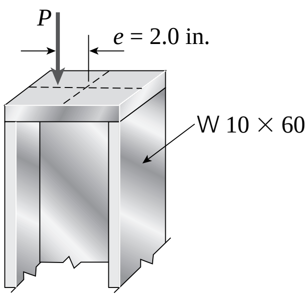
$$\sigma_{max} = 20 \text{ MPa} = \frac{120{,}620}{d_2^2} + \left[1 + \frac{220.99}{d_2}\sec\left(\frac{6688.2}{d_2^2}\right)\right] \qquad (2)$$

SOLVE EQ. (2) NUMERICALLY:

$d_2 = 131 \text{ mm} \quad \leftarrow$

Problem 11.6-7 A steel column ($E = 30 \times 10^3$ ksi) with pinned ends is constructed of W 10 × 60 wide-flange shape (see figure). The column is 24 ft long.The resultant of axial loads acting on the column is a force P acting with eccentricity $e = 2.0$ in.

(a) If $P = 120$ k, determine the maximum compressive stress σ_{max} in the column.
(b) Determine the allowable load P_{allow} if the yield stress is $\sigma_Y = 42$ ksi and the factor of safety with respect to yielding of the material is $n = 2.5$.



Solution 11.6-7 Steel column with pinned ends

$E = 30 \times 10^3$ ksi $\quad L = 24$ ft $= 288$ in.

$e = 2.0$ in.

W 10 × 60 wide-flange shape

$A = 17.6 \text{ in.}^2 \quad I = 341 \text{ in.}^4 \quad d = 10.22$ in..

$$r^2 = \frac{I}{A} = 19.38 \text{ in.}^2 \quad r = 4.402 \text{ in.} \quad c = \frac{d}{2} = 5.11 \text{ in.}$$

$$\frac{L}{r} = 65.42 \quad \frac{ec}{r^2} = 0.5273$$

(a) Maximum compressive stress ($P = 120$ k)

Secant formula (Eq. 11-59):

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right] \qquad (1)$$

$$\frac{P}{A} = 6.818 \text{ ksi} \qquad \frac{L}{2r}\sqrt{\frac{P}{EA}} = 0.4931$$

Substitute into Eq. (1): $\quad \sigma_{max} = 10.9$ ksi $\quad\leftarrow$

(b) Allowable load

$\sigma_Y = 42$ ksi $\quad n = 2.5 \quad$ Find P_{allow}

Substitute into Eq. (1):

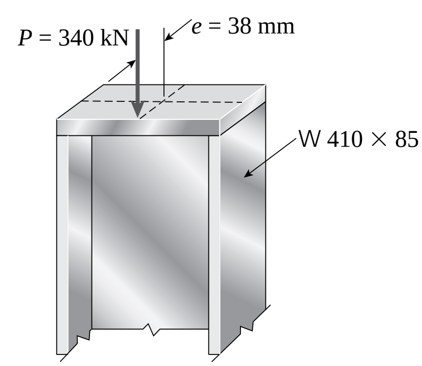
$$42 = \frac{P}{17.6}[1 + 0.5273 \sec(0.04502\sqrt{P})]$$

Solve numerically: $P = P_Y = 399.9$ k

$P_{allow} = P_Y/n = 160$ k $\quad\leftarrow$

Problem 11.6-8 A W 410 × 85 steel column is compressed by a force $P = 340$ kN acting with an eccentricity $e = 38$ mm., as shown in the figure.The column has pinned ends and length L. Also,the steel has modulus of elasticity $E = 200$ GPa and yield stress $\sigma_Y = 250$ MPa.

(a) If the length $L = 3$ m, what is the maximum compressive stress σ_{max} in the column?
(b) If a factor of safety $n = 2.0$ is required with respect to yielding, what is the longest permissible length L_{max} of the column?



Solution 11.6-8

W 410 × 85 $\qquad A = 10800\ \text{mm}^2 \qquad I = I_2$

$I = 17.9 \times 10^6\ \text{mm}^4 \qquad b = 181\ \text{mm} \qquad c = \dfrac{b}{2}$

$c = 90.5\ \text{mm}$

$e = 38\ \text{mm} \qquad r = \sqrt{\dfrac{I}{A}} \qquad r = 40.711\ \text{mm}$

$P = 340\ \text{kN} \quad E = 200\ \text{GPa} \quad L = 3\ \text{m}$

(a) Maximum compression stress

$$\sigma_{max} = \frac{P}{A}\left(1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right)$$

$\sigma_{max} = 104.5\ \text{MPa} \quad \leftarrow$

(b) Maximum length for

$\sigma_y = 250\ \text{MPa} \qquad n = 2.0$

$P_y = nP \qquad P_y = 680\ \text{kN}$

from $$\sigma_y = \frac{P_y}{A}\left(1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P_y}{EA}}\right)\right)$$

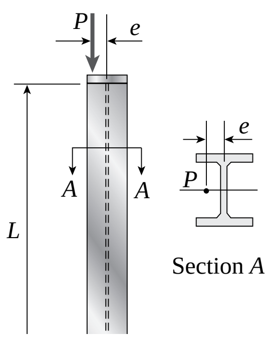
solve for the length L

$$L_{max} = 2\sqrt{\frac{EI}{P_y}}\,\text{acos}\left[\frac{P_y\left(\frac{ec}{r^2}\right)}{\sigma_y A - P_y}\right]$$

$L_{max} = 3.66\ \text{m} \quad \leftarrow$

Problem 11.6-9 A steel column ($E = 30 \times 10^3$ ksi) that is fixed at the base and free at the top is constructed of a W 8 × 35 wide-flange member (see figure). The column is 9.0 ft long. The force P acting at the top of the column has an eccentricity $e = 1.25$ in.

(a) If $P = 40$ k,what is the maximum compressive stress in the column?
(b) If the yield stress is 36 ksi and the required factor of safety with respect to yielding is 2.1,what is allowable load P_{allow}?



Probs. 11.6-9 and 11.6-10

Solution 11.6-9 Steel column (fixed-free)

$E = 30 \times 10^3\ \text{ksi} \qquad e = 1.25\ \text{in.}$
$L_e = 2L = 2(9.0\ \text{ft}) = 18\ \text{ft} = 216\ \text{in.}$

W 8 × 35 Wide-flange shape

$A = 10.3\ \text{in.}^2 \qquad I = I_2 = 42.6\ \text{in.}^4 \qquad b = 8.020\ \text{in.}$

$r^2 = \dfrac{I}{A} = 4.136\ \text{in.}^2 \qquad r = 2.034\ \text{in.}$

$c = \dfrac{b}{2} = 4.010\ \text{in.} \qquad \dfrac{L_e}{r} = 106.2 \qquad \dfrac{ec}{r^2} = 1.212$

(a) Maximum compressive stress ($P = 40$ k)

Secant formula (Eq. 11-59):

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L_e}{2r}\sqrt{\frac{P}{EA}}\right)\right] \qquad (1)$$

$\dfrac{P}{A} = 3.883\ \text{ksi} \qquad \dfrac{L_e}{2r}\sqrt{\dfrac{P}{EA}} = 0.6042$

Substitute into Eq. (1): $\sigma_{max} = 9.60\ \text{ksi} \quad \leftarrow$

(b) Allowable load

$\sigma_Y = 36\ \text{ksi} \qquad n = 2.1 \qquad$ Find P_{allow}

Substitute into Eq. (1):

$$36 = \frac{P}{10.3}\left[1 + 1.212\sec\left(0.09552\sqrt{P}\right)\right]$$

Solve numerically: $\qquad P = P_Y = 112.6\ \text{k}$

$P_{allow} = P_Y/n = 53.6\ \text{k} \quad \leftarrow$

Problem 11.6-10 A W 310 × 74 wide-flange steel column with length $L = 3.8$ m is fixed at the base and free at the top (see figure). The load P acting on the column is intended to be centrally applied,but because of unavoidable discrepancies in construction, an eccentricity ratio of 0.25 is specified. Also, the following data are supplied: $E = 200$ GPa, $\sigma_Y = 290$ MPa and $P = 310$ kN.

(a) What is the maximum compressive stress σ_{max} in the column?
(b) What is the factor of safety n with respect to yielding of the steel?

Solution 11.6-10

W 310 × 74 $\quad A = 9420\ \text{mm}^2 \quad I = I_2$

$I = 23.4 \times 10^6\ \text{mm}^4 \quad r = \sqrt{\dfrac{I}{A}} \quad r = 49.841\ \text{mm}$

$\dfrac{ec}{r^2} = 0.25 \quad P = 310\ \text{kN} \quad E = 200\ \text{GPa}$

$L = 3.8\ \text{m} \quad L_e = 2L \quad L_e = 7.6\ \text{m}$

(a) Maximum compression stress

$$\sigma_{max} = \frac{P}{A}\left(1 + \frac{ec}{r^2}\sec\left(\frac{L_e}{2r}\sqrt{\frac{P}{EA}}\right)\right)$$

$$\sigma_{max} = 47.6\ \text{MPa} \quad \longleftarrow$$

(b) Factor of safety with respect to yielding $\quad \sigma_y = 290$ MPa

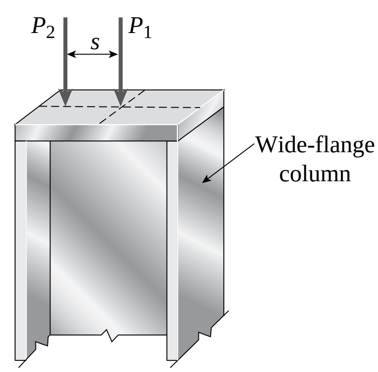
from $\quad \sigma_y = \dfrac{P_y}{A}\left(1 + \dfrac{ec}{r^2}\sec\left(\dfrac{L_e}{2r}\sqrt{\dfrac{P_y}{EA}}\right)\right)$

solve numerically for P_y

$P_y = 712\ \text{kN} \quad n = \dfrac{P_y}{P} \quad n = 2.30 \quad \longleftarrow$

Problem 11.6-11 A pinned-end column with length $L = 18$ ft is constructed from a W 12 × 87 wide-flange shape (see figure). The column is subjected to centrally applied load $P_1 = 180$ k and an eccentrically applied load $P_2 = 75$ k .The load P_2 acts at distance $s = 5.0$ in. from the centroid of the cross section. The properties of the steel are $E = 29{,}000$ ksi and $\sigma_Y = 36$ ksi.

(a) Calculate the maximum compressive stress in the column.
(b) Determine the factor of safety with respect to yielding.



Probs. 11.6.11 and 11.6.12

Solution 11.6-11 Column with two loads

Pinned-end column. W 12 × 87

DATA

$L = 18 \text{ ft} = 216 \text{ in.}$
$P_1 = 180 \text{ k} \quad P_2 = 75 \text{ k} \quad s = 5.0 \text{ in.}$
$E = 29{,}000 \text{ ksi} \quad \sigma_Y = 36 \text{ ksi}$

$P = P_1 + P_2 = 255 \text{ k} \quad e = \frac{P_2 s}{P} = 1.471 \text{ in.}$

$A = 25.6 \text{ in.}^2 \quad I = I_1 = 740 \text{ in.}^4 \quad d = 12.53 \text{ in.}$

$r^2 = \frac{I}{A} = 28.91 \text{ in.}^2 \quad r = 5.376 \text{ in.}$

$c = \frac{d}{2} = 6.265 \text{ in.} \quad \frac{ec}{r^2} = 0.3188$

$\frac{P}{A} = 9.961 \text{ ksi} \quad \frac{L}{2r}\sqrt{\frac{P}{EA}} = 0.3723$

(a) MAXIMUM COMPRESSIVE STRESS

Secant formula (Eq. 11-59):

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right] \quad (1)$$

Substitute into Eq. (1): $\sigma_{max} = 13.4 \text{ ksi}$ ←

(b) FACTOR OF SAFETY WITH RESPECT TO YIELDING

$\sigma_{max} = \sigma_Y = 36 \text{ ksi} \quad P = P_Y$

Substitute into Eq. (1):

$$36 = \frac{p_Y}{25.6}[1 + 0.3188\sec(0.02332\sqrt{P_Y})]$$

Solve numerically: $P_Y = 664.7 \text{ k}$

$P = 255 \text{ k} \quad n = \frac{p_Y}{P} = \frac{664.7 \text{ k}}{255 \text{ k}} = 2.61$ ←

Problem 11.6-12 The wide-flange pinned-end column shown in the figure carries two loads, a force $P_1 = 450$ kN acting at the centroid and a force $P_2 = 270$ kN acting at distance $s = 100$ mm, from the centroid. The column is a W 250 × 67 shape with $L = 4.2$ m, $E = 200$ GPa, and $\sigma_Y = 290$ MPa.

(a) What is the maximum compressive stress in the column?
(b) If the load P_1 remain at 450 kN, what is the largest permissible value of the load P_2 in order to maintain a factor of safety of 2.0 with respect to yielding?

Solution 11.6-12

W 250 × 67 $\quad L = 4.2 \text{ m}$

$P_1 = 450 \text{ kN} \quad P_2 = 270 \text{ kN}$

$s = 100 \text{ mm} \quad E = 200 \text{ GPa}$

$\sigma_y = 290 \text{ MPa} \quad P = P_1 + P_2$

$P = 720 \text{ kN} \quad e = \frac{P_2 s}{P} \quad e = 37.5 \text{ mm}$

$A = 8580 \text{ mm}^2 \quad I = I_1 \quad I = 103 \times 10^6 \text{ mm}^4$

$d = 257 \text{ mm} \quad r = \sqrt{\frac{I}{A}} \quad r = 109.6 \text{ mm}$

$c = \frac{d}{2} \quad c = 128.5 \text{ mm}$

(a) MAXIMUM COMPRESSION STRESS

$$\sigma_{max} = \frac{P}{A}\left(1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right)$$

$\sigma_{max} = 120.4 \text{ MPa}$ ←

(b) LARGEST VALUE OF LOAD P_2 WHEN
$P_1 = 450 \text{ kN} \quad n = 2.0$

$P_y = n(P_1 + P_2)$

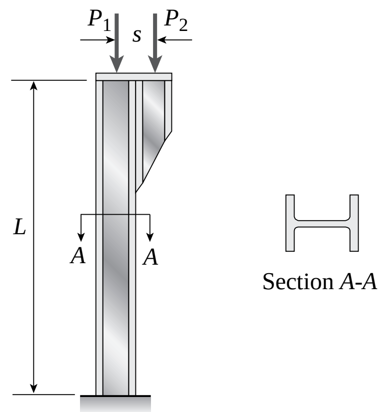
from $\quad \sigma_y = \frac{P_y}{A}\left(1 + \frac{ec}{r^2}sec\left(\frac{L}{2r}\sqrt{\frac{P_y}{EA}}\right)\right)$

$$\sigma_y = \frac{n(P_1+P_2)}{A} \times \left[1 + \frac{ec}{r^2}\sec\left[\frac{L}{2r}\sqrt{\frac{n(P_1+P_2)}{EA}}\right]\right]$$

Solve for $P_2 \quad P_2 = 387 \text{ kN}$ ←

Problem 11.6-13 A W 14 × 53 wide-flange column of length $L = 15$ ft is fixed at the base and free at the top (see figure). The column supports a centrally applied load $P_1 = 120$ k and a load $P_2 = 40$ k supported on a bracket. The distance from the centroid of the column to the load P_2 is $s = 12$ in. Also, the modulus of elasticity is $E = 29{,}000$ ksi and yield stress is $\sigma_Y = 36$ ksi.

(a) Calculate the maximum compressive stress in the column.
(b) Determine the factor of safely with respect to yielding.



Probs. 11.6-13 and 11.6-14

Solution 11.6-13 Column with two loads

Fixed-free column. W 14 × 53

DATA

$L = 15 \text{ ft} = 180 \text{ in.} \qquad L_e = 2L = 360 \text{ in.}$
$P_1 = 120 \text{ k} \qquad P_2 = 40 \text{ k} \qquad s = 12 \text{ in.}$
$E = 29{,}000 \text{ ksi} \qquad \sigma_Y = 36 \text{ ksi}$

$$P = P_1 + P_2 = 160 \text{ k} \qquad e = \frac{P_2 s}{P} = 3.0 \text{ in.}$$

$$A = 15.6 \text{ in.}^2 \qquad I = I_1 = 541 \text{ in.}^4 \qquad d = 13.92 \text{ in.}$$

$$r^2 = \frac{I}{A} = 34.68 \text{ in.}^2 \qquad r = 5.889 \text{ in.}$$

$$c = \frac{d}{2} = 6.960 \text{ in.} \qquad \frac{ec}{r^2} = 0.6021$$

$$\frac{P}{A} = 10.26 \text{ ksi} \qquad \frac{L_e}{2r}\sqrt{\frac{P}{EA}} = 0.5748$$

(a) MAXIMUM COMPRESSIVE STRESS

Secant formula (Eq. 11-59):

$$\sigma_{\max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L_e}{2r}\sqrt{\frac{P}{EA}}\right)\right] \qquad (1)$$

Substitute into Eq. (1): $\sigma_{\max} = 17.6$ ksi ←

(b) FACTOR OF SAFETY WITH RESPECT TO YIELDING

$$\sigma_{\max} = \sigma_Y = 36 \text{ ksi} \qquad P = P_Y$$

Substitute into Eq. (1):

$$36 = \frac{P_Y}{15.6}\left[1 + 0.6021 \sec\left(0.04544\sqrt{P_Y}\right)\right]$$

Solve numerically: $P_Y = 302.6$ k

$$P = 160 \text{ k} \qquad n = \frac{P_Y}{P} = \frac{302.6 \text{ k}}{160 \text{ k}} = 1.89 \quad \leftarrow$$

Problem 11.6-14 A wide-flange column with a bracket is fixed at the base and free at the top (see figure). The column supports a load $P_1 = 340$ kN acting at the centroid and a load $P_2 = 110$ kN acting on the bracket at distance $s = 250$ mm, from the load P_1. The column is a W 310 × 52 shape with $L = 5$ m, $E = 200$ GPa, and $\sigma_Y = 290$ MPa.

(a) What is the maximum compressive stress in the column?
(b) If the load P_1 remains at 340 kN, what is the largest permissible value of the load P_2 in order to maintain a factor of safety of 1.8 with respect to yielding?

Solution 11.6-14

W 310 $\times$ 52 $\quad L = 5.0$ m

$P_1 = 340$ kN $\quad P_2 = 110$ kN

$s = 250$ mm $\quad E = 200$ GPa

$\sigma_y = 290$ MPa $\quad P = P_1 + P_2$

$P = 450$ kN $\quad e = \dfrac{P_2 s}{P} \quad e = 61.1$ mm

$A = 6650\ \text{mm}^2 \quad I = I_1 \quad I = 119 \times 10^6\ \text{mm}^4$

$d = 318$ mm $\quad r = \sqrt{\dfrac{I}{A}} \quad r = 133.8$ mm

$c = \dfrac{d}{2} \quad c = 159.0$ mm

$L_e = 2L \quad L_e = 10.0$ m

(a) Maximum compression stress

$$\sigma_{max} = \frac{P}{A}\left(1 + \frac{ec}{r^2}\sec\left(\frac{L_e}{2r}\sqrt{\frac{P}{EA}}\right)\right)$$

$\sigma_{max} = 115.2$ MPa $\leftarrow$

(b) Largest value of load P_2 when
$P_1 = 340$ kN $\quad n = 1.8$

$P_y = n(P_1 + P_2)$

from $\quad \sigma_y = \dfrac{P_y}{A}\left(1 + \dfrac{ec}{r^2} sec\left(\dfrac{L_e}{2r}\sqrt{\dfrac{P_y}{EA}}\right)\right)$

$$\sigma_y = \frac{n(P_1 + P_2)}{A} \times \left[1 + \frac{ec}{r^2}\sec\left[\frac{L_e}{2r}\sqrt{\frac{n(P_1+P_2)}{EA}}\right]\right]$$

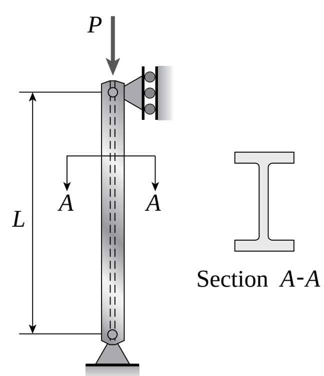
Solve for $P_2 \quad P_2 = 193$ kN

Design Formulas for Columns

The problems for Section 11.9 are to be solved assuming that the axial loads are centrally applied at the ends of the columns. Unless otherwise stated, the columns may buckle in any direction.

Steel Columns

Problem 11.9-1 Determine the allowable axial load P_{allow} for a W 10 $\times$ 45 steel wide-flange column with pinned ends (see figure) for each of the following lengths: $L = 8$ ft, 16 ft, 24 ft, and 32 ft. (Assume $E = 29{,}000$ ksi and $\sigma_Y = 36$ ksi.)



Probs 11.9-1 through 11.9-6

Solution 11.9.1 Steel wide-flange column

Pinned ends ($K = 1$).
Bucking about axis 2-2 (see Table E-1a).
Use AISC formulas.

W 10 $\times$ 45 $\quad A = 13.3\ \text{in.}^2 \quad r_2 = 2.01$ in.

$E = 29{,}000$ ksi $\quad \sigma_Y = 36$ ksi $\quad \left(\dfrac{L}{r}\right)_{max} = 200$

Eq. (11 − 76): $\left(\dfrac{L}{r}\right)_c = \sqrt{\dfrac{2\pi^2 E}{\sigma_Y}} = 126.1$

$L_c = 126.1\,r = 253.5$ in. $= 21.1$ ft.

L	8 ft	16 ft	24 ft	32 ft
L/r	47.76	95.52	143.3	191.0
n_1 (Eq. 11-79)	1.802	1.896	-	-
n_2 (Eq. 11-80)	-	-	1.917	1.917
σ_{allow}/σ_Y (Eq. 11-81)	0.5152	0.3760	-	
σ_{allow}/σ_Y (Eq. 11-82)	-	-	0.2020	0.1137
σ_{allow} (ksi)	18.55	13.54	7.274	4.091
$P_{allow} = A\,\sigma_{allow}$	247 k	180 k	96.7 k	54.4 k

Problem 11.9-2 Determine the allowable axial load P_{allow} for a W 310 × 129 steel wide-flange column with pinned ends (see figure) for each of the following lengths: $L = 3$ m, 6 m, 9 m, and 12 m. (Assume $E = 200$ GPa and $\sigma_Y = 340$ MPa.)

Solution 11.9-2

Pinned ends $K = 1$

Buckling about axis 2-2

W 310 × 129

$A = 16500\ \text{mm}^2 \qquad r_2 = 78.0\ \text{mm} \qquad r = r_2$

$E = 200\ \text{GPa} \qquad \sigma_y = 340\ \text{MPa} \qquad L_{max} = 200 \cdot r$

$$L_{max} = 15.6\ \text{m} \qquad L_c = r\sqrt{\frac{2\pi^2 E}{\sigma_y}} \qquad L_c = 8.405\ \text{m}$$

$$L = \begin{pmatrix} 3\ \text{m} \\ 6\ \text{m} \\ 9\ \text{m} \\ 12\ \text{m} \end{pmatrix} \qquad \frac{L}{r} = \begin{pmatrix} 38.462 \\ 76.923 \\ 115.385 \\ 153.846 \end{pmatrix}$$

$i = 1 .. 4$

$$n_{1_i} = \left| \begin{array}{ll} \text{"NA"} & \text{if } \dfrac{KL_i}{r} > \dfrac{KL_c}{r} \\ \dfrac{5}{3} + \dfrac{3\left(\dfrac{KL_i}{r}\right)}{8\left(\dfrac{KL_c}{r}\right)} - \dfrac{\left(\dfrac{KL_i}{r}\right)^3}{8\left(\dfrac{KL_c}{r}\right)^3} & \text{otherwise} \end{array} \right.$$

$$n_1 = \begin{pmatrix} 1.795 \\ 1.889 \\ \text{"NA"} \\ \text{"NA"} \end{pmatrix}$$

$$n_{2_i} = \left| \begin{array}{ll} \dfrac{23}{12} & \text{if } \dfrac{KL_i}{r} > \dfrac{KL_c}{r} \\ \text{"NA"} & \text{otherwise} \end{array} \right.$$

$$n_2 = \begin{pmatrix} \text{"NA"} \\ \text{"NA"} \\ 1.917 \\ 1.917 \end{pmatrix}$$

$$\sigma_{allow_i} = \sigma_y \left| \begin{array}{ll} \dfrac{1}{n_{1_i}}\left[1 - \dfrac{\left(\dfrac{KL_i}{r}\right)^2}{2\left(\dfrac{KL_c}{r}\right)^2}\right] & \text{if } \dfrac{KL_i}{r} \le \dfrac{KL_c}{r} \\ \dfrac{\left(\dfrac{KL_c}{r}\right)^2}{2n_{2_i}\left(\dfrac{KL_i}{r}\right)^2} & \text{otherwise} \end{array} \right.$$

$$\sigma_{allow} = \begin{pmatrix} 177.366 \\ 134.135 \\ 77.355 \\ 43.512 \end{pmatrix} \text{MPa}$$

$$P_{allow_i} = A\sigma_{allow_i}$$

$$P_{allow} = \begin{pmatrix} 2927 \\ 2213 \\ 1276 \\ 718 \end{pmatrix} \text{kN} \quad \text{for} \begin{pmatrix} 3\ \text{m} \\ 6\ \text{m} \\ 9\ \text{m} \\ 12\ \text{m} \end{pmatrix} \qquad \leftarrow$$

Problem 11.9-3 Determine the allowable axial load P_{allow} for a W 10 × 60 steel wide-flange column with pinned ends (see figure) for each of the following lengths: L = 10 ft, 20 ft, 30 ft, and 40 ft. (Assume E = 29,000 ksi and σ_Y = 36 ksi.)

Solution 11.9-3 Steel wide-flange column

Pinned ends ($K = 1$).
Bucking about axis 2-2 (see Table E-1a).
Use AISC formulas.

W 10 × 60 $\quad A = 17.6 \text{ in.}^2 \quad r_2 = 2.57$ in.

$$E = 29{,}000 \text{ ksi} \qquad \sigma_Y = 36 \text{ ksi} \qquad \left(\frac{L}{r}\right)_{max} = 200$$

$$\text{Eq. (11-76): } \left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 126.1$$

$$L_c = 126.1\, r = 324.\text{in.} = 27.0 \text{ ft}$$

L	10 ft	20 ft	30 ft	40 ft
L/r	46.69	93.39	140.1	186.8
n_1 (Eq. 11-79)	1.799	1.894	-	-
n_2 (Eq. 11-80)	-	-	1.917	1.917
σ_{allow}/σ_Y (Eq. 11-81)	0.5177	0.3833	-	-
σ_{allow}/σ_Y (Eq. 11-82)	-	-	0.2114	0.1189
σ_{allow}(ksi)	18.64	13.80	7.610	4.281
$P_{allow} = A\,\sigma_{allow}$	328 k	243 k	134 k	75.3 k

Problem 11.9-4 Select a steel wide-flange column of nominal depth 250 mm. (W 250 shape) to support an axial load P = 800 kN (see figure). The column has pinned ends and length L = 4.25 m. Assume E = 200 GPa and σ_Y = 250 MPa. (*Note*: The selection of columns is limited to those listed in Table E-1(b), Appendix E.)

Solution 11.9-4

$$K = 1 \qquad P = 800 \text{ kN} \qquad L = 4.25 \text{ m}$$

$$\sigma_y = 250 \text{ MPa} \qquad E = 200 \text{ GPa}$$

$$\frac{L_c}{r} = \sqrt{\frac{2\pi^2 E}{\sigma_y}} \qquad \sqrt{\frac{2\pi^2 E}{\sigma_y}} = 125.664$$

(1) TRIAL VALUE OF σ_{allow}

$$\text{Upper limit: with } \frac{L}{r} = 0 \qquad n_1 = \frac{5}{3}$$

$$\sigma_{allow\_max} = \frac{\sigma_y}{n_1}$$

$$\sigma_{allow\_max} = 150 \text{ MPa}$$

$$\text{Try} \qquad \sigma_{allow} = 110 \text{ MPa}$$

(2) TRIAL VALUE OF AREA

$$A = \frac{P}{\sigma_{allow}} \qquad A = 7273 \text{ mm}^2$$

(3) TRIAL COLUMN W 250 × 67

$$A = 8580 \text{ mm}^2 \quad r = 51.1 \text{ mm}$$

(4) ALLOWABLE STRESS FOR TRIAL COLUMN

$$\frac{L}{r} = 83.170 \qquad \frac{L}{r} < \frac{L_c}{r}$$

$$n_1 = \frac{5}{3} + \frac{3\left(\frac{KL}{r}\right)}{8\left(\sqrt{\frac{2\pi^2 E}{\sigma_y}}\right)} - \frac{\left(\frac{KL}{r}\right)^3}{8\left(\sqrt{\frac{2\pi^2 E}{\sigma_y}}\right)^3}$$

$$n_1 = 1.879$$

$$\sigma_{allow} = \sigma_y \frac{1}{n_1}\left[1 - \frac{\left(\frac{KL}{r}\right)^2}{2\left(\sqrt{\frac{2\pi^2 E}{\sigma_y}}\right)^2}\right]$$

$$\sigma_{allow} = 103.9 \text{ MPa}$$

(5) ALLOWABLE LOAD FOR TRIAL COLUMN

$P_{allow} = \sigma_{allow} A \qquad P_{allow} = 891.7 \text{ kN}$

$P_{allow} > P \qquad \text{(OK)}$

(W 250 × 67)

(6) NEXT SMALLER SIZE COLUMN

W 250 × 44.8 $\qquad A = 5700 \text{ mm}^2 \qquad r = 34.8 \text{ mm}$

$\frac{L}{r} = 122.126 \qquad \frac{L}{r} < \frac{L_c}{r}$

$$n_1 = \frac{5}{3} + \frac{3\left(\frac{KL}{r}\right)}{8\left(\sqrt{\frac{2\pi^2 E}{\sigma_y}}\right)} - \frac{\left(\frac{KL}{r}\right)^3}{8\left(\sqrt{\frac{2\pi^2 E}{\sigma_y}}\right)^3}$$

$n_1 = 1.916$

$$\sigma_{allow} = \sigma_y \frac{1}{n_1}\left[1 - \frac{\left(\frac{KL}{r}\right)^2}{2\left(\sqrt{\frac{2\pi^2 E}{\sigma_y}}\right)^2}\right]$$

$\sigma_{allow} = 68.85 \text{ MPa}$

$P_{allow} = A\sigma_{allow} \qquad P_{allow} = 392.4 \text{ kN}$

$P_{allow} < P \qquad \text{(Not Satisfactory)}$

$\therefore$ Select W250 × 67 $\leftarrow$

Problem 11.9-5 Select a steel wide-flange column of nominal depth 12 in. (W 12 shape) to support an axial load $P = 175$ k (see figure). The column has pinned ends and length $L = 35$ ft. Assume $E = 29{,}000$ ksi and $\sigma_Y = 36$ ksi. (*Note*: The selection of columns is limited to those listed in Table E-1a, Appendix E.)

Solution 11.9-5 Select a column of W 12 shape

$P = 175 \text{ k} \qquad L = 35 \text{ ft} = 420 \text{ in.} \qquad K = 1$

$\sigma_Y = 36 \text{ ksi} \qquad E = 29{,}000 \text{ ksi}$

Eq. (11-76): $\left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 126.1$

(1) TRIAL VALUE OF σ_{allow}

Upper limit: use Eq. (11-81) with $L/r = 0$

Max. $\sigma_{allow} = \frac{\sigma_Y}{n_1} = \frac{\sigma_Y}{5/3} = 21.6 \text{ ksi}$

Try $\sigma_{allow} = 8$ ksi (Because column is very long)

(2) TRIAL VALUE OF AREA

$$A = \frac{P}{\sigma_{allow}} = \frac{175 \text{ k}}{8 \text{ ksi}} = 22 \text{ in.}^2$$

(3) TRIAL COLUMN $\qquad$ W 12 × 87

$A = 25.6 \text{ in.}^2 \qquad r = 3.07 \text{ in.}$

(4) ALLOWABLE STRESS FOR TRIAL COLUMN

$\frac{L}{r} = \frac{4.20 \text{ in.}}{3.07 \text{ in.}} = 136.8 \qquad \frac{L}{r} > \left(\frac{L}{r}\right)_c$

Eqs. (11-80) and (11-82): $\quad n_2 = 1.917$

$\frac{\sigma_{allow}}{\sigma_Y} = 0.2216 \qquad \sigma_{allow} = 7.979 \text{ ksi}$

(5) ALLOWABLE LOAD FOR TRIAL COLUMN

$P_{allow} = \sigma_{allow} A = 204 \text{ k} > 175 \text{ k} \quad \text{(ok)}$

(6) NEXT SMALLER SIZE COLUMN

W 12 × 50 $\quad A = 14.7 \text{ in.}^2 \quad r = 1.96 \text{ in.}$

$\frac{L}{r} = 214$ Since the maximum permissible value of L/r is 200, this section is not satisfactory.

Select W 12 × 87 $\leftarrow$

Problem 11.9-6 Select a steel wide-flange column of nominal depth 360 mm (W 360 shape) to support an axial load $P = 1100$ kN (see figure). The column has pinned ends and length $L = 6$ m. Assume $E = 200$ GPa and $\sigma_Y = 340$ MPa. (*Note*: The selection of columns is limited to those listed in Table E-1 (b), Appendix E.)

Solution 11.9-6

$K = 1 \qquad P = 1100 \text{ kN} \qquad L = 6 \text{ m}$

$\sigma_y = 340 \text{ MPa} \qquad E = 200 \text{ GPa}$

$$\frac{L_c}{r} = \sqrt{\frac{2\pi^2 E}{\sigma_y}} = 107.756$$

(1) Trial value of σ_{allow}

Upper limit: with $\frac{L}{r} = 0 \qquad n_1 = \frac{5}{3}$

$$\sigma_{allow\_max} = \frac{\sigma_y}{n_1} \qquad \sigma_{allow\_max} = 204 \text{ MPa}$$

Try $\sigma_{allow} = 110$ MPa

(2) Trial value of area

$$A = \frac{P}{\sigma_{allow}} \qquad A = 10000 \text{ mm}^2$$

(3) Trial column W 360 × 79

$A = 10100 \text{ mm}^2 \qquad r = 48.8 \text{ mm}$

(4) Allowable stress for trial column

$$\frac{L}{r} = 122.951 \qquad \frac{L}{r} > \frac{L_c}{r}$$

$$n_2 = \frac{23}{12} \qquad n_2 = 1.917$$

$$\sigma_{allow} = \sigma_y \frac{\left(\sqrt{\frac{2\pi^2 E}{\sigma_y}}\right)^2}{2n_2\left(\frac{KL}{r}\right)^2}$$

$\sigma_{allow} = 68.1$ MPa

(5) Allowable load for trial column

$P_{allow} = \sigma_{allow} A \qquad P_{allow} = 688.1 \text{ KN}$

$P_{allow} < P$ (Not Satisfactory)

(W 360 × 79)

(6) Next larger size column

W 360 × 122 $\quad A = 15500 \text{ mm}^2 \quad = 63.0 \text{ mm}$

$$\frac{L}{r} = 95.238 \qquad \frac{L}{r} < \frac{L_c}{r}$$

$$n_1 = \frac{5}{3} + \frac{3\left(\frac{KL}{r}\right)}{8\left(\sqrt{\frac{2\pi^2 E}{\sigma_y}}\right)} - \frac{\left(\frac{KL}{r}\right)^3}{8\left(\sqrt{\frac{2\pi^2 E}{\sigma_y}}\right)^3}$$

$n_1 = 1.912$

$$\sigma_{allow} = \sigma_y \frac{1}{n_1}\left[1 - \frac{\left(\frac{KL}{r}\right)^2}{2\left(\sqrt{\frac{2\pi^2 E}{\sigma_y}}\right)^2}\right]$$

$\sigma_{allow} = 108.38$ MPa

$P_{allow} = A\sigma_{allow} \qquad P_{allow} = 1679.9 \text{ kN}$

$P_{allow} > P$ (OK) (W 360 × 122)

$\therefore$ Select W 360 × 122

Problem 11.9-7 Determine the allowable axial load P_{allow} for a steel *pipe column with pinned ends* having an outside diameter of 4.5 in. and wall thickness of 0.237 in. for each of the following lengths: $L = 6$ ft, 12 ft, 18 ft, and 24 ft. (Assume $E = 29{,}000$ ksi and $\sigma_Y = 36$ ksi.)

Solution 11.9.7 Steel pipe column

Pinned ends ($K = 1$).
Use AISC formulas.

$d_2 = 4.5$ in. $\quad t = 0.237$ in. $\quad d_1 = 4.026$ in.

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 3.1740 \text{ in.}^2$$

$$I = \frac{\pi}{64}(d_2{}^4 - d_1{}^4) = 7.2326 \text{ in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 1.5095 \text{ in.} \qquad \left(\frac{L}{r}\right)_{max} = 200$$

$E = 29{,}000$ ksi $\quad \sigma_Y = 36$ ksi

Eq. (11-76): $$\left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 126.1$$

$L_c = 126.1\, r = 190.4$ in. $= 15.9$ ft

L	6 ft	12 ft	18 ft	24 ft
L/r	47.70	95.39	143.1	190.8
n_1 (Eq. 11-79)	1.802	1.896	-	-
n_2 (Eq. 11-80)	-	-	1.917	1.917
σ_{allow}/σ_Y (Eq. 11-81)	0.5153	0.3765	-	-
σ_{allow}/σ_Y (Eq. 11-82)	-	-	0.2026	0.1140
σ_{allow} (ksi)	18.55	13.55	7.293	4.102
$P_{allow} = A\,\sigma_{allow}$	58.9 k	43.0 k	23.1 k	13.0 k

Problem 11.9-8 Determine the allowable axial load P_{allow} for a steel *pipe column with pinned ends* having an outside diameter of 220 mm and wall thickness of 12 mm for each of the following lengths: $L = 2.5$ m, 5 m, 7.5 m, and 10 m. (Assume $E = 200$ GPa and $\sigma_Y = 250$ MPa.)

Solution 11.9.8 Steel pipe column

Pinned ends ($K = 1$).
Use AISC formulas.

$d_2 = 220$ mm $\quad t = 12$ mm $\quad d_1 = 196$ mm

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 7841.4 \text{ mm}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 42.548 \text{ mm} \times 10^6 \text{ mm}^4$$

$$r = \sqrt{\frac{I}{A}} = 73.661 \text{ mm} \qquad \left(\frac{L}{r}\right)_{max} = 200$$

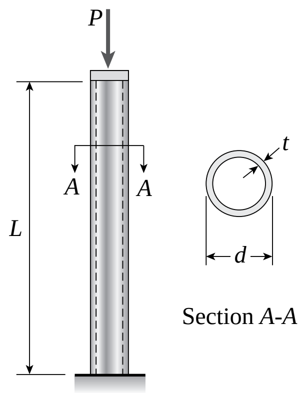
$E = 200$ GPa $\quad \sigma_Y = 250$ MPa

Eq. (11-76): $$\left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 125.7$$

$L_c = 125.7\, r = 9257$ mm $= 9.26$ m

L	2.5 m	5.0 m	7.5 m	10.0 m
L/r	33.94	67.88	101.8	135.8
n_1 (Eq. 11-79)	1.765	1.850	1.904	-
n_2 (Eq. 11-80)	-	-	-	1.917
σ_{allow}/σ_Y (Eq. 11-81)	0.5458	0.4618	0.3528	-
σ_{allow}/σ_Y (Eq. 11-82)	-	-	-	0.2235
σ_{allow} (MPa)	136.4	115.5	88.20	55.89
$P_{allow} = A\,\sigma_{allow}$	1070 kN	905 kN	692 kN	438 kN

Problem 11.9-9 Determine the allowable axial load P_{allow} for a steel pipe column that is fixed at the base and free at the top (see figure) for each of the following lengths: $L = 6$ ft, 9 ft, 12 ft, and 15 ft. The column has outside diameter $d = 6.625$ in. and wall thickness $t = 0.280$ in. (Assume $E = 29{,}000$ ksi and $\sigma_Y = 36$ ksi.)



Probs. 11.9-9 through 11.9-12

Solution 11.9-9 Steel pipe column

Fixed-free column ($K = 2$).
Use AISC formulas.

$d_2 = 6.625$ in. $\quad t = 0.280$ in. $\quad d_1 = 6.065$ in.

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 5.5814 \text{ in.}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 28.142 \text{ in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 2.2455 \qquad \left(\frac{KL}{r}\right)_{max} = 200$$

$E = 29{,}000$ ksi $\quad \sigma_Y = 36$ ksi

Eq. (11-76): $$\left(\frac{KL}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 126.1$$

$$L_c = 126.1\frac{r}{K} = 141.6 \text{ in.} = 11.8 \text{ ft}$$

L	6 ft	9 ft	12 ft	15 ft
KL/r	64.13	96.19	128.3	160.3
n_1 (Eq. 11-79)	1.841	1.897	-	-
n_2 (Eq. 11-80)	-	-	1.917	1.917
σ_{allow}/σ_Y (Eq. 11-81)	0.4730	0.3737	-	-
σ_{allow}/σ_Y (Eq.11-82)	-	-	0.2519	0.1614
σ_{allow} (Ksi)	17.03	13.45	9.078	5.810
$P_{allow} = A\,\sigma_{allow}$	95.0 k	75.1 k	50.7 k	32.4 k

Problem 11.9-10 Determine the allowable axial load P_{allow} for a steel pipe column that is fixed at the base and free at the top (see figure) for each of the following lengths: $L = 2.6$ m, 2.8 m, 3.0 m, and 3.2 m. The column has outside diameter $d = 140$ mm and wall thickness $t = 7$ mm. (Assume $E = 200$ GPa and $\sigma_Y = 250$ MPa.)

Solution 11.9-10 Steel pipe column

Fixed-free column ($K = 2$).
Use AISC formulas.

$d_2 = 140$ mm $\quad t = 7.0$ mm $\quad d_1 = 126$ mm

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 2924.8 \text{ mm}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 6.4851 \times 10^6 \text{ mm}^4$$

$$r = \sqrt{\frac{I}{A}} = 47.09 \text{ mm} \qquad \left(\frac{KL}{r}\right)_{max} = 200$$

$E = 200$ GPa $\quad \sigma_Y = 250$ MPa

Eq. (11-76): $$\left(\frac{KL}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 125.7$$

$$L_c = 125.7\frac{r}{K} = 2959 \text{ mm} = 2.959 \text{ m}$$

L	2.6 m	2.8 m	3.0 m	3.2 m
KL/r	110.4	118.9	127.4	135.9
n_1 (Eq. 11-79)	1.911	1.916	-	-
n_2 (Eq. 11-80)	-	-	1.917	1.917
σ_{allow}/σ_Y (Eq. 11-81)	0.3212	0.2882	-	-
σ_{allow}/σ_Y (Eq. 11-82)	-	-	0.2537	0.2230
σ_{allow} (MPa)	80.29	72.06	63.43	55.75
$P_{allow} = A\,\sigma_{allow}$	235 kN	211 kN	186 kN	163 kN

Problem 11.9-11 Determine the maximum permissible length L_{max} for a steel pipe column that is fixed at the base and free at the top and must support an axial load $P = 40$ k (see figure). The column has outside diameter $d = 4.0$ in. wall thickness $t = 0.226$ in., $E = 29{,}000$ ksi, and $\sigma_Y = 42$ ksi.

Solution 11.9-11 Steel pipe column

Fixed-free column ($K = 2$). $\quad P = 40$ k
Use AISC formulas.

$d_2 = 4.0$ in. $\quad t = 0.226$ in. $\quad d_1 = 3.548$ in.

$$A = \frac{\pi}{4}(d_2{}^2 - d_1{}^2) = 2.6795 \text{ in.}$$

$$I = \frac{\pi}{64}(d_2{}^4 - d_1{}^4) = 4.7877 \text{ in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 1.3367 \qquad \left(\frac{KL}{r}\right)_{max} = 200$$

$E = 29{,}000$ ksi $\quad \sigma_Y = 42$ ksi

Eq. (11-76): $$\left(\frac{KL}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 116.7$$

$$L_c = 116.7\frac{r}{K} = 78.03 \text{ in.} = 6.502 \text{ ft}$$

Select trial values of the length L and calculate the corresponding values of P_{allow} (see table). Interpolate between the trial values to obtain the value of L that produces $P_{allow} = P$.
Note: If $L < L_c$, use Eqs.(11-79) and (11-81).
If $L > L_c$, use Eqs.(11-80) and (11-82).

L(ft)	5.20	5.25	5.90
KL/r	93.86	94.26	93.90
n_1 (Eq. 11-79)	1.903	1.904	1.903
n_2 (Eq. 11-80)	-	-	-
σ_{allow}/σ_Y (Eq. 11-81)	0.3575	0.3541	0.3555
σ_{allow}/σ_Y (Eq. 11-82)	-	-	-
σ_{allow} (ksi)	15.02	14.87	14.93
$P_{allow} = A\,\sigma_{allow}$	40.2 k	39.8 k	40.0 k

For $P = 40$ k, $\quad L_{max} = 5.23$ ft $\quad \leftarrow$

Problem 11.9-12 Determine the maximum permissible length L_{max} for a steel pipe column that is fixed at the base and free at the top and must support an axial load $P = 500$ kN (see figure). The column has outside diameter $d = 200$ mm, wall thickness $t = 10$ mm, $E = 200$ GPa, and $\sigma_Y = 250$ MPa.

Solution 11.9-12 Steel pipe column

Fixed-free column ($K = 2$). $P = 500$ kN
Use AISC formulas.

$d_2 = 200$ mm $t = 10$ mm $d_1 = 180$ mm

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 5{,}969.0 \text{ mm}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 27.010 \times 10^6 \text{ mm}^4$$

$$r = \sqrt{\frac{I}{A}} = 67.27 \text{ mm} \qquad \left(\frac{KL}{r}\right)_{max} = 200$$

$E = 200$ GPa $\sigma_Y = 250$ GPa

$$\text{Eq. (11-76):} \quad \left(\frac{KL}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 125.7$$

$$L_c = 125.7 \frac{r}{K} = 4.226 \text{ m}$$

Select trial values of the length L and calculate the corresponding values of P_{allow} (see table). Interpolate between the trial values to obtain the value of L that produces $P_{allow} = P$.

Note: If $L < L_c$, use Eqs. (11-79) and (11-81).
If $L > L_c$, use Eqs. (11-80) and (11-82).

L(m)	3.55	3.60	3.59
KL/r	105.5	107.0	106.7
n_1 (Eq. 11-79)	1.908	1.909	1.909
n_2 (Eq. 11-80)	-	-	-
σ_{allow}/σ_Y (Eq. 11-81)	0.3393	0.3338	0.3349
σ_{allow}/σ_Y (Eq. 11-82)	-	-	-
σ_{allow} (MPa)	84.83	83.46	83.74
$P_{allow} = A\,\sigma_{allow}$	506 kN	498 kN	500 kN

For $P = 500$ kN, $L = 3.59$ m ←

Problem 11.9-13 A steel pipe column with *Pinned ends* supports an axial load $P = 21$ k. The pipe has outside and inside diameters of 3.5 in. and 2.9 in., respectively. What is the maximum permissible length L_{max} of the column if $E = 29{,}000$ ksi and $\sigma_Y = 36$ ksi?

Solution 11.9-13 Steel pipe column

Pinned ends ($K = 1$). $P = 21$ k
Use AISC formulas.

$d_2 = 3.5$ in. $t = 0.3$ in. $d_1 = 2.9$ in.

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 3.0159 \text{ in.}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 3.8943 \text{ in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 1.1363 \text{ in.} \qquad \left(\frac{L}{r}\right)_{max} = 200$$

$E = 29{,}000$ ksi $\sigma_Y = 36$ ksi

$$\text{Eq. (11-76):} \quad \left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 126.1$$

$$L_c = 126.1\, r = 143.3 \text{ in.} = 11.9 \text{ ft}$$

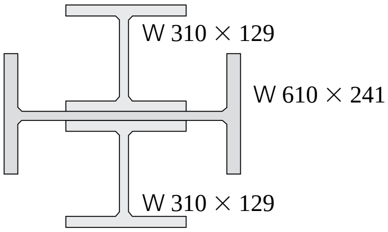
Select trial values of the length L and calculate the corresponding values of P_{allow} (see table). Interpolate between the trial values to obtain the value of L that produces $P_{allow} = P$.
Note: If $L < L_c$, use Eqs. (11-79) and (11-81).
If $L > L_c$, use Eqs. (11-80) and (11-82).

L(ft)	13.8	13.9	14.0
L/r	145.7	146.8	147.8
n_1 (Eq. 11-79)	-	-	-
n_2 (Eq. 11-80)	1.917	1.917	1.917
σ_{allow}/σ_Y (Eq. 11-81)	-	-	-
σ_{allow}/σ_Y (Eq. 11-82)	0.1953	0.1925	0.1898
σ_{allow} (ksi)	7.031	6.931	6.832
$P_{allow} = A\,\sigma_{allow}$	21.2 k	20.9 k	20.6 k

For $P = 21$ k, $L = 13.9$ ft ←

Problem 11.9-14 A steel column used in a college recreation center are 16.75 m long and are formed by welding three wide-flange sections (see figure). The columns are pin-supported at the ends and may buckle in any direction.

Calculate the allowable load P_{allow} for one column, assuming $E = 200$ GPa and $\sigma_Y = 250$ MPa.



Solution 11.9-14

$L = 16.75\text{ m} \quad E = 200\text{ GPa} \quad \sigma_y = 250\text{ MPa} \quad K = 1$

W 310 × 129 $\quad A_1 = 16500\text{ mm}^2 \quad d_1 = 318\text{ mm}$

$I_{1-1} = 308 \times 10^6\text{ mm}^4 \quad I_{2-1} = 100 \times 10^6\text{ mm}^4$

W 610 × 241 $\quad A_2 = 30800\text{ mm}^2 \quad t_w = 17.9\text{ mm}$

$I_{1-2} = 2150 \times 10^6\text{ mm}^4 \quad I_{2-2} = 184 \times 10^6\text{ mm}^4$

For built-up column:

$A = 2A_1 + A_2 \quad A = 63800\text{ mm}^2$

$I_y = 2\,I_{2\text{–}1} + I_{1\text{–}2} \quad I_y = 2.350 \times 10^9\text{ mm}^4$

$$h = \frac{d_1}{2} + \frac{t_w}{2} \quad h = 168\text{ mm}$$

$I_z = I_{2\text{–}2} + 2\,(I_{1\text{–}1+}A_1h^2) \quad I_z = 1.731 \times 10^9\text{ mm}^4$

$$r = \sqrt{\frac{I_z}{A}} \quad r = 165\text{ mm}$$

$$\frac{L_c}{r} = \sqrt{\frac{2\pi^2 E}{\sigma_y}} = 125.664$$

$$\frac{L}{r} = 101.694 \qquad \frac{L}{r} < \frac{L_c}{r}$$

$$n_1 = \frac{5}{3} + \frac{3\left(\frac{KL}{r}\right)}{8\left(\sqrt{\frac{2\pi^2 E}{\sigma_y}}\right)} - \frac{\left(\frac{KL}{r}\right)^3}{8\left(\sqrt{\frac{2\pi^2 E}{\sigma_y}}\right)^3}$$

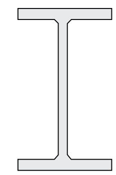
$n_1 = 1.904$

$$\sigma_{\text{allow}} = \sigma_y \frac{1}{n_1}\left[1 - \frac{\left(\frac{KL}{r}\right)^2}{2\left(\sqrt{\frac{2\pi^2 E}{\sigma_y}}\right)^2}\right]$$

$\sigma_{\text{allow}} = 88.31\text{ MPa}$

$P_{\text{allow}} = A\sigma_{\text{allow}} \qquad P_{\text{allow}} = 5634\text{ kN} \quad \leftarrow$

Problem 11.9-15 A W 8 $\times$ 28 steel wide-flange column with pinned ends carries an axial load P. What is the maximum permissible length L_{max} of the column if (a) $P = 50$ k, and (b) $P = 100$ k? (Assume $E = 29{,}000$ ksi and $\sigma_Y = 36$ ksi.)



Probs. 11.9-15 and 11.9-16

Solution 11.9-15 Steel wide-flange column

Pinned ends ($K = 1$).
Buckling about axis 2-2 (see Table E-1a).
Use AISC formulas.

W 8 $\times$ 28 $\quad A = 8.25 \text{ in.}^2 \quad r_2 = 1.62$ in.

$$E = 29{,}000 \text{ ksi} \quad \sigma_Y = 36 \text{ ksi} \quad \left(\frac{L}{r}\right)_{max} = 200$$

$$\text{Eq.(11-76):} \quad \left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 126.1$$

$$L_c = 126.1\, r = 204.3 \text{ in.} = 17.0 \text{ ft}$$

For each load P, select trial values of the length L and calculate the corresponding values of P_{allow} (see table). Interpolate between the trial values to obtain the value of L that produces $P_{allow} = P$.

Note: If $L < L_c$, use Eqs. (11-79) and (11-81).

If $L > L_c$, use Eqs. (11-80) and (11-82).

(a) $P = 50$ k

L(ft)	21.0	21.5	21.2
L/r	155.6	159.3	157.0
n_1 (Eq. 11-79)	-	-	-
n_2 (Eq. 11-80)	1.917	1.917	1.917
σ_{allow}/σ_Y (Eq. 11-81)	-	-	-
σ_{allow}/σ_Y (Eq. 11-82)	0.1714	0.1635	0.1682
σ_{allow} (ksi)	6.171	5.888	6.056
$P_{allow} = A\,\sigma_{allow}$	50.9 k	48.6 k	50.0 k

For $P = 50$ k, $\quad L_{max} = 21.2$ ft $\quad \leftarrow$

(b) $P = 100$ k

L(ft)	14.3	14.4	14.5
L/r	105.9	106.7	107.4
n_1 (Eq. 11-79)	1.908	1.908	1.909
n_2 (Eq. 11-80)	-	-	-
σ_{allow}/σ_Y (Eq. 11-81)	0.3393	0.3366	0.3338
σ_{allow}/σ_Y (Eq. 11-82)	-	-	-
σ_{allow} (ksi)	12.21	12.12	12.02
$P_{allow} = A\,\sigma_{allow}$	100.8 k	100.0 k	99.2 k

For $P = 100$ k, $\quad L_{max} = 14.4$ ft $\quad \leftarrow$

Problem 11.9-16 A W 250 × 67 steel wide-flange column with pinned ends carries an axial load P. What is the maximum permissible length L_{max} of the column if (a) $P = 560$ kN, and (b) $P = 890$ kN? (Assume $E = 200$ GPa and $\sigma_Y = 290$ MPa.)

Solution 11.9-16

$E = 200\text{ GPa} \quad \sigma_y = 290\text{ MPa} \quad K = 1$

W 250 × 67 $\quad A = 8580\text{ mm}^2$

$r_2 = 51.1\text{ mm} \quad r = r_2$

$L_{max} = 200r$

$$L_c = r\sqrt{\frac{2\pi^2 E}{\sigma_y}} \quad L_c = 5.962\text{ m} \quad \frac{L_c}{r} = 116.676$$

For each load, select trial values of the length L and calculate the corresponding values of P_{allow}

This solution show only the successful trial.

(a) $P = 560$ kN

$$\text{Try} \quad L = 6.41\text{ m} \quad \frac{L}{r} = 125.440 \quad \frac{L}{r} > \frac{L_c}{r}$$

$$n_2 = \frac{23}{12}$$

$$\sigma_{allow} = \sigma_y \frac{\left(\frac{KL_c}{r}\right)^2}{2n_2\left(\frac{KL}{r}\right)^2}$$

$\sigma_{allow} = 65.45\text{ MPa}$

$P_{allow} = A\sigma_{allow} \quad P_{allow} = 561.6\text{ kN}$

Therefore $\quad L_{max} = L \quad L_{max} = 6.41\text{ m} \quad \leftarrow$

(b) $P = 890$ kN

$$\text{Try} \quad L = 4.76\text{ m} \quad \frac{L}{r} = 93.151 \quad \frac{L}{r} < \frac{L_c}{r}$$

$$n_1 = \frac{5}{3} + \frac{3\left(\frac{KL}{r}\right)}{8\left(\sqrt{\frac{2\pi^2 E}{\sigma_y}}\right)} - \frac{\left(\frac{KL}{r}\right)^3}{8\left(\sqrt{\frac{2\pi^2 E}{\sigma_y}}\right)^3}$$

$n_1 = 1.902$

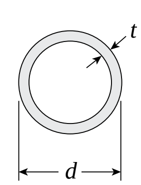
$$\sigma_{allow} = \sigma_y \frac{1}{n_1}\left[1 - \frac{\left(\frac{KL}{r}\right)^2}{2\left(\sqrt{\frac{2\pi^2 E}{\sigma_y}}\right)^2}\right]$$

$\sigma_{allow} = 103.85\text{ MPa}$

$P_{allow} = A\sigma_{allow} \quad P_{allow} = 891\text{ kN}$

Therefore $\quad L_{max} = L \quad L_{max} = 4.76\text{ m} \quad \leftarrow$

Problem 11.9-17 Find the required outside diameter d for a steel pipe column (see figure) of length $L = 20$ ft that is pinned at both ends and must support an axial load $P = 25$ k. Assume that the wall thickness t is equal to $d/20$. (Use $E = 29{,}000$ ksi and $\sigma_Y = 36$ ksi.)



Probs. 11.9-17 through 11.9-20

Solution 11.9-17 Pipe column

Pinned ends ($K = 1$).

$L = 20 \text{ ft} = 240 \text{ in.} \quad P = 25 \text{ k}$

$d =$ outside diameter $\quad t = d/20$

$E = 29{,}000 \text{ ksi} \quad \sigma_Y = 36 \text{ ksi}$

$$A = \frac{\pi}{4}[d^2 - (d - 2t)^2] = 0.14923\, d^2$$

$$I = \frac{\pi}{64}[d^4 - (d - 2t)^4] = 0.016881\, d^4$$

$$r = \sqrt{\frac{I}{A}} = 0.33634\, d$$

$$\left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 126.1 \quad L_c = (126.1)r$$

Seltect various values of diameter d until we obtain $P_{allow} = P$.

If $L \le L_c$, Use Eqs. (11-79) and (11-81).

If $L \ge L_c$, Use Eqs. (11-80) and (11-82).

d (in.)	4.80	4.90	5.00
A (in.^{2})	3.438	3.583	3.731
I (in.^{4})	8.961	9.732	10.551
r (in.)	1.614	1.648	1.682
L_c (in.)	204	208	212
L/r	148.7	145.6	142.7
n_2 (Eq. 11-80)	23/12	23/12	23/12
σ_{allow}/σ_Y (Eq. 11-82)	0.1876	0.1957	0.2037
σ_{allow} (ksi)	6.754	7.044	7.333
$P_{allow} = A\,\sigma_{allow}$	23.2 k	25.2 k	27.4 k

For $P = 25$ k, $\quad d = 4.89$ in. $\quad \leftarrow$

Problem 11.9-18 Find the required outside diameter d for a steel pipe column (see figure) of length $L = 3.5$ m that is pinned at both ends and must support an axial load $P = 130$ kN. Assume that the wall thickness t is equal to d/20. (Use $E = 200$ GPa and $\sigma_Y = 275$ MPa.)

Solution 11.9-18 Pipe column

Pinned ends ($K = 1$).

$L = 3.5 \text{ m} \quad P = 130 \text{ kN}$

$d =$ outside diameter $\quad t = d/20$

$E = 200 \text{ GPa} \quad \sigma_Y = 275 \text{ MPa}$

$$A = \frac{\pi}{4}[d^2 - (d - 2t)^2] = 0.14923\, d^2$$

$$I = \frac{\pi}{64}[d^4 - (d - 2t)^4] = 0.016881\, d^4$$

$$r = \sqrt{\frac{I}{A}} = 0.33634\, d$$

$$\left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 119.8 \quad L_c = (119.8)r$$

Select various values of diameter d until we obtain $P_{allow} = P$.
If $L \le L_c$, Use Eqs. (11-79) and (11-81).
If $L \ge L_c$, Use Eqs. (11-80) and (11-82).

d (mm)	98	99	100
A (mm^2)	1433	1463	1492
I (mm^4)	1557×10^3	1622×10^3	1688×10^3
r (mm)	32.96	33.30	33.64
L_c (mm)	3950	3989	4030
L/r	106.2	105.1	104.0
n_1 (Eq. 11-79)	1.912	1.911	1.910
σ_{allow}/σ_Y (Eq. 11-81)	0.3175	0.3219	0.3263
σ_{allow} (MPa)	87.32	88.53	89.73
$P_{allow} = A\,\sigma_{allow}$	125.1 kN	129.5 kN	133.9 kN

For $P = 130$ kN, $\quad d = 99$ mm $\quad \leftarrow$

Problem 11.9-19 Find the required outside diameter d for a steel pipe column (see figure) of length $L = 11.5$ ft that is pinned at both ends and must support an axial load $P = 80$ k. Assume that the wall thickness t is 0.30 in. (Use $E = 29{,}000$ ksi and $\sigma_Y = 42$ ksi.)

Solution 11.9-19 Pipe column

Pinned ends ($K = 1$).

$L = 11.5 \text{ ft} = 138 \text{ in.} \quad P = 80 \text{ k}$

d = outside diameter $\quad t = 0.30$ in.

$E = 29{,}000 \text{ ksi} \quad \sigma_Y = 42 \text{ ksi}$

$$A = \frac{\pi}{4}[d^2 - (d - 2t)^2]$$

$$I = \frac{\pi}{64}[d^4 - (d - 2t)^4] \quad r = \sqrt{\frac{I}{A}}$$

$$\left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 116.7 \quad L_c = (116.7)r$$

Seltect various values of diameter d until we obtain $P_{allow} = P$.

If $L \le L_c$, Use Eqs. (11-79) and (11-81).
If $L \ge L_c$, Use Eqs. (11-80) and (11-82).

d (in.)	5.20	5.25	5.30
A (in.^{2})	4.618	4.665	4.712
I (in.^{4})	13.91	14.34	14.78
r (in.)	1.736	1.753	1.771
L_c (in.)	203	205	207
L/r	79.49	78.72	77.92
n_1 (Eq. 11-79)	1.883	1.881	1.880
σ_{allow}/σ_Y (Eq. 11-81)	0.4079	0.4107	0.4133
σ_{allow} (ksi)	17.13	17.25	17.36
$P_{allow} = A\,\sigma_{allow}$	79.1 k	80.5 k	81.8 k

For $P = 80$ k, $\quad d = 5.23$ in. $\leftarrow$

Problem 11.9-20 Find the required outside diameter d for a steel pipe column (see figure) of length $L = 3.0$ m that is pinned at both ends and must support an axial load $P = 800$ kN. Assume that the wall thickness t is 9 mm. (Use $E = 200$ GPa and $\sigma_Y = 300$ MPa.)

Solution 11.9-20 Pipe column

Pinned ends ($K = 1$).

$L = 3.0 \text{ m} \quad P = 800 \text{ kN}$

d = outside diameter $\quad t = 9.0$ mm

$E = 200 \text{ GPa} \quad \sigma_Y = 300 \text{ MPa}$

$$A = \frac{\pi}{4}[d^2 - (d - 2t)^2]$$

$$I = \frac{\pi}{64}[d^4 - (d - 2t)^4] \quad r = \sqrt{\frac{I}{A}}$$

$$\left(\frac{L}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 114.7 \quad L_c = (114.7)r$$

Seltect various values of diameter d until we obtain $P_{allow} = P$.
If $L \le L_c$, Use Eqs. (11-79) and (11-81).
If $L \ge L_c$, Use Eqs. (11-80) and (11-82).

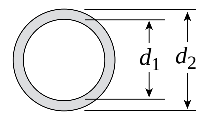
d (mm)	193	194	195
A (mm^2)	5202	5231	5259
I (mm^4)	20.08×10^6	22.43×10^6	22.80×10^6
r (mm)	65.13	65.48	65.84
L_c (mm)	7470	7510	7550
L/r	46.06	45.82	45.57
n_1 (Eq. 11-79)	1.809	1.809	1.808
σ_{allow}/σ_Y (Eq. 11-81)	0.5082	0.5087	0.5094
σ_{allow} (MPa)	152.5	152.6	152.8
$P_{allow} = A\,\sigma_{allow}$	793.1 kN	798.3 kN	803.8 kN

For $P = 800$ kN, $\quad d = 194$ mm $\leftarrow$

Aluminum Columns

Problem 11.9-21 An aluminum pipe column (alloy 2014-T6) with pinned ends has outside diameter $d_2 = 5.60$ in. and inside diameter $d_1 = 4.80$ in. (see figure).

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 6$ ft, 8 ft, 10 ft, and 12 ft.



Probs. 11.9-21 through 11.9-24

Solution 11.9-21 Aluminum pipe column

Alloy 2014-T6
Pinned ends ($K = 1$).
$d_2 = 5.60$ in.
$d_1 = 4.80$ in.

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 6.535 \text{ in.}^2$$

$$I = \frac{\pi}{4}(d_2^4 - d_1^4) = 22.22 \text{ in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 1.844 \text{ in.}$$

Use Eqs. (11-84 *a* and *b*):

$$\sigma_{allow} = 30.7 - 0.23\,(L/r) \text{ ksi} \quad L/r \le 55$$

$$\sigma_{allow} = 54{,}000/(L/r)^2 \text{ ksi} \quad L/r \ge 55$$

L (ft)	6 ft	8 ft	10 ft	12 ft
L/r	39.05	52.06	65.08	78.09
σ_{allow} (ksi)	21.72	18.73	12.75	8.86
$P_{allow} = \sigma_{allow} A$	142 k	122 k	83 k	58 k

Problem 11.9-22 An aluminum pipe column (alloy 2014-T6) with pinned ends has outside diameter $d_2 = 120$ mm and inside diameter $d_1 = 110$ mm (see figure).

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 1.0$ m, 2.0 m, 3.0 m , and 4.0 m.

(*Hint*: Convert the given data to USCS units, determine the required quantities, and then convert back to SI units.)

Solution 11.9-22 Aluminum pipe column

Alloy 2014-T6
Pinned ends ($K = 1$).

$d_2 = 120 \text{ mm} = 4.7244$ in.

$d_1 = 110 \text{ mm} = 4.3307$ in.

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 2.800 \text{ in.}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 7.188 \text{ in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 40.697 \text{ mm} = 1.6022 \text{ in.}$$

Use Eqs. (11-84 *a* and *b*):

$$\sigma_{allow} = 30.7 - 0.23\,(L/r) \text{ ksi} \quad L/r \le 55$$

$$\sigma_{allow} = 54{,}000/(L/r)^2 \text{ ksi} \quad L/r \ge 55$$

L (m)	1.0 m	2.0 m	3.0 ft	4.0 m
L (in.)	39.37	78.74	118.1	157.5
L/r	24.58	49.15	73.73	98.30
σ_{allow} (ksi)	25.05	19.40	9.934	5.588
$P_{allow} = \sigma_{allow} A$	70.14 k	54.31 k	27.81 k	15.65 k
P_{allow}(kN)	312 kN	242 kN	124 kN	70 kN

Problem 11.9-23 An aluminum pipe column (alloy 6061-T6) that is fixed at the base and free at the top has outside diameter $d_2 = 3.25$ in. and inside diameter $d_1 = 3.00$ in. (see figure).

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 2$ ft, 3 ft, 4 ft, and 5 ft.

Solution 11.9-23 Aluminum pipe column

Alloy 6061-T6
Pinned ends ($K = 2$).

$d_2 = 3.25$ in.

$d_1 = 3.00$ in.

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 1.227 \text{ in.}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 1.500 \text{ in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 1.106 \text{ in.}$$

Use Eqs. (11-85 *a* and *b*):

$\sigma_{allow} = 20.2 - 0.126\,(KL/r)$ ksi $\quad KL/r \leq 66$

$\sigma_{allow} = 51{,}000/(KL/r)^2$ ksi $\quad KL/r \geq 66$

L (ft)	2 ft	3 ft	4 ft	5 ft
KL/r	43.40	65.10	86.80	108.5
σ_{allow} (ksi)	14.73	12.00	6.77	4.33
$P_{allow} = \sigma_{allow} A$	18.1 k	14.7 k	8.3 k	5.3 k

Problem 11.9-24 An aluminum pipe column (alloy 6061-T6) that is fixed at the base and free at the top has outside diameter $d_2 = 80$ mm and inside diameter $d_1 = 72$ mm (see figure).

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 0.6$ m, 0.8 m, 1.0 m, and 1.2 m.

(*Hint:* Convert the given data to USCS units, determine the required quantities, and then convert back to SI units.)

Solution 11.9-24 Aluminum pipe column

Alloy 6061-T6
Pinned ends ($K = 2$).

$d_2 = 80$ mm $= 3.1496$ in.

$d_1 = 72$ mm $= 2.8346$ in.

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 1.480 \text{ in.}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 1.661 \text{ in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 26.907 \text{ mm} = 1.059 \text{ in.}$$

Use Eqs. (11-85 *a* and *b*):

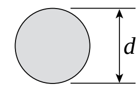
$\sigma_{allow} = 20.2 - 0.126\,(L/r)$ ksi $\quad L/r \leq 66$

$\sigma_{allow} = 51{,}000/(L/r)^2$ ksi $\quad KL/r \geq 66$

L (m)	0.6 m	0.8 m	1.0 ft	1.2 m
KL (in.)	47.24	62.99	78.74	94.49
KL/r	44.61	59.48	74.35	89.23
σ_{allow} (ksi)	14.58	12.71	9.226	6.405
$P_{allow} = \sigma_{allow} A$	21.58 k	18.81 k	13.65 k	9.48 k
P_{allow}(kN)	96 kN	84 kN	61 kN	42 kN

Problem 11.9-25 A solid round bar of aluminum having diameter d (see figure) is compressed by an axial force $P = 60$ k. The bar has pinned supports and is made of alloy 2014-T6.

(a) If the diameter $d = 2.0$ in., what is the maximum allowable length L_{max} of the bar?
(b) If the length $L = 30$ in., what is the minimum required diameter d_{min}?



Probs. 11.9-25 through 11.9-28

Solution 11.9-25 Aluminum bar

Alloy 2014-T6
Pinned ends ($K = 1$). $P = 60$ k

(a) FIND L_{max} IF $d = 2.0$ in.

$$A = \frac{\pi d^2}{4} = 3.142 \text{ in.}^2 \quad I = \frac{\pi d^4}{64}$$

$$r = \sqrt{\frac{I}{A}} = \frac{d}{4} = 0.5 \text{ in.}$$

$$\sigma_{allow} = \frac{P}{A} = \frac{60 \text{ k}}{3.142 \text{ in.}^2} = 19.10 \text{ ksi}$$

Assume L/r is less than 55:

Eq. (11-84a): $\sigma_{allow} = 30.7 - 0.23(L/r)$ ksi

or $19.10 = 30.7 - 0.23(L/r)$

Solve for L/r: $\frac{L}{r} = 50.43 \quad \frac{L}{r} < 55 \quad \therefore$ ok

$$L_{max} = (50.43)r = 25.2 \text{ in.} \quad \leftarrow$$

(b) FIND d_{min} IF $L = 30$ in.

$$A = \frac{\pi d^2}{4} \quad r = \frac{d}{r} \quad \frac{L}{r} = \frac{30 \text{ in.}}{d/4} = \frac{120 \text{ in.}}{d}$$

$$\sigma_{allow} = \frac{P}{A} = \frac{60 \text{ k}}{\pi d^2/4} = \frac{76.39}{d^2} \text{ (ksi)}$$

Assume L/r is greater than 55:

$$\text{Eq.(11-84}b\text{): } \sigma_{allow} = \frac{54{,}000 \text{ ksi}}{(L/r)^2}$$

$$\text{or} \quad \frac{76.39}{d^2} = \frac{54{,}000}{(120/d)^2}$$

$$d^4 = 20.37 \text{ in.}^4 \quad d_{min} = 2.12 \text{ in.} \quad \leftarrow$$

$$L/r = 120/d = 120/2.12 = 56.6 > 55 \quad \therefore \text{ ok}$$

Problem 11.9-26 A solid round bar of aluminum having diameter d (see figure) is compressed by an axial force $P = 175$ kN. The bar has pinned supports and is made of alloy 2014-T6.

(a) If the diameter $d = 40$ mm, what is the maximum allowable length L_{max} of the bar?
(b) If the length $L = 0.6$ m, what is the minimum required diameter d_{min}?

(*Hint:* Convert the given data to USCS units, determine the required quantities, and then convert back to SI units.)

Solution 11.9-26 Aluminum bar

Alloy 2014-T6
Pinned supports ($K = 1$). $P = 175 \text{ kN} = 39.34$ k

(a) FIND L_{max} IF $d = 40 \text{ mm} = 1.575$ in.

$$A = \frac{\pi d^2}{4} = 1.948 \text{ in.}^2 \quad I = \frac{\pi d^4}{64}$$

$$r = \sqrt{\frac{I}{A}} = \frac{d}{4} = 0.3938 \text{ in.}$$

$$\sigma_{allow} = \frac{P}{A} = \frac{39.34 \text{ k}}{1.948 \text{ in.}^2} = 20.20 \text{ ksi}$$

Assume L/r is less than 55:

Eq. (11-84a): $\sigma_{allow} = 30.7 - 0.23\,(L/r)$ ksi

or $20.20 = 30.7 - 0.23\,(L/r)$

Solve for L/r: $\dfrac{L}{r} = 45.65 \quad \dfrac{L}{r} < 55 \quad \therefore$ ok

$L_{max} = (45.65)r = 17.98$ in. $= 457$ mm $\leftarrow$

(b) FIND d_{min} IF $L = 0.6$ m $= 23.62$ in.

$$A = \frac{\pi d^2}{4} \quad r = \frac{d}{4} \quad \frac{L}{r} = \frac{23.62 \text{ in.}}{d/4} = \frac{94.48 \text{ in.}}{d}$$

$$\sigma_{allow} = \frac{P}{A} = \frac{39.34 \text{ k}}{\pi d^2/4} = \frac{50.09}{d^2} \quad \text{(ksi)}$$

Assume L/r is greater than 55:

Eq. (11-84b): $\sigma_{allow} = \dfrac{54{,}000 \text{ ksi}}{(L/r)^2}$

or $\dfrac{50.09}{d^2} = \dfrac{54{,}000}{(94.48/d)^2}$

$d^4 = 8.280$ in.^{4}

$d_{min} = 1.696$ in. $= 43.1$ mm $\leftarrow$

$L/r = 94.48/d = 94.48/1.696$

$= 55.7 > 55 \quad \therefore$ ok

Problem 11.9-27 A solid round bar of aluminum having diameter d (see figure) is compressed by an axial force $P = 10$ k. The bar has pinned supports and is made of alloy 6061-T6.

(a) If the diameter $d = 1.0$ in., what is the maximum allowable length L_{max} of the bar?
(b) If the length $L = 20$ in., what is the minimum required diameter d_{min}?

Solution 11.9-27 Aluminum bar

Alloy 6061-T6
Pinned supports ($K = 1$). $P = 10$ k

(a) FIND L_{max} IF $d = 1.0$ in.

$$A = \frac{\pi d^2}{4} = 0.7854 \text{ in.}^2 \quad I = \frac{\pi d^4}{64}$$

$$r = \sqrt{\frac{I}{A}} = \frac{d}{4} = 0.2500 \text{ in.}$$

$$\sigma_{allow} = \frac{P}{A} = \frac{10 \text{ k}}{0.7854 \text{ in.}^2} = 12.73 \text{ ksi}$$

Assume L/r is less than 66:

Eq.(11-85a): $\sigma_{allow} = 20.2 - 0.126\,(L/r)$ ksi

or $12.73 = 20.2 - 0.126\,(L/r)$

Solve For L/r: $\dfrac{L}{r} = 59.29 \quad \dfrac{L}{r} < 66 \quad \therefore$ ok

$L_{max} = (59.29)r = 14.8$ in. $\leftarrow$

(b) FIND d_{min} IF $L = 20$ in.

$$A = \frac{\pi d^2}{4} \quad r = \frac{d}{4} \quad \frac{L}{r} = \frac{20 \text{ in.}}{d/4} = \frac{80 \text{ in.}}{d}$$

$$\sigma_{allow} = \frac{P}{A} = \frac{10 \text{ k}}{\pi d^2/4} = \frac{12.73}{d^2} \text{ (ksi)}$$

Assume L/r is greater than 66:

Eq. (11-85b): $\sigma_{allow} = \dfrac{51{,}000 \text{ ksi}}{(L/r)^2}$

or $\dfrac{12.73}{d^2} = \dfrac{51{,}000}{(80/d)^2}$

$d^4 = 1.597$ in.^{4} $\quad d_{min} = 1.12$ in. $\leftarrow$

$L/r = 80/d = 80/1.12 = 71 > 66 \quad \therefore$ ok

Problem 11.9-28 A solid round bar of aluminum having diameter d (see figure) is compressed by an axial force $P = 60$ kN. The bar has pinned supports and is made of alloy 6061-T6.

(a) If the diameter $d = 30$ mm, what is the maximum allowable length L_{max} of the bar?
(b) If the length $L = 0.6$ m, what is the minimum required diameter d_{min}?

(*Hint:* Convert the given data to USCS units, determine the required quantities, and the convert back to SI units.)

Solution 11.9-28 Aluminum bar

Alloy 6061-T6

Pinned supports ($K = 1$). $P = 60 \text{ kN} = 13.49 \text{ k}$

(a) FIND L_{max} IF $d = 30 \text{ mm} = 1.181$ in.

$$A = \frac{\pi d^2}{4} = 1.095 \text{ in.}^2 \quad I = \frac{\pi d^4}{64}$$

$$r = \sqrt{\frac{I}{A}} = \frac{d}{4} = 0.2953 \text{ in.}$$

$$\sigma_{allow} = \frac{P}{A} = \frac{13.49 \text{ k}}{1.095 \text{ in.}^2} = 12.32 \text{ ksi}$$

Assume L/r is less than 66:

Eq. (11-85*a*): $\sigma_{allow} = 20.2 - 0.126\,(L/r)$ ksi

or $12.32 = 20.2 - 0.126\,(L/r)$

Solve For L/r: $\frac{L}{r} = 62.54 \quad \frac{L}{r} < 66 \quad \therefore$ ok

$L_{max} = (62.54)r = 18.47 \text{ in.} = 469 \text{ mm} \quad \leftarrow$

(b) FIND d_{min} IF $L = 0.6 \text{ m} = 23.62$ in.

$$A = \frac{\pi d^2}{4} \quad r = \frac{d}{4} \quad \frac{L}{r} = \frac{23.62 \text{ in.}}{d/4} = \frac{94.48 \text{ in.}}{d}$$

$$\sigma_{allow} = \frac{P}{A} = \frac{13.48 \text{ k}}{\pi d^2/4} = \frac{17.18}{d^2} \text{(ksi)}$$

Assume L/r is greater than 66:

Eq. (11-85*b*): $\sigma_{allow} = \dfrac{51{,}000 \text{ ksi}}{(L/r)^2}$

or $\dfrac{17.18}{d^2} = \dfrac{51{,}000}{(94.48/d)^2}$

$d^4 = 3.007 \text{ in.}^4$

$d_{min} = 1.317 \text{ in.} = 33.4 \text{ mm} \quad \leftarrow$

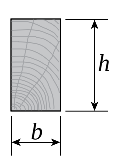
$L/r = 94.48/d = 94.48/1.317 = 72 > 66 \quad \therefore$ ok

Wood Columns

When solving the problems for wood columns, assume that the columns are constructed of sawn lumber ($c = 0.8$ and $K_{cE} = 0.3$) and have pinned-end conditions. Also, buckling may occur about either principal axis of the cross section.

Problem 11.9-29 A wood post of rectangular cross section (see figure) is constructed of 4 in. × 6 in. structural grade, Douglas fir lumber ($F_c = 2{,}000$ psi, $E = 1{,}800{,}00$ psi). The net cross-sectional dimensions of the post are $b = 3.5$ in. (see Appendix F).

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 5.0$ ft, and 10.0 ft.



Probs. 11.9-29 through 11.9-32

Solution 11.9-29 Wood post (rectangular cross section)

$F_c = 2{,}000$ psi $E = 1{,}800{,}000$ psi $c = 0.8$

$K_{cE} = 0.3$ $b = 3.5$ in. $h = 5.5$ in. $d = b$

Find P_{allow}

Eq. (11-94): $\phi = \dfrac{K_{cE}E}{F_c(L_e/d)^2}$

Eq. (11-95): $C_P = \dfrac{1+\phi}{2c} - \sqrt{\left[\dfrac{1+\phi}{2c}\right]^2 - \dfrac{\phi}{c}}$

Eq. (11-92): $P_{\text{allow}} = F_cC_PA = F_cC_Pbh$

L_e	5 ft	7.5 ft	10.0 ft	
L_e/d	17.14	25.71	34.29	
ϕ	0.9188	0.4083	0.2297	
C_P	0.6610	0.3661	0.2176	
P_{allow}	25.4 k	14.1 k	8.4 k	←

Problem 11.9-30 A wood post of rectangular cross section (see figure) is constructed of structural grade, southern pine lumber ($F_c = 14$ MPa, $E = 12$ GPa). The cross-sectional dimensions of the post (actual dimensions) are $b = 100$ mm and $h = 150$ mm.

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 1.5$ m, 2.0 m, and 2.5 m.

Solution 11.9-30 Wood post (rectangular cross section)

$F_c = 14$ MPa $E = 12$ GPa

$c = 0.8$ $K_{cE} = 0.3$

$b = 100$ mm $h = 150$ mm $d = b$

Find P_{allow}

Eq. (11-94): $\phi = \dfrac{K_{cE}E}{F_c(L_e/d)^2}$

Eq. (11-95): $C_P = \dfrac{1+\phi}{2c} - \sqrt{\left[\dfrac{1+\phi}{2c}\right]^2 - \dfrac{\phi}{c}}$

Eq. (11-92): $P_{\text{allow}} = F_cC_PA = F_cC_Pbh$

L_e	1.5 m	2.0 m	2.5 m	
L_e/d	15	20	25	
ϕ	1.1429	0.6429	0.4114	
C_P	0.7350	0.5261	0.3684	
P_{allow}	154 kN	110 kN	77 kN	←

Problem 11.9-31 A wood post column of rectangular cross section (see figure) is constructed of 4 in. × 8 in. construction grade, western hemlock lumber ($F_c = 1{,}000$ psi, $E = 1{,}300{,}000$ psi). The net cross-sectional dimensions of the column are $b = 3.5$ in. and $h = 7.25$ in. (see Appendix F).

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 6$ ft 8 ft, and 10.0 ft.

Solution 11.9-31 Wood column (rectangular cross section)

$F_c = 1{,}000$ psi $\quad E = 1{,}300{,}000$ psi $\quad c = 0.8$

$K_{cE} = 0.3 \quad b = 3.5$ in. $\quad h = 7.25$ in. $\quad d = b$

Find P_{allow}

Eq. (11-94): $\phi = \dfrac{K_{cE}E}{F_c(L_e/d)^2}$

Eq. (11-95): $C_P = \dfrac{1+\phi}{2c} - \sqrt{\left[\dfrac{1+\phi}{2c.}\right]^2 - \dfrac{\phi}{c}}$

Eq. (11-92): $P_{\text{allow}} = F_c C_P A = F_c C_P bh$

L_e	6 ft	8 ft	10 ft	
L_e/d	20.57	27.43	34.29	
ϕ	0.9216	0.5184	0.3318	
C_P	0.6621	0.4464	0.3050	
P_{allow}	16.8 k	11.3 k	7.7 k	←

Problem 11.9-32 A wood column of rectangular cross section (see figure) is constructed of structural grade, Douglas fir lumber ($F_c = 12$ MPa, $E = 10$ GPa). The cross-sectional dimensions of the column (actual dimensions) are $b = 140$ mm and $h = 210$ mm.

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 2.5$ m, 3.5 m, and 4.5 m.

Solution 11.9-32 Wood column (rectangular cross section)

$F_c = 12$ MPa $\quad E = 10$ GPa $\quad c = 0.8 \quad K_{cE} = 0.3$

$b = 140$ mm $\quad h = 210$ mm $\quad d = b$

Find P_{allow}

Eq. (11-94): $\phi = \dfrac{K_{cE}E}{F_c(L_e/d)^2}$

Eq. (11-95): $C_P = \dfrac{1+\phi}{2c} - \sqrt{\left[\dfrac{1+\phi}{2c}\right]^2 - \dfrac{\phi}{c}}$

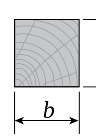
Eq. (11-92): $P_{\text{allow}} = F_c C_P A = F_c C_P bh$

L_e	2.5 m	3.5 m	4.5 m	
L_e/d	17.86	25.00	32.14	
ϕ	0.7840	0.4000	0.2420	
C_P	0.6019	0.3596	0.2284	
P_{allow}	212 kN	127 kN	81 kN	←

Problem 11.9-33 A square wood column with side dimensions b (see figure) is constructed of a structural grade of Douglas fir for which $F_c = 1{,}700$ psi and $E = 1{,}400{,}000$ psi. An axial force $P = 40$ k acts on the column.

(a) If the dimension $b = 55$ in., what is the maximum allowable length $L_{\max}$ of the column?

(b) If the length $L = 11$ ft, what is the minimum required dimension $b_{\min}$?



Probs. 11.9-33 through 11.9-36

Solution 11.9-33 Wood column (square cross section)

$F_c = 1{,}700$ psi $\quad E = 1{,}400{,}000$ psi $\quad c = 0.8$

$K_{cE} = 0.3 \quad P = 40$ k

(a) MAXIMUM LENGTH L_{max} FOR $b = d = 5.5$ in.

From Eq. (11-92): $C_P = \dfrac{P}{F_C b^2} = 0.77783$

From Eq. (11-95):

$$C_P = 0.77783 = \frac{1+\phi}{1.6} - \sqrt{\left[\frac{1+\phi}{1.6}\right]^2 - \frac{\phi}{0.8}}$$

Trial and error: $\phi = 1.3225$

From Eq. (11-94): $\dfrac{L}{d} = \sqrt{\dfrac{K_{cE}E}{\phi F_c}} = 13.67$

$\therefore L_{max} = 13.67d = (13.67)(5.5 \text{ in.})$

$= 75.2$ in.

(b) MINIMUM DIMENSION b_{min} for $L = 11$ ft

Trial and error: $\dfrac{L}{d} = \dfrac{L}{b}$

$$\phi = \frac{K_{cE}E}{F_c(L/d)^2}$$

$$C_P = \frac{1+\phi}{1.6} - \sqrt{\left[\frac{1+\phi}{1.6}\right]^2 - \frac{\phi}{0.8}}$$

$P = F_c C_P b^2$

Given load: $P = 40$ k

Trial b (in.)	$\dfrac{L}{d} = \dfrac{L}{b}$	ϕ	C_P	P (kips)
6.50	20.308	0.59907	0.49942	35.87
6.70	19.701	0.63651	0.52230	39.86
6.71	19.672	0.63841	0.52343	40.06

$b_{min} = 6.71$ in. $\leftarrow$

Problem 11.9-34 A square wood column with side dimensions b (see figure) is constructed of a structural grade of southern pine for which $F_c = 10.5$ MPa and $E = 12$ GPa. An axial force $P = 200$ kN acts on the column.

(a) If the dimension b =150 mm, what is the maximum allowable length L_{max} of the column?
(b) If the length $L = 4.0$ m, what is the minimum required dimension b_{min}?

Solution 11.9-34 Wood column(square cross section)

$F_c = 10.5$ MPa $\quad E = 12$ GPa $\quad c = 0.8$

$K_{cE} = 0.3 \quad P = 200$ kN

(a) MAXIMUN LENGTH L_{max}FOR $b = d = 150$ mm

From Eq. (11-92): $C_P = \dfrac{P}{F_c b^2} = 0.84656$

From Eq. (11-95):

$$C_P = 0.84656 = \frac{1+\phi}{1.6} - \sqrt{\left[\frac{1+\phi}{1.6}\right]^2 - \frac{\phi}{0.8}}$$

Trial and error: $\phi = 1.7807$

From Eq. (11-94): $\dfrac{L}{d} = \sqrt{\dfrac{K_{cE}E}{\phi F_c}} = 13.876$

$\therefore L_{max} = 13.876\, d = (13.876)(150 \text{ mm})$

$= 2.08$ m $\leftarrow$

(b) MINIMUM DIMENSION b_{min} FOR $L = 4.0$ m

Trial and error: $\frac{L}{d} = \frac{L}{b}$ $\quad \phi = \frac{K_{cE}E}{F_c(L/d)^2}$

$$C_P = \frac{1+\phi}{1.6} - \sqrt{\left[\frac{1+\phi}{1.6}\right]^2 - \frac{\phi}{0.8}}$$

$P = F_c C_P b^2$

Given load: $P = 200$ kN

Trial b (mm)	$\frac{L}{d} = \frac{L}{b}$	ϕ	C_P	P (kN)
180	22.22	0.69429	0.55547	189.0
182	21.98	0.70980	0.56394	196.1
183	21.86	0.71762	0.56814	199.8
184	21.74	0.72549	0.57231	203.5

$\therefore\ b_{min} = 184$ mm $\leftarrow$

Problem 11.9-35 A square wood column with side dimensions b (see figure) is constructed of a structural grade of spruce for which $F_c = 900$ psi and $E = 1{,}500{,}000$ psi. An axial force $P = 8.0$ k acts on the column.

(a) If the dimension $b = 3.5$ in., what is the maximum allowable length L_{max} of the column?
(b) If the length $L = 10$ ft, what is the minimum required dimension b_{min}?

Solution 11.9-35 Wood column(square cross section)

$F_c = 900$ psi $\quad E = 1{,}500{,}000$ psi $\quad c = 0.8$

$K_{cE} = 0.3 \quad P = 8.0\ k$

(a) MAXIMUM LENGTH L_{max} FOR $b = d = 3.5$ in.

From Eq. (11-92): $C_P = \frac{P}{F_c b^2} = 0.72562$

From Eq. (11-95):

$$C_P = 0.72562 = \frac{1+\phi}{1.6} - \sqrt{\left[\frac{1+\phi}{1.6}\right]^2 - \frac{\phi}{0.8}}$$

Trial and error: $\phi = 1.1094$

From Eq. (11-94): $\frac{L}{d} = \sqrt{\frac{K_{cE}E}{\phi F_c}} = 21.23$

$\therefore\ L_{max} = 21.23\ d$

$= (21.23)(3.5 \text{ in.}) = 74.3 \text{ in.} \leftarrow$

(b) MINIMUM DIMENSION b_{min} FOR $L = 10$ ft

Trial and error. $\frac{L}{d} = \frac{L}{b}$ $\quad \phi = \frac{K_{cE}E}{F_c(L/d)^2}$

$$C_P = \frac{1+\phi}{1.6} - \sqrt{\left[\frac{1+\phi}{1.6}\right]^2 - \frac{\phi}{0.8}}$$

$P = F_c C_P b^2$

Given load: $P = 8000$ lb

Trial b (in.)	$\frac{L}{d} = \frac{L}{b}$	ϕ	C_P	P (lb)
4.00	30.00	0.55556	0.47145	6789
4.20	28.57	0.61250	0.50775	8061
4.19	28.64	0.60959	0.50596	7994

$\therefore\ b_{min} = 4.20$ in. $\leftarrow$

Problem 11.9-36 A square wood column with side dimensions b (see figure) is constructed of a structural grade of eastern white pine for which $F_c = 8.0$ MPa and $E = 8.5$ GPa. An axial force $P = 100$ kN acts on the column.

(a) If the dimension $b = 120$ mm, what is the maximum allowable length L_{max} of the column?
(b) If the length $L = 4.0$ m, what is the minimum required dimension b_{min}?

Solution 11.9-36 Wood column (square cross section)

$F_c = 8.0$ MPa $\quad E = 8.5$ GPa $\quad c = 0.8$

$K_{cE} = 0.3 \quad P = 100$ kN

(a) Maximum length L_{max} for $b = d = 120$ mm

From Eq. (11-92): $C_P = \dfrac{P}{F_c b^2} = 0.86806$

From Eq. (11.95):

$$C_p = 0.86806 = \frac{1+\phi}{1.6} - \sqrt{\left[\frac{1+\phi}{1.6}\right]^2 - \frac{\phi}{0.8}}$$

Trial and error: $\phi = 2.0102$

From Eq. (11-94): $\dfrac{L}{d} = \sqrt{\dfrac{K_{cE}E}{\phi F_c}} = 12.592$

$\therefore L_{max} = 12.592\, d = (12.592)(120 \text{ mm})$

$= 1.51 \text{ m} \quad \leftarrow$

(b) Minimum dimension b_{min} for $L = 4.0$ m

Trial and error. $\dfrac{L}{d} = \dfrac{L}{b} \quad \phi = \dfrac{K_{cE}E}{F_c(L/d)^2}$

$$C_p = \frac{1+\phi}{1.6} - \sqrt{\left[\frac{1+\phi}{1.6}\right]^2 - \frac{\phi}{0.8}}$$

$P = F_c C_P b^2$

Given load: $P = 100$ kN

Trial b (mm)	$\frac{L}{d} = \frac{L}{b}$	ϕ	C_P	P (kN)
160	25.00	0.51000	0.44060	90.23
164	24.39	0.53582	0.45828	98.61
165	24.24	0.54237	0.46269	100.77

$\therefore b_{min} = 165 \text{ mm} \quad \leftarrow$

12

Review of Centroids and Moments of Inertia

Centroids of Plane Areas

The problems for Section 12.2 are to be solved by integration.

Problem 12.2-1 Determine the distances $\bar{x}$ and $\bar{y}$ to the centroid C of a right triangle having base b and altitude h (see Case 6, Appendix D).

Solution 12.2-1 **Centroid of a right triangle**

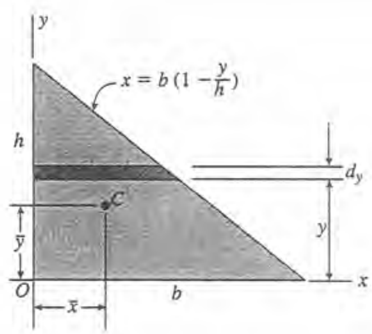
$dA = x\,dy = b(1 - y/h)\,dy$

$$A = \int dA = \int_0^h b(1 - y/h)\,dy = \frac{bh}{2}$$

$$Q_x = \int y\,dA = \int_0^h yb(1 - y/h)\,dy = \frac{bh^2}{6}$$

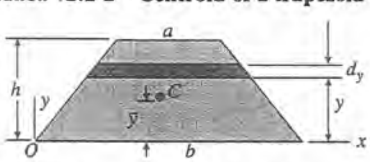
$$\bar{y} = \frac{Q_x}{A} = \frac{h}{3} \longleftarrow$$

Similarly, $\bar{x} = \dfrac{b}{3} \longleftarrow$



Problem 12.2-2 Determine the distance $\bar{y}$ to the centroid C of a trapezoid having bases a and b and altitude h (see Case 8, Appendix D).

Solution 12.2-2 **Centroid of a trapezoid**



Width of element $= b + (a - b)y/h$

$dA = [b + (a - b)y/h]\,dy$

$$A = \int dA = \int_0^h [b + (a - b)y/h]\,dy = \frac{h(a + b)}{2}$$

$$Q_x = \int y\,dA = \int_0^h y[b + (a - b)\,y/h]\,dy$$

$$= \frac{h^2}{6}(2a + b)$$

$$\bar{y} = \frac{Q_x}{A} = \frac{h(2a + b)}{3(a + b)} \longleftarrow$$

Problem 12.2-3 Determine the distance $\bar{y}$ to the centroid C of a semicircle of radius r (see Case 10, Appendix D).

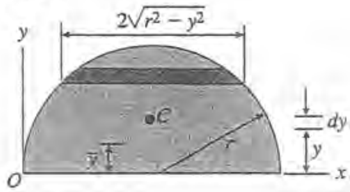
Solution 12.2-3 **Centroid of a semicircle**

$$dA = 2\sqrt{r^2 - y^2}\,dy$$

$$A = \int dA = \int_0^r 2\sqrt{r^2 - y^2}\,dy = \frac{\pi r^2}{2}$$

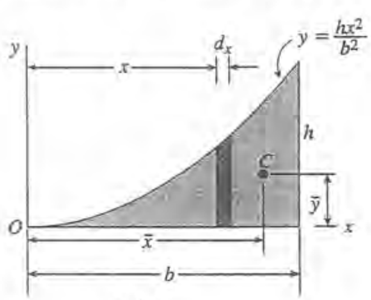
$$Q_x = \int y\,dA = \int_0^r 2y\sqrt{r^2 - y^2}\,dy = \frac{2r^3}{3}$$

$$\bar{y} = \frac{Q_x}{A} = \frac{4r}{3\pi} \longleftarrow$$



Problem 12.2-4 Determine the distances $\bar{x}$ and $\bar{y}$ to the centroid C of a parabolic spandrel of base b and height h (see Case 18, Appendix D).

Solution 12.2-4 **Centroid of a parabolic spandrel**



$$dA = y\,dx = \frac{hx^2\,dx}{b^2}$$

$$A = \int dA = \int_0^b \frac{hx^2}{b^2}dx = \frac{bh}{3}$$

$$Q_y = \int x\,dA = \int_0^b \frac{hx^3}{b^2}dx = \frac{b^2h}{4}$$

$$\bar{x} = \frac{Q_y}{A} = \frac{3b}{4} \longleftarrow$$

$$Q_x = \int y/2\,dA = \int_0^b \frac{1}{2}\left(\frac{hx^2}{b^2}\right)\left(\frac{hx^2}{b^2}\right)dx = \frac{bh^2}{10}$$

$$\bar{y} = \frac{Q_x}{A} = \frac{3h}{10} \longleftarrow$$

Problem 12.2-5 Determine the distances $\bar{x}$ and $\bar{y}$ to the centroid C of a semisegment of nth degree having base b and height h (see Case 19, Appendix D).

Solution 12.2-5 **Centroid of a semisegment of nth degree**

$$dA = y\,dx = h\left(1 - \frac{x^n}{b^n}\right)dx$$

$$A = \int dA = \int_0^b h\left(1 - \frac{x^n}{b^n}\right)dx = bh\left(\frac{n}{n+1}\right)$$

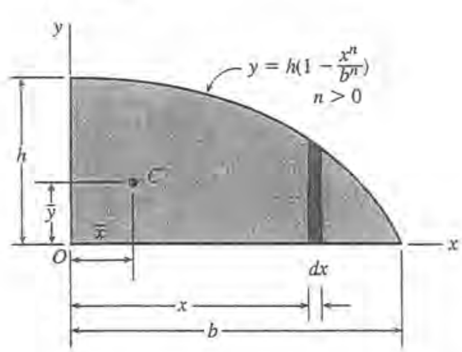
$$Q_y = \int x\,dA = \int_0^b xh\left(1 - \frac{x^n}{b^n}\right)dx = \frac{hb^2}{2}\left(\frac{n}{n+2}\right)$$

$$\bar{x} = \frac{Q_y}{A} = \frac{b(n+1)}{2(n+2)} \longleftarrow$$

$$Q_x = \int \frac{y}{2}dA = \int_0^b \frac{1}{2}h\left(1 - \frac{x^n}{b^n}\right)(h)\left(1 - \frac{x^n}{b^n}\right)dx$$

$$= bh^2\left[\frac{n^2}{(n+1)(2n+1)}\right]$$

$$\bar{y} = \frac{Q_x}{A} = \frac{hn}{2n+1} \longleftarrow$$

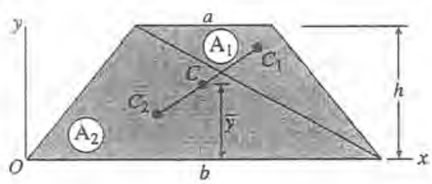


Centroids of Composite Areas

The problems for Section 12.3 are to be solved by using the formulas for composite areas.

Problem 12.3-1 Determine the distance $\bar{y}$ to the centroid C of a trapezoid having bases a and b and altitude h (see Case 8, Appendix D) by dividing the trapezoid into two triangles.

Solution 12.3-1 Centroid of a trapezoid



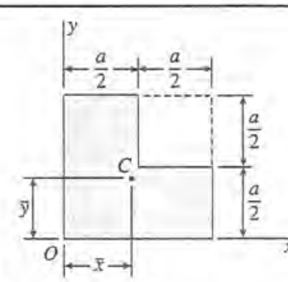
$$A_1 = \frac{ah}{2} \qquad \bar{y}_1 = \frac{2h}{3} \qquad A_2 = \frac{bh}{2} \qquad \bar{y}_2 = \frac{h}{3}$$

$$A = \sum A_i = \frac{ah}{2} + \frac{bh}{2} = \frac{h}{2}(a + b)$$

$$Q_x = \sum \bar{y}_i A_i = \frac{2h}{3}\left(\frac{ah}{2}\right) + \frac{h}{3}\left(\frac{bh}{2}\right) = \frac{h^2}{6}(2a + b)$$

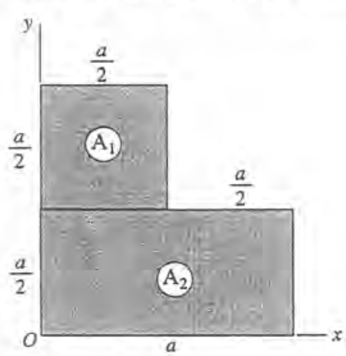
$$\bar{y} = \frac{Q_x}{A} = \frac{h(2a + b)}{3(a + b)} \longleftarrow$$

Problem 12.3-2 One quarter of a square of side a is removed (see figure). What are the coordinates $\bar{x}$ and $\bar{y}$ of the centroid C of the remaining area?



PROBS. 12.3-2 and 12.5-2

Solution 12.3-2 Centroid of a composite area



$$A_1 = \frac{a^2}{4} \qquad \bar{y}_1 = \frac{3a}{4}$$

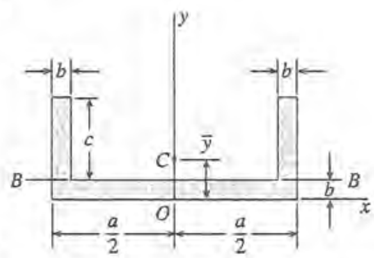
$$A_2 = \frac{a^2}{2} \qquad \bar{y}_2 = \frac{a}{4}$$

$$A = \sum A_i = \frac{3a^2}{4}$$

$$Q_x = \sum \bar{y}_i A_i = \frac{3a}{4}\left(\frac{a^2}{4}\right) + \frac{a}{4}\left(\frac{a^2}{2}\right) = \frac{5a^3}{16}$$

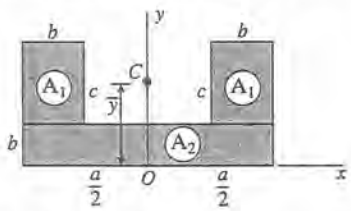
$$\bar{x} = \bar{y} = \frac{Q_x}{A} = \frac{5a}{12} \longleftarrow$$

Problem 12.3-3 Calculate the distance $\bar{y}$ to the centroid C of the channel section shown in the figure if $a = 6$ in., $b = 1$ in., and $c = 2$ in.



PROBS. 12.3-3, 12.3-4, and 12.5-3

Solution 12.3-3 **Centroid of a channel section**



$a = 6$ in. $\quad b = 1$ in. $\quad c = 2$ in.

$A_1 = bc = 2 \text{ in.}^2 \qquad \bar{y}_1 = b + c/2 = 2 \text{ in.}$

$A_2 = ab = 6 \text{ in.}^2 \qquad \bar{y}_2 = \dfrac{b}{2} = 0.5 \text{ in.}$

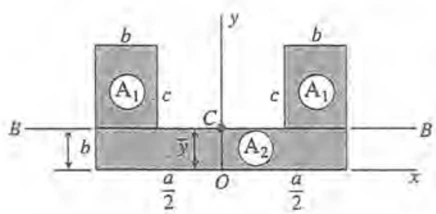
$A = \sum A_i = 2A_1 + A_2 = 10 \text{ in.}^2$

$Q_x = \sum \bar{y}_i A_i = 2\bar{y}_1 A_1 + \bar{y}_2 A_2 = 11.0 \text{ in.}^3$

$\bar{y} = \dfrac{Q_x}{A} = 1.10 \text{ in.} \quad \longleftarrow$

Problem 12.3-4 What must be the relationship between the dimensions a, b, and c of the channel section shown in the figure in order that the centroid C will lie on line BB?

Solution 12.3-4 **Dimensions of channel section**



$A_1 = bc \qquad \bar{y}_1 = b + c/2$

$A_2 = ab \qquad \bar{y}_2 = b/2$

$A = \sum A_i = 2A_1 + A_2 = b(2c + a)$

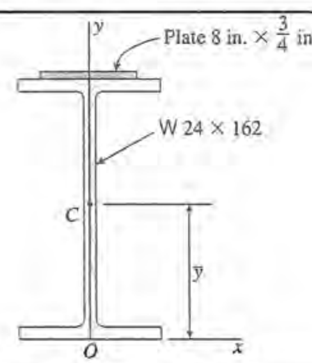
$Q_x = \sum \bar{y}_i A_i = 2\bar{y}_1 A_1 + \bar{y}_2 A_2 = b/2(4bc + 2c^2 + ab)$

$\bar{y} = \dfrac{Q_x}{A} = \dfrac{4bc + 2c^2 + ab}{2(2c + a)}$

Set $\bar{y} = b$ and solve: $2c^2 = ab \quad \longleftarrow$

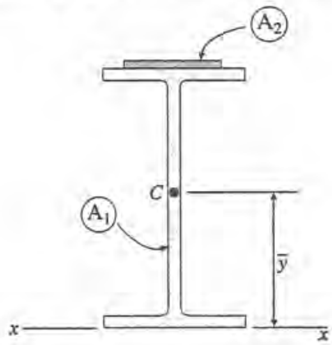
Problem 12.3-5 The cross section of a beam constructed of a W 24 × 162 wide-flange section with an 8 in. × 3/4 in. cover plate welded to the top flange is shown in the figure.

Determine the distance $\bar{y}$ from the base of the beam to the centroid C of the cross-sectional area.



PROBS. 12.3-5 and 12.5-5

Solution 12.3-5 Centroid of beam cross section



W 24 × 162 $A_1 = 47.7 \text{ in.}^2$ $d = 25.00$ in.

$\bar{y}_1 = d/2 = 12.5$ in.

PLATE: 8.0 × 0.75 in. $A_2 = (8.0)(0.75) = 6.0 \text{ in.}^2$

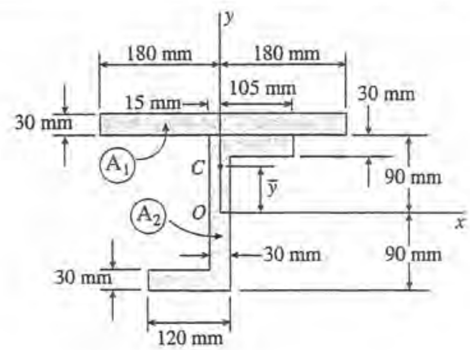
$\bar{y}_2 = 25.00 + 0.75/2 = 25.375$ in.

$A = \sum A_i = A_1 + A_2 = 53.70 \text{ in.}^2$

$Q_x = \sum \bar{y}_i A_i = \bar{y}_1 A_1 + \bar{y}_2 A_2 = 748.5 \text{ in.}^3$

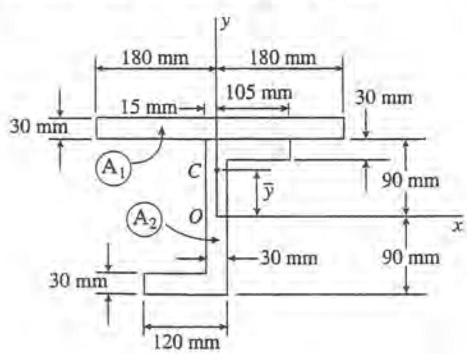
$\bar{y} = \dfrac{Q_x}{A} = 13.94$ in. ←

Problem 12.3-6 Determine the distance $\bar{y}$ to the centroid C of the composite area shown in the figure.



PROBS. 12.3-6, 12.5-6 and 12.7-6

Solution 12.3-6 Centroid of composite area



$A_1 = (360)(30) = 10{,}800 \text{ mm}^2$

$\bar{y}_1 = 105$ mm

$A_2 = 2(120)(30) + (120)(30) = 10{,}800 \text{ mm}^2$

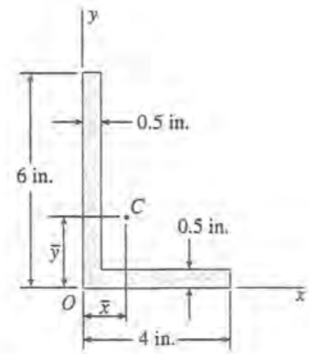
$\bar{y}_2 = 0$

$A = \sum A_i = A_1 + A_2 = 21{,}600 \text{ mm}^2$

$Q_x = \sum \bar{y}_i A_i = \bar{y}_1 A_1 + \bar{y}_2 A_2 = 1.134 \times 10^6 \text{ mm}^3$

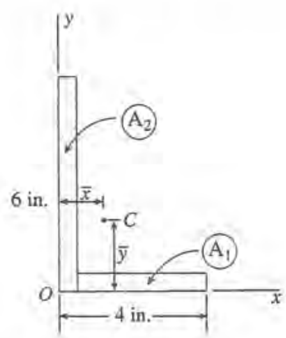
$\bar{y} = \dfrac{Q_x}{A} = 52.5$ mm ←

Problem 12.3-7 Determine the coordinates $\bar{x}$ and $\bar{y}$ of the centroid C of the L-shaped area shown in the figure.



PROBS. 12.3-7, 12.4-7, 12.5-7 and 12.7-7

Solution 12.3-7 **Centroid of L-shaped area**



Thickness $t = 0.5$ in.

$A_1 = (3.5)(0.5) = 1.75 \text{ in.}^2$

$\bar{y}_1 = 0.25$ in. $\quad \bar{x}_1 = 2.25$ in.

$A_2 = (6)(0.5) = 3.0 \text{ in.}^2$

$\bar{y}_2 = 3.0$ in. $\quad \bar{x}_2 = 0.25$ in.

$$A = \sum A_i = A_1 + A_2 = 4.75 \text{ in.}^2$$

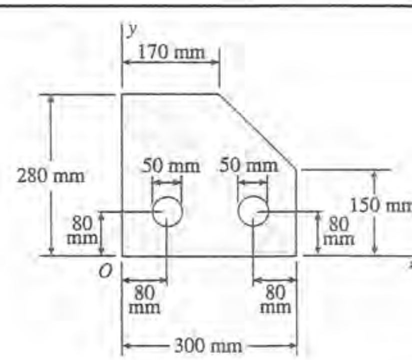
$$Q_y = \sum \bar{x}_i A_i = \bar{x}_1 A_1 + \bar{x}_2 A_2 = 4.688 \text{ in.}^3$$

$$\bar{x} = \frac{Q_y}{A} = 0.99 \text{ in.} \quad \longleftarrow$$

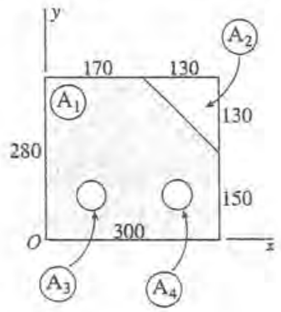
$$Q_x = \sum \bar{y}_i A_i = \bar{y}_1 A_1 + \bar{y}_2 A_2 = 9.438 \text{ in.}^3$$

$$\bar{y} = \frac{Q_x}{A} = 1.99 \text{ in.} \quad \longleftarrow$$

Problem 12.3-8 Determine the coordinates $\bar{x}$ and $\bar{y}$ of the centroid C of the area shown in the figure.



Solution 12.3-8 **Centroid of composite area**



A_1 = large rectangle
A_2 = triangular cutout
$A_3 = A_4$ = circular holes
All dimensions are in millimeters.
Diameter of holes = 50 mm
Centers of holes are 80 mm from edges.

$A_1 = (280)(300) = 84{,}000\ \text{mm}^2$

$\bar{x}_1 = 150\ \text{mm} \qquad \bar{y}_1 = 140\ \text{mm}$

$A_2 = 1/2(130)^2 = 8450\ \text{mm}^2$

$\bar{x}_2 = 300 - 130/3 = 256.7\ \text{mm}$

$\bar{y}_2 = 280 - 130/3 = 236.7\ \text{mm}$

$$A_3 = \frac{\pi d^2}{4} = \frac{\pi}{4}(50)^2 = 1963\ \text{mm}^2$$

$\bar{x}_3 = 80\ \text{mm} \qquad \bar{y}_3 = 80\ \text{mm}$

$A_4 = 1963\ \text{mm}^2 \qquad \bar{x}_4 = 220\ \text{mm} \qquad \bar{y}_4 = 80\ \text{mm}$

$$A = \sum A_i = A_1 - A_2 - A_3 - A_4 = 71{,}620\ \text{mm}^2$$

$$Q_y = \sum \bar{x}_i A_i = \bar{x}_1 A_1 - \bar{x}_2 A_2 - \bar{x}_3 A_3 - \bar{x}_4 A_4$$
$$= 9.842 \times 10^6\ \text{mm}^3$$

$$\bar{x} = \frac{Q_y}{A} = \frac{9.842 \times 10^6}{71{,}620} = 137\ \text{mm} \quad \longleftarrow$$

$$Q_x = \sum \bar{y}_i A_i = \bar{y}_1 A_1 - \bar{y}_2 A_2 - \bar{y}_3 A_3 - \bar{y}_4 A_4$$
$$= 9.446 \times 10^6\ \text{mm}^3$$

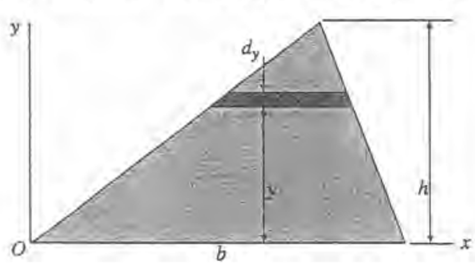
$$\bar{y} = \frac{Q_x}{A} = \frac{9.446 \times 10^6}{71{,}620} = 132\ \text{mm} \quad \longleftarrow$$

Moments of Inertia of Plane Areas

Problems 12.4-1 through 12.4-4 are to be solved by integration.

Problem 12.4-1 Determine the moment of inertia I_x of a triangle of base b and altitude h with respect to its base (see Case 4, Appendix D).

Solution 12.4-1 **Moment of inertia of a triangle**



Width of element

$$= b\left(\frac{h-y}{h}\right)$$

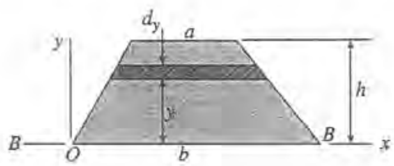
$$dA = \frac{b(h-y)}{h}dy$$

$$I_x = \int y^2 dA = \int_0^h y^2 b \frac{(h-y)}{h} dy$$

$$= \frac{bh^3}{12} \quad \longleftarrow$$

Problem 12.4-2 Determine the moment of inertia I_{BB} of a trapezoid having bases a and b and altitude h with respect to its base (see Case 8, Appendix D).

Solution 12.4-2 Moment of inertia of a trapezoid



Width of element

$$= a + (b - a)\left(\frac{h - y}{h}\right)$$

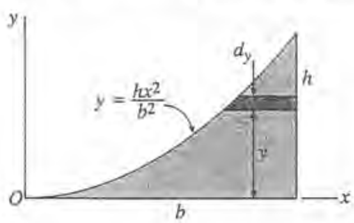
$$dA = \left[a + (b - a)\left(\frac{h - y}{h}\right)\right] dy$$

$$I_{BB} = \int y^2 dA = \int_0^h y^2\left[a + (b - a)\left(\frac{h - y}{h}\right)\right] dy$$

$$= \frac{h^3(3a + b)}{12} \longleftarrow$$

Problem 12.4-3 Determine the moment of inertia I_x of a parabolic spandrel of base b and height h with respect to its base (see Case 18, Appendix D).

Solution 12.4-3 Moment of inertia of a parabolic spandrel



Width of element

$$= b - x = b - b\sqrt{\frac{y}{h}}$$

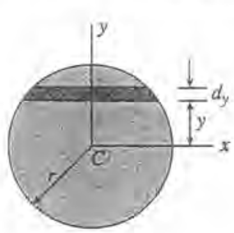
$$= b(1 - \sqrt{y/h})$$

$$dA = b(1 - \sqrt{y/h})\, dy$$

$$I_x = \int y^2 dA = \int_0^h y^2 b\,(1 - \sqrt{y/h})\, dy = \frac{bh^3}{21} \longleftarrow$$

Problem 12.4-4 Determine the moment of inertia I_x of a circle of radius r with respect to a diameter (see Case 9, Appendix D).

Solution 12.4-4 Moment of inertia of a circle



Width of element $= 2\sqrt{r^2 - y^2}$

$$dA = 2\sqrt{r^2 - y^2}\, dy$$

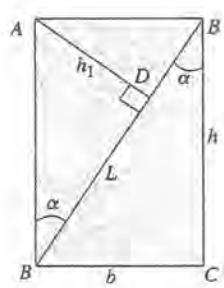
$$I_x = \int y^2 dA = \int_{-r}^{r} y^2(2\sqrt{r^2 - y^2})\, dy$$

$$= \frac{\pi r^4}{4} \longleftarrow$$

Problems 12.4-5 through 12.4-9 are to be solved by considering the area to be a composite area.

Problem 12.4-5 Determine the moment of inertia I_{BB} of a rectangle having sides of lengths b and h with respect to a diagonal of the rectangle (see Case 2, Appendix D).

Solution 12.4-5 Moment of inertia of a rectangle with respect to a diagonal



L = length of diagonal BB

$L = \sqrt{b^2 + h^2}$

h_1 = distance from A to diagonal BB

Triangle BBC: $\sin\alpha = \dfrac{b}{L}$

Triangle ADB: $\sin\alpha = \dfrac{h_1}{h}$ $\quad h_1 = h\sin\alpha = \dfrac{bh}{L}$

I_1 = moment of inertia of triangle ABB with respect to its base BB

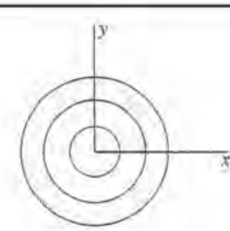
From Case 4, Appendix D:

$$I_1 = \frac{Lh_1^3}{12} = \frac{L}{12}\left(\frac{bh}{L}\right)^3 = \frac{b^3h^3}{12L^2}$$

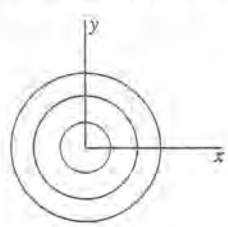
For the rectangle:

$$I_{BB} = 2I_1 = \frac{b^3h^3}{6(b^2 + h^2)} \quad \longleftarrow$$

Problem 12.4-6 Calculate the moment of inertia I_x for the composite circular area shown in the figure. The origin of the axes is at the center of the concentric circles, and the three diameters are 20, 40, and 60 mm.



Solution 12.4-6 Moment of inertia of composite area



Diameters = 20, 40, and 60 mm

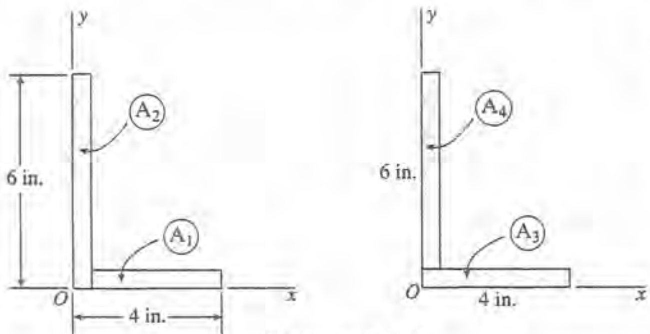
$$I_x = \frac{\pi d^4}{64} \text{ (for a circle)}$$

$$I_x = \frac{\pi}{64}[(60)^4 - (40)^4 + (20)^4]$$

$$I_x = 518 \times 10^3 \text{ mm}^4 \quad \longleftarrow$$

Problem 12.4-7 Calculate the moments of inertia I_x and I_y with respect to the x and y axes for the L-shaped area shown in the figure for Prob. 12.3-7.

Solution 12.4-7 Moments of inertia of composite area

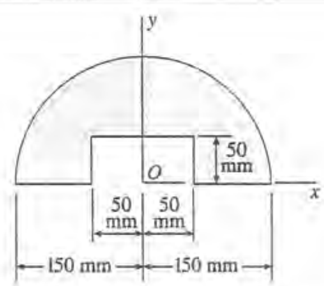


Thickness $t = 0.5$ in.

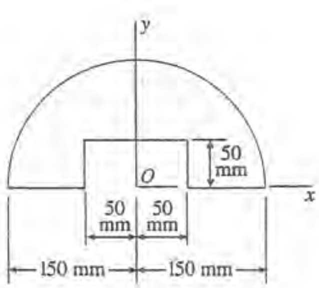
$$I_x = I_1 + I_2$$
$$= \frac{1}{3}(3.5)(0.5)^3 + \frac{1}{3}(0.5)(6)^3$$
$$= 36.1 \text{ in.}^4 \quad \leftarrow$$
$$I_y = I_3 + I_4$$
$$= \frac{1}{3}(0.5)(4)^3 + \frac{1}{3}(5.5)(0.5)^3$$
$$= 10.9 \text{ in.}^4 \quad \leftarrow$$

Problem 12.4-8 A semicircular area of radius 150 mm has a rectangular cutout of dimensions 50 mm × 100 mm (see figure).

Calculate the moments of inertia I_x and I_y with respect to the x and y axes. Also, calculate the corresponding radii of gyration r_x and r_y.



Solution 12.4-8 Moments of inertia of composite area



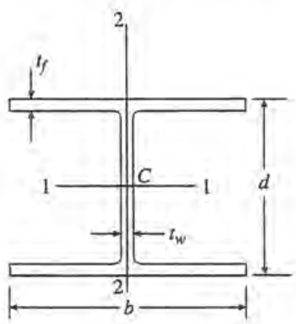
All dimensions in millimeters

$r = 150$ mm $\quad b = 100$ mm $\quad h = 50$ mm

$$I_x = (I_x)_{\text{semicircle}} - (I_x)_{\text{rectangle}} = \frac{\pi r^4}{8} - \frac{bh^3}{3}$$
$$= 194.6 \times 10^6 \text{ mm}^4 \quad \leftarrow$$
$$I_y = I_x \quad \leftarrow$$
$$A = \frac{\pi r^2}{2} - bh = 30.34 \times 10^3 \text{ mm}^2$$
$$r_x = \sqrt{I_x/A} = 80.1 \text{ mm} \quad \leftarrow$$
$$r_y = r_x \quad \leftarrow$$

Problem 12.4-9 Calculate the moments of inertia I_1 and I_2 of a W 16 × 100 wide-flange section using the cross-sectional dimensions given in Table E-1, Appendix E. (Disregard the cross-sectional areas of the fillets.) Also, calculate the corresponding radii of gyration r_1 and r_2, respectively.

Solution 12.4-9 **Moments of inertia of a wide-flange section**



W 16 × 100 $\quad d = 16.97$ in.

$t_w = t_{\text{web}} = 0.585$ in.

$b = 10.425$ in.

$t_f = t_{\text{flange}} = 0.985$ in.

All dimensions in inches.

$$I_1 = \frac{1}{12}bd^3 - \frac{1}{12}(b - t_w)(d - 2t_f)^3$$

$$= \frac{1}{12}(10.425)(16.97)^3 - \frac{1}{12}(9.840)(15.00)^3$$

$$= 1478 \text{ in.}^4 \qquad \text{say,} \quad I_1 = 1480 \text{ in.}^4 \longleftarrow$$

$$I_2 = 2\left(\frac{1}{12}\right)t_f b^3 + \frac{1}{12}(d - 2t_f)t_w^3$$

$$= \frac{1}{6}(0.985)(10.425)^3 + \frac{1}{12}(15.00)(0.585)^3$$

$$= 186.3 \text{ in.}^4 \qquad \text{say,} \quad I_2 = 186 \text{ in.}^4 \longleftarrow$$

$$A = 2(bt_f) + (d - 2t_f)t_w$$

$$= 2(10.425)(0.985) + (15.00)(0.585)$$

$$= 29.31 \text{ in.}^2$$

$$r_1 = \sqrt{I_1/A} = 7.10 \text{ in.} \longleftarrow$$

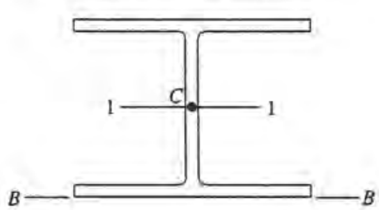
$$r_2 = \sqrt{I_2/A} = 2.52 \text{ in.} \longleftarrow$$

Note that these results are in close agreement with the tabulated values.

Parallel-Axis Theorem

Problem 12.5-1 Calculate the moment of inertia I_b of a W 12 × 50 wide-flange section with respect to its base. (Use data from Table E-1, Appendix E.)

Solution 12.5-1 **Moment of inertia**



W 12 × 50 $\quad I_1 = 394$ in.^{4} $\quad A = 14.7$ in.^{2}

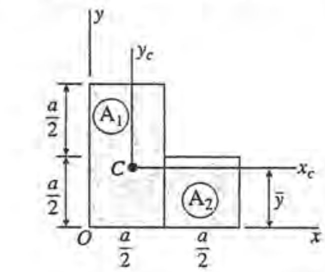
$d = 12.19$ in.

$$I_b = I_1 + A\left(\frac{d}{2}\right)^2$$

$$= 394 + 14.7(6.095)^2 = 940 \text{ in.}^4 \longleftarrow$$

Problem 12.5-2 Determine the moment of inertia I_c with respect to an axis through the centroid C and parallel to the x axis for the geometric figure described in Prob. 12.3-2.

Solution 12.5-2 Moment of inertia



From Prob. 12.3-2:

$A = 3a^2/4$

$\bar{y} = 5a/12$

$$I_x = \frac{1}{3}\left(\frac{a}{2}\right)(a^3) + \frac{1}{3}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)^3 = \frac{3a^4}{16}$$

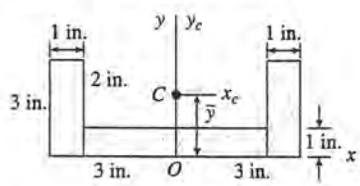
$$I_x = I_{x_c} + A\bar{y}^2$$

$$I_c = I_{x_c} = I_x - A\bar{y}^2 = \frac{3a^4}{16} - \frac{3a^2}{4}\left(\frac{5a}{12}\right)^2$$

$$= \frac{11a^4}{192} \leftarrow$$

Problem 12.5-3 For the channel section described in Prob. 12.3-3, calculate the moment of inertia I_{x_c} with respect to an axis through the centroid C and parallel to the x axis.

Solution 12.5-3 Moment of inertia



From Prob. 12.3-3:

$A = 10.0 \text{ in.}^2$

$\bar{y} = 1.10 \text{ in.}$

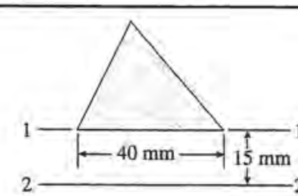
$I_x = 1/3(4)(1)^3 + 2(1/3)(1)(3)^3 = 19.33 \text{ in.}^4$

$I_x = I_{x_c} + A\bar{y}^2$

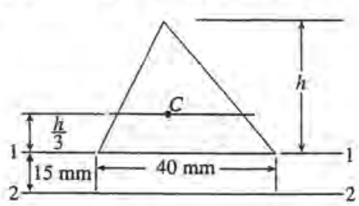
$I_{x_c} = I_x - A\bar{y}^2 = 19.33 - (10.0)(1.10)^2$

$= 7.23 \text{ in.}^4 \leftarrow$

Problem 12.5-4 The moment of inertia with respect to axis 1-1 of the scalene triangle shown in the figure is $90 \times 10^3 \text{ mm}^4$. Calculate its moment of inertia I_2 with respect to axis 2-2.



Solution 12.5-4 Moment of inertia



$b = 40 \text{ mm}$ $\quad I_1 = 90 \times 10^3 \text{ mm}^4$ $\quad I_1 = bh^3/12$

$$h = \sqrt[3]{\frac{12I_1}{b}} = 30 \text{ mm}$$

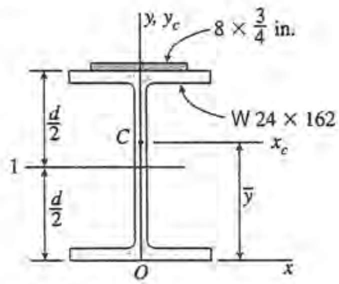
$I_c = bh^3/36 = 30 \times 10^3 \text{ mm}^4$

$$I_2 = I_c + Ad^2 = I_c + (bh/2)\,d^2 = 30 \times 10^3$$

$$+ \frac{1}{2}(40)\,(30)\,(25)^2 = 405 \times 10^3 \text{ mm}^4 \leftarrow$$

Problem 12.5-5 For the beam cross section described in Prob. 12.3-5, calculate the centroidal moments of inertia I_{x_c} and I_{y_c} with respect to axes through the centroid C such that the x_c axis is parallel to the x axis and the y_c axis coincides with the y axis.

Solution 12.5-5 **Moment of inertia**



From Prob. 12.3-5:

$\bar{y} = 13.94$ in.

W 24 × 162 $\quad d = 25.00$ in. $\quad d/2 = 12.5$ in.

$I_1 = 5170 \text{ in.}^4 \quad A = 47.7 \text{ in.}^2$

$I_2 = I_y = 443 \text{ in.}^4$

$I'_{xc} = I_1 + A(\bar{y} - d/2)^2 = 5170 + (47.7)(1.44)^2$
$= 5269 \text{ in.}^4$

$I'_{yc} = I_2 = 443 \text{ in.}^4$

PLATE

$I''_{xc} = 1/12(8)(3/4)^3 + (8)(3/4)(d + 3/8 - \bar{y})^2$
$= 0.2813 + 6(25.00 + 0.375 - 13.94)^2$
$= 0.2813 + 6(11.44)^2 = 785 \text{ in.}^4$

$I''_{yc} = 1/12(3/4)(8)^3 = 32.0 \text{ in.}^4$

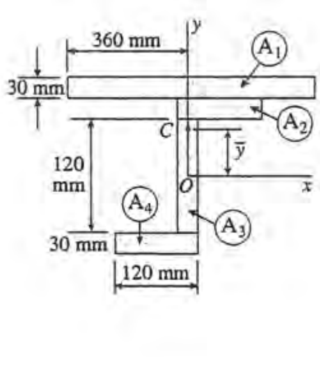
ENTIRE CROSS SECTION

$I_{xc} = I'_{xc} + I''_{xc} = 5269 + 785 = 6050 \text{ in.}^4$ ←

$I_{yc} = I'_{yc} + I''_{yc} = 443 + 32 = 475 \text{ in.}^4$ ←

Problem 12.5-6 Calculate the moment of inertia I_{x_c} with respect to an axis through the centroid C and parallel to the x axis for the composite area shown in the figure for Prob. 12.3-6.

Solution 12.5-6 **Moment of inertia**



From Prob. 12.3-6:

$\bar{y} = 52.50$ mm $\quad t = 30$ mm $\quad A = 21{,}600 \text{ mm}^2$

A_1: $I_x = 1/12(360)(30)^3 + (360)(30)(105)^2$
$= 119.9 \times 10^6 \text{ mm}^4$

A_2: $I_x = 1/12(120)(30)^3 + (120)(30)(75)^2$
$= 20.52 \times 10^6 \text{ mm}^4$

A_3: $I_x = 1/12(30)(120)^3 = 4.32 \times 10^6 \text{ mm}^4$

A_4: $I_x = 20.52 \times 10^6 \text{ mm}^4$

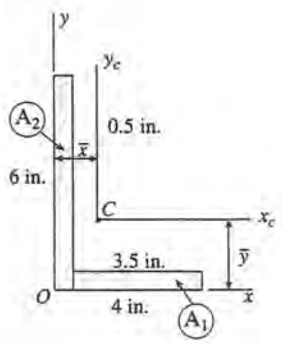
ENTIRE AREA:

$I_x = \sum I_x = 165.26 \times 10^6 \text{ mm}^4$

$I_{xc} = I_x - A\bar{y}^2 = 165.26 \times 10^6 - (21{,}600)(52.50)^2$
$= 106 \times 10^6 \text{ mm}^4$ ←

Problem 12.5-7 Calculate the centroidal moments of inertia I_{x_c} and I_{y_c} with respect to axes through the centroid C and parallel to the x and y axes, respectively, for the L-shaped area shown in the figure for Prob. 12.3-7.

Solution 12.5-7 **Moments of inertia**



From Prob. 12.3-7:

$t = 0.5$ in. $\quad A = 4.75$ in.^{2}

$\bar{y} = 1.987$ in.

$\bar{x} = 0.9869$ in.

From Problem 12.4-7:

$I_x = 36.15$ in.^{4}

$I_y = 10.90$ in.^{4}

$I_{x_c} = I_x - A\bar{y}^2 = 36.15 - (4.75)(1.987)^2$

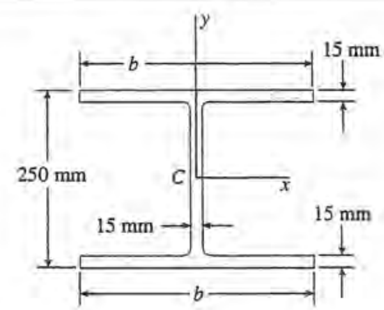
$= 17.40$ in.^{4} ←

$I_{y_c} = I_y - A\bar{x}^2 = 10.90 - (4.75)(0.9869)^2$

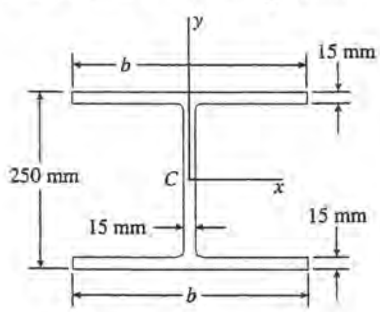
$= 6.27$ in.^{4} ←

Problem 12.5-8 The wide-flange beam section shown in the figure has a total height of 250 mm and a constant thickness of 15 mm.

Determine the flange width b if it is required that the centroidal moments of inertia I_x and I_y be in the ratio 3 to 1, respectively.



Solution 12.5-8 **Wide-flange beam**



$t = 15$ mm $\quad b =$ flange width

All dimensions in millimeters.

$$I_x = \frac{1}{12}(b)(250)^3 - \frac{1}{12}(b - 15)(220)^3$$

$$= 0.4147 \times 10^6 b + 13.31 \times 10^6 \text{ (mm)}^4$$

$$I_y = 2\left(\frac{1}{12}\right)(15)(b)^3 + \frac{1}{12}(220)(15)^3$$

$$= 2.5 b^3 + 61{,}880 \text{ (mm}^4)$$

Equate I_x to $3I_y$ and rearrange:

$$7.5 b^3 - 0.4147 \times 10^6 b - 13.12 \times 10^6 = 0$$

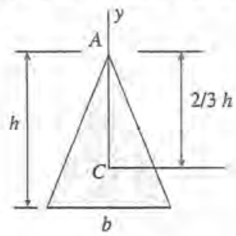
Solve numerically:

$b = 250$ mm ←

Polar Moments of Inertia

Problem 12.6-1 Determine the polar moment of inertia I_P of an isosceles triangle of base b and altitude h with respect to its apex (see Case 5, Appendix D)

Solution 12.6-1 **Polar moment of inertia**



POINT C (CENTROID) FROM CASE 5:

$$(I_P)_c = \frac{bh}{144}(4h^2 + 3b^2)$$

POINT A (APEX):

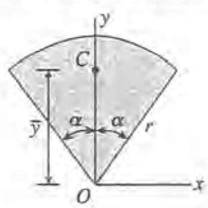
$$I_P = (I_P)_c + A\left(\frac{2h}{3}\right)^2$$

$$= \frac{bh}{144}(4h^2 + 3b^2) + \frac{bh}{2}\left(\frac{2h}{3}\right)^2$$

$$I_P = \frac{bh}{48}(b^2 + 12h^2) \quad \longleftarrow$$

Problem 12.6-2 Determine the polar moment of inertia $(I_P)_C$ with respect to the centroid C for a circular sector (see Case 13, Appendix D).

Solution 12.6-2 **Polar moment of inertia**



POINT O (ORIGIN) FROM CASE 13:

$$(I_P)_o = \frac{\alpha r^4}{2} \qquad (\alpha = \text{radians})$$

$$A = \alpha r^2$$

$$\bar{y} = \frac{2r \sin \alpha}{3\alpha}$$

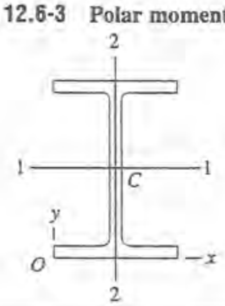
POINT C (CENTROID):

$$(I_P)_C = (I_P)_O - A\bar{y}^2 = \frac{\alpha r^4}{2} - \alpha r^2\left(\frac{2r \sin \alpha}{3\alpha}\right)^2$$

$$= \frac{r^4}{18\alpha}(9\alpha^2 - 8\sin^2\alpha) \quad \longleftarrow$$

Problem 12.6-3 Determine the polar moment of inertia I_P for a W 8 × 21 wide-flange section with respect to one of its outermost corners.

Solution 12.6-3 **Polar moment of inertia**



W 8 × 21 $\quad I_1 = 75.3 \text{ in.}^4 \quad I_2 = 9.77 \text{ in.}^4$

$A = 6.16 \text{ in.}^2$

Depth $d = 8.28$ in.

Width $b = 5.27$ in.

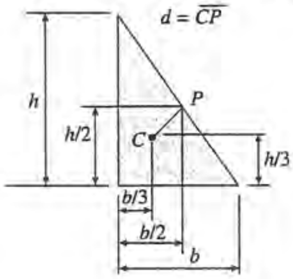
$I_x = I_1 + A(d/2)^2 = 75.3 + 6.16(4.14)^2 = 180.9 \text{ in.}^4$

$I_y = I_2 + A(b/2)^2 = 9.77 + 6.16(2.635)^2 = 52.5 \text{ in.}^4$

$I_P = I_x + I_y = 233 \text{ in.}^4 \quad \longleftarrow$

Problem 12.6-4 Obtain a formula for the polar moment of inertia I_P with respect to the midpoint of the hypotenuse for a right triangle of base b and height h (see Case 6, Appendix D).

Solution 12.6-4 Polar moment of inertia



POINT C FROM CASE 6:

$$(I_P)_c = \frac{bh}{36}(h^2 + b^2)$$

POINT P:

$$I_P = (I_P)_c + Ad^2$$

$$A = \frac{bh}{2}$$

$$d^2 = \left(\frac{b}{2} - \frac{b}{3}\right)^2 + \left(\frac{h}{2} - \frac{h}{3}\right)^2$$

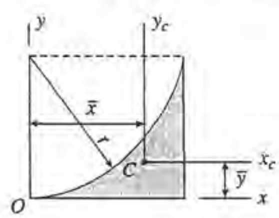
$$= \frac{b^2}{36} + \frac{h^2}{36} = \frac{b^2 + h^2}{36}$$

$$I_P = \frac{bh}{36}(h^2 + b^2) + \frac{bh}{2}\left(\frac{b^2 + h^2}{36}\right)$$

$$= \frac{bh}{24}(b^2 + h^2) \quad \longleftarrow$$

Problem 12.6-5 Determine the polar moment of inertia $(I_P)_C$ with respect to the centroid C for a quarter-circular spandrel (see Case 12, Appendix D).

Solution 12.6-5 Polar moment of inertia



POINT O FROM CASE 12:

$$I_x = \left(1 - \frac{5\pi}{16}\right)r^4$$

$$\bar{y} = \frac{(10 - 3\pi)r}{3(4 - \pi)}$$

$$A = \left(1 - \frac{\pi}{4}\right)r^2$$

POINT C (CENTROID):

$$I_{x_c} = I_x - A\bar{y}^2 = \left(1 - \frac{5\pi}{16}\right)r^4 - \left(1 - \frac{\pi}{4}\right)(r^2)\left[\frac{(10 - 3\pi)r}{3(4 - \pi)}\right]^2$$

COLLECT TERMS AND SIMPLIFY:

$$I_{x_c} = \frac{r^4}{144}\left(\frac{176 - 84\pi + 9\pi^2}{4 - \pi}\right)$$

$$I_{y_c} = I_{x_c} \quad \text{(by symmetry)}$$

$$(I_P)_C = 2I_{x_c} = \frac{r^4}{72}\left(\frac{176 - 84\pi + 9\pi^2}{4 - \pi}\right) \quad \longleftarrow$$

Products of Inertia

Problem 12.7-1 Using integration, determine the product of inertia I_{xy} for the parabolic semisegment shown in Fig. 12-5 (see also Case 17 in Appendix D).

Solution 12.7-1 Product of inertia

Product of inertia of element dA with respect to axes through its own centroid equals zero.

$$dA = y\,dx = h\left(1 - \frac{x^2}{b^2}\right)dx$$

dI_{xy} = product of inertia of element dA with respect to xy axes

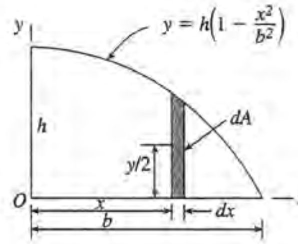
$d_1 = x \quad d_2 = y/2$

Parallel-axis theorem applied to element dA:

$$dI_{xy} = 0 + (dA)(d_1 d_2) = (y\,dx)(x)(y/2)$$

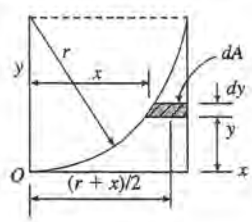
$$= \frac{h^2 x}{2}\left(1 - \frac{x^2}{b^2}\right)^2 dx$$

$$I_{xy} = \int dI_{xy} = \frac{h^2}{2}\int_0^b x\left(1 - \frac{x^2}{b^2}\right)^2 dx = \frac{b^2h^2}{12} \longleftarrow$$



Problem 12.7-2 Using integration, determine the product of inertia I_{xy} for the quarter-circular spandrel shown in Case 12, Appendix D.

Solution 12.7-2 Product of inertia



EQUATION OF CIRCLE:

$x^2 + (y - r)^2 = r^2$

or $r^2 - x^2 = (y - r)^2$

ELEMENT dA:

d_1 = distance to its centroid in x direction $= (r + x)/2$

d_2 = distance to its centroid in y direction $= y$

dA = area of element $= (r - x)\,dy$

Product of inertia of element dA with respect to axes through its own centroid equals zero.

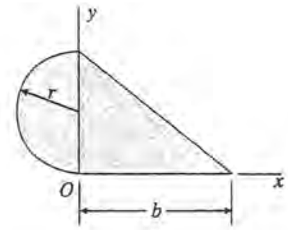
Parallel-axis theorem applied to element dA:

$$dI_{xy} = 0 + (dA)(d_1 d_2) = (r - x)(dy)\left(\frac{r + x}{2}\right)(y)$$

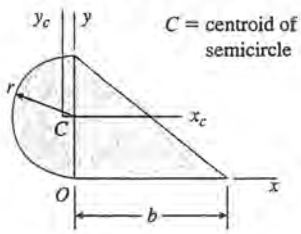
$$= \frac{1}{2}(r^2 - x^2)\,y\,dy = \frac{1}{2}(y - r)^2 y\,dy$$

$$I_{xy} = 1/2\int_0^r y(y - r)^2\,dy = \frac{r^4}{24} \longleftarrow$$

Problem 12.7-3 Find the relationship between the radius r and the distance b for the composite area shown in the figure in order that the product of inertia I_{xy} will be zero.



Solution 12.7-3 Product of inertia



TRIANGLE (CASE 7):

$$I_{xy} = \frac{b^2h^2}{24} = \frac{b^2(2r)^2}{24} = \frac{b^2r^2}{6}$$

SEMICIRCLE (CASE 10):

$$I_{xy} = I_{x_cy_c} + Ad_1d_2$$

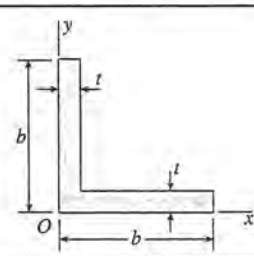
$$I_{x_cy_c} = 0 \quad A = \frac{\pi r^2}{2} \quad d_1 = r \quad d_2 = -\frac{4r}{3\pi}$$

$$I_{xy} = 0 + \left(\frac{\pi r^2}{2}\right)(r)\left(-\frac{4r}{3\pi}\right) = -\frac{2r^4}{3}$$

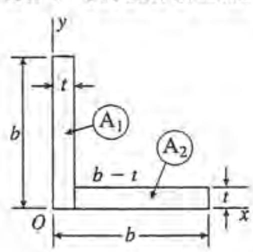
COMPOSITE AREA ($I_{xy} = 0$)

$$I_{xy} = \frac{b^2r^2}{6} - \frac{2r^4}{3} = 0 \quad \therefore b = 2r \quad \longleftarrow$$

Problem 12.7-4 Obtain a formula for the product of inertia I_{xy} of the symmetrical L-shaped area shown in the figure.



Solution 12.7-4 Product of inertia



AREA 1:

$$(I_{xy})_1 = \frac{t^2b^2}{4}$$

AREA 2:

$$(I_{xy})_2 = I_{x_cy_c} + A_2d_1d_2$$

$$= 0 + (b - t)(t)(t/2)\left(\frac{b + t}{2}\right)$$

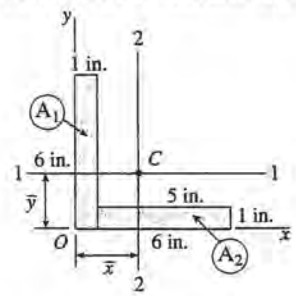
$$= \frac{t^2}{4}(b^2 - t^2)$$

COMPOSITE AREA:

$$I_{xy} = (I_{xy})_1 + (I_{xy})_2 = \frac{t^2}{4}(2b^2 - t^2) \quad \longleftarrow$$

Problem 12.7-5 Calculate the product of inertia I_{12} with respect to the centroidal axes 1-1 and 2-2 for an L 6 × 6 × 1 in. angle section (see Table E-4, Appendix E). (Disregard the cross-sectional areas of the fillet and rounded corners.)

Solution 12.7-5 Product of inertia



All dimensions in inches.

$A_1 = (6)(1) = 6.0 \text{ in.}^2$
$A_2 = (5)(1) = 5.0 \text{ in.}^2$
$A = A_1 + A_2 = 11.0 \text{ in.}^2$

With respect to the x axis:

$$Q_1 = (6.0 \text{ in.}^2)\left(\frac{6 \text{ in.}}{2}\right) = 18.0 \text{ in.}^3$$

$$Q_2 = (5.0 \text{ in.}^2)\left(\frac{1.0 \text{ in.}}{2}\right) = 2.5 \text{ in.}^3$$

$$\bar{y} = \frac{Q_1 + Q_2}{A} = \frac{20.5 \text{ in.}^3}{11.0 \text{ in.}^2} = 1.8636 \text{ in.}$$

$\bar{x} = \bar{y} = 1.8636$ in.

Coordinates of centroid of area A_1 with respect to 1–2 axes:

$d_1 = -(\bar{x} - 0.5) = -1.3636$ in.

$d_2 = 3.0 - \bar{y} = 1.1364$ in.

Product of inertia of area A_1 with respect to 1-2 axes:

$I'_{12} = 0 + A_1 d_1 d_2$

$= (6.0 \text{ in.}^2)(-1.3636 \text{ in.})(1.1364 \text{ in.}) = -9.2976 \text{ in.}^4$

Coordinates of centroid of area A_2 with respect to 1–2 axes:

$d_1 = 3.5 - \bar{x} = 1.6364$ in.

$d_2 = -(\bar{y} - 0.5) = -1.3636$ in.

Product of inertia of area A_2 with respect to 1-2 axes:

$I''_{12} = 0 + A_2 d_1 d_2$

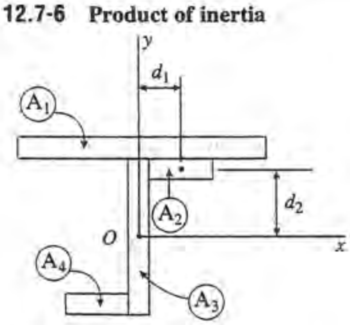
$= (5.0 \text{ in.}^2)(1.6364 \text{ in.})(-1.3636 \text{ in.})$

$= -11.1573 \text{ in.}^4$

ANGLE SECTION: $I_{12} = I'_{12} + I''_{12} = -20.5 \text{ in.}^4$ ⟵

Problem 12.7-6 Calculate the product of inertia I_{xy} for the composite area shown in Prob. 12.3-6.

Solution 12.7-6 Product of inertia



All dimensions in millimeters

$A_1 = 360 \times 30$ mm $\quad A_2 = 90 \times 30$ mm
$A_3 = 180 \times 30$ mm $\quad A_4 = 90 \times 30$ mm
$d_1 = 60$ mm $\quad d_2 = 75$ mm

AREA A_1: $(I_{xy})_1 = 0$ (By symmetry)

AREA A_2: $(I_{xy})_2 = 0 + A_2 d_1 d_2 = (90 \times 30)(60)(75)$
$= 12.15 \times 10^6 \text{ mm}^4$

AREA A_3: $(I_{xy})_3 = 0$ (By symmetry)

AREA A_4: $(I_{xy})_4 = (I_{xy})_2 = 12.15 \times 10^6 \text{ mm}^4$

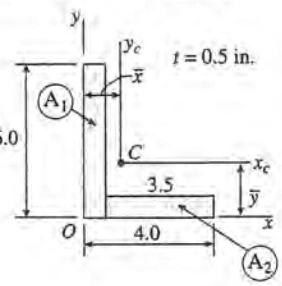
$I_{xy} = (I_{xy})_1 + (I_{xy})_2 + (I_{xy})_3 + (I_{xy})_4$

$= (2)(12.15 \times 10^6 \text{ mm}^4)$

$= 24.3 \times 10^6 \text{ mm}^4$ ⟵

Problem 12.7-7 Determine the product of inertia $I_{x_c y_c}$ with respect to centroidal axes x_c and y_c parallel to the x and y axes, respectively, for the L-shaped area shown in Prob. 12.3-7.

Solution 12.7-7 Product of inertia



All dimensions in inches.

$A_1 = (6.0)(0.5) = 3.0 \text{ in.}^2$

$A_2 = (3.5)(0.5) = 1.75 \text{ in.}^2$

$A = A_1 + A_2 = 4.75 \text{ in.}^2$

With respect to the x axis:

$Q_1 = A_1\bar{y}_1 = (3.0 \text{ in.}^2)(3.0 \text{ in.}) = 9.0 \text{ in.}^3$

$Q_2 = A_2\bar{y}_2 = (1.75 \text{ in.}^2)(0.25 \text{ in.}) = 0.4375 \text{ in.}^3$

$$\bar{y} = \frac{Q_1 + Q_2}{A} = \frac{9.4375 \text{ in.}^3}{4.75 \text{ in.}^2} = 1.9868 \text{ in.}$$

With respect to the y axis:

$Q_1 = A_1\bar{x}_1 = (3.0 \text{ in.}^2)(0.25 \text{ in.}) = 0.75 \text{ in.}^3$

$Q_2 = A_2\bar{x}_2 = (1.75 \text{ in.}^2)(2.25 \text{ in.}) = 3.9375 \text{ in.}^3$

$$\bar{x} = \frac{Q_1 + Q_2}{A} = \frac{4.6875 \text{ in.}^3}{4.75 \text{ in.}^2} = 0.98684 \text{ in.}$$

Product of inertia of area A_1 with respect to xy axes:

$$(I_{xy})_1 = (I_{xy})_{\text{centroid}} + A_1 d_1 d_2 = 0 + (3.0 \text{ in.}^2)(0.25 \text{ in.})(3.0 \text{ in.}) = 2.25 \text{ in.}^4$$

Product of inertia of area A_2 with respect to xy axes:

$$(I_{xy})_2 = (I_{xy})_{\text{centroid}} + A_2 d_1 d_2 = 0 + (1.75 \text{ in.}^2)(2.25 \text{ in.})(0.25 \text{ in.}) = 0.98438 \text{ in.}^4$$

ANGLE SECTION

$$I_{xy} = (I_{xy})_1 + (I_{xy})_2 = 3.2344 \text{ in.}^4$$

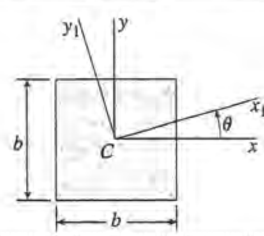
CENTROIDAL AXES

$$I_{x_c y_c} = I_{xy} - A\bar{x}\,\bar{y}$$
$$= 3.2344 \text{ in.}^4 - (4.75 \text{ in.}^2)(0.98684 \text{ in.})(1.9868 \text{ in.})$$
$$= -6.079 \text{ in.}^4 \quad \longleftarrow$$

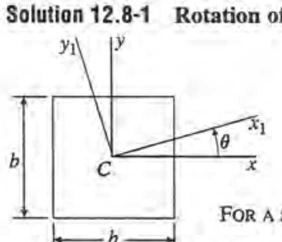
Rotation of Axes

The problems for Section 12.8 are to be solved by using the transformation equations for moments and products of inertia.

Problem 12.8-1 Determine the moments of inertia I_{x_1} and I_{y_1} and the product of inertia $I_{x_1y_1}$ for a square with sides b, as shown in the figure. (Note that the x_1y_1 axes are centroidal axes rotated through an angle θ with respect to the xy axes.)



Solution 12.8-1 Rotation of axes



FOR A SQUARE:

$$I_x = I_y = \frac{b^4}{12} \qquad I_{xy} = 0$$

EQ. (12-25):

$$I_{x_1} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$
$$= \frac{I_x + I_y}{2} + 0 - 0 = \frac{b^4}{12} \quad \longleftarrow$$

EQ. (12-29):

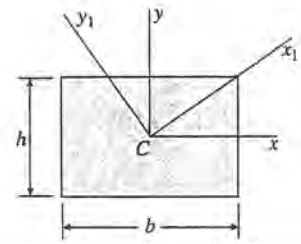
$$I_{x_1} + I_{y_1} = I_x + I_y \qquad \therefore I_{y_1} = \frac{b^4}{12} \quad \longleftarrow$$

EQ. (12-27):

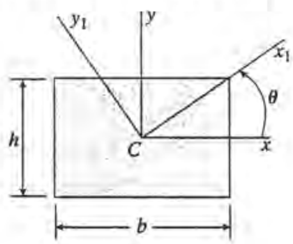
$$I_{x_1y_1} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta = 0 \quad \longleftarrow$$

Since θ may be any angle, we see that all moments of inertia are the same and the product of inertia is always zero (for axes through the centroid C).

Problem 12.8-2 Determine the moments and product of inertia with respect to the x_1y_1 axes for the rectangle shown in the figure. (Note that the x_1 axis is a diagonal of the rectangle.)



Solution 12.8-2 **Rotation of axes (rectangle)**



APPENDIX D, CASE 1:

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{hb^3}{12} \qquad I_{xy} = 0$$

ANGLE OF ROTATION:

$$\cos\theta = \frac{b}{\sqrt{b^2+h^2}} \qquad \sin\theta = \frac{h}{\sqrt{b^2+h^2}}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{b^2-h^2}{b^2+h^2}$$

$$\sin 2\theta = 2\sin\theta\cos\theta = \frac{2bh}{b^2+h^2}$$

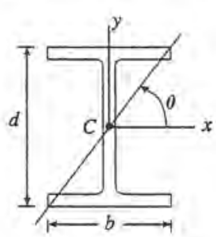
SUBSTITUTE INTO EQS. (12-25), (12-29), AND (12-27) AND SIMPLIFY:

$$I_{x_1} = \frac{b^3h^3}{6(b^2+h^2)} \longleftarrow \qquad I_{y_1} = \frac{bh(b^4+h^4)}{12(b^2+h^2)} \longleftarrow$$

$$I_{x_1y_1} = \frac{b^2h^2(h^2-b^2)}{12(b^2+h^2)} \longleftarrow$$

Problem 12.8-3 Calculate the moment of inertia I_d for a W 12 × 50 wide-flange section with respect to a diagonal passing through the centroid and two outside corners of the flanges. (Use the dimensions and properties given in Table E-1.)

Solution 12.8-3 **Rotation of axes**



W 12 × 50 $I_x = 394 \text{ in.}^4$
$I_y = 56.3 \text{ in.}^4$ $I_{xy} = 0$
Depth $d = 12.19$ in.
Width $b = 8.080$ in.

$$\text{Tan }\theta = \frac{d}{b} = \frac{12.19}{8.080} = 1.509$$

$$\theta = 56.46° \qquad 2\theta = 112.92°$$

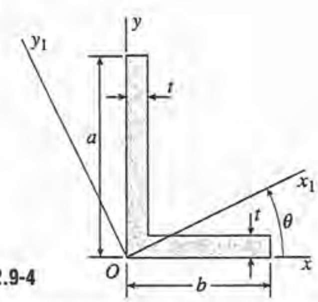
EQ. (12-25):

$$I_d = \frac{I_x+I_y}{2} + \frac{I_x-I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$= \frac{394+56.3}{2} + \frac{394-56.3}{2}\cos(112.92°) - 0$$

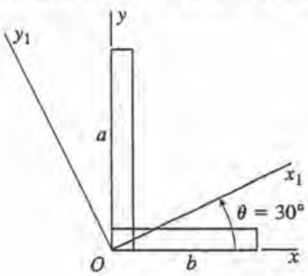
$$= 225 \text{ in.}^4 - 66 \text{ in.}^4 = 159 \text{ in.}^4 \longleftarrow$$

Problem 12.8-4 Calculate the moments of inertia I_{x_1} and I_{y_1} and the product of inertia $I_{x_1y_1}$ with respect to the x_1y_1 axes for the L-shaped area shown in the figure if $a = 150$ mm, $b = 100$ mm, $t = 15$ mm, and $\theta = 30°$.



Probs. 12.8-4 and 12.9-4

Solution 12.8-4 **Rotation of axes**



All dimensions in millimeters.

$a = 150$ mm $b = 100$ mm

$t = 15$ mm $\theta = 30°$

$$I_x = \frac{1}{3}ta^3 + \frac{1}{3}(b-t)t^3$$
$$= \frac{1}{3}(15)(150)^3 + \frac{1}{3}(85)(15)^3$$
$$= 16.971 \times 10^6 \text{ mm}^4$$

$$I_y = \frac{1}{3}(a-t)t^3 + \frac{1}{3}tb^3$$
$$= \frac{1}{3}(135)(15)^3 + \frac{1}{3}(15)(100)^3$$
$$= 5.152 \times 10^6 \text{ mm}^4$$

$$I_{xy} = \frac{1}{4}t^2a^2 + Ad_1d_2 \qquad A = (b-t)(t)$$
$$d_1 = t + \frac{b-t}{2} \qquad d_2 = \frac{t}{2}$$

$$I_{xy} = \frac{1}{4}(15)^2(150)^2 + (85)(15)(57.5)(7.5)$$
$$= 1.815 \times 10^6 \text{ mm}^4$$

SUBSTITUTE into Eq. (12-25) with $\theta = 30°$:

$$I_{x_1} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$
$$= 12.44 \times 10^6 \text{ mm}^4 \longleftarrow$$

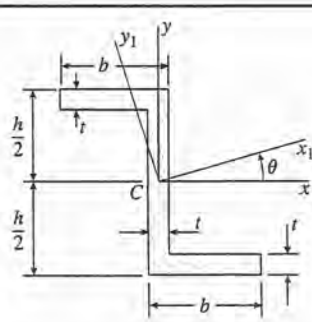
SUBSTITUTE into Eq. (12-25) with $\theta = 120°$:

$$I_{y_1} = 9.68 \times 10^6 \text{ mm}^4 \longleftarrow$$

SUBSTITUTE into Eq. (12-27) with $\theta = 30°$:

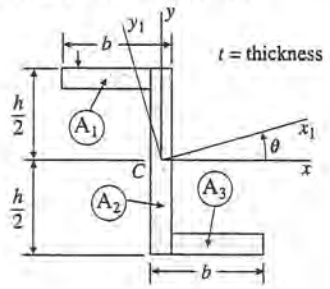
$$I_{x_1y_1} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$
$$= 6.03 \times 10^6 \text{ mm}^4 \longleftarrow$$

Problem 12.8-5 Calculate the moments of inertia I_{x_1} and I_{y_1} and the product of inertia $I_{x_1y_1}$ with respect to the x_1y_1 axes for the Z-section shown in the figure if $b = 3$ in., $h = 4$ in., $t = 0.5$ in., and $\theta = 60°$.



Probs. 12.8-5, 12.8-6, 12.9-5 and 12.9-6

Solution 12.8-5 Rotation of axes



All dimensions in inches.

$b = 3.0$ in. $h = 4.0$ in. $t = 0.5$ in. $\theta = 60°$

MOMENT OF INERTIA I_x

Area A_1: $I'_x = \frac{1}{12}(b-t)(t^3) + (b-t)(t)\left(\frac{h}{2} - \frac{t}{2}\right)^2$
$= 3.8542 \text{ in.}^4$

Area A_2: $I''_x = \frac{1}{12}(t)(h^3) = 2.6667 \text{ in.}^4$

Area A_3: $I'''_x = I'_x = 3.8542 \text{ in.}^4$

$I_x = I'_x + I''_x + I'''_x = 10.3751 \text{ in.}^4$

MOMENT OF INERTIA I_y

Area A_1: $I'_y = \frac{1}{12}(t)(b-t)^3 + (b-t)(t)\left(\frac{b}{2}\right)^2$
$= 3.4635 \text{ in.}^4$

Area A_2: $I''_y = \frac{1}{12}(h)(t^3) = 0.0417 \text{ in.}^4$

Area A_3: $I'''_y = I'_y = 3.4635 \text{ in.}^4$

$I_y = I'_y + I''_y + I'''_y = 6.9688 \text{ in.}^4$

PRODUCT OF INERTIA I_{xy}

Area A_1: $I'_{xy} = 0 + (b-t)(t)\left(-\frac{b}{2}\right)\left(\frac{h}{2} - \frac{t}{2}\right)$
$= -\frac{1}{4}(bt)(b-t)(h-t) = -3.2813 \text{ in.}^4$

Area A_2: $I''_{xy} = 0$ Area A_3: $I'''_{xy} = I'_{xy}$

$I_{xy} = I'_{xy} + I''_{xy} + I'''_{xy} = -6.5625 \text{ in.}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 60°$:

$$I_{x_1} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$
$$= 13.50 \text{ in.}^4 \longleftarrow$$

SUBSTITUTE into Eq. (12-25) with $\theta = 150°$:

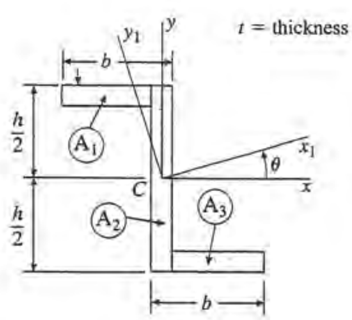
$I_{y_1} = 3.84 \text{ in.}^4 \longleftarrow$

SUBSTITUTE into Eq. (12-27) with $\theta = 60°$:

$$I_{x_1y_1} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta = 4.76 \text{ in.}^4 \longleftarrow$$

Problem 12.8-6 Solve the preceding problem if $b = 80$ mm, $h = 120$ mm, $t = 12$ mm, and $\theta = 30°$.

Solution 12.8-6 Rotation of axes



All dimensions in millimeters.

$b = 80$ mm $h = 120$ mm
$t = 12$ mm $\theta = 30°$

MOMENT OF INERTIA I_x

Area A_1: $I'_x = \frac{1}{12}(b-t)(t^3) + (b-t)(t)\left(\frac{h}{2} - \frac{t}{2}\right)^2$
$= 2.3892 \times 10^6 \text{ mm}^4$

Area A_2: $I''_x = \frac{1}{12}(t)(h^3) = 1.7280 \times 10^6 \text{ mm}^4$

Area A_3: $I'''_x = I'_x = 2.3892 \times 10^6 \text{ mm}^4$

$I_x = I'_x + I''_x + I'''_x = 6.5065 \times 10^6 \text{ mm}^4$

(Continued)

MOMENT OF INERTIA I_y

Area A_1: $I'_y = \frac{1}{12}(t)(b-t)^3 + (b-t)(t)\left(\frac{b}{2}\right)^2$

$= 1.6200 \times 10^6 \text{ mm}^4$

Area A_2: $I''_y = \frac{1}{12}(h)(t^3) = 0.01728 \times 10^6 \text{ mm}^4$

Area A_3: $I'''_y = I'_y = 1.6200 \times 10^6 \text{ mm}^4$

$I_y = I'_y + I''_y + I'''_y = 3.2573 \times 10^6 \text{ mm}^4$

PRODUCT OF INERTIA I_{xy}

Area A_1: $I'_{xy} = 0 + (b-t)(t)\left(-\frac{b}{2}\right)\left(\frac{h}{2} - \frac{t}{2}\right)$

$= -\frac{1}{4}(bt)(b-t)(h-t) = -1.7626 \times 10^6 \text{ mm}^4$

Area A_2: $I''_{xy} = 0$ Area A_3: $I'''_{xy} = I'_{xy}$

$I_{xy} = I'_{xy} + I''_{xy} + I'''_{xy} = -3.5251 \times 10^6 \text{ mm}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 30°$:

$$I_{x_1} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$= 8.75 \times 10^6 \text{ mm}^4$ ←

SUBSTITUTE into Eq. (12-25) with $\theta = 120°$:

$I_{y_1} = 1.02 \times 10^6 \text{ mm}^4$ ←

SUBSTITUTE into Eq. (12-27) with $\theta = 30°$:

$$I_{x_1y_1} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

$= -0.356 \times 10^6 \text{ mm}^4$ ←

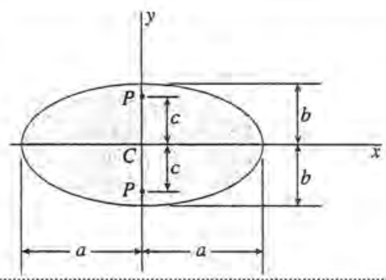
Principal Axes, Principal Points, and Principal Moments of Inertia

Problem 12.9-1 An ellipse with major axis of length $2a$ and minor axis of length $2b$ is shown in the figure.

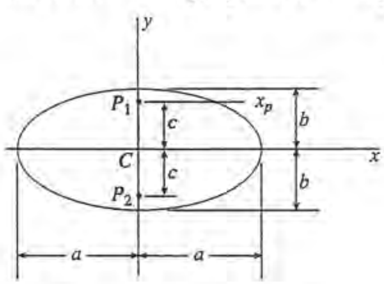
(a) Determine the distance c from the centroid C of the ellipse to the principal points P on the minor axis (y axis).

(b) For what ratio a/b do the principal points lie on the circumference of the ellipse?

(c) For what ratios do they lie inside the ellipse?



Solution 12.9-1 Principal points of an ellipse



(a) LOCATION OF PRINCIPAL POINTS

At a principal point, all moments of inertia are equal.

At point P_1: $I_{x_p} = I_y$ Eq. (1)

From Case 16: $I_y = \frac{\pi b a^3}{4}$

$I_x = \frac{\pi a b^3}{4}$ $A = \pi ab$

Parallel-axis theorem:

$$I_{x_p} = I_x + Ac^2 = \frac{\pi a b^3}{4} + \pi abc^2$$

Substitute into Eq. (1):

$$\frac{\pi a b^3}{4} + \pi abc^2 = \frac{\pi b a^3}{4}$$

Solve for c: $c = \frac{1}{2}\sqrt{a^2 - b^2}$ ←

(b) PRINCIPAL POINTS ON THE CIRCUMFERENCE

$\therefore c = b$ and $b = \dfrac{1}{2}\sqrt{a^2 - b^2}$

Solve for ratio $\dfrac{a}{b}$: $\quad \dfrac{a}{b} = \sqrt{5} \quad \longleftarrow$

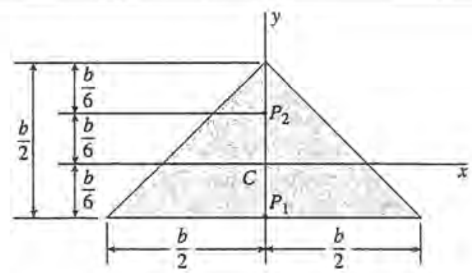
(c) PRINCIPAL POINTS INSIDE THE ELLIPSE

$\therefore 0 \le c < b \qquad$ For $c = 0$: $\quad a = b$ and $\dfrac{a}{b} = 1$

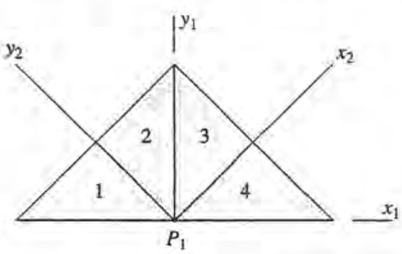
For $c = b$: $\quad \dfrac{a}{b} = \sqrt{5}$

$\therefore 1 \le \dfrac{a}{b} < \sqrt{5} \quad \longleftarrow$

Problem 12.9-2 Demonstrate that the two points P_1 and P_2, located as shown in the figure, are the principal points of the isosceles right triangle.



Solution 12.9-2 **Principal points of an isosceles right triangle**



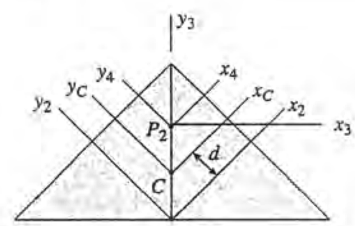
CONSIDER POINT P_1:

$I_{x_1y_1} = 0$ because y_1 is an axis of symmetry.

$I_{x_2y_2} = 0$ because areas 1 and 2 are symmetrical about the y_2 axis and areas 3 and 4 are symmetrical about the x_2 axis.

Two different sets of principal axes exist at point P_1.

$\therefore P_1$ is a principal point $\longleftarrow$



CONSIDER POINT P_2:

$I_{x_3y_3} = 0$ because y_3 is an axis of symmetry.

$I_{x_2y_2} = 0$ (see above).

Parallel-axis theorem:

$I_{x_2y_2} = I_{x_cy_c} + Ad_1d_2 \qquad A = \dfrac{b^2}{4} \qquad d = d_1 = d_2 = \dfrac{b}{6\sqrt{2}}$

$I_{x_cy_c} = -\left(\dfrac{b^2}{4}\right)\left(\dfrac{b}{6\sqrt{2}}\right)^2 = -\dfrac{b^4}{288}$

Parallel-axis theorem:

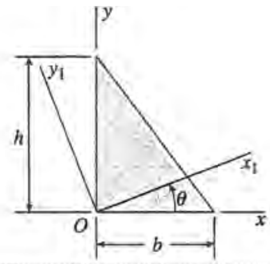
$I_{x_4y_4} = I_{x_cy_c} + Ad_1d_2 \qquad d_1 = d_2 = -\dfrac{b}{6\sqrt{2}}$

$I_{x_4y_4} = -\dfrac{b^4}{288} + \dfrac{b^2}{4}\left(-\dfrac{b}{6\sqrt{2}}\right)^2 = 0$

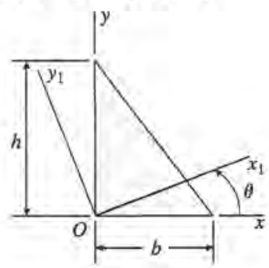
Two different sets of principal axes (x_3y_3 and x_4y_4) exist at point P_2.

$\therefore P_2$ is a principal point $\longleftarrow$

Problem 12.9-3 Determine the angles θ_{p_1} and θ_{p_2} defining the orientations of the principal axes through the origin O for the right triangle shown in the figure if $b = 6$ in. and $h = 8$ in. Also, calculate the corresponding principal moments of inertia I_1 and I_2.



Solution 12.9-3 **Principal axes**



RIGHT TRIANGLE

$b = 6.0$ in. $\quad h = 8.0$ in.

CASE 7:

$$I_x = \frac{bh^3}{12} = 256 \text{ in.}^4$$

$$I_y = \frac{hb^3}{12} = 144 \text{ in.}^4$$

$$I_{xy} = \frac{b^2h^2}{24} = 96 \text{ in.}^4$$

EQ. (12-30): $\tan 2\theta_p = -\dfrac{2I_{xy}}{I_x - I_y} = -1.71429$

$2\theta_p = -59.744°$ and $120.256°$

$\theta_p = -29.872°$ and $60.128°$

SUBSTITUTE into Eq. (12-25) with $\theta = -29.872°$:

$I_{x_1} = 311.1 \text{ in.}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 60.128°$:

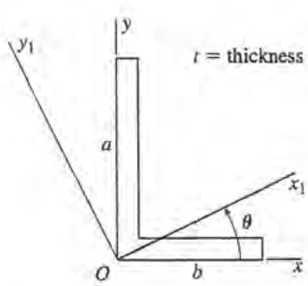
$I_{x_1} = 88.9 \text{ in.}^4$

THEREFORE, $I_1 = 311.1 \text{ in.}^4 \quad \theta_{p_1} = -29.87°$
$I_2 = 88.9 \text{ in.}^4 \quad \theta_{p_2} = 60.13°$ ←

NOTE: The principal moments of inertia can be verified with Eqs. (12-33a and b) and Eq. (12-29).

Problem 12.9-4 Determine the angles θ_{p_1} and θ_{p_2} defining the orientations of the principal axes through the origin O and the corresponding principal moments of inertia I_1 and I_2 for the L-shaped area described in Prob. 12.8-4 ($a = 150$ mm, $b = 100$ mm, and $t = 15$ mm).

Solution 12.9-4 **Principal axes**



ANGLE SECTION

$a = 150$ mm $\quad b = 100$ mm $\quad t = 15$ mm

FROM PROB. 12.8-4:

$I_x = 16.971 \times 10^6 \text{ mm}^4$

$I_y = 5.152 \times 10^6 \text{ mm}^4 \quad I_{xy} = 1.815 \times 10^6 \text{ mm}^4$

EQ. (12-30): $\quad \tan 2\theta_p = -\dfrac{2I_{xy}}{I_x - I_y} = -0.3071$

$2\theta_p = -17.07°$ and $162.93°$

$\theta_p = -8.54°$ and $81.46°$

SUBSTITUTE into Eq. (12-25) with $\theta = -8.54°$:

$I_{x_1} = 17.24 \times 10^6 \text{ mm}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 81.46°$:

$I_{x_1} = 4.88 \times 10^6 \text{ mm}^4$

THEREFORE,

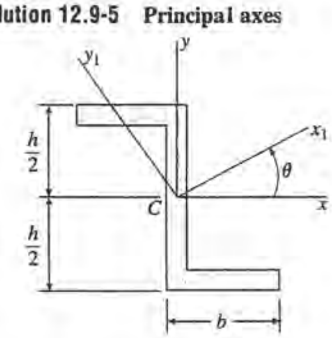
$I_1 = 17.24 \times 10^6 \text{ mm}^4 \quad \theta_{p_1} = -8.54°$

$I_2 = 4.88 \times 10^6 \text{ mm}^4 \quad \theta_{p_2} = 81.46°$

NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-33*a* and *b*) and Eq. (12-29).

Problem 12.9-5 Determine the angles θ_{p_1} and θ_{p_2} defining the orientations of the principal axes through the centroid *C* and the corresponding principal centroidal moments of inertia I_1 and I_2 for the Z-section described in Prob. 12.8-5 (b = 3 in., h = 4 in., and t = 0.5 in.).

Solution 12.9-5 Principal axes



Z-SECTION

t = thickness = 0.5 in.
b = 3.0 in $\quad h$ = 4.0 in

FROM PROB. 12.8-5:

$I_x = 10.3751 \text{ in.}^4 \quad I_y = 6.9688 \text{ in.}^4$
$I_{xy} = -6.5625 \text{ in.}^4$

EQ. (12-30): $\quad \tan 2\theta_p = -\dfrac{2I_{xy}}{I_x - I_y} = 3.8532$

$2\theta_p = 75.451° \quad$ and $\quad 255.451°$

$\theta_p = 37.726° \quad$ and $\quad 127.726°$

SUBSTITUTE into Eq. (12-25) with $\theta = 37.726°$:

$I_{x_1} = 15.452 \text{ in.}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 127.726°$:

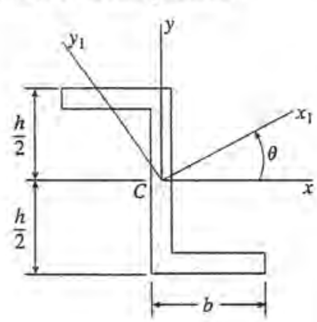
$I_{x_1} = 1.892 \text{ in.}^4$

THEREFORE, $\left.\begin{array}{ll} I_1 = 15.45 \text{ in.}^4 & \theta_{p_1} = 37.73° \\ I_2 = 1.89 \text{ in.}^4 & \theta_{p_2} = 127.73° \end{array}\right\} \longleftarrow$

NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-33*a* and *b*) and Eq. (12-29).

Problem 12.9-6 Solve the preceding problem for the Z-section described in Prob. 12.8-6 (b = 80 mm, h = 120 mm, and t = 12 mm).

Solution 12.9-6 Principal axes



Z-SECTION

t = thickness
= 12 mm
b = 80 mm
h = 120 mm

FROM PROB. 12.8-6:

$I_x = 6.5065 \times 10^6 \text{ mm}^4 \quad I_y = 3.2573 \times 10^6 \text{ mm}^4$
$I_{xy} = -3.5251 \times 10^6 \text{ mm}^4$

(Continued)

Eq. (12-30): $\tan 2\theta_p = -\dfrac{2I_{xy}}{I_x - I_y} = 2.1698$

$2\theta_p = 65.257°$ and $245.257°$
$\theta_p = 32.628°$ and $122.628°$

SUBSTITUTE into EQ. (12-25) with $\theta = 32.628°$:

$I_{x_1} = 8.763 \times 10^6\ \text{mm}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 122.628°$:

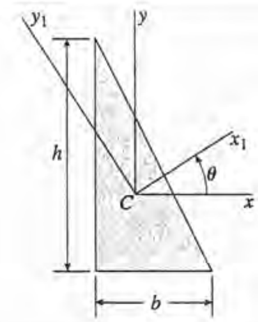
$I_{x_1} = 1.000 \times 10^6\ \text{mm}^4$

THEREFORE,

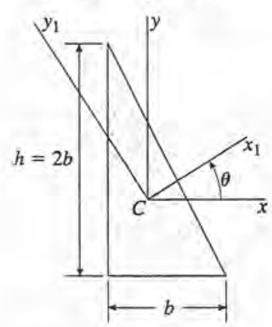
$$\left.\begin{array}{l} I_1 = 8.76 \times 10^6\ \text{mm}^4 \quad \theta_{p_1} = 32.63° \\ I_2 = 1.00 \times 10^6\ \text{mm}^4 \quad \theta_{p_2} = 122.63° \end{array}\right\} \longleftarrow$$

NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-33*a* and *b*) and Eq. (12-29).

Problem 12.9-7 Determine the angles θ_{p_1} and θ_{p_2} defining the orientations of the principal axes through the centroid C for the right triangle shown in the figure if $h = 2b$. Also, determine the corresponding principal centroidal moments of inertia I_1 and I_2.



Solution 12.9-7 **Principal axes**



RIGHT TRIANGLE

$h = 2b$

CASE 6

$$I_x = \frac{bh^3}{36} = \frac{2b^4}{9}$$

$$I_y = \frac{hb^3}{36} = \frac{b^4}{18}$$

$$I_{xy} = -\frac{b^2h^2}{72} = -\frac{b^4}{18}$$

EQ. (12-30): $\tan 2\theta_p = -\dfrac{2I_{xy}}{I_x - I_y} = \dfrac{2}{3}$

$2\theta_p = 33.6901°$ and $213.6901°$
$\theta_p = 16.8450°$ and $106.8450°$

SUBSTITUTE into Eq. (12-25) with $\theta = 16.8450°$:

$I_{x_1} = 0.23904\, b^4$

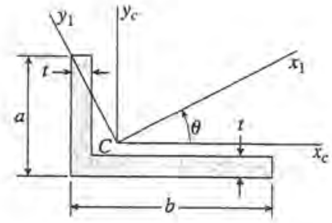
SUBSTITUTE into Eq. (12-25) with $\theta = 106.8450°$:

$I_{x_1} = 0.03873\, b^4$

THEREFORE, $$\left.\begin{array}{l} I_1 = 0.2390\, b^4 \quad \theta_{p_1} = 16.85° \\ I_2 = 0.0387\, b^4 \quad \theta_{p_2} = 106.85° \end{array}\right\} \longleftarrow$$

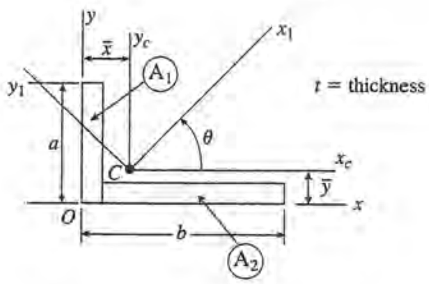
NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-33*a* and *b*) and Eq. (12-29).

Problem 12.9-8 Determine the angles θ_{p_1} and θ_{p_2} defining the orientations of the principal centroidal axes and the corresponding principal moments of inertia I_1 and I_2 for the L-shaped area shown in the figure if $a = 80$ mm, $b = 150$ mm, and $t = 16$ mm.



Probs. 12.9-8 and 12.9-9

Solution 12.9-8 **Principal axes (angle section)**



$a = 80$ mm $\quad b = 150$ mm $\quad t = 16$ mm

$A_1 = at = 1280$ mm^2

$A_2 = (b - t)(t) = 2144$ mm^2

$A = A_1 + A_2 = t\,(a + b - t) = 3424$ mm^2

LOCATION OF CENTROID C

$$Q_x = \sum A_i \bar{y}_i = (at)\left(\frac{a}{2}\right) + (b - t)(t)\left(\frac{t}{2}\right)$$
$$= 68{,}352 \text{ mm}^3$$
$$\bar{y} = \frac{Q_x}{A} = \frac{68{,}352 \text{ mm}^3}{3{,}424 \text{ mm}^2} = 19.9626 \text{ mm}$$
$$Q_y = \sum A_i \bar{x}_i = (at)\left(\frac{t}{2}\right) + (b - t)(t)\left(\frac{b + t}{2}\right)$$
$$= 188{,}192 \text{ mm}^3$$
$$\bar{x} = \frac{Q_y}{A} = \frac{188{,}192 \text{ mm}^3}{3{,}424 \text{ mm}^2} = 54.9626 \text{ mm}$$

MOMENTS OF INERTIA (xy AXES)

Use parallel-axis theorem.

$$I_x = \frac{1}{12}(t)(a^3) + A_1\left(\frac{a}{2}\right)^2 + \frac{1}{12}(b - t)(t^3) + A_2\left(\frac{t}{2}\right)^2$$
$$= \frac{1}{12}(16)(80)^3 + (1280)(40)^2 + \frac{1}{12}(134)(16)^3$$
$$+ (2144)(8)^2$$
$$= 2.91362 \times 10^6 \text{ mm}^4$$

$$I_y = \frac{1}{12}(a)(t^3) + A_1\left(\frac{t}{2}\right)^2 + \frac{1}{12}(t)(b - t^3)$$
$$+ A_2\left(\frac{b + t}{2}\right)^2$$
$$= \frac{1}{12}(80)(16)^3 + (1280)(8)^2 + \frac{1}{12}(16)(134)^3$$
$$+ (2144)\left(\frac{166}{2}\right)^2$$
$$= 18.08738 \times 10^6 \text{ mm}^4$$

MOMENTS OF INERTIA ($x_c y_c$ AXES)

Use parallel-axis theorem.

$$I_{x_c} = I_x - A\bar{y}^2 = 2.91362 \times 10^6 - (3424)(19.9626)^2$$
$$= 1.54914 \times 10^6 \text{ mm}^4$$
$$I_{y_c} = I_y - A\bar{x}^2 = 18.08738 \times 10^6 - (3424)(54.9626)^2$$
$$= 7.74386 \times 10^6 \text{ mm}^4$$

PRODUCT OF INERTIA

Use parallel-axis theorem: $\quad I_{xy} = I_{\text{centroid}} + A\,d_1 d_2$

$$\text{Area } A_1\text{: } I'_{x_c y_c} = 0 + A_1\left[-\left(\bar{x} - \frac{t}{2}\right)\right]\left[\frac{e}{2} - \bar{y}\right]$$
$$= (1280)(8 - 54.9626)(40 - 19.9626)$$
$$= -1.20449 \times 10^6 \text{ mm}^4$$
$$\text{Area } A_2\text{: } I''_{x_c y_c} = 0 + A_2\left[\frac{b + t}{2} - \bar{x}\right]\left[-\left(\bar{y} - \frac{t}{2}\right)\right]$$
$$= (2144)(83 - 54.9626)(8 - 19.9626)$$
$$= -0.71910 \times 10^6 \text{ mm}^4$$
$$I_{x_c y_c} = I'_{x_c y_c} + I''_{x_c y_c} = -1.92359 \times 10^6 \text{ mm}^4$$

SUMMARY

$I_{x_c} = 1.54914 \times 10^6$ mm^4 $\quad I_{y_c} = 7.74386 \times 10^6$ mm^4

$I_{x_c y_c} = -1.92359 \times 10^6$ mm^4

(Continued)

PRINCIPAL AXES

Eq. (12-30): $\tan 2\theta_p = -\dfrac{2I_{xy}}{I_x - I_y} = -0.621041$

$2\theta_p = -31.8420°$ and $148.1580°$

$\theta_p = -15.9210°$ and $74.0790°$

SUBSTITUTE into Eq. (12-25) with $\theta = -15.9210°$

$I_{x_1} = 1.0004 \times 10^6\ \text{mm}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 74.0790°$

$I_{x_1} = 8.2926 \times 10^6\ \text{mm}^4$

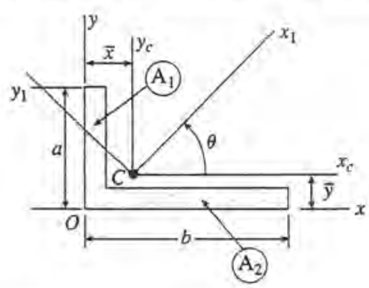
THEREFORE,

$$\left.\begin{array}{ll} I_1 = 8.29 \times 10^6\ \text{mm}^4 & \theta_{p_1} = 74.08° \\ I_2 = 1.00 \times 10^6\ \text{mm}^4 & \theta_{p_2} = -15.92° \end{array}\right\} \longleftarrow$$

NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-33a and b) and Eq. (12-29).

Problem 12.9-9 Solve the preceding problem if $a = 3$ in., $b = 6$ in., and $t = 5/8$ in.

Solution 12.9-9 Principal axes (angle section)



$a = 3.0$ in.

$b = 6.0$ in.

$t = 5/8$ in.

$A_1 = at = 1.875\ \text{in.}^2$

$A_2 = (b - t)(t) = 3.35938\ \text{in.}^2$

$A = A_1 + A_2 = t(a + b - t) = 5.23438\ \text{in.}^2$

LOCATION OF CENTROID C

$$Q_x = \sum A_i\bar{y}_i = (at)\left(\frac{a}{2}\right) + (b - t)(t)\left(\frac{t}{2}\right)$$

$$= 3.86230\ \text{in.}^3$$

$$\bar{y} = \frac{Q_x}{A} = \frac{3.86230\ \text{in.}^3}{5.23438\ \text{in.}^2} = 0.73787\ \text{in.}$$

$$Q_y = \sum A_i\bar{x}_i = (at)\left(\frac{t}{2}\right) + (b - t)(t)\left(\frac{b + t}{2}\right)$$

$$= 11.71387\ \text{in.}^3$$

$$\bar{x} = \frac{Q_y}{A} = \frac{11.71387\ \text{in.}^3}{5.23438\ \text{in.}^2} = 2.23787\ \text{in.}$$

MOMENTS OF INERTIA (xy AXES)

Use parallel-axis theorem.

$$I_x = \frac{1}{12}(t)(a^3) + A_1\left(\frac{a}{2}\right)^2 + \frac{1}{12}(b - t)(t^3) + A_2\left(\frac{t}{2}\right)^2$$

$$= \frac{1}{12}\left(\frac{5}{8}\right)(3.0)^3 + (1.875)(1.5)^2 + \frac{1}{12}(5.375)\left(\frac{5}{8}\right)^3$$

$$+ (3.35938)\left(\frac{5}{16}\right)^2$$

$$= 6.06242\ \text{in.}^4$$

$$I_y = \frac{1}{12}(a)(t^3) + A_1\left(\frac{t}{2}\right)^2 + \frac{1}{12}(t)(b - t^3)$$

$$+ A_2\left(\frac{b + t}{2}\right)^2$$

$$= \frac{1}{12}(3.0)\left(\frac{5}{8}\right)^3 + (1.875)\left(\frac{5}{16}\right)^2 + \frac{1}{12}\left(\frac{5}{8}\right)(5.375)^3$$

$$+ (3.35938)\left(\frac{6.625}{2}\right)^2$$

$$= 45.1933\ \text{in.}^4$$

MOMENTS OF INERTIA (x_cy_c AXES)

Use parallel-axis theorem.

$$I_{x_c} = I_x - A\bar{y}^2 = 6.06242 - (5.23438)(0.73787)^2$$

$$= 3.21255\ \text{in.}^4$$

$$I_{y_c} = I_y - A\bar{x}^2 = 45.1933 - (5.23438)(2.23787)^2$$

$$= 18.97923\ \text{in.}^4$$

PRODUCT OF INERTIA

Use parallel-axis theorem: $I_{xy} = I_{\text{centroid}} + A\,d_1 d_2$

Area A_1: $I'_{x_c y_c} = 0 + A_1\left[-\left(\bar{x} - \frac{t}{2}\right)\right]\left[\frac{a}{2} - \bar{y}\right]$

$= (1.875)(-1.92537)(0.76213)$

$= -2.75134 \text{ in.}^4$

Area A_2: $I''_{x_c y_c} = 0 + A_2\left[\frac{b+t}{2} - \bar{x}\right]\left[-\left(\bar{y} - \frac{t}{2}\right)\right]$

$= (3.35938)(1.07463)(-0.42537)$

$= -1.53562 \text{ in.}^4$

$I_{x_c y_c} = I'_{x_c y_c} + I''_{x_c y_c} = -4.28696 \text{ in.}^4$

SUMMARY

$I_{x_c} = 3.21255 \text{ in.}^4 \qquad I_{y_c} = 18.97923 \text{ in.}^4$

$I_{x_c y_c} = -4.28696 \text{ in.}^4$

PRINCIPAL AXES

EQ. (12-30): $\tan 2\theta_p = -\dfrac{2I_{xy}}{I_x - I_y} = -0.54380$

$2\theta_p = -28.5374°$ and $151.4626°$
$\theta_p = -14.2687°$ and $75.7313°$

SUBSTITUTE into Eq. (12-25) with $\theta = -14.2687°$

$I_{x_1} = 2.1223 \text{ in.}^4$

SUBSTITUTE into Eq. (12-25) with $\theta = 75.7313°$

$I_{x_1} = 20.0695 \text{ in.}^4$

THEREFORE,

$$\left.\begin{array}{ll} I_1 = 20.07 \text{ in.}^4 & \theta_{p_1} = 75.73° \\ I_2 = 2.12 \text{ in.}^4 & \theta_{p_2} = -14.27° \end{array}\right\} \longleftarrow$$

NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-33a and b) and Eq. (12-29).